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SOME REMARKS ON A PROBLEM OF J. D. MONK

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Abstract

In this paper we give a partial answer to a question of J. D. Monk on the size of independent families in product of interval algebras and for some related questions.

1. Introduction

Given a Boolean algebra B let $\mathrm{ind}(B) = \sup\{|X| : X \subset B, X \text{ is independent}\}$. In [2, Problem 3] J.D. Monk raised the following question:

(1) *If A_α is an interval algebra for each $\alpha < \kappa$ is* $\mathrm{ind}(\prod_{\alpha<\kappa} A_\alpha) = 2^\kappa$ *?*

Years later, getting no answer for his original question, he asked whether one can get large independent families having some special form in the product of certain interval algebras. To present his new questions we must introduce some notions.

Given an ordering $\prec$ on a set X and $y, z \in X$ take $(y,z)_\prec = \{x \in X : y \prec x \prec z \text{ or } z \prec x \prec y\}$. For $Y, Z \subset X$ we write $Y \prec Z$ to mean that "*$y \prec z$ for each $y \in Y$ and $z \in Z$*".

A countable set $\mathcal{L}$ of orderings on some cardinal κ is called *very nice* if given two disjoint finite subset Σ and Δ of κ there is a $\prec \in \mathcal{L}$ such that either $\Delta \prec \Sigma$ or $\Sigma \prec \Delta$.

First Monk asked whether there exist very nice sets of orderings on ω_2 under CH. Of course, a very nice set of orderings of size $(2^\omega)^+$ could have given a counterexample for (1). But we can make the following observation.

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OBSERVATION 1.1. *There are no very nice sets of orderings on* $(2^\omega)^+$.

PROOF. Assume on the contrary that $\mathcal{L} = \{\prec_n : n < \omega\}$ is very nice on $(2^\omega)^+$. For each $\alpha < (2^\omega)^+$ define the function $f_\alpha : \omega \to 2$ by setting $f_\alpha(i) = 1$ iff $\alpha \prec_i \alpha+1$. Pick $\alpha < \alpha+1 < \beta$ with $f_\alpha = f_\beta$. Take $\Delta = \{\alpha, \beta+1\}$ and $\Sigma = \{\alpha+1, \beta\}$. Then $\alpha \prec_i \alpha + 1$ iff $\beta \prec_i \beta + 1$, so neither $\Delta \prec_i \Sigma$ nor $\Sigma \prec_i \Delta$ for any $i \in \omega$. ■

Getting this negative answer Monk introduced the notion of *nice set of orderings*, which is a weakening of the notion of *very nice set of orderings*, but it is still strong enough to produce a counterexample for (1). In section 2 we answer in negative, disprove the existence of large, nice set of orderings.

A. Hajnal considered an other generalization of Monk's first question on very nice orderings. Let Σ be a set of some orderings on a fixed natural number n. We say that Σ is (κ, λ)*-representable* iff there exists a set $\mathcal{L}$ of orderings on κ with $|\mathcal{L}| = \lambda$ such that for each $\alpha_0 < \alpha_1 < \ldots < \alpha_{n-1} < \kappa$ there exist $\ll \in \Sigma$ and $\prec \in \mathcal{L}$ such that $i \ll j$ iff $\alpha_i \prec \alpha_j$ for any $i < j < n$. In this case we will take $\text{tp}(\alpha_0, \ldots, \alpha_{n-1}, \prec) = \ll$. We will write *representable* for $((2^\omega)^+, \omega)$-representable.

Now Hajnal's problem can be formulated as follows: *Characterize the representable sets of orderings!*

In section 3 we investigate some special cases of this question and show that there are non-trivial representable sets.

We use the standard set-theoretical notation throughout, cf [1]. The usual ordering of ordinals will be denoted by $<_{\text{On}}$. Given a set A of ordinals write type(A) for the order type of $\langle A, <_{\text{On}} \rangle$.

Theorem 3.1 was also proved by J.E Baumgartner independently and approximately in the same time.

