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WAS SIERPINSKI RIGHT? IV

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Abstract. We prove for any $\mu = \mu^{<\mu} < \theta < \lambda$, λ large enough (just strongly inaccessible Mahlo) the consistency of $2^\mu = \lambda \rightarrow [\theta]^2_3$ and even $2^\mu = \lambda \rightarrow [\theta]^2_{\sigma,2}$ for $\sigma < \mu$. The new point is that possibly $\theta > \mu^+$.

Introduction. An important theme in modern set theory is to prove the consistency of "small cardinals" having "a large cardinal property". Probably the dominant interpretation concerns large ideals (with reflection properties or connected to generic embedding). But here we deal with another important interpretation: partition properties. We continue here [6, §2], [8], [7], [9], [10] but generally do not rely on them except in the end (of the proof of 25) when it becomes like the proof of [6, §2]. This work is continued in Rabus and Shelah [3].

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Preliminaries.

1. Let $<^*_\chi$ be a well ordering of

$$\mathscr{H}(\chi) = \{\, x : \text{the transitive closure of } x \text{ has cardinality} < \chi \,\}$$

agreeing with the usual well ordering of the ordinals. P (and Q, R) will denote forcing notions, i.e., quasi orders with a minimal element $\emptyset = \emptyset_P$.

A forcing notion P is λ-closed or λ-complete if every increasing sequence of members of P, of length less than λ, has an upper bound.

2. If $P \in \mathscr{H}(\chi)$, then for a sequence $\bar{p} = \langle p_i : i < \gamma \rangle$ of members of P (not necessarily increasing) let

$$\alpha = \alpha_{\bar{p}} =: \sup\{\, j : \{\, p_i : i < j \,\} \text{ has an upper bound in } P \,\}$$

and define the canonical upper bound of $\bar{p}$, denoted by $\&\bar{p}$ as follows:

(a) the least upper bound of $\{\, p_i : i < \alpha_{\bar{p}} \,\}$ in P if there exists such an element
(b) the $<^*_\chi$-first upper bound of $\bar{p}$ if (a) can't be applied but there is an upper bound of $\{\, p_i : i < \alpha_{\bar{p}} \,\}$,

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(c) p_0 if (a), (b) fail, $\gamma > 0$,
(d) $\emptyset_P$ if $\gamma = 0$.

Let p_0 & p_1 be the canonical upper bound of $\langle p_\ell : \ell < 2 \rangle$.
Take

$$[a]^\kappa = \{ b \subseteq a : |b| = \kappa \} \qquad \text{and} \qquad [a]^{<\kappa} = \bigcup_{\theta<\kappa} [a]^\theta.$$

3. For sets of ordinals, A and B, define $\mathrm{OP}_{B,A}$ as the maximal order preserving one-to-one function between initial segments of A and B, i.e., it is the function with domain

$$\{ \alpha \in A : \mathrm{otp}(\alpha \cap A) < \mathrm{otp}(B) \}$$

and $\mathrm{OP}_{B,A}(\alpha) = \beta$ if and only if $\alpha \in A$, $\beta \in B$ and

$$\mathrm{otp}(\alpha \cap A) = \mathrm{otp}(\beta \cap B).$$

If A, B are sets of ordinals, let $A \lhd B$ mean A is a proper initial segment of B. If η, ν are sequences let $\eta \lhd \nu$ mean ν is an initial segment of ν. If we write $\trianglelefteq$ (rather than $\lhd$) we allow equality.

Let

$$S^\lambda_\kappa = \{ \delta < \lambda : \mathrm{cf}(\delta) = \kappa \}.$$

Definition 4. $\lambda \to [\alpha]^n_\theta$ holds provided that whenever F is a function from $[\lambda]^n$ to θ, then there is $A \subseteq \lambda$ of order type α and $t < \theta$ such that

$$[w \in [A]^n \Longrightarrow F(w) \neq t].$$

Definition 5. $\lambda \to [\alpha]^n_{\kappa,\theta}$ if for every function F from $[\lambda]^n$ to κ there is $A \subseteq \lambda$ of order type α such that $\{ F(w) : w \in [A]^n \}$ has power $\leq \theta$. If we write "$< \theta$" instead of θ we mean that the set above has cardinality $< \theta$.

