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STATIONARY SETS AND INFINITARY LOGIC

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Abstract. Let K_λ^0 be the class of structures $\langle \lambda, <, A \rangle$, where $A \subseteq \lambda$ is disjoint from a club, and let K_λ^1 be the class of structures $\langle \lambda, <, A \rangle$, where $A \subseteq \lambda$ contains a club. We prove that if $\lambda = \lambda^{<\kappa}$ is regular, then no sentence of $L_{\lambda^+\kappa}$ separates K_λ^0 and K_λ^1. On the other hand, we prove that if $\lambda = \mu^+$, $\mu = \mu^{<\mu}$, and a forcing axiom holds (and $\aleph_1^L = \aleph_1$ if $\mu = \aleph_0$), then there is a sentence of $L_{\lambda\lambda}$ which separates K_λ^0 and K_λ^1.

One of the fundamental properties of $L_{\omega_1\omega}$ is that although every countable ordinal itself is definable in $L_{\omega_1\omega}$, the class of all countable well-ordered structures is not. In particular, the classes

$$K^0 = \{\langle \omega, R \rangle : R \text{ well-orders } \omega\}$$
$$K^1 = \{\langle \omega, R \rangle : \langle \omega, R \rangle \text{ contains a copy of the rationals}\}$$

cannot be separated by any $L_{\omega_1\omega}$-sentence. In this paper we consider infinite quantifier languages $L_{\kappa\lambda}$, $\lambda > \omega$. Here well-foundedness is readily definable, but we may instead consider the class

$$T_\lambda = \{\langle \lambda, R \rangle : \langle \lambda, R \rangle \text{ is a tree with no branches of length } \lambda\}.$$

If $\lambda = \lambda^{<\lambda}$, then a result of Hyttinen [1] implies that T_λ cannot be defined in $L_{\lambda^+\lambda}$.

The main topic of this paper is the question whether the classes

$$K_\lambda^0 = \{\langle \lambda, <, A \rangle : A \text{ is disjoint from a club of } \lambda\}$$
$$K_\lambda^1 = \{\langle \lambda, <, A \rangle : A \text{ contains a club of } \lambda\}$$

can be separated in $L_{\lambda^+\lambda}$ and related languages. Note that a set $A \subseteq \lambda$ contains a club if and only if the tree $T(A)$ of continuously ascending sequences of elements of A has a branch of length λ. We show (Theorem 1) that the classes K_λ^0 and K_λ^1 cannot be separated by a sentence of $L_{\lambda^+\kappa}$, if $\lambda = \lambda^{<\kappa}$ is regular. The proof of this result uses forcing in a way which seems to be new in the model theory of infinitary languages. It follows that the class

$$S_\lambda = \{\langle \lambda, <, A \rangle : A \text{ is stationary on } \lambda\},$$

that separates K_λ^0 and K_λ^1, is undefinable in $L_{\lambda^+\kappa}$, if $\lambda = \lambda^{<\kappa}$ is regular. We complement this result by showing (Theorem 10) that if $\lambda = \mu^+$, $\mu = \mu^{<\mu}$, and a

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forcing axiom holds (and $\aleph_1^L = \aleph_1$ if $\mu = \aleph_0$), then there is a sentence of $L_{\lambda\lambda}$ which defines S_λ and thereby separates K_λ^0 and K_λ^1.

Hyttinen [1] actually proves more than undefinability of T_λ in $L_{\lambda^+\lambda}$. He shows that T_λ is undefinable — assuming $\lambda = \lambda^{<\lambda}$ — in $\mathrm{PC}(L_{\lambda^+\lambda})$. We show (Theorems 5 and 6) that the related statement that S_{ω_1} is definable in $\mathrm{PC}(L_{\omega_2\omega_1})$ is independent of ZFC + CH.

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§1. The case $\lambda = \lambda^{<\mu}$.

THEOREM 1. *If $\lambda = \lambda^{<\kappa}$ is regular, then the classes K_λ^0 and K_λ^1 cannot be separated by a sentence of $L_{\lambda^+\kappa}$.*

PROOF. Assume $\lambda = \lambda^{<\kappa}$ is regular and $\psi \in L_{\lambda^+\kappa}$. Let $\mathscr{P}$ be the forcing notion for adding a Cohen subset to λ. Thus $p \in \mathscr{P}$ if p is a mapping $p : \alpha_p \to 2$ for some $\alpha_p < \lambda$. A condition p extends another condition q, in symbols $p \geq q$, if $\alpha_p \geq \alpha_q$ and $p \restriction \alpha_q = q$. Let G be $\mathscr{P}$-generic and $g = \bigcup G$. Thus

$$V[G] \models g^{-1}(1) \text{ is bi-stationary on } \lambda.$$

Now either ψ or $\neg\psi$ is true in $\langle \lambda, <, g^{-1}(1)\rangle$ in $V[G]$. We may assume, by symmetry, that it is ψ. Let $p \in G$ such that

$$p \Vdash_{\mathscr{P}} \langle \lambda, <, \tilde{g}^{-1}(1)\rangle \models \psi,$$

where $\tilde{g}$ is the canonical name for g. It is easy to use $\lambda = \lambda^{<\kappa}$ and regularity of λ to construct an elementary chain $\langle M_\xi : \xi < \lambda\rangle$ such that

(i) $M_\xi \prec \langle H(\beth_7(\lambda)), \in, <^*\rangle$, where $<^*$ is a well-ordering of $H(\beth_7(\lambda))$.
(ii) $\lambda + 1 \cup \{p\} \cup \{\mathscr{P}\} \cup \mathrm{TC}(\{\psi\}) \subseteq M_0$.
(iii) $\langle M_\eta : \eta < \xi\rangle \in M_{\xi+1}$.
(iv) $M_\nu = \bigcup_{\xi<\nu} M_\xi$ for limit ν.
(v) $(M_\xi)^{<\kappa} \subseteq M_{\xi+1}$.
(vi) $|M_\xi| = \lambda$.