A negative result

To present Monk's second question we need to introduce some notations. Given a set X let $\mathcal{I}(X)$ be the set of finite subsets of $[X]^2$. If $\prec$ is an ordering on X and $I \in \mathcal{I}(X)$ take

$$(I)_\prec = \bigcup \{ (y, z)_\prec : \{y, z\} \in I \}.$$

Given a cardinal κ a set $\{\prec_n : n < \omega\}$ of orderings on κ is called κ*-nice* if there exist sequences $\langle I_\alpha : \alpha < \kappa \rangle \subset \mathcal{I}(\kappa)$ and $\langle J_\alpha : \alpha < \kappa \rangle \subset \mathcal{I}(\kappa)$ with the properties that

(A) $\forall \alpha \in \kappa\ \forall n \in \omega\ (I_\alpha)_{\prec_n} \cap (J_\alpha)_{\prec_n} = \emptyset$,

(B) $\forall \Delta, \Sigma \in [\kappa]^{<\omega}$ if $\Delta \cap \Sigma = \emptyset$ then

$$\bigcap_{\delta \in \Delta} (I_\delta)_{\prec_n} \cap \bigcap_{\sigma \in \Sigma} (J_\sigma)_{\prec_n} \neq \emptyset$$

for some $n \in \omega$.

Now the Monk's second question can be formulated as follows: *Is ZFC + CH consistent with the existence of ω_2-nice sequences?* It is easy to see that a $(2^\omega)^+$-nice set of orderings could have given a counterexample for (1). But we answer in negative proving that

THEOREM 2.1. *There are no $(2^\omega)^+$-nice sequences.*

PROOF. Write $\kappa = (2^\omega)^+$. Assume on the contrary that the sequences $\langle I_\alpha : \alpha < \kappa\rangle \subset \mathcal{I}(\kappa)$ and $\langle J_\alpha : \alpha < \kappa\rangle \subset \mathcal{I}(\kappa)$ show that $\{\prec_n : n < \omega\}$ is κ-nice. We can assume that $\prec_n$ is a dense ordering for each n. We can also assume that there are fixed natural numbers, r and s, such that $|I_\alpha| = r$ and $|J_\alpha| = s$ for each $\alpha \in \kappa$. Write $I_\alpha = \{\{a^\alpha_{i,0}, a^\alpha_{i,1}\} : i < r\}$ and $J_\alpha = \{\{b^\alpha_{i,0}, b^\alpha_{i,1}\} : i < s\}$. Take $\mathrm{Lim} = \{\nu \in \kappa : \nu \text{ is limit}\}$.

LEMMA 2.2 $\forall m \in \omega\ \exists A \in [\kappa]^m\ \exists n \in \omega\ \forall \Delta \subset A$

$$\bigcap_{\delta\in\Delta}(I_\delta)_{\prec_n} \cap \bigcap_{\sigma\in(A\setminus\Delta)}(J_\sigma)_{\prec_n} \neq \emptyset. \tag{2}$$

PROOF. For each $\alpha \in \mathrm{Lim}$ take

$$M_\alpha = \bigcup\{\cup(I_{\alpha+i} \cup J_{\alpha+i}) : i < m\}$$

and

$$\mathcal{M}_\alpha = \left\langle M_\alpha, <_{On}, \prec_n : n < \omega, a^{\alpha+i}_{j,l}, b^{\alpha+i}_{k,l} : i < m, j < r, k < s, l < 2\right\rangle.$$

