The Existence of Coding Sets.

Lately Zwicker (see [Z]), generalizing theorems on regular κ (and the filter of $\mathcal{D}_\kappa$ generated by the closed unbounded subsets) to $\mathcal{P}_{<\kappa}(\lambda)$ (and $\mathcal{D}_\kappa(\lambda)$) find that many times we can generalize if we restrict ourselves to a coding subset of $\mathcal{P}_{<\kappa}(\lambda)$. He shows existence for κ super-compact, (see Solovay [So] and Menas [M]) and Stanley and Vellman (independently) show existence for $\lambda = \kappa^+$ assuming a suitable morass exists.

During the meeting in Colorado, I heard about this (and the two other variants) and prove some existence theorems mainly that: for $\kappa > \aleph_2$ regular there is a coding set for $\mathcal{P}_{<\kappa}(\lambda)$. We present here somewhat improved versions written in Aug 83 announed in [Sh 2]. Here is a summary.

More information will appear in "More on Stationary Coding".

Definition : (1) We call S a weak (κ,λ)-stationary coding set (or weak (κ,λ)-SC or (κ,λ)-WSC) if S is a stationary subset of $\mathcal{P}_{<\kappa}(\lambda) = \{a \subset \lambda : |A| < \kappa\}$, and for no $a \neq b$ in S, $a \cap \kappa = b \cap \kappa$, $a \subseteq b$.

(2) We call S a (κ,λ)-stationary coding set (or (κ,λ)-SC) if S is a stationary subset of $\mathcal{P}_{<\kappa}(\lambda)$, and for some one-to-one function $h : S \to \lambda$, for every $a,b \in S$, $[a \neq b \wedge a \subset b \Longrightarrow h(a) \in b]$.

(3) We call S a strong (κ,λ)-stationary coding set (or strong (κ,λ)-SC or (κ,λ)-CD) if for $h(a) = \sup(a)$, (2) holds.

The simplest cases of our results are:

Theorem A: If $\kappa \geq \aleph_{n+1}$, then there is a (κ,κ^{+n})-WSC (on e.g. $\kappa^{+(\omega+1)}$ we could have weaker results.) (see (19).)

Theorem B: 1) If κ is an ineffable cardinal (or just $\diamondsuit_{\{\alpha\mu<\kappa:\mu \text{ inaccessible}\}}$), $\kappa < \lambda = \lambda^{<\kappa}$, and $\lambda \overset{w}{\not\to} (\omega)^{<\omega}_{\kappa}$ (Silver's relation) (see 11). then there is a (κ,λ)-SC

2) If $\lambda > \kappa > \aleph_1$, $\diamondsuit_{\{\delta<\lambda:\, cf\ \delta<\kappa\}}$ *then* there is a (κ,λ)-CD (see 7).

3) If $\lambda > \aleph_1$, $\diamondsuit_S$, $S \subset \lambda$, $(\forall \delta \in S)\ cf\ \delta = \aleph_0$, S does not reflect in any α of cofinality $\aleph_1$ then there is an $(\aleph_1,\lambda)$-CD (see 7).

Theorem C: If $\vartheta^{\mu} = \vartheta$, $\lambda < \vartheta^{+(\mu^+)}$, $\kappa = \vartheta^+$ and in $\kappa^{\kappa}/\mathcal{D}_{\kappa}$ there is an increasing sequence of length $\kappa^+ + 1$, then there is a (κ,λ)-WSC (see (13).

For Theorem 1 we use:

Theorem D: If D is a normal fine filter on $\mathcal{P}_{<\kappa}(\lambda)$, $\kappa = \mu^+,\lambda$ regular and $\{a \in \mathcal{P}_{<\kappa}(\lambda) : cf\ (\sup a) \neq cf\ \mu\} \in D$ *then* D is not λ^+-saturated.

By Lemma 20, and later result of Foreman, Magidor and Shelah [FMS], it is consistent then that there is no $(\aleph_1,\aleph_2)$-WSC.

1 Notation: κ will be a regular uncountable cardinal, λ a cardinal $\geq \kappa$, μ,χ infinite cardinals, $\mathcal{D}$ a fine normal filter on some $\mathcal{P}_{<\kappa}(A)$.

2 Definition : 1) $\mathcal{P}_{<\mu}(A) = \{a : a$ a subset of A of power $< \mu\}$.

2) A filter $\mathcal{D}$ on $\mathcal{P}_{<\kappa}(A)$ is *fine* if for $x \in A$ $\{a : a \in \mathcal{P}_{<\kappa}(A),\ x \in A\}$ belong to $\mathcal{D}$ We say $\mathcal{D}$ is finer if for $b \subset \mathcal{P}_{<\kappa}(A)$, $\{a : \alpha \subseteq a,\ a \in \mathcal{P}_{<\kappa}(\lambda)\}$ belongs to $\mathcal{D}$.

3) A filter $\mathcal{D}$ on $\mathcal{P}_{<\kappa}(A)$ is normal if for any $C_x \in \mathcal{D}$ (for $x \in A$) the set

$$\{a : a \in \mathcal{P}_{<\kappa}(A),\ a \in \bigcap_{x\in a} C_x\}$$

belongs to $\mathcal{D}$

4) $\mathcal{D}_{<\kappa}(A)$ is the minimal finer normal filter on $\mathcal{P}_{<\kappa}(A)$ (if $|A| \leq \kappa$ it is trivial) (normal include κ-complete).

3 Fact: 1) The set of ordinals $< \kappa$ belong to $\mathcal{D}_{\kappa}(\kappa)$ so $\mathcal{D}_{\kappa}(\kappa)$ can be identified with the filter of $\mathcal{D}_{\kappa}$ of closed unbounded subsets of κ.

2) If $\mathcal{D}$ is a (fine normal) filter on $\mathcal{P}_{<\kappa}(A)$, F a function from $\mathcal{P}_{<\aleph_0}(A)$

to $\mathcal{D}$ then the set $\{a : a \in \mathcal{P}_{<\kappa}(A)$ and for every $w \in \mathcal{P}_{<\kappa}(a)$, $a \supseteq F(w)\}$ belongs to $\mathcal{D}$.

3) $\mathcal{D}_\kappa(A)$ is the filter of closed unbounded subsets of $\mathcal{P}_{<\kappa}(A)$, hence $\phi \notin \mathcal{D}_\kappa(A)$.