Definition 6. A forcing notion P satisfies the Knaster condition (has property K) if for any $\{ p_i : i < \omega_1 \} \subseteq P$ there is an uncountable $A \subseteq \omega_1$ such that the conditions p_i and p_j are compatible whenever $i, j \in A$.

What problems do [6], [8], [7], [9] and [10] raise? The most important "minimal open", as suggested in [10] were:

Question A.

(1) Can we get, e.g., $\mathrm{CON}(2^{\aleph_0} \to [\aleph_2]^2_3)$ (generally raise μ^+ in part (3) below to higher cardinals)? We solve it here.
(2) Can we get $\mathrm{CON}(\aleph_\omega > 2^{\aleph_0} \to [\aleph_1]^2_3)$ (the exact $\aleph_n$ seems to me less exciting)?
(3) Can we get, e.g., $\mathrm{CON}(2^\mu > \lambda \to [\mu^+]^2_3)$?

Also

Question B.

(1) Can we get the continuity on a non-meagre set for functions $f : {}^\kappa 2 \to {}^\kappa 2$? (Solved in [9].)
(2) What can we say on continuity of 2-place functions (dealt with in Rabus-Shelah [3])?

(3) What about n-place functions (continuing in this respect [8] probably just combined with [3])?

QUESTION C.

(1) [10] for $\mu > \aleph_0$.

(2) Can we get, e.g.,

$$\mathrm{CON}(2^{\aleph_0} \geq \aleph_2, \text{ and if } P \text{ is } 2^{\aleph_0}\text{-c. c., } Q \text{ is } \aleph_2\text{-c. c., then } P \times Q \text{ is } 2^{\aleph_0}\text{-c. c.})?$$

(3) Can we get, e.g.,

$$\mathrm{CON}(2^{\aleph_0} > \lambda > \aleph_0, \text{ and if } P \text{ is } \lambda\text{-c. c., } Q \text{ is } \aleph_1\text{-c. c. then } P \times Q \text{ is } \lambda\text{-c. c.});$$

more general is

$$\mathrm{CON}(\mu = \mu^{<\mu} > \aleph_0 + \text{ if } P \text{ is } 2^{\mu}\text{-c. c. } Q \text{ is } \mu^{+}\text{-c. c. then } P \times Q \text{ is } 2^{\mu}\text{-c. c})?$$

So several are solved. But, of course, solving two or more of those problems does not necessarily solve their natural combinations, though probably it does.

§1. We return here to consistency of statements of the form $\chi \to [\theta]^2_{\sigma,2}$ (i.e., for every $c : [\chi]^2 \to \sigma$ there is $A \in [\chi]^\theta$ such that on $[A]$, c has at most two values), (when $2^\mu \geq \chi > \theta^{<\mu} > \mu$, of course). In [6, §2] this was done for $\mu = \aleph_0$, $\chi = 2^\mu$, $\theta = \aleph_1$, $2 < \sigma < \omega$ and χ quite large (in the original universe χ is an Erdős cardinal). Originally, [6, §2] was written for any $\mu = \mu^{<\mu}$ (χ measurable in the original universe) but because of the referee urging it is written up there for $\mu = \aleph_0$ only; though with an eye on the more general result which is only stated. In [8] the main objective is to replace colouring of pairs by colouring of n-tuples (and even $(<\omega)$-tuples) but we also say somewhat more on the $\mu > \aleph_0$ case (in [8, 1.4]) and using only k^2_2-Mahlo (for a specific natural number k^2_2)(an improvement for $\mu = \aleph_0$ too), explaining that it is like [7]. A side benefit of the present paper is giving a full self-contained proof of this theorem even for 1-Mahlo. The main point of this work is to increase θ, and this time write it for $\mu = \mu^{<\mu} > \aleph_0$, too.