Since there are only (2^ω)-many pairwise non-isomorphic $\mathcal{M}_\alpha$, we can find a set $X \in [\mathrm{Lim}]^\kappa$ such that $\mathcal{M}_\alpha \cong \mathcal{M}_\beta$ for each $\alpha, \beta \in X$. Pick 2^m-many different elements of X, $\{\xi_i : i < 2^m\}$. Fix an enumeration $\{D_i : i < 2^m\}$ of the subsets of m. Take $\Delta_i = \{\xi_i + l : l \in D_i\}$ and $\Sigma_i = \{\xi_i + l : l \in m \setminus D_i\}$ for $i < 2^m$. Let $\Delta^* = \bigcup\{\Delta_i : i < 2^m\}$ and $\Sigma^* = \bigcup\{\Sigma_i : i < 2^m\}$. By (B) there is an $n \in \omega$ with

$$\bigcap_{\delta\in\Delta^*}(I_\delta)_{\prec_n} \cap \bigcap_{\sigma\in\Sigma^*}(J_\sigma)_{\prec_n} \neq \emptyset. \tag{3}$$

We claim that the set $A = \{\xi_0 + i : i < m\}$ and the ordering $\prec_n$ works. Indeed, if $\Delta \subset A$, then there is an $i < 2^m$ with $\Delta = \{\xi_0 + l : l \in D_i\}$. But by (3) it follows that

$$\bigcap_{\delta\in\Delta_i}(I_\delta)_{\prec_n} \cap \bigcap_{\sigma\in\Sigma_i}(J_\sigma)_{\prec_n} \neq \emptyset,$$

so $\mathcal{M}_{\xi_0} \cong \mathcal{M}_{\xi_i}$ implies that

$$\bigcap_{\delta\in\Delta}(I_\delta)_{\prec_n} \cap \bigcap_{\sigma\in A\setminus\Delta}(J_\sigma)_{\prec_n} \neq \emptyset$$

as well, which was to be proved. ■

If $n \in \omega$, $\alpha, \beta \in \kappa$, $\alpha < \beta$, take $N_{\alpha,\beta} = \bigcup(I_\alpha \cup I_\beta \cup J_\alpha \cup J_\beta)$ and $\mathcal{N}^n_{\alpha,\beta} = \left\langle N_{\alpha,\beta}, <_{\mathrm{On}}, \prec_n : n < \omega, a^\alpha_{j,l}, b^\alpha_{k,l}, a^\beta_{j,l}, b^\beta_{k,l} : j < r, k < s, l < 2\right\rangle$.

LEMMA 2.3. *$\forall m \in \omega\ \exists A \in [\kappa]^m\ \exists n \in \omega$ such that*
1. $\forall \Delta \subset A$

$$\bigcap_{\delta \in \Delta} (I_\delta)_{\prec_n} \cap \bigcap_{\sigma \in (A \setminus \Delta)} (J_\sigma)_{\prec_n} \neq \emptyset,$$

2. $\forall \alpha, \beta, \gamma, \delta \in A$ if $\alpha < \beta$ and $\gamma < \delta$ then $\mathcal{N}^n_{\alpha,\beta} \cong \mathcal{N}^n_{\gamma,\delta}$.

PROOF. The number of the isomorphism classes of $\mathcal{N}^n_{\alpha,\beta}$ is finite, say u. Fix $m' \in \omega$ with $m' \to (m)^2_u$. Applying lemma 2.2 for m' we obtain an $A' \in [\kappa]^{m'}$ and $n \in \omega$ satisfying the first requirement. Coloring the pair $\{\alpha, \beta\} \in [A']^2$ with $\mathcal{N}^n_{\alpha,\beta}$ provided $\alpha < \beta$ and using $m' \to (m)^2_u$ we get an $A \in [A']^m$ and an $\mathcal{N}$ such that $\mathcal{N}^n_{\alpha,\beta} = \mathcal{N}$ whenever $\alpha, \beta \in A$ with $\alpha < \beta$. ■

Fix now $A \in [\kappa]^4$ and $n \in \omega$ satisfying 2.3.1–2.