4 Definition : 1) $S \subseteq \mathcal{P}_{<\kappa}(A)$ is called $\mathcal{D}$- stationary if $\mathcal{P}_{<\kappa}(A)-S \notin \mathcal{D}$. If $\mathcal{D} = \mathcal{D}_\kappa(A)$ (and the identity of A,κ is clear) we omit $\mathcal{D}$.

2) S_1,S_2 are $\mathcal{D}$-almost disjoint if $(\mathcal{P}_{<\kappa}(A)-S_1 \cap S_2) \in \mathcal{D}$.

3) We say $\mathcal{D}$ is μ-saturated if there are no μ $\mathcal{D}$-stationary pairwise $\mathcal{D}$-almost disjoint subsets of $\cup\, \mathcal{D}$.

5 Main Definition : Let $\mathcal{D}$ be a filter on $\mathcal{P}_{<\kappa}(\lambda)$ and $S \subseteq \mathcal{P}_{<\kappa}(\lambda)$ is $\mathcal{D}$-stationary.

1) S is a weak stationary coding set (WSC) if $a,b \in S$, $a \neq b$, $a \cap \kappa = b \cap \kappa$ implies $a \not\subset b$.

2) S is a stationary coding set (SC) if there is a one-to-one function h from S into λ such that: $[a \in S \wedge b \in S \wedge a \neq b \wedge a \subset b \Longrightarrow h(a) \in b]$; We call h a witness.

3) S is a strong stationary coding set (CD) if the function $h(a) = \sup(a)$ is a witness (for its being a stationary coding set).

4) We shall say S is a $\mathcal{D}$-WSC if $S \neq \phi$ *mod* $\mathcal{D}$, and similarly for SC, CD; we may also say: for $\mathcal{D}$, there is a WSC; when $\mathcal{D} = \mathcal{D}_\kappa(\lambda)$ we may write (κ,λ) instead of $\mathcal{D}$.

6 Fact: 1) For $\mathcal{D}_\kappa(\kappa)$ there is a strong stationary coding set (κ itself!).

2) A strong stationary coding set is a stationary coding set, and a stationary coding set is a weak stationary coding set (for any fixed $\mathcal{D}$).

3) If λ is a singular, $\mathcal{D}$ a fine normal filter on $\mathcal{P}_{<\kappa}(\lambda)$ then there is no strong stationary coding set for $\mathcal{D}$.

4) If $2^{<\kappa} \le \lambda < \lambda^{<\kappa}$, $\mathcal{D}$ a fine normal filter on $\mathcal{P}_{<\kappa}(\lambda)$ then there is no SC for $\mathcal{D}$.

Proof : 3) Suppose S is a $\mathcal{D}$-CD, and let $C \subseteq \lambda$ be closed unbounded of cardinality $< \lambda$. Clearly $S_1 = \{a \in \mathcal{P}_{<\kappa}(\lambda)$: $a \cap c$ is unbounded in $a\} \in \mathcal{D}$, hence $S_2 = \{a \in S : \sup a \in c \cup \{\lambda\}\} \neq \phi \bmod \mathcal{D}$, but $|S_2| < \lambda$ so we get contradiction to "$\mathcal{D}$ is fine".

4) If S is such a set, if h witnesses its being SC then it shows $|S| \le \lambda$, but every $a \in \mathcal{P}_{<\kappa}(\lambda)$ is a subset of some $b \in S$ hence $\lambda^{<\kappa} \le |\mathcal{P}_{<\kappa}(\lambda)| = \sum_{b \in S} |\mathcal{P}(b))\le |S| \cdot 2^{<\kappa} \le \lambda$ contradiction.

7 Claim: Suppose $\lambda > \kappa$ is regular, $\mathcal{D} = \mathcal{D}_\kappa(\lambda)$, $T \subseteq \lambda$ is stationary, $(\forall \delta \in T)$ $[cf\ \delta < \kappa]$, and $\diamondsuit_T$ holds. Suppose also: $\kappa > \aleph_1$ or for some normal filter $\mathcal{D}$ over $\aleph_1$ for every increasing continuous $h : \omega_1 \to \lambda$, $\{i < \delta : h(i) \notin T\} \in \mathcal{D}$. *Then* there is a CD for $\mathcal{D}$.

7A Remark: Really (κ,λ)-CD for $\lambda > \kappa$ is a weak form of $\diamondsuit_{\{\delta<\lambda : cf\ \delta<\kappa\}}$. In fact if (κ,λ)-CD exists, $\lambda = \lambda^{<\lambda} > 2^{<\kappa}$ then $\diamondsuit_{\{\delta<\lambda : cf\ \delta < \kappa\}}$ holds.

Proof: We know that $\mathcal{D}_0 = \{A \subseteq \lambda : \diamondsuit_{\lambda - A}$ does not hold$\}$ is a normal (fine) filter on λ. As $\diamondsuit_T$ holds, $T \neq \phi \bmod \mathcal{D}_0$, hence for some $\vartheta < \kappa$, $T_0 = \{\delta \in T : cf\ \delta = \vartheta\} \neq \phi \bmod \mathcal{D}_0$. So $\diamondsuit_{T_0}$ holds hence there are $\langle M_\delta : \delta \in T_0 \rangle$, M_δ a model with universe δ and $\aleph_0$ functions such that or every model M with universe λ and $\aleph_0$ functions $\{\delta \in T_0 : M \restriction \delta = M_\delta\}$ is stationary. Now we define by induction on $\delta \in T_0$, a set $A_\delta \subseteq \delta$ such that:

(a) A_δ is closed under the functions of $M_\delta\}$,

(b) $\sup A_\delta = \delta$.

(c) $|A_\delta| < \kappa$.

(d) if $\delta_1 \in \delta \cap T_0$, $A_{\delta_1} \subseteq A_\delta$ then $\delta_1 \in A_\delta$.

(This clearly suffices). We can even strengthen (d) to

(d') if $\delta_1 \in T_0 \cap \delta$ is in the closure of A_δ then $\delta_1 \in A_\delta$.