Let $M = \bigcup_{\xi<\lambda} M_\xi$. Note, that $M^{<\kappa} \subseteq M$ because λ is regular. We shall construct two $\mathscr{P}$-generic sets, G^0 and G^1, over M. For this end, list open dense $D \subseteq \mathscr{P}$ with $D \in M$ as $\langle D_\xi : \xi < \lambda\rangle$. Define $G^l = \{p_\xi^l : \xi < \lambda\}$ so that $p_0^l = p$, $p_{\xi+1}^l \geq p_\xi^l$ with $p_{\xi+1}^l \in D_\xi \cap M$, $p_{\xi+1}^l(\alpha_{p_\xi^l}) = l$, and $p_\nu^l = \bigcup_{\xi<\nu} p_\xi^l$ for limit ν. Clearly, G^l is $\mathscr{P}$-generic over M and

$$M[G^l] \models [\langle \lambda, <, (g^l)^{-1}(1)\rangle \models \psi],$$

where $g^l = \bigcup G^l$. Note also that $M[G^l]^{<\kappa} \subseteq M[G^l]$, because $M^{<\kappa} \subseteq M$ and $\mathscr{P}$ is $< \kappa$-closed.

LEMMA 2. *If $\varphi(\vec{x}) \in L_{\lambda^+\kappa}$ such that $\mathrm{TC}(\{\varphi(\vec{x})\}) \subseteq M$, $X \in M$, and $\vec{a} \in \lambda^{<\kappa}$, then*

$$\langle \lambda, <, X\rangle \models \varphi(\vec{a}) \iff M[G^l] \models [\langle \lambda, <, X\rangle \models \varphi(\vec{a})].$$

PROOF. Easy induction on $\varphi(\vec{x})$. ⊣

By the lemma, $\langle \lambda, <, (g^l)^{-1}(1)\rangle \models \psi$. By construction, $\langle \lambda, <, (g^l)^{-1}(1)\rangle \in K_\lambda^l$. Now we can finish the proof. Suppose $K_\lambda^0 \subseteq \mathrm{Mod}(\psi)$ and $K_\lambda^1 \cap \mathrm{Mod}(\psi) = \emptyset$. This contradicts the fact that $\langle \lambda, <, (g^1)^{-1}(1) \rangle \in K^1_\lambda \cap \mathrm{Mod}(\psi)$. Suppose $K^1_\lambda \subseteq \mathrm{Mod}(\psi)$ and $K^0_\lambda \cap \mathrm{Mod}(\psi) = \emptyset$. This contradicts $\langle \lambda, <, (g^0)^{-1}(1) \rangle \in K^0_\lambda \cap \mathrm{Mod}(\psi)$. ⊣

COROLLARY 3. *If $\lambda = \lambda^{<\kappa}$ is regular, then there is no $\varphi \in L_{\lambda^+\kappa}$ such that for all $A \subseteq \lambda$: $\langle \lambda, <, A \rangle \models \varphi \iff A$ is stationary.*

Theorem 1 gives a new proof of the result, referred to above, that if $\lambda = \lambda^{<\lambda}$, then T_λ is not definable in $L_{\lambda^+\lambda}$. Our proof does not give the stronger result that T_λ is not definable in $\mathrm{PC}(L_{\lambda^+\lambda})$, and there is a good reason: S_{ω_1} may be $\mathrm{PC}(L_{\omega_2\omega_1})$-definable, even if $2^{\aleph_0} = \aleph_1$. This is the topic of the next section.

§2. An application of Canary trees.

A tree $\mathscr{C}$ is a *Canary tree* if $\mathscr{C}$ has cardinality $\leq 2^\omega$, $\mathscr{C}$ has no uncountable branches, but if a stationary subset of ω_1 is killed by forcing which does not add new reals, then this forcing adds an uncountable branch to $\mathscr{C}$. By [4], this is equivalent to the statement that

($\star$) For every co-stationary $A \subseteq \omega_1$ there is a mapping f with $\mathrm{Rng}(f) \subseteq \mathscr{C}$ such that for all increasing closed sequences s, s' of elements of A, if s is an initial segment of s', then $f(s) <_{\mathscr{C}} f(s')$.

THEOREM 4. (i) $\mathrm{Con}(\mathrm{ZF}) \to \mathrm{Con}(\mathrm{ZFC} + \mathrm{CH} +$ *there is a Canary tree*$)$ [3].

(ii) $V{=}L \to$ *there are no Canary trees* [6].

Thus the non-existence of Canary trees is consistent with CH, relative to the consistency of ZF. This result was first proved in [3] by the method of forcing.