To simplify our notations, if $\alpha \in A$, $i < r$, $j < s$ we will write $I^\alpha_i = (a^\alpha_{i,0}, a^\alpha_{i,1})_{\prec_n}$ and $J^\alpha_j = (b^\alpha_{j,0}, b^\alpha_{j,1})_{\prec_n}$. If $l < r + s$ and $i < 2$ let

$$K^\alpha_l = \begin{cases} I^\alpha_l & \text{if } l < r \\ J^\alpha_{l-r} & \text{if } r \le l < r + s \end{cases}$$

and

$$c^\alpha_{l,i} = \begin{cases} a^\alpha_{l,i} & \text{if } l < r \\ b^\alpha_{l-r,i} & \text{if } r \le l < r + s\,. \end{cases}$$

LEMMA 2.4. *If $\alpha, \beta \in A$, $i, j < r + s$, $\{c^\alpha_{i,0}, c^\alpha_{i,1}\} \cap K^\beta_j \neq \emptyset$ then*

$$K^\gamma_j \cap K^\delta_j \neq \emptyset \tag{4}$$

for each $\gamma, \delta \in A$.

PROOF. Without loss of generality we can assume that $\alpha < \beta$ and $c^\alpha_{i,0} \in K^\beta_j$. By 2.3.2 this implies that $c^\nu_{i,0} \in K^\delta_j$ for each $\delta \in A \setminus \{\nu\}$, where $\nu = \min_{<_{\mathrm{On}}} A$. So (4) really holds. ■

A natural number $i < r+s$ is called *well-met* iff $K^\gamma_i \cap K^\delta_i \neq \emptyset$ for each $\gamma, \delta \in A$.

LEMMA 2.5. *If i is well-met and $\alpha < \beta < \gamma \in A$, then $K^\beta_i \subset K^\alpha_i \cup K^\gamma_i$.*

PROOF. By the 2-dimensional Hall theorem for convex sets there is an $x \in$ $\in K^\alpha_i \cap K^\beta_i \cap K^\gamma_i$. By 2.3.2 we have $c^\beta_{i,0} \in (c^\alpha_{i,0}, c^\gamma_{i,0})_{\prec_n}$ and $c^\beta_{i,1} \in (c^\alpha_{i,1}, c^\gamma_{i,1})_{\prec_n}$. So $(c^\beta_{i,0}, x)_{\prec_n} \subset (c^\alpha_{i,0}, x)_{\prec_n} \cup (c^\gamma_{i,0}, x)_{\prec_n}$ and $(c^\beta_{i,1}, x)_{\prec_n} \subset (c^\alpha_{i,1}, x)_{\prec_n} \cup (c^\gamma_{i,1}, x)_{\prec_n}$. ■

Write now $A = \{\alpha_0, \ldots, \alpha_3\}$ where $\alpha_0 < \ldots \alpha_3$. Take $\Delta = \{\alpha_0, \alpha_2\}$ and $\Sigma = \{\alpha_1, \alpha_3\}$. We will show that (1) fails. It is enough to show that

$$I_i^{\alpha_2} \bigcap \left((I_{\alpha_0})_{\prec_n} \cap (J_{\alpha_1})_{\prec_n} \cap (J_{\alpha_3})_{\prec_n} \right) = \emptyset \tag{5}$$

for each $i < r$. If i is well-met then, by lemma 2.5, it follows that $I_i^{\alpha_2} \subset (I_{\alpha_1})_{\prec_n} \cup \cup (I_{\alpha_3})_{\prec_n}$ so (5) holds. If i is not well-met, then $I_i^\gamma \cap I_i^\delta = \emptyset$ whenever $\gamma, \delta \in A$ by lemma 2.4. This implies that if $j < r + s$, $j \neq i$, then either (i) $I_i^{\alpha_2} \subset K_j^\beta$ for each $\beta \in A$, or (ii) $I_i^{\alpha_2} \cap K_j^\beta = \emptyset$ for each $\beta \in A$. If there is a j satisfying (i) then $I_i^{\alpha_2} \subset K_j^{\alpha_0} \cap K_j^{\alpha_1}$. But $K_j^{\alpha_0} \cap K_j^{\alpha_1} \cap (I_{\alpha_0})_{\prec_n} \cap (J_{\alpha_1})_{\prec_n} = \emptyset$, so (5) is satisfied. If (ii) holds for all j then $I_i^{\alpha_2} \cap \left((I_{\alpha_0})_{\prec_n} \cup (J_{\alpha_0})_{\prec_n}\right) = \emptyset$, so (5) holds. It means that (*) fails, that is, the theorem is proved. ■

A positive result

Let Σ be a set of orderings on some natural number n.