If $\kappa > \aleph_1$ let $\sigma < \kappa$ be regular cardinal such that $\sigma \neq \vartheta$, $(\sigma \geq \aleph_0)$, and then define by induction on $\xi < \sigma$, a set $A^\delta_\xi \subseteq \delta$, such that A^δ_ξ has cardinality $< \kappa$, is increasing with ξ, the closure of A^δ_ξ under the functions of M_δ is $\subseteq A^\delta_{\xi+1}$ and also every accumulation point of A^δ_ξ which is $< \delta$ is in $A^\delta_{\xi+1}$ and $\sup(A^\delta_0) = \delta$. Then $\bigcup \{A^\delta_\xi : \xi < \sigma\}$ is as required. The case $\kappa = \aleph_1$ is similar.

The following (8,9) is a variant of Silver [Si].

8 Definition: 1) $Si(\kappa,\lambda)$ means that for every algebra M with universe λ and countably many functions there are isomorphic subalgebras M_1, M_2 of power $< \kappa$, $M_1 \subseteq M_2$, $M_1 \neq M_2$ and $M_1 \cap \kappa = M_2 \cap \kappa =$ an ordinal.

2) For $\mathcal{D}$ a fine normal filter on $\mathcal{P}_{<\kappa}(\lambda)$, $Si(\mathcal{D})$ $[SSi(\mathcal{D})]$ means that for every $T \in \mathcal{D}$ $[T \neq \phi \bmod \mathcal{D}]$ and M above we can find M_1, M_2 as above $M_1 \in T$, $M_2 \in T$.

3) The negation of $Si(\kappa,\lambda)$, $Si(S), SSi(\mathcal{D})$ are denoted by $N\ Si(\kappa,\lambda)$, $NSi(\mathcal{D})$, $NSSi(\kappa,\lambda)$ $NSSi(\mathcal{D})$ resp.

9 Fact: 1) If $Si(\kappa,\lambda)$ and $\lambda \leq \lambda^*$ then $Si(\kappa,\lambda^*)$.

2) If $Si(\mathcal{D})$ then $Si(\mathcal{D}_\kappa(\lambda))$

3) The first $\lambda \geq \kappa$ for which $Si(\kappa,\lambda)$ holds, is a strongly inacessible cardinal.

4) $Si(\mathcal{D}_\kappa(\lambda))$ is equivalent to $Si(\kappa,\lambda)$.

5) If $Si(\mathcal{D})$ then there is a minimal normal $\mathcal{D}_1$ extending $\mathcal{D}$ for which $SSi(\mathcal{D})$.

10 Claim: 1) Suppose $\diamondsuit_\kappa$ and $N\ Si(\kappa,\lambda)$, then there is a WSC for $\mathcal{D}_\kappa(\lambda)$.

2) If in 1), T is a set of strongly inaccessible cardinals (hence κ is Mahlo), $\diamondsuit_T$ and $\lambda = \lambda^{<\kappa}$ *then* there is a SC for $\mathcal{D}$.

3) Suppose $\mathcal{D}$ is a normal fine filter on $\mathcal{P}_{<\kappa}(\lambda)$, $\mathcal{D}_0 = \{A \subseteq \kappa : \{a \in \mathcal{P}_{<\kappa}(\lambda): a \cap \kappa \in A\} \in \mathcal{D}\}$ (so $\mathcal{D}_0$ is necessarily a normal filter on κ). Suppose further $\diamondsuit(\mathcal{D}_0)$ which means: there is $\langle A_\delta : \delta < \kappa \rangle$ such that for every $A \subseteq \kappa$,

$\{\delta < \kappa : A \cap \delta = A_\delta\} \neq mod\ \mathcal{D}_0$. If $NSi(\mathcal{D})$ then there is a $\mathcal{D}$-WSC; and when in addition $\{\vartheta < \kappa : \vartheta$ strongly inaccessible$\} \neq mod\ \mathcal{D}_0$ then there is a $\mathcal{D}$-CD.

Proof : 1) Let $T \subseteq \kappa$ be stationary such that $\diamondsuit_T$ holds. There is an algebra M with countably many functions exemplifying $NSi(\mathcal{D}(\kappa,\lambda))$. As $\diamondsuit_T$, we can find models M_α $(\alpha \in T)$ such that:

i) M_α is an algebra with countably many functions, and universe γ_α $\alpha \leq \gamma_\alpha < \kappa$.

ii) *if* $\langle N_i : i < \kappa \rangle$ is an increasing continuous sequence of algebras with countably many functions, $||N_i|| < \kappa$, and $\kappa \subseteq \bigcup_{i<\kappa} N_i$ *then* for stationary many $i \in T$, N_i, M_i are isomorphic over i.

Let $S^* = \{a \in \mathcal{P}_{<\kappa}(\lambda) : M \restriction a$ is isomorphic to $N_{a \cap \kappa}$ where $a \cap \kappa \in T\}$. Now S^* is a WSC, in fact if $a \in S^*$, $b \in S^*$, $a \subset b$ (but $a \neq b$) then $a \cap \kappa < b \cap \kappa$.

2) Straightforward, let h be any one-to-one function from $\mathcal{P}_{<\kappa}(\lambda)$ into λ, then $S^{**} = \{a \in \mathcal{P}_{<\kappa}(\lambda): a \cap \kappa \in T, \quad |a| \leq |a \cap \kappa| \quad$ and $(\forall b)(b \subseteq a \wedge |b| < a \cap \kappa \rightarrow h(b) \in a)\} \neq \phi\ mod\ \mathcal{D}_\kappa(\lambda)$. Now also $S^* \cap S^{**} \neq \phi\ mod\ \mathcal{D}_\kappa(\lambda)$ and $S^* \cap S^{**}$ is a strong stationary coding set.

3) Left to the reader.

11 Conclusion: If κ is an ineffable cardinal (see e.g. [KM]), $N\ Si(\mathcal{D}_\kappa(\lambda))$ and $\lambda = \lambda^{<\kappa}$ then there is a SC for $\mathcal{D}_\kappa(\lambda)$.

Proof : It is known that for ineffable κ, $\diamondsuit_\kappa$, holds, moreover $\diamondsuit_T$, where $T = \{\mu < \kappa : \mu$ strongly inaccessible$\}$. By 10(2) we finish.