THEOREM 5. *Assuming* CH *and the existence of a Canary tree, there is a* $\Phi \in \mathrm{PC}(L_{\omega_2\omega_1})$ *such that for all $A \subseteq \omega_1$: $\langle \omega_1, <, A \rangle \models \Phi \iff A$ is stationary.*

PROOF. Let $\mathscr{C}$ be a Canary tree. It is easy to construct a $\mathrm{PC}(L_{\omega_2\omega_1})$-sentence Ψ such that the following conditions are equivalent for all $A \subseteq \omega_1$:

(i) $\langle \omega_1, <, A \rangle \models \Psi$.

(ii) There is a mapping f with $\mathrm{Rng}(f) \subseteq \mathscr{C}$ such that for all increasing closed sequences s, s' of elements of A, if s is an initial segment of s', then $f(s) <_{\mathscr{C}} f(s')$.

We allow predicate symbols with ω-sequences of variables in the $\mathrm{PC}(L_{\omega_2\omega_1})$-sentence Ψ. Now the claim follows from the property ($\star$) of Canary trees. ⊣

THEOREM 6. $\mathrm{Con}(\mathrm{ZF})$ *implies* $\mathrm{Con}\big(\mathrm{ZFC} + \mathrm{CH} +$ *there is no $\Phi \in \mathrm{PC}(L_{\omega_2\omega_1})$ such that for all $A \subseteq \omega_1$: $\langle \omega_1, <, A \rangle \models \Phi \iff A$ is stationary*$\big)$.

PROOF. We start with a model of GCH and add $\aleph_2$ Cohen subsets to ω_1. In the extension GCH continues to hold. Suppose there is in the extension a $\Phi \in \mathrm{PC}(L_{\omega_2\omega_1})$ such that for all $A \subseteq \omega_1$:

$$\langle \omega_1, <, A \rangle \models \Phi \iff A \text{ is stationary.}$$

Since the forcing to add $\aleph_2$ Cohen subsets of ω_1 satisfies the $\aleph_2$-c.c., Φ belongs to the extension of the universe by $\aleph_1$ of the subsets. By first adding all but one of the subsets we can work in $V[A]$ where A is a Cohen subset of ω_1 and Φ is in V. Note that A is a bi-stationary subset of ω_1. Let $\mathscr{P}$ be in V the forcing for adding a Cohen generic subset of ω_1 and let $\tilde{A}$ be the $\mathscr{P}$-name for A. Let p force $\langle \omega_1, <, \tilde{A} \rangle \models \Phi$. By arguing as in the proof of Theorem 1, we can construct in V a model M of cardinality $\aleph_1$ containing $\mathscr{P}$ such that $M^\omega \subseteq M$,

$$M \models [p \Vdash \langle \omega_1, <, \tilde{A} \rangle \models \Phi],$$

and, furthermore, we can extend p to a $\mathscr{P}$-generic set $H \subseteq \omega_1$ over M such that H is non-stationary. Thus $M[H]$ satisfies

$$\langle \omega_1, <, H \rangle \models \Phi. \tag{1}$$

Now (1) is true in V, because $M[H]^\omega \subseteq M[H]$. Since $\mathscr{P}$ is countably closed, we have (1) in $V[A]$, whence H is stationary in $V[A]$, contrary to the fact that H is non-stationary in V. ⊣

§3. An application to the topological space $^{\omega_1}\omega_1$. Let $\mathscr{N}_1$ denote the generalized Baire space consisting of all functions $f : \omega_1 \to \omega_1$, with the sets

$$N_s = \{f \in \mathscr{N}_1 : f \restriction \mathrm{Dom}(s) = s\},$$

where $s \in {}^{<\omega_1}\omega_1$, as basic open sets. We call open sets Σ^0_1 and closed sets Π^0_1. A set of the form $\bigcup_{\xi<\omega_1} A_\xi$, where each A_ξ is in $\bigcup_{\beta<\alpha} \Pi^0_\beta$, is called Σ^0_α. Respectively, a set of the form $\bigcap_{\xi<\omega_1} A_\xi$, where each A_ξ is in $\bigcup_{\beta<\alpha} \Sigma^0_\beta$, is called Π^0_α. In $\mathscr{N}_1$ it is natural to define Borel sets as follows: A subset of $\mathscr{N}_1$ is *Borel* if it is Σ^0_α or Π^0_α for some $\alpha < \omega_2$. A set $A \subseteq \mathscr{N}_1$ is Π^1_1 if there is an open set $B \subseteq \mathscr{N}_1 \times \mathscr{N}_1$ such that $\forall f \big(f \in A \iff \forall g ((f,g) \in B)\big)$. A set is Σ^1_1 if its complement is Π^1_1.

Let CUB be the set of characteristic functions of closed unbounded subsets of ω_1, and NON-STAT the set of characteristic functions of non-stationary subsets of ω_1. Clearly, CUB and NON-STAT are disjoint Σ^1_1. It was proved in [4] that, assuming CH, CUB and NON-STAT are Π^1_1 if and only if there is a Canary tree. Another result of [4] says that the sets CUB and NON-STAT cannot be separated by any Π^0_3 or Σ^0_3 set.