We present two simple and well-known arguments which can be used to show that certain Σ-s are not representable. Assume that the set $\mathcal{L} = \{\prec_n : n < \omega\}$ of orderings on $(2^\omega)^+$ represents Σ.

Take

$$\mathrm{Bad}_0(\Sigma) = \{\ll \in \Sigma : \exists i < j < k < n \; (i \ll j \not\leftrightarrow i \ll k)\}$$

and

$$\mathrm{Bad}_1(\Sigma) = \{\ll \in \Sigma : \exists i < j < \frac{n-1}{2} \; (2i \ll 2i + 1 \not\leftrightarrow 2j \ll 2j + 1)\}.$$

We show that $\Sigma \neq \mathrm{Bad}_0(\Sigma)$ and $\Sigma \neq \mathrm{Bad}_1(\Sigma)$.

For $\alpha < \beta < \kappa$ define the function $f_{\alpha,\beta} : \omega \to 2$ by setting $f_{\alpha,\beta}(n) = 1$ iff $\alpha \prec_n \beta$.

Choose $\alpha_0 < \alpha_1 < \ldots < \alpha_{n-1} < (2^\omega)^+$ such that $f_{\alpha_i,\alpha_j} = f_{\alpha_i \alpha_k}$ whenever $i < j < k < n$.

If $\ll \in \mathrm{Bad}_0(\Sigma)$ then there is no $\prec \in \mathcal{L}$ with $\mathrm{tp}(\alpha_0, \ldots, \alpha_{n-1}, \prec) = \ll$.

Choose $\beta_0 < \beta_0 + 1 < \beta_1 < \ldots < \beta_{\frac{n-1}{2}}$ such that $f_{\beta_i,\beta_i+1} = f_{\beta_j,\beta_j+1}$ for $i < j < \frac{n-1}{2}$.

If $\ll \in \mathrm{Bad}_1(\Sigma)$ then there is no $\prec \in \mathcal{L}$ with $\mathrm{tp}(\beta_0, \beta_0+1, \beta_1, \beta_1+1, \ldots, \prec) = \ll$.

One can expect that if Σ is not representable then this can be proved by using the above-mentioned two simple arguments or their combinations. It is true if $|\Sigma| = 1$. Indeed, let $n \in \omega$ and $\ll$ be an ordering on n. Write $\Sigma = \{\ll\}$. Without loss of generality we can assume that $0 \ll 1$ and that Σ is not representable by the natural ordering of $(2^\omega)^+$. Since $\ll \notin \mathrm{Bad}_0$, $0 \ll i$ for each $i \in \{1, 2, \ldots, n-1\}$. If there were $i \geq 2$ with $i + 1 \ll i$, then the second argument would show that $\ll$ is not representable. So

$$0 \ll 2 \ll 1 \ll 3 \ll 4 \ldots \ll n - 1.$$

If $n \geq 5$ then $2 \ll 1$ but $3 \ll 4$, so the first argument gives that $\ll$ can not be represented. For $n = 3, 4$ it is consistent with ZFC that this $\ll$ can be represented. The following theorem claims it only for $n = 3$, but similar arguments work for $n = 4$. For $|\Sigma| \geq 2$ the characterization of the representable Σ-s is still open.