12 Observation : The following properties for a succesor cardinal κ and stationary $T \subseteq \kappa$ are equivalent:

i) in $\kappa^\kappa / (\mathcal{D}_\kappa + T)$ there is an increasing sequence sequence $\langle g_\alpha / (\mathcal{D}_\kappa + T) : \alpha < \kappa^+ \rangle$ and $g / (\mathcal{D}_\kappa + T)$ such that $g / (\mathcal{D}_\kappa + T) \nleq g_\alpha / (\mathcal{D}_\kappa + T)$ for every $\alpha < \kappa^+$.

ii) there is $g : \kappa \rightarrow \kappa$ such that for any well-ordering $<^*$ of

κ, $\{\alpha \in T : (\alpha, <^* \restriction \alpha)$ has order-type $< g(\alpha)\}$ is stationary.

iii) there is $g : \kappa \to \kappa$ such that the set $\{a \in \mathcal{P}_{<\kappa}(\kappa^+) : a \cap \kappa \in T$, $(a, < \restriction a)$ has order-type exactly $g(a \cap \kappa)\}$ is stationary.

iv) *for any* cardinal $\mu, 1 \le \mu < \kappa$ such that $(\forall \delta \in T)[cf\ \delta < \mu \wedge |\delta|^\mu < \kappa]$, cardinal $\lambda > \kappa$ and subsets $P_i \subseteq \lambda (i < \mu)$, *there are* functions $g_i : \kappa \to \kappa\ (i < \mu)$, such that the set

$\{a : a \in \mathcal{P}_{<\kappa}(\lambda)$, $a \cap \kappa$ an ordinal from T and for $i < \mu$ the order type of $a \cap P_i$ is $g_i(a \cap \kappa)\}$

is stationary.

Proof: Trivially (iv) $\Longrightarrow$ (iii). Next we show (iii) $\Longrightarrow$ (ii): if g exemplify (iii), $<^*$ a well ordering of κ, then for some α, $\kappa \le \alpha < \kappa^+$, $(\alpha, <)$ is isomorphic to $(\kappa, <^*)$, and let h be such an isomorphism. Let $\alpha + 1 = \bigcup_{i<\kappa} a_i$, a_i increasing continuous, $|a_i| < \kappa$, so for some closed unbounded $C \subseteq \kappa$, $(\forall \delta \in C)$ $[a_\delta \cap \kappa = \delta$, and h is isomorphism from a_δ onto $(\delta, <^* \restriction \delta)]$. If (ii) fail (for this g, for this $<^*$) we can assume $(\forall j \in C \cap T)[(\delta, <^* \restriction \delta)$ has order type $> g(\alpha)$, but then $\{a \in \mathcal{P}_{<\kappa}(\kappa^+) : \alpha + 1 \in a, a \cap (\alpha+1)$ is a_δ for some $\delta \in C\}$ belongs to $\mathcal{D}_\kappa(\kappa^+)$ contradicting the choice of g.

Now we show (ii) $\Longrightarrow$ (i), let for $\alpha \in (\kappa, \kappa^+)$, $<_\alpha$ be a well ordering of κ of order type α, and let $g_\alpha(i) \stackrel{\text{def}}{=}$ "order type of $(i, <_\alpha \restriction i)$", the checking is easy.

Now if (i) holds for g, $g_\alpha (\alpha < \kappa^+)$, also (ii) holds for g: let $<_\alpha$ be any well ordering of κ of order type α (for $\kappa \le \alpha < \kappa^+$), $g^*_\alpha : \kappa \to \kappa$ be defined by $g^*_\alpha(i) =$ "the order type of $(i, <_\alpha \restriction i)$", for $\alpha < \kappa$ let $g^*_\alpha(i) = i$; we can proved by induction on $\alpha < \kappa^+$ that $g^*_\alpha / \mathcal{D}_\kappa \le g_\alpha / \mathcal{D}_\kappa$ so (ii) is clear.

Lastly assume (ii) holds for g, $\mu < \kappa$ $(\forall \delta \in T)[cf\ \delta > \mu \wedge |\delta|^\mu < \kappa]$, $P_i \subseteq \lambda (i < \mu)$ and we shall prove (iv) (for those $P_i (i < \mu)$). Let for $\delta \in T$ (w.l.o.g. $|g(\delta)| \le |\delta|$), h_δ be a one-to-one function from δ onto $g(\delta)$. For any sequence $\bar{\beta} = \langle \beta_j : j < \mu \rangle$ of ordinals $< \kappa$, we define function $g_{\bar{\beta}, j} : \kappa \to \kappa$ by $g_{\bar{\beta}, j}(i) = h_i(\beta_j)$. If $\langle g_{\bar{\beta}, i} : i < \mu \rangle$ is not as required then there is a

$C_{\bar\beta} \in \mathcal{D}_\kappa(\lambda)$ such that $(\forall a \in C_{\bar\beta})(\exists j < \mu)$ $[a \cap P_j$ has order-type $\neq g_{\bar\beta,j}(a \cap \kappa)]$. Let $\{\bar\beta^\xi : \xi < \kappa\}$ list all such sequences $\bar\beta$, then

$C = \{a \in \mathcal{P}_{<\kappa}(\lambda)$: if $\xi \in a \cap \kappa$ then $a \in C_{\bar\beta^\xi}$ and if $\beta_j < a \cap \kappa$ for $j < \mu$ then for some $\xi < a \cap \kappa$, $\bar\beta^\xi = \langle \beta_j : j < \mu \rangle\}$

is in $\mathcal{D}_\kappa(\lambda)$. But $S = \{a \in \mathcal{P}_{<\kappa}(\lambda) : a \cap \kappa \in T$, and the order-type of a is $< g(a \cap \kappa)\}$ is $\mathcal{D}$-stationary, so there is $a^* \in S \cap C$. Let $\beta_j^* =$ order-type of $P_j \cap a$, so necessarily $\beta_j^* < g(a \cap \kappa)$, hence for some $\beta_j < a \cap \kappa$, $h_{a\cap\kappa}(\beta_j) = \beta_j^*$; now we get contradiction to $a^* \in C_{\langle \beta_j : j < \mu\rangle}$.

Remark: At least (i) $\Longleftrightarrow$ (ii) is well known.

12A Remark: We can omit the assumption "κ successor" if we add in (i), (ii), (iii) $g(\alpha) < \omega + |\alpha|^+$, and in (iv) $g_i(\alpha) < \omega + |\alpha|^+$.

13 Claim: Suppose (i) of Fact 12 holds so κ is a successor; or at least (i) of 12A holds. Suppose further that $\lambda = \kappa^{+\alpha}$ (i.e. $\kappa = \aleph_\beta$, $\lambda = \aleph_{\beta+\alpha}$) and $|\alpha|^+ < \kappa$, $(\forall \gamma < \kappa)\ \gamma^{|\alpha|} < \kappa$. Then there is a WSC for $\mathcal{D}_{<\kappa}(\lambda)$.