THEOREM 7. *Assuming* CH, *the sets* CUB *and* NON-STAT *cannot be separated by a Borel set.*

PROOF. Let $\{s_\alpha : \alpha < \omega_1\}$ enumerate all $s \in {}^{<\omega_1}\omega_1$. Let $C = \bigcup_{\alpha<\omega_2} C_\alpha$, where

$$C_0 = \{0,1\} \times \mathscr{N}_1$$
$$C_\delta = \{2,3\} \times {}^{\omega_1}\Big(\bigcup_{\alpha<\delta} C_\alpha\Big).$$

Now we define a Borel set B_c for each $c \in C$ as follows:

$$B_{\langle 0,f\rangle} = \bigcup_{\alpha<\omega_1} N_{s_{f(\alpha)}}, \; B_{\langle 1,f\rangle} = \bigcap_{\alpha<\omega_1} \mathscr{N}_1 \setminus N_{s_{f(\alpha)}},$$
$$B_{\langle 2,f\rangle} = \bigcup_{\alpha<\omega_1} B_{f(\alpha)}, \; B_{\langle 3,f\rangle} = \bigcap_{\alpha<\omega_1} B_{f(\alpha)}.$$

Clearly, every Borel subset X of $\mathscr{N}_1$ is of the form B_c for some $c \in C$. Then we call c a *Borel code* of X. Below we consider $f \in B_c$ as a property of f and c.

Assume A is a Borel set which separates CUB and NON-STAT. Let c be a Borel code of A. Let $\mathscr{P}$ be the forcing notion for adding a Cohen subset to ω_1. Let G be $\mathscr{P}$-generic and $g = \bigcup G$. Thus

$$V[G] \models g^{-1}(1) \text{ is bi-stationary.}$$

Now either $g^{-1}(1) \in B_c$ or $g^{-1}(1) \in B_c$ in $V[G]$. We may assume, by symmetry, that $g^{-1}(1) \in B_c$. Let $p \in G$ such that

$$p \Vdash_{\mathscr{P}} \tilde{g}^{-1}(\check{1}) \in B_c,$$

where $\tilde{g}$ is the canonical name for g. Let $M \prec \langle H(\beth_7(\omega_1)), \in, <^* \rangle$, where $<^*$ is a well-ordering of $H(\beth_7(\lambda))$, such that $\omega_1 + 1 \cup \{p\} \cup \{\mathscr{P}\} \cup \mathrm{TC}(\{c\}) \subseteq M$, $M^\omega \subseteq M$ and $|M| = \omega_1$.

We shall construct two $\mathscr{P}$-generic sets, G^0 and G^1, over M. For this end, list open dense $D \subseteq \mathscr{P}$ with $D \in M$ as $\langle D_\xi : \xi < \omega_1 \rangle$. Define $G^l = \{p^l_\xi : \xi < \omega_1\}$ so that $p^l_0 = p$, $p^l_{\xi+1} \geq p^l_\xi$ with $p^l_{\xi+1} \in D_\xi \cap M$, $p^l_{\xi+1}(\alpha_{p^l_\xi}) = l$, and $p^l_\nu = \bigcup_{\xi<\nu} p^l_\xi$ for limit ν. Clearly, G^l is $\mathscr{P}$-generic over M and

$$M[G^l] \models (g^l)^{-1}(1) \in B_c,$$

where $g^l = \bigcup G^l$. Note also that $M[G^l]^\omega \subseteq M[G^l]$.

Lemma 8. *If $c \in C$ such that $\mathrm{TC}(\{c\}) \subseteq M$, and $f \in M$, then*

$$f \in B_c \iff M[G^l] \models [f \in B_c].$$

Proof. Easy induction on c. ⊣

By the lemma, $(g^l)^{-1}(1) \in B_c$. By construction, $(g^0)^{-1}(1) \in$ NON-STAT and $(g^1)^{-1}(1) \in$ CUB. Now we can finish the proof. Suppose CUB $\subseteq A$ and NON-STAT $\cap A = \emptyset$. This contradicts the fact that $(g^0)^{-1}(1) \in$ NON-STAT $\cap A$. Suppose NON-STAT $\subseteq A$ and CUB $\cap A = \emptyset$. This contradicts the fact that $(g^1)^{-1}(1) \in$ CUB $\cap A$. ⊣

§4. The case $\lambda^\mu > \lambda$.

Let μ be a cardinal. Sets $A, B \subseteq \mu$ are called *almost disjoint (on μ)* if $\sup(A \cap B) < \mu$. An *almost disjoint λ-sequence* of subsets of μ is a sequence $\mathscr{B} = \langle B_\alpha : \alpha < \lambda \rangle$ such that for all $\alpha \neq \beta$, $|B_\alpha| = \mu$ and the sets B_α and B_β are almost disjoint. The sequence $\mathscr{B}$ is said to be *definable on L_λ* if there is a sequence $\langle \delta_\alpha : \alpha < \lambda \rangle$ such that $\limsup_{\alpha<\lambda} \delta_\alpha = \lambda$ and the predicate $x \in B_y \wedge y < \delta_\alpha$ is definable on every structure $\langle L_\alpha, \in \rangle$, where $\alpha < \lambda$, that is, there is a first order formula $\varphi_0(x, y)$ of the language of set theory such that for $x, y < \alpha < \lambda$:

$$x \in B_y \wedge y < \delta_\alpha \iff \langle L_\alpha, \in \rangle \models \varphi_0(x, y).$$