THEOREM 3.1. *If ZF is consistent then so is ZFC + GCH + "there are orderings $\langle \prec_n : n < \omega \rangle$ on ω_2 with property that*

$$\forall \alpha < \beta < \gamma < \omega_2 \; \exists n \in \omega \; \gamma \in (\alpha, \beta)_{\prec_n} \text{"}.$$

PROOF. Assume GCH in the ground model and define a poset $\mathcal{P}$ as follows. The underlying set of $\mathcal{P}$ consists of triples $\langle A, \langle \prec_n : n < \omega \rangle, F \rangle$ with properties that:
1. $A \in [\omega_2]^{\leq \omega}$,
2. $\prec_n$ is an ordering on A for each $n \in \omega$,
3. $\forall \alpha, \beta, \gamma \in A$ if $\alpha < \beta < \gamma$ then there is an $n \in \omega$ with $\gamma \in (\alpha, \beta)_{\prec_n}$,
4. F is a function, $F : A \times A \to [\omega]^{\omega}$,
5. $\forall \alpha, \beta \in A \; \forall n \in F(\alpha, \beta) \; (\alpha, \beta)_{\prec_n} \cap \alpha \cap \beta = \emptyset$.

We write $p = \langle A^p, \langle \prec_n^p : n < \omega \rangle, F^p \rangle$ for $p \in \mathcal{P}$. The ordering of $\mathcal{P}$ is as expected:

$$\begin{aligned} p \leq q \quad \text{iff} \quad & A^q \subset A^p, \\ & F^q \subset F^p, \\ & \prec_n^q \subset \prec_n^p \quad \text{for each } n \in \omega. \end{aligned}$$

Clearly $\mathcal{P}$ is a σ-complete poset with cardinality ω_2.

If $A, B \subset \mathrm{On}$ with $\mathrm{type}(A) = \mathrm{type}(B)$ then denote by $\pi_{A,B}$ the unique $<_{\mathrm{On}}$-preserving bijection between A and B.

DEFINITION 3.2. Given $p, q \in \mathcal{P}$ we write $p \ll q$ to mean that (1)–(4) below are satisfied:
1. $\mathrm{type}(A^p) = \mathrm{type}(A^q)$,
2. $A^p \cap A^q <_{\mathrm{On}} A^p \setminus A^q <_{\mathrm{On}} A^q \setminus A^p$,
3. $x \prec_n^p y$ iff $\pi_{A^p,A^q}(x) \prec_n^q \pi_{A^p,A^q}(y)$ for each $n \in \omega$ and $x, y \in A^p$,
4. $F^p(x, y) = F^q(\pi_{A^p,A^q}(x), \pi_{A^p,A^q}(y))$ for each, $x, y \in A^p$.

LEMMA 3.3 *If $p \ll q$ then p and q are compatible in $\mathcal{P}$.*

PROOF. Write $\pi = \pi_{A^p,A^q}$, $A = A^p \cap A^q$ and $U = A^p \cup A^q$. For $\alpha \in U$ take

$$\tilde{\alpha} = \begin{cases} \alpha & \text{if } \alpha \in A^p, \\ \pi^{-1}(\alpha) & \text{if } \alpha \in A^q. \end{cases}$$

For each $n \in \omega$ and $r \in \{p, q\}$ define the equivalence relation $\cong_n^r$ on $A^r \setminus A$ as follows:

$$\beta \cong_n^r \gamma \text{ iff } (\beta, \gamma)_{\prec_n^r} \cap A = \emptyset.$$

For $\beta \in A^r \setminus A$ take
$$[\beta]_n = \{\gamma \in A^r \setminus A : \gamma \cong^r_n \beta\}.$$
The set of equivalence classes of $\cong^r_n$ will be denoted by $(A^r \setminus A)/\cong^r_n$.