Proof : Clearly we can find $g_i (i \leq \alpha)$ as in 12 (iv) replacing μ by α, and P_i by μ_i.

Let $M = (\lambda, f, g)$ f a two place function such that for every $i < \lambda$, $i = \{f(i,j) : j < |i|\}$, and for $j < |i|$, $g(i, f(i,j)) = j$, and $S = \{a : a \in \mathcal{P}_{<\kappa}(\lambda)$, a closed under f and g, $a \cap \kappa$ an ordinal, and for every $i \leq \alpha$, the order type of $a \cap \kappa^{+i}$ is $g_i(a \cap \kappa)\}$. By 12 (iv) S is $\mathcal{D}_\kappa(\lambda)$-stationary. Suppose $a \neq b$ are in S, $a \cap \kappa = b \cap \kappa$, and $a \subseteq b$. We know (as $a, b \in S$, $a \cap \kappa = b \cap \kappa$) that for each $i \leq \alpha$, $a \cap \kappa^{+i}$, $b \cap \kappa^{+i}$ has the same order type. Now we prove by induction on i that $a \cap \kappa^{+i} = b \cap \kappa^{+i}$. For $i = 0$ this is given; for i limit by the induction hypothesis; for $i = j + 1$: $a \cap \kappa^{+i}$ is unbounded in $b \cap \kappa^{+i}$ as they have the same order type, now apply the functions f, g under which a, b are closed (and $a \cap \kappa^{+j} = b \cap \kappa^{+j}$). For $i = \alpha$ we get the desired conclusion.

14 Lemma : 1) If $\kappa = \mu^+$, $\mathcal{D}$ a fine normal filter on $\mathcal{P}_{<\kappa}(\lambda)$, λ regular $\{a : cf\ (\sup a) \neq cf\ (\mu)\} \in \mathcal{D}$, then $\mathcal{D}$ is not λ^+-saturated (see Definition 4(3).).

2) $\mathcal{P}_{<\kappa}(\lambda) - \{a : cf\ (\sup\ a) \neq cf\ \mu\} \neq \mathcal{D}$ is enough in 14(1).

Proof: 1) Let P be the set of $\mathcal{D}$-stationary sets ordered by inverse inclusion. Suppose $\mathcal{D}$ is λ^+-saturated, so P satisfies the λ^+-Chain condition. We shall prove that λ^+ is a cardinal of V^P, all V-cardinals $< \kappa$ are V^P- cardinals, $V^P \models$ "$|\lambda| = \mu$, $cf\ \lambda \neq cf\ \mu$", thus contradicting [Sh 1], XIII 4.9, p. 440. The following facts fulfilling the above, are folklore at least for $\lambda = \kappa$, and straightforward generalization generally.

Fact A: For every P-name $\underset{\sim}{\alpha}$ of an ordinal $< \alpha^*$, there is a function $g : \mathcal{P}_{<\kappa}(\lambda) \to \alpha^*$, such that $[a \neq \phi \wedge \alpha^* \leq \lambda \Longrightarrow g(a) \in a]$, and ($\underset{\sim}{G}_P$ - the P-name of the generic set) $\Vdash_P$ "$\underset{\sim}{\alpha}$ is the unique α such that $\{a : g(a) = \alpha\} \in \underset{\sim}{G}_P$".

Proof : Let $\langle S_i : i < \beta \rangle$ be a maximal antichain of P such that for each i for some α_i, $S_i \Vdash_P$ "$\underset{\sim}{\alpha} = \alpha_i$". Now $|\beta| \leq \lambda$ (by the λ^+-chain condition) so w.l.o.g. $\beta = \lambda$, (we allow $S_i = \phi$) so for $i \neq j$, $S^{i,j} \overset{\text{def}}{=} \mathcal{P}_{<\kappa}(\lambda) - S_i \cap S_j$ belong to $\mathcal{D}$. Let $C = \{a \in \mathcal{P}_{<\kappa}(\lambda)$: for every $i \in a[\alpha_i < \lambda \Longrightarrow \alpha_i \in a]$, and for every $i \neq j \in a$, $a \in S^{i,j}\}$.

By the normality of $\mathcal{D}$, $C \in \mathcal{D}$.

Define a partial function h on C: $h(a) = i$ if $a \in S_i$, $i \in a$. By C's definition h has at most one value.

If $S \overset{\text{def}}{=} C - \text{Dom}\ h$ is $\mathcal{D}$-stationary then remember $S \cap S_i \subseteq \mathcal{P}_{<\kappa}(\lambda) - \{a : i \in a\}$, so S contradict the "$\langle S_i : i < \alpha \rangle$ is a maximal antichain". So h is defined on some $C^* \in \mathcal{D}$, and is as required (it does not matter how we complete it on $\mathcal{P}_{<\kappa}(\lambda) - \{\phi\}$, as long as $\alpha^* \leq \lambda$, $a \neq \phi \Longrightarrow h(a) \in a$.

Fact B: Forcing by P does not collapse any $\vartheta < \kappa$.

Proof : Let $S \in P$, $S \Vdash_P$" ϑ is collapsed". Choose minimal ϑ, so ϑ is regular in V and (maybe changing S) for some regular $\sigma < \vartheta$:

$\Vdash_P$"$\underset{\sim}{f}$ is a function from σ to ϑ"

$S \Vdash_P$"$\underset{\sim}{f}$ has an unbounded range".

W.l.o.g. for each $i < \vartheta$, we apply Fact A (with $\alpha^* = \vartheta \le \lambda$) on the P-name $\underset{\sim}{f}(i)$ and get g_i. For every $a \in \mathcal{P}_{<\kappa}(\lambda) - \{\phi\}$, as (in V) ϑ is regular $> \sigma$, clearly $g(a) = \sup\{g_i(a) : i < \alpha\}$ is an ordinal $<\vartheta$. As $\mathcal{D}$ is ϑ^+-complete and fine, $\{a : \vartheta \subseteq a\} \in \mathcal{D}$ and as Rang $(g_i) \subseteq \vartheta$ and $\mathcal{D}$ is normal clearly for some $\gamma < \vartheta$, $S^* = \{a \in S : g(a) = \gamma\}$ is $\mathcal{D}$-stationary; trivially

$$S^* \Vdash_P \text{" Rang } \underset{\sim}{f} \subseteq \gamma\text{"}$$

contradiction.