Lemma 9. *If $\aleph_1^L = \aleph_1$, then there is an almost disjoint ω_1-sequence of subsets of ω, which is definable on L_{ω_1}.*

Proof. Let $\langle X_i : i < \omega_1 \rangle$ be the set of constructible subsets of ω enumerated in the order of the canonical well-order $<_L$ of L. Let $\langle s_n : n < \omega \rangle$ be a canonical enumeration of all functions s with $\mathrm{Dom}(s) \in \omega$ and $\mathrm{Rng}(s) \subseteq \{0, 1\}$. Let

$$B_i = \{n \in \omega : \forall m \in \mathrm{Dom}(s_n)(m \in X_i \iff s_n(m) = 1)\}.$$

Let δ_α be the largest limit ordinal $\leq \alpha$, when $\alpha < \omega_1$. Now $\langle B_i : i < \omega_1 \rangle$ is an almost disjoint ω_1-sequence which is definable on L_{ω_1}. ⊣

Theorem 10. *Suppose*

(i) $\lambda = \mu^+$.

(ii) *There is an almost disjoint λ-sequence $\mathscr{B} = \langle B_\alpha : \alpha < \lambda \rangle$ of subsets of μ which is definable on L_λ.*

(iii) *For all club subsets C of λ there is a subset X of μ such that for all $\alpha < \lambda$ we have*

$$\alpha \in C \iff \sup(B_\alpha \setminus X) < \mu.$$

Then there is a sentence $\varphi \in L_{\lambda\lambda}$ so that for all $A \subseteq \lambda$:

$$\langle \lambda, <, A \rangle \models \varphi \iff A \text{ is stationary}.$$

PROOF. Suppose φ_0 defines the almost disjoint sequence, as above. We define a sequence of formulas of $L_{\lambda\lambda}$. The variable vectors $\vec{x}$ in these formulas are always sequences of the form $\langle x_i : i < \mu \rangle$. Let Φ be the conjunction of a large but finite number of axioms of ZFC + $V = L$. If $\psi(\vec{z})$ is a formula of set theory, let $\psi'(\vec{z}, \vec{x}, \vec{u}, \vec{v})$ be the result of replacing every quantifier $\forall y \ldots$ in Φ by $\forall y\big((\bigvee_{i<\mu} y = x_i) \to \ldots\big)$, every quantifier $\exists y \ldots$ in Φ by $\exists y\big((\bigvee_{i<\mu} y = x_i) \wedge \ldots\big)$, and $y \in z$ everywhere in Φ by $\bigvee_{i<\mu}(y = u_i \wedge z = v_i)$. The following formulas pick μ from $\langle \lambda, < \rangle$:

$$\begin{aligned}
\varphi_{\approx\mu}(y) &\iff \exists \vec{x}\big((\textstyle\bigwedge_{i<j<\mu} x_i < x_j) \wedge \forall z(z < y \leftrightarrow \bigvee_{i<\mu} z = x_i)\big),\\
\varphi_{\in\mu}(y) &\iff \forall u\big(\varphi_{\approx\mu}(u) \to y < u\big),\\
\psi_{\in\mu}(\vec{y}) &\iff \textstyle\bigwedge_{i<\mu} \varphi_{\in\mu}(y_i).
\end{aligned}$$

The following formulas are needed to refer to well-founded models of set theory:

$$\begin{aligned}
\varphi_{\mathrm{uni}}(\vec{x}, z) &\iff \textstyle\bigvee_{i<\mu} z = x_i\\
\varphi_{\mathrm{eps}}(\vec{x}, \vec{u}, \vec{v}, z, y) &\iff \varphi_{\mathrm{uni}}(\vec{x}, z) \wedge \varphi_{\mathrm{uni}}(\vec{x}, y) \wedge \textstyle\bigvee_{i<\mu}(z = u_i \wedge y = v_i)\\
\varphi_{\mathrm{wf}}(\vec{x}, \vec{u}, \vec{v}) &\iff \Phi'(\vec{x}, \vec{u}, \vec{v}) \wedge \forall \vec{y}\Big(\big(\textstyle\bigwedge_{i<\mu} \varphi_{\mathrm{uni}}(\vec{x}, y_i)\big)\\
&\qquad \to \textstyle\bigvee_{i<\mu} \neg\varphi_{\mathrm{eps}}(\vec{x}, \vec{u}, \vec{v}, y_{i+1}, y_i)\Big)\\
\varphi_{\mathrm{cor}}(\vec{x}, \vec{u}, \vec{v}, z) &\iff \forall s\big(s < z \leftrightarrow \textstyle\bigvee_{i<\mu}(s = u_i \wedge z = v_i)\big).
\end{aligned}$$

Let

$$\begin{aligned}
\varphi_B(z, y) \iff\ & \exists \vec{x} \exists \vec{u} \exists \vec{v}\big(\varphi_{\mathrm{wf}}(\vec{x}, \vec{u}, \vec{v}) \wedge \varphi_{\mathrm{cor}}(\vec{x}, \vec{u}, \vec{v}, z)\\
& \wedge \varphi_{\mathrm{cor}}(\vec{x}, \vec{u}, \vec{v}, y) \wedge \phi_0'(z, y, \vec{x}, \vec{u}, \vec{v})\big).
\end{aligned}$$