An ordering $\prec_n \supset \prec^p_n \cup \prec^q_n$ on U is called *n-nice* if for each $\beta, \gamma \in A^q \setminus A$
$$\beta \cong^q_n \gamma \text{ implies } (\beta,\gamma)_{\prec_n} \cap A^p = \emptyset$$

The following sublemma obviously holds:

SUBLEMMA 3.3.1. *If $\prec_n$ is n-nice, $r \in \{p,q\}$, $\alpha, \beta \in A^r$ then*
$$(\alpha,\beta)_{\prec^r_n} \cap \alpha \cap \beta = \emptyset \text{ implies } (\alpha,\beta)_{\prec_n} \cap \alpha \cap \beta = \emptyset. \quad \blacksquare$$

For $n \in \omega$ and $\beta \in A^p \setminus A$ a partition (I_0, I_1) of $[\beta]_n$ is called *n-cut* iff $I_0 \prec^p_n I_1$. A sequence $\mathcal{I} = \{(I^x_0, I^x_1) : x \in (A^p \setminus A)/\cong^p_n\}$ is called *n-cut sequence* provided every (I^x_0, I^x_1) is an n-cut. Given an n-cut sequence $\mathcal{I} = \{(I^x_0, I^x_1) : x \in (A^p \setminus A)/\cong^p_n\}$ define $\prec^{\mathcal{I}}_n$ as the unique ordering on U satisfying (i)–(ii) below:

(i) $\prec^{\mathcal{I}}_n \supset \prec^p_n \cup \prec^q_n$,

(ii) $I^{[\tilde{\gamma}]_n}_0 \prec^{\mathcal{I}}_n [\gamma]_n \prec^{\mathcal{I}}_n I^{[\tilde{\gamma}]_n}_1$ for each $\gamma \in A^q \setminus A$.

SUBLEMMA 3.3.2. *$\prec^{\mathcal{I}}_n$ is n-nice.*

PROOF. Straightforward. $\blacksquare$

Fix an $\alpha \in A^p \setminus A$ and an n-cut $\mathcal{J} = (J_0, J_1)$ of $[\alpha]_n$. An ordering $\prec$ on U is called *n-compatible with $\mathcal{J}$* provided that there exists an n-cut sequence $\mathcal{I} = \{(I^x_0, I^x_1) : x \in (A^p \setminus A)/\cong^p_n\}$ such that $(I^{[\alpha]_n}_0, I^{[\alpha]_n}_1) = (J_0, J_1)$ and $\prec = \prec^{\mathcal{I}}_n$.

SUBLEMMA 3.3.3. *If $\alpha, \beta \in U$, $n \in \omega$ and $\tilde{\alpha} \cong^p_n \tilde{\beta}$ then there is an n-nice ordering $\prec_n$ with $(\alpha,\beta)_{\prec_n} \cap \alpha \cap \beta = \emptyset$.*

PROOF. We can assume that $\alpha \in A^p \setminus A$ and $\beta \in A^q \setminus A$. Take $J_0 = \{\nu \in [\alpha]_n : \nu \prec^p_n \alpha\} \cup \{\{\alpha\}\}$ and $J_1 = \{\nu \in [\alpha]_n : \alpha \prec^p_n \nu\}$. If $\prec_n$ is any ordering which is n-compatible with (J_0, J_1) then $\prec_n$ satisfies our requirements. $\blacksquare$

Consider a triple $\{\alpha,\beta,\gamma\} \in [U]^3 \setminus [A^p]^3 \cup [A^q]^3$ with $\alpha < \beta < \gamma$. We say that an ordering $\prec_n$ on U is n-*(α,β,γ)-good* iff $\prec_n$ is n-nice and $\gamma \in (\alpha,\beta)_{\prec_n}$. Take
$$K(\alpha,\beta,\gamma) = \{n \in \omega : \text{there exists an n-}(\alpha,\beta,\gamma)\text{-good ordering}\}.$$

SUBLEMMA 3.3.4. *$K(\alpha,\beta,\gamma)$ is infinite.*

PROOF. First remark that $\gamma \in A^q \setminus A$. We distinguish two cases.

Case 1. $\beta \in A^p \setminus A$.

For any $n \in F(\beta, \tilde{\gamma})$ we can choose an n-cut $\mathcal{J} = (J_0, J_1)$ of $[\beta]_n$ with properties that

- if $\alpha \prec^p_n \beta$ then $\beta \in J_1$ and $\alpha \notin J_1$,

– if $\beta \prec_n^p \alpha$ then $\beta \in J_0$ and $\alpha \notin J_0$.