Fact C: $S \Vdash_P$" $cf\ (\chi) = \vartheta$" if χ is a regular cardinal $\le \lambda$, $S \in P$, and $S \subseteq \{a \in \mathcal{P}_{<\kappa}(\lambda) : \vartheta = cf\ (\sup\ (a \cap \chi))\}$.

For every $a \in S$ let $\langle \beta(a,i) : i < \vartheta \rangle$ be an increasing sequence of ordinals from $a \cap \chi$ with limit $\sup(a \cap \chi)$. We know by $\mathcal{D}$'s normality that for every $i < \vartheta$, and $S' \subseteq S$, $S' \in P$, for some $S'' \subseteq S'$, $S'' \in P$, and $\beta(a,i)$ is constant for $a \in S''$.

Let $\underset{\sim i}{\beta}$ be the unique ordinal β such that $\{a : \beta(a,i) = \beta\} \in \underset{\sim P}{G}$ (this is a P-name). So $S \Vdash_P$"$\underset{\sim i}{\beta}$ is an ordinal $< \kappa$ and $\underset{\sim i}{\beta} < \underset{\sim j}{\beta}$ for $i < j < \vartheta$."

Also we shall show that $S \Vdash_P$"$\{\underset{\sim i}{\beta} : i < \vartheta\}$ is unbounded below χ (hence $S \Vdash$" $cf\,(\chi) = \vartheta$" and we shall finish). This holds because for every $S' \in P, (S' \subseteq S)$ and $\beta < \chi$ w.l.o.g. $(\forall a \in S)(\beta \in a)$, and so for every $a \in S'$ there is $i_a < \vartheta$ such that $\beta(a,i_a) > \beta$, and the function i_a has a constant value j on some $\mathcal{D}$-stationary $S'' \subseteq S'$ and $S'' \Vdash$ "$\underset{\sim j}{\beta} > \beta$". So we finish.

Remark: This is essentially Ulam argument for "$\mathcal{D}_{\omega_1}$ is not $\aleph_1$-saturated".

Fact D: $\Vdash_P$" the power of λ is μ". It is enough to prove that every regular $\chi, \kappa \le \chi \le \lambda$ is collapsed. As the number of possible $cf\ (\sup(a \cap \chi))$ is

$\leq |\{\vartheta : \vartheta < \kappa\}| = |\{\vartheta : \vartheta \leq \mu < \kappa\}| < \kappa$ we can use Fact C.

Fact E: $\Vdash_P$" $cf\ \lambda \neq cf\ \mu$".

By a hypothesis $S_0 = \{a \in \mathcal{P}_{<\kappa}(\lambda) : cf(\sup a) \neq cf\ \mu\} \in \mathcal{D}$ is in $\mathcal{D}$. As $\{cf \sup a : a \in S_0\} \subseteq \{\vartheta : \vartheta \leq \mu\}$ has power $< \kappa$ for every $S \in P$, for some ϑ, $S = \{a \in S \cap S_0 : cf(\sup a) = \vartheta\} \neq \phi \bmod \mathcal{D}$. As $S \subseteq S_0$, necessarily $\vartheta \neq cf\ \mu$. By Fact C $S_1 \Vdash_P$ "$cf\ \lambda = \vartheta$" and by Fact C, $(cf\,\mu)^V = (cf\ \mu)^{V^P}$ so $S_1 \Vdash_P$"$cf\ \lambda \neq \mu$". Hence $S \not\Vdash^P$"$cf\ \lambda = cf\ \mu$".

As $S \in P$ was arbitarily $\Vdash_P$" $cf\ \mu \neq cf\ \lambda$". So we finish the proof of 1.14.

2) Trivial by 14(1).

15 Definition : For W a set of regular cardinals $< \kappa$, let $\mathcal{D}^W_{<\kappa}(A)$ be the minimal (fine normal) filter on $\mathcal{P}_{<\kappa}(A)$, such that for every well ordering $<^*$ of A the following set belongs to $\mathcal{D}$:

$S(<^*) = S^W(<^*,A) = \{a \in \mathcal{P}_{<\kappa}(A)$: if $\mu \in W$, $\bar{x} = \langle x_i : i < \mu \rangle$ is an $<^*$-increasing but bounded (in $(A,<^*)$ but not necessarily in a) and $x_i \in a$ then the limit of $\bar{x}$ belong to $a\}$.

Remark: 1) We can ask only that each $\mathcal{D}^{\{\mu\}}_{<\kappa}(A)$ is included; the difference is small.

2) $\mathcal{D}^W_{<\kappa}(A)$ may be trivial, i.e. $\phi \in \mathcal{D}^W_{<\kappa}(A)$.

16 Fact: For every $S \in \mathcal{D}^W_{<\kappa}(A)$ there is a function $G : \mathcal{P}_{<\aleph_0}(A) \to \mathcal{P}_{<\kappa}(A)$ and well orderings $<^*_x$ of A for $x \in A$ such that S includes:

$Gn(G,<^*) = \{a \in \mathcal{P}_{<\kappa}(A)$: for every $w \in \mathcal{P}_{\aleph_0}(a), G(w) \subseteq a$ and for every $x \in A$, $a \in S^W(<^*_x,A)\}$.

17 Fact: If $\lambda \geq \kappa, W$ is a set of regular cardinals $< \kappa$, $\vartheta < \kappa$ is regular, $\vartheta \not\in W$ then $\phi \not\in \mathcal{D}^W_{<\kappa}(A)$.

Proof : Let G, $<^*_x(x \in A)$ be as in 16.

It suffices to prove that $\{a : (\forall x \le a)[a \in Gn(G,<_x^*)\} \neq \phi$. We define by induction on $i < \vartheta$, a set $a_i \in \mathcal{P}_{<\kappa}(A)$.

Let $a_0 = \{0\}$.

For i limit $a_i = \bigcup_{j<i} a_j$.

For $i = j + 1$, let $a_i = a_j \cup \bigcup\{G(w) : w \in \mathcal{P}_{<\kappa}(a_j)\} \cup \{y$: for some $x \in a_j$, y is an accumulation point of a_j in $(A,<_x^*)\}$, the last part contributes $\le \aleph_0 + |a_i|$ elements as each $<_x^*$ is well ordering.