The point is that if $\alpha \in \mu$ and $\beta \in \lambda$, then $\alpha \in B_\beta$ if and only if $\langle \lambda, < \rangle \models \varphi_B(\alpha, \beta)$. The following formula says that the element y of μ is in the subset of λ coded by $\vec{x}$:

$$\varphi_\varepsilon(y, \vec{x}) \iff \exists u\Big(\varphi_{\in\mu}(u) \wedge \forall z\Big(\big(\varphi_B(z, y) \wedge \textstyle\bigwedge_{i<\mu} z \neq x_i\big) \to z < u\Big)\Big),$$

Finally, if:

$$\begin{aligned}
\varphi_{\mathrm{ub}}(\vec{x}) &\iff \forall y \exists z\big(y < z \wedge \varphi_\varepsilon(z, \vec{x})\big),\\
\varphi_{\mathrm{cl}}(\vec{x}) &\iff \forall y\Big(\forall z\Big(z < y \to \exists u\big(z < u \wedge u < y \wedge \varphi_\varepsilon(u, \vec{x})\big)\Big)\\
&\qquad \to \varphi_\varepsilon(y, \vec{x})\Big)\\
\varphi_{\mathrm{cub}}(\vec{x}) &\iff \varphi_{\mathrm{ub}}(\vec{x}) \wedge \varphi_{\mathrm{cl}}(\vec{x})\\
\varphi_{\mathrm{stat}} &\iff \forall \vec{x}\Big(\big(\psi_{\in\mu}(\vec{x}) \wedge \varphi_{\mathrm{cub}}(\vec{x})\big) \to \exists y\big(A(y) \wedge \varphi_\varepsilon(y, \vec{x})\big)\Big),
\end{aligned}$$

then $\langle \lambda, <, A \rangle \models \varphi_{\mathrm{stat}}$ if and only if A is stationary. ⊣

COROLLARY 11. *If* $2^{\aleph_0} > \aleph_1$, $\aleph_1^L = \aleph_1$ *and* MA, *then there is a* $\varphi \in L_{\omega_1\omega_1}$ *such that for all* $A \subseteq \omega_1$:

$$\langle \omega_1, <, A\rangle \models \varphi \iff A \text{ is stationary.}$$

PROOF. We choose $\lambda = \omega_1$ and $\mu = \omega$ in Theorem 10. Condition (ii) holds by Lemma 9. Condition (iii) is a consequence of MA + $\neg$CH by [2]. ⊣

NOTE. The proof of Theorem 10 shows that we actually get the following stronger result: If $2^{\aleph_0} > \aleph_1$, $\aleph_1^L = \aleph_1$ and MA, then the full second order extension $L^{II}_{\omega_1\omega_1}$ of $L_{\omega_1\omega_1}$ is reducible to $L_{\omega_1\omega_1}$ in expansions of $\langle\omega_1, <\rangle$. Then, in particular, $T_{\aleph_1}$ is PC($L_{\omega_1\omega_1}$)-definable. This kind of reduction cannot hold on all models. For example, ω_1-like dense linear orders with a first element are all $L_{\infty\omega_1}$-equivalent, but not $L^{II}_{\omega\omega}$-equivalent.

For $\alpha < \lambda = \mu^+$, let $\langle a_i^\alpha : i < \mu\rangle$ be a continuously increasing sequence of subsets of α with $\alpha = \bigcup_{i<\mu} a_i^\alpha$ and $|a_i^\alpha| < \mu$. Define $f_\alpha : \mu \to \mu$ by

$$f_\alpha(i) = \mathrm{otp}(a_i^\alpha) \ (= \text{the order-type of } a_i^\alpha).$$

Let D_μ be the club-filter on μ. Define for $f, g \in {}^\mu\mu$;

$$f \sim_{D_\mu} g \iff \{i : f(i) = g(i)\} \in D_\mu.$$

LEMMA 12. f_α/D_μ *is independent of the choice of the sequence* $\langle a_i^\alpha : i < \mu\rangle$.

THEOREM 13. *Suppose*

(i) $\lambda = \mu^+$, *where* $\mu = \mu^{<\mu} > \aleph_0$.
(ii) *For every club* $C \subseteq \lambda$ *there is some* $X \subseteq \mu \times \mu$ *such that*

$$\alpha \in C \to \{i < \mu : \langle i, f_\alpha(i)\rangle \in X\} \text{ contains a club}$$
$$\alpha \notin C \to \{i < \mu : \langle i, f_\alpha(i)\rangle \notin X\} \text{ contains a club.}$$

Then there is a sentence $\varphi \in L_{\lambda\lambda}$ *such that for all* $A \subseteq \lambda$:

$$\langle \lambda, <, A\rangle \models \varphi \iff A \text{ is stationary.}$$

PROOF. This is like the proof of Theorem 10. One uses Lemma 12 to refer to the functions f_α. We leave the details to the reader. ⊣

The *Generalized Martin's Axiom for* μ (GMA$_\mu$) from [5] is the following principle:

Generalized Martin's Axiom GMA$_\mu$: Suppose $\mathscr{P}$ is a forcing notion with the properties:

(GMA1) Every descending sequence of length $< \mu$ in $\mathscr{P}$ has a greatest lower bound.