If $\prec$ is any ordering which is n-compatible with $\mathcal{J}$, then $\gamma \in (\alpha, \beta)_{\prec_n}$ and so $n \in K(\alpha, \beta, \gamma)$. It means that $F(\beta, \tilde{\gamma}) \subset K(\alpha, \beta, \gamma)$.

Case 2. $\beta \in A^q \setminus A$.

Then $\alpha \in A^p \setminus A$. Let $n \in F(\alpha, \tilde{\gamma})$ be arbitrary. Choose an n-cut $\mathcal{J} = (J_0, J_1)$ of $[\alpha]_n$ with properties that

– if $\beta \prec_n^q \gamma$ then $\alpha \in J_1$,

– if $\gamma \prec_n^q \beta$ then $\alpha \in J_0$.

If $\prec$ is any ordering which is n-compatible with $\mathcal{J}$, then $\gamma \in (\alpha, \beta)_{\prec_n}$ and so so $n \in K(\alpha, \beta, \gamma)$. Therefore $F(\alpha, \tilde{\gamma}) \subset K(\alpha, \beta, \gamma)$. ■

Now we conclude the proof of lemma 3.3. Choose pairwise distinct natural numbers $n(\alpha, \beta, \gamma) \in K(\alpha, \beta, \gamma)$ for $\{\alpha, \beta, \gamma\} \in [U]^3 \setminus [A^p]^3 \cup [A^q]^3$ with $\alpha < \beta < \gamma$ and select pairwise disjoint sets $G(\nu, \mu) \in [F(\tilde{\nu}, \tilde{\mu})]^\omega$ for $\{\nu, \mu\} \in [U]^2 \setminus [A^p]^2 \cup [A^q]^2$ in such a way that $n(\alpha, \beta, \gamma) \notin G(\nu, \mu)$. This is possible by sublemma 3.3.4.

For each $n \in \omega$ choose an n-nice ordering $\prec_n$ such that (i) if $n \in G(\alpha, \beta)$ then $(\alpha, \beta)_{\prec_n} \cap \alpha \cap \beta = \emptyset$,

(ii) if $n = n(\alpha, \beta, \gamma)$ then $\gamma \in (\alpha, \beta)_{\prec_n}$.

These requirements can be satisfied by sublemma 3.3.3. and by $n(\alpha, \beta, \gamma) \in \in K(\alpha, \beta, \gamma)$.

Define the function $F : [U]^2 \to [\omega]^\omega$ by setting

$$F(\alpha, \beta) = \begin{cases} F^p(\alpha, \beta) & \text{if } \alpha, \beta \in A^p, \\ F^q(\alpha, \beta) & \text{if } \alpha, \beta \in A^q, \\ G(\alpha, \beta) & \text{otherwise.} \end{cases}$$

Take $r = \langle A^p \cup A^q, \langle \prec_n : n < \omega \rangle, F \rangle$. Then $r \in \mathcal{P}$ and r is a common extension of p and q. ■

LEMMA 3.4. *$\mathcal{P$ satisfies ω_2-c.c.*

PROOF. The previous lemma implies this because CH holds. ■

LEMMA 3.5. *For each $\alpha \in \omega_2$ the set $D_\alpha = \{p \in P : \alpha \in A^p\}$ is dense in $\mathcal{P}$.*

PROOF. Straightforward. ■

So the forcing with $\mathcal{P}$ preserves the cardinalities and GCH. Let $\mathcal{G}$ be a $\mathcal{P}$-generic filter. For $n \in \omega$ take $\prec_n = \bigcup\{\prec_n^p : p \in \mathcal{G}\}$. By lemma 3.5 $\prec_n$ is an ordering on ω_2 and by the property (3) of $\mathcal{P}$ the sequence $\langle \prec_n : n \in \omega \rangle$ satisfies the requirements of the theorem. ■

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