Now $\bigcup_{i<\vartheta} a_i$ is as required.

18 Lemma : Suppose $W \cup \{\vartheta\}$ is a set of regular cardinals $< \kappa$, $\vartheta \notin W$, the set S is a WSC for $\mathcal{D}_{<\kappa}^{W\cup\{\vartheta\}}(\lambda)$ (which is not trivial), and $\mathcal{D}_{<\kappa}^{W\cup\{\vartheta\}}(\lambda) + S$ is not λ^+-saturated. Then there is a WSC set for $\mathcal{D}_{<\kappa}^{W}(\lambda^+)$.

Proof : Let $\langle S_\alpha : \alpha < \lambda^+ \rangle$ exemplify $\mathcal{D}_{<\kappa}^{W\cup\{\vartheta\}} + S$ is not λ^+-saturated, and w.l.o.g. $S_\alpha \subseteq S$ for each α.

W.l.o.g. let when $\lambda \le \alpha < \lambda^+$, g_α be a one-to-one function from α onto λ, let for a set a and function h, $h''(a) = \{h(x) : x \in a\}$.

We now define for every $\alpha \in T \overset{def}{=} \{\beta : \lambda \le \beta < \lambda^+$ and $cf\ \beta = \vartheta\}$ a subset S^α of $\mathcal{P}_{<\kappa}(\alpha)$ such that:

(i) S^α is stationary for $\mathcal{D}_{<\kappa}^{W\cup\{\vartheta\}}(\alpha)$.

(ii) $\{g''_\alpha(a) : a \in S^\alpha\} \subseteq S_\alpha$

(iii) each $a \in S^\alpha$ is an unbounded subset of α and $(\forall a \in S^\alpha)$ $(a \cap \lambda = g''_\alpha(a)$ and $a \cap \kappa \in \kappa)$.

(iv) if $a \neq b \in \bigcup\{S^\beta : \beta \in T, \beta \le \alpha\}$ and $a \cap \kappa = b \cap \kappa$ then $a \not\subseteq b$.

If we succeed then $S^* = \bigcup_{\alpha\in T} S^\alpha$ is as required. As T is stationary (in λ^+), $\beta \in T \to cf(\beta) \notin W$, by (i) and 16 easily S^* is $\mathcal{D}_{<\kappa}^{W}(\lambda^+)$-stationary. The other requirement for being a WSC of $\mathcal{D}_{<\kappa}^{W}(\lambda^+)$ follows by iv).

So we concentrate on the induction step.

Let $S_0^{\alpha} = \{a \in \mathcal{P}_{<\kappa}(\alpha) : g''_{\alpha}(a) \in S_{\alpha}\}$ and $S_1^{\alpha} = \{a \in \mathcal{P}_{<\kappa}(\alpha) : g''_{\alpha}(a) \in S\}$, clearly $S_0^{\alpha} \subset S_1^{\alpha}$ and both are clearly $\mathcal{D}_{<\kappa}^{W \cup \{\vartheta\}}(\alpha)$-stationary.

Now the set

$C^{\alpha} = \{a \in \mathcal{P}_{<\kappa}(\alpha)$: $a \cap \kappa$ an ordinal $< \kappa$, every accumulation point $\delta < \alpha$ of a, $\delta < \alpha$, $cf\ \delta \in W \cup \{\vartheta\}$ belong to a, a is closed under $g_{\beta}, g_{\beta}^{-1}$ for $\beta \in a \cup \{\alpha\}$, and a is unbounded in $\alpha\}$

belong to $\mathcal{D}_{<\kappa}^{W \cup \{\vartheta\}}(\alpha)$. As $S_{\alpha} \cap S_{\beta}$ is not $\mathcal{D}_{<\kappa}^{W \cup \{\vartheta\}}(\lambda)$-stationary by (ii) easily for $\beta < \alpha$, $\beta \in T$ $S^{\beta} \cap \{a \cap \beta : a \in S^{\alpha}\}$ is not $\mathcal{D}_{<\kappa}(\beta)$-stationary hence $S^{\alpha} \cap S^{\beta,\alpha}$ is not $\mathcal{D}_{<\kappa}(\beta)$-stationary where

$$S^{\alpha,\beta} = \{a \in \mathcal{P}_{<\kappa}(\alpha) : a \cap \beta \in S^{\beta}\}.$$

Hence

$$S^{\alpha} = \{a \in \mathcal{P}_{<\kappa}(\alpha) : a \in S_0^{\alpha},\ a \in C^{\alpha} \text{ and } (\forall \beta \in a)\ a \notin S^{\alpha,\beta}\}$$

in $\mathcal{D}_{<\kappa}^{W}(\alpha)$-stationary. We shall show that (iv) holds, thus finishing. [(ii) holds as $S^{\alpha} \subseteq S_0^{\alpha}$, (iii) as $S^{\alpha} \subseteq C^{\alpha}$).

If $a,b \in \bigcup\{S^{\beta} : \beta < \alpha,\ \beta \in T\}$ this is by the induction hypothesis. If $a,b \in S^{\alpha}$, then (remembering $a,b \in C^{\alpha}$), $a \cap \lambda = g''_{\alpha}(a) \cap \lambda$, $b \cap \lambda = g''_{\alpha}(b) \cap \lambda$, hence we use the assumption "S_{α} in a WSC for $\mathcal{D}_{<\kappa}^{W \cup \{\vartheta\}}(\lambda)$" and $S^{\alpha} \subset S_0^{\alpha}$. Now $a \in S^{\alpha}$, $b \in S^{\beta}$, $\beta < \alpha, a \subseteq b$ is impossible as a is unbounded in α.

We are left with the care $a \in S^{\beta}$, $b \in S^{\alpha}$, $\beta < \alpha$, and assume $a \subseteq b$; as $\sup a = \beta$ and $cf\ \beta = \vartheta \in W \cup \{\vartheta\}$ clearly $\beta \in b$. But by the definition of S^{α}, $\beta \in b \Longrightarrow b \notin S^{\alpha,\beta} \Longrightarrow b \cap \beta \notin S^{\beta}$, hence $b \cap \beta \neq a$. But as $b \in C^{\alpha}$, $\beta \in b$ clearly $b \cap \beta \in S_1^{\beta}$; so $a \subset b \cap \beta$, $a \neq b \cap \beta$ are both in S_1^{β}, and then $g''(a)$, $g''(b \cap \beta)$ will contradict "S is a WSC".