(GMA2) If $p_\alpha \in \mathscr{P}$ for $\alpha < \mu^+$, then there is a club $C \subseteq \mu^+$ and a regressive function $f : \mu^+ \to \mu^+$ such that if $\alpha \in C$ and $\mathrm{cf}(\alpha) = \mu$, then the set

$$A = \{p_\beta : \mathrm{cf}(\beta) = \mu, f(\alpha) = f(\beta)\}$$

is well-met (i.e., $p, q \in A \to p \vee q \in a$).

Then for any dense open sets $D_\alpha \subseteq \mathscr{P}$, $\alpha < \kappa$, where $\kappa < 2^\mu$, there is a filter in $\mathscr{P}$ which meets every D_α.

PROPOSITION 14. *Suppose $\lambda = \mu^+$, where $\mu = \mu^{<\mu} > \aleph_0$, and* GMA_μ. *Then for every club $C \subseteq \lambda$ there is some $X \subseteq \mu \times \mu$ such that*

$$\alpha \in C \to \{i < \mu : \langle i, f_\alpha(i)\rangle \in X\} \text{ contains a club}$$
$$\alpha \notin C \to \{i < \mu : \langle i, f_\alpha(i)\rangle \notin X\} \text{ contains a club.}$$

PROOF. Let a club $C \subseteq \lambda$ be given. For $\alpha < \beta < \lambda$, let $C_{\alpha\beta} \in D_\mu$ so that $f_\alpha \restriction C_{\alpha\beta} < f_\beta \restriction C_{\alpha\beta}$. (To get such a set $C_{\alpha\beta}$ one first constructs a club of ordinals i with $a_i^\alpha = a_i^\beta \cap \alpha$.) Let $\mathscr{P}$ consist of conditions

$$p = \langle B^p, f^p, \boldsymbol{c}^p, g^p, \delta^p\rangle,$$

where

(i) $B^p \subseteq \lambda$, $|B^p| < \mu$, $\delta^p < \mu$.
(ii) f^p is a partial mapping with $\mathrm{Dom}(f^p) \subseteq \mu \times \mu$, $|\mathrm{Dom}(f^p)| < \mu$, and $\mathrm{Rng}(f^p) \subseteq \{0, 1\}$.
(iii) If $\alpha \in B^p$, then $\{i < \mu : \langle i, f_\alpha(i)\rangle \in \mathrm{Dom}(f^p)\}$ is an ordinal $j_\alpha^p > \delta^p$.
(iv) $\boldsymbol{c}^p = \langle c_\alpha^p : \alpha \in B^p\rangle$, where c_α^p is a closed subset of j_α^p with $\max(c_\alpha^p) = \delta^p$.
(v) If $\alpha \in B^p \cap C$ and $i \in c_\alpha^p$, then $f^p(i, f_\alpha(i)) = 1$. If $\alpha \in B^p \setminus C$ and $i \in c_\alpha^p$, then $f^p(i, f_\alpha(i)) = 0$.
(vi) $g^p : [B^p]^2 \to \mu$.
(vii) If $\alpha, \beta \in B^p$ with $\alpha < \beta$, then $\emptyset \neq c_\alpha^p \setminus g^p(\{\alpha, \beta\}) \subseteq C_{\alpha\beta}$.

The partial ordering "q extends p" is defined as follows:

$$p \leq q \Leftrightarrow B^p \subseteq B^q, f^p \subseteq f^q, g^p \subseteq g^q, \delta^p \leq \delta^q,$$
$$\forall \alpha \in B^p (c_\alpha^p \text{ is an initial segment of } c_\alpha^q).$$

We show now that $\mathscr{P}$ satisfies conditions (GMA1) and (GMA2).

LEMMA 15. *$\mathscr{P}$ satisfies* (GMA1).

PROOF. Let $p_0 \leq \cdots \leq p_i \leq \cdots (i < \gamma)$ in $\mathscr{P}$ with $\gamma < \mu$. We may assume $\delta^{p_0} < \delta^{p_1} < \cdots$. Let $\delta = \sup\{\delta^{p_i} : i < \gamma\}$. Let $B = \bigcup_{i<\gamma} B^{p_i}$. We extend $\bigcup_i f^{p_i}$ to f by defining

$$f(\delta, f_\alpha(\delta)) = \begin{cases} 1 & \text{if } \alpha \in B \cap C \\ 0 & \text{if } \alpha \in B \setminus C. \end{cases}$$

We have to check that this definition is coherent, i.e., if $\alpha \in B \cap C$ and $\beta \in B \setminus C$, then $f_\alpha(\delta) \neq f_\beta(\delta)$. Suppose $\alpha \in B^{p_i}$ and $\beta \in B^{p_{i'}}$ with $\alpha < \beta$ and $i < i'$. Now $\emptyset \neq c_\alpha^{p_i} \setminus g^{p_{i'}}(\{\alpha, \beta\}) \subseteq C_{\alpha\beta}$. Hence $\delta \in C_{\alpha\beta}$, whence $f_\alpha(\delta) < f_\beta(\delta)$. Let $\boldsymbol{c} = \langle c_\alpha : \alpha \in B\rangle$, where $c_\alpha = \bigcup_i c_\alpha^{p_i} \cup \{\delta\}$, and $g = \bigcup_i g_i$. Now the condition $p = (B, f, \boldsymbol{c}, g, \delta)$ is the needed l.u.b. of $(p_i)_{i<\mu}$. ⊣

LEMMA 16. *$\mathscr{P}$ satisfies* (GMA2).

PROOF. For a start we define the restriction of $p \restriction \alpha$ of a condition $p \in \mathscr{P}$ in the following natural way:

$$p \restriction \alpha = \langle B^p \cap \alpha, f^p, \langle c_\gamma^p : \gamma \in B^p \cap \alpha\rangle, g^p \restriction [B^p \cap \alpha]^2, \delta^p\rangle.$$

Obviously, $p \restriction \alpha \leq p$. Suppose p_α, $\alpha < \lambda$, are in $\mathscr{P}$. Let h be a one-one mapping from $\mathscr{P}$ to odd ordinals $< \lambda$. By $\mu^{<\mu} = \mu$ there is a club $C \subseteq \lambda$ such that if $\alpha \in C$, $\mathrm{cf}(\alpha) = \mu$, and $B^p \subseteq \alpha$, then $h(p) < \alpha$, and if $\alpha < \beta$, $\alpha, \beta \in C$, then $B^{p_\alpha} \subseteq \beta$. Choose a regressive function g from the complement of C to the even ordinals that is one-one on ordinals of cofinality μ. Suppose $\mathrm{cf}(\alpha) = \mu$. Let $f(\alpha) = g(\alpha)$ if $\alpha \notin C$, and $f(\alpha) = h(p_\alpha \restriction \alpha)$ if $\alpha \in C$. Suppose now $\alpha < \beta$, $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta) = \mu$, and $f(\alpha) = f(\beta)$. W.l.o.g. $\alpha, \beta \in C$. Thus $h(p_\alpha \restriction \alpha) = h(p_\beta \restriction \beta)$, whence $p_\alpha \restriction \alpha = p_\beta \restriction \beta$. It follows that p_α and p_β have a l.u.b. ⊣

Let

$$D_{\alpha\beta} = \{p \in \mathscr{P} : \alpha \in B^p \text{ and } \delta^p \geq \beta\}$$

where $\alpha < \lambda$, $\beta < \mu$. We show that $D_{\alpha\beta}$ is dense open. Suppose therefore $p \in \mathscr{P}$ is given. We construct a condition $q \in D_{\alpha\beta}$ with $p \leq q$. We may assume $\alpha \notin B^p$. Let $B^q = B^p \cup \{\alpha\}$. Let

$$E = \bigcap\{C_{\xi\eta} : \xi, \eta \in B^q, \xi < \eta\} (\in D_\mu).$$

Let $\delta^q \in E \setminus \beta$ such that $\delta^q \geq \sup\{j_\xi^p : \xi \in B^p\}$. Define $c^q = \langle c_\xi^q : \xi \in B^q \rangle$ by

$$c_\xi^q = \begin{cases} c_\xi^p \cup \{\delta^q\}, & \text{if } \xi \neq \alpha \\ \{\delta^q\}, & \text{if } \xi = \alpha. \end{cases}$$

Let f^q extend f^p so that if $\xi \in B^q$ and $j \leq \delta^q$ but $\langle j, f_\xi(j) \rangle \notin \mathrm{Dom}(f^p)$, then

$$f^q(j, f_\xi(j)) = \begin{cases} 1 & \text{if } \alpha \in C \\ 0 & \text{if } \alpha \notin C. \end{cases}$$

Let $q = (B^q, f^q, g^q, \delta^q)$, where

$$g^q = g^p \cup \left\{\langle \{\xi, \alpha\}, \delta^p \rangle : \xi \in B^p \right\}.$$

Then $q \in D_{\alpha\beta}$, and $p \leq q$.

Let G be a filter that meets every $D_{\alpha\beta}$. Let

$$B = \bigcup\{B^p : p \in G\}$$
$$f = \bigcup\{f^p : p \in G\}$$
$$c_\alpha = \bigcup\{c_\alpha^p : p \in G\}.$$

Then $B = \lambda$ and each c_α is a club of μ. Let $X = \{\langle \alpha, \beta \rangle \in \mu \times \mu : f(\alpha, \beta) = 1\}$. Suppose $\alpha \in C$ and $i \in c_\alpha$. Then $f\left(i, f_\alpha(i)\right) = 1$ whence $\langle i, f_\alpha(i) \rangle \in X$. Suppose $\alpha \notin C$ and $i \in c_\alpha$. Then $f\left(i, f_\alpha(i)\right) = 0$ whence $\langle i, f_\alpha(i) \rangle \notin X$. ⊣

Corollary 17. *Suppose $\lambda = \mu^+$, where $\mu = \mu^{<\mu} > \aleph_0$, and* GMA_μ. *Then there is a sentence $\varphi \in L_{\lambda\lambda}$ such that for all $A \subseteq \lambda$:*

$$\langle \lambda, <, A \rangle \models \varphi \iff A \text{ is stationary.}$$

Proof. The claim follows from Theorem 13 and Proposition 14. ⊣

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