19 Conclusion; If $\kappa > \aleph_n, \kappa$ successor then there is a WSC for $\mathcal{D}_{\kappa}(\kappa^{+n})$.

Proof : By 18 and 14. Let $\kappa = \mu^+$, let $\alpha = 0$, if $cf\ \mu \geq \aleph_n$, and α be n otherwise. So in both cases (as $\kappa > \aleph_n$), $\aleph_{\alpha+n} < \kappa$, and $\aleph_\alpha$ is regular and $cf\ \mu \neq \{\aleph_\alpha, \ldots, \aleph_{\alpha+n-1}\}$.

Let for $0 \leq \ell \leq n$ $W^\ell = \{\vartheta : \vartheta < \kappa, \vartheta$ regular $\} - \{\aleph_\alpha, \ldots, \aleph_{\alpha+\ell}\}$.

Note that $\mathcal{D}^{W^\ell}_{<\kappa}(\kappa^{+\ell})$ is a proper (fine normal) ideal as $\aleph_0 < \kappa$, $\aleph_0 \not\in W$, $W \subseteq \{\mu : \mu < \kappa$ and μ is regular$\}$ also for each $\ell < n$ it is not $\kappa^{+\ell}$- saturated: by 14 as κ is a successor and $cf\ \mu \in W$ (otherwise by inspecting the definition of W, clearly $cf\ \mu \geq \aleph_\alpha$, hence $\alpha = 0$ so hence $cf\ \mu \geq \aleph_{\alpha+n}$ hence $cf\ \mu > \aleph_{\alpha+\ell}$).

Now we prove by induction on $\ell \leq n$ that there is a WSC set S_ℓ for $\mathcal{D}^{W^\ell}_{<\kappa}(\kappa^{+\ell})$ with $cf(\sup a) = \aleph_{\alpha+\ell}$ for $a \in S_\ell$. For $\ell = 0$ $S = \{\delta < \kappa : cf\ \delta = \aleph_0\}$ is O.K. and for $\ell = m + 1$ use 18 with W^m, $\aleph_{\alpha+\ell}, \kappa^{+m}$ standing W, ϑ, λ. [on the problematic assumption "$\mathcal{D}^{W \cup \{\vartheta\}}_{<\kappa}(\lambda) + S_m$ is not λ^+-saturated"; this holds by 14 as $cf(\sup a) = \aleph_{\alpha+m}$ for $a \in S_m$ $\aleph_{\alpha+m} \neq cf\ \mu$ as $0 \leq m < n$. We still have to show $cf(\sup a) = \aleph_\ell$ for $a \in S_\ell$, but this holds by the construction in 18]. So we get the result for $\ell = n$.

20 Lemma: Suppose

(i) $\mathcal{D}_\kappa$ is κ^+-saturated.

(ii) every $\mathcal{D}_{<\kappa}(\kappa^+)$-stationary set S is reflected i.e., for some $\alpha < \kappa^+, S \cap \mathcal{P}_{<\kappa}(\alpha)$ is $\mathcal{D}_{<\kappa}(\alpha)$-stationary.

Then there is no WSC set S for $\mathcal{D}_{<\kappa}(\kappa^+)$.

Remark: Later the assumptions were proved consistent for $\kappa = \aleph_1$ in Foreman Magidor and Shelah [FMS].

Proof : Suppose S is a counterexample, let $g_\alpha(\alpha < \kappa^+)$ be a one-to-one function from $\kappa + \alpha$ onto κ and let $S_\alpha = \{g_\alpha''(a) : a \in S, a \in \mathcal{P}_{<\kappa}(\alpha)\}$.

By (i) for some $\alpha(*) < \kappa^+$, $\kappa + \alpha(*) = \alpha(*)$ and for every $\alpha, \alpha(*) \leq \alpha < \kappa^+$, and stationary $S^* \subseteq S_\alpha$ for some $\gamma < \alpha(*)$: $S^* \cap S_\gamma \neq \phi \bmod \mathcal{D}_{<\kappa}(\kappa)$. Now

$S^a \stackrel{def}{=} \{a \in S : \alpha(*) \in a\}$ is $\mathcal{D}_{<\kappa}(\kappa^+)$- stationary (as $S - S^a$ is not). So by (ii) for some α, $\alpha(*) < \alpha < \kappa^+$, and $S^b = S^a \cap \mathcal{P}_{<\kappa}(\alpha)$ is $\mathcal{D}_\kappa(\alpha)$-stationary; by the choice of $\alpha(*)$ there are $a \in S^b$, $\gamma < \alpha(*)$ such that $a \cap \gamma \in S_\gamma \subseteq S$. But $a \cap \gamma \subset a$, (not equal as $\alpha(*) \in a$ because $a \in S^a$) and we get contradiction to "S is WSC".

21 Lemma : Suppose $\lambda \to (\kappa)^{<\omega}_\kappa$, then there is a fine normal filter $\mathcal{D}$ on $\mathcal{P}_{<\kappa}(\lambda)$ for which there is no WSC.

Proof : For every model M with universe λ and $<\kappa$ functions let $G(M) = \{A : A$ a submodel of M, and some expansion of A is generated of a sequence of length α of indiscernible $\omega \le \alpha < \kappa\}$.

References

[FMS]
Foreman, M. Magidor, and S. Shelah, in preparation.

[BTW]
J. Baumgartner, A. Taylor, S. Wagon, Structural properties of ideal, Dissertation Math. 17 (182).

[KM]A. Kanamori, and M. Magidor, The development of large cardinals.

[M] T. K. Menas, On strong compactness and super-compactness, *Annals of Math. Logic*, 7 (1974), 327.

[Sh1]
S. Shelah, Proper Forcing, Springer Verlag Lecture Notes, 940 (1982).

[Sh2]
S. Shelah, Abstracts of A.M.S. (1983).

[Si] J. Silver, A large cardinal in the constructible Universe, 69 *Fund. Math.* 93-100.

[So] Solovay, Strongly compact cardinals and the G.C.H. proc. of The Tarski Symp. Berkeley 1971, ed. Henlien; Symp. in Pure Math. Vol. XXV (1974), 187- 204.

[Z] B. Zwicker, $\mathcal{P}_\kappa(\lambda)$ combinatorics, I: Stationary Coding sets, *Comtemporary Mathematics*, 31 (1984), 243-259.