The case $\theta = \mu^+$ is easier as it enables us to separate the forcing producing the sets admitting few colours: each appear for some $\delta < \chi$, $\mathrm{cf}(\delta) = \mu^+$, is connected to a closed subset a_δ of δ unbounded in δ of order type μ^+, so that below $\alpha < \delta$ in P_α we get little information on the colouring on the relevant set. Here there is less separation, as names of such colouring can have long common initial segments, but they behave like a tree and in each node we divide the set to μ sets, each admitting only 2 colours.

As we would like to prove the theorem also for $\mu > \aleph_0$, we repeat material on μ^+-c. c., essentially from [4], [12], [8].

DEFINITION 7.

(1) Let D be a normal filter on μ^+ to which

$$\{\, \delta < \mu^+ : \mathrm{cf}(\delta) = \mu \,\}$$

belongs. A forcing notion Q satisfies $*^{\varepsilon}_{D}$ where ε is a limit ordinal $< \mu$, if player I has a winning strategy in the following game $*^{\varepsilon}_{D}[Q]$ defined as follows:

PLAYING. The play finishes after ε moves. In the ζth move:

Player I. If $\zeta \neq 0$ he chooses $\langle q_i^\zeta : i < \mu^+ \rangle$ such that $q_i^\zeta \in Q$ and

$$(\forall \xi < \zeta)(\forall i < \mu^+)\ p_i^\xi \leq q_i^\zeta$$

and he chooses a function $f_\zeta : \mu^+ \to \mu^+$ such that for a club of $i < \mu^+$, $f_\zeta(i) < i$; if $\zeta = 0$ let $q_i^\zeta = \emptyset_Q$, f_ζ is identically zero.

Player II. He chooses $\langle p_i^\zeta : i < \mu^+ \rangle$ such that $(\forall i)\ q_i^\zeta \leq p_i^\zeta$ and $p_i^\zeta \in Q$.

Outcome. Player I wins provided that for some $E \in D$: if $\mu < i < j < \mu^+$, i, $j \in E$, $\mathrm{cf}(i) = \mathrm{cf}(j) = \mu$ and

$$\bigwedge_{\xi < \varepsilon} f_\xi(i) = f_\xi(j)$$

then the set

$$\{ p_i^\zeta : \zeta < \varepsilon \} \cup \{ p_j^\zeta : \zeta < \varepsilon \}$$

has an upper bound in Q; also, if player I has no legal move for some $\zeta < i$ he loses.

(1′) If D is

$$\{ A \subseteq \mu^+ : \text{for some club } E \text{ of } \mu^+ \text{ we have } i \in E \text{ and } \mathrm{cf}(i) = \mu \Longrightarrow i \in A \}$$

we may write μ instead of D (in $*_D^\varepsilon$ and in the related notions defined below and above).

(2) A strategy for a player is a sequence $\bar{F} = \langle F_\zeta : \zeta < \varepsilon \rangle$, F_ζ telling him what to do in the ζth move depending only on the previous moves of the other player. But here a play according to the strategy $\bar{F}$ will mean the player chooses in the ζth move for each $i < \mu^+$ an element of Q which is above and possibly strictly above (in $\leq_Q$'s sense) of what F_ζ dictates and a function f_ζ such that on some $E \in D$, the equivalence relation $f_\zeta(\alpha) = f_\zeta(\beta)$ induce on E refine the one which the strategy induces (this change does not change the truth value of "player X has a winning strategy"). This applies to the game $\otimes_Q^\varepsilon$ in part (5) below too.

(3) We define $**_\mu^\varepsilon$ similarly but for ζ limit q_i^ζ is not chosen (so player II has to satisfy for limit ζ just $\forall \xi < \zeta \Longrightarrow (\forall i)\ (p_i^\xi \leq p_i^\zeta)$).

(4) We may allow the strategy to be non-deterministic, e.g., choose not f_ζ just f_ζ / D_{μ^+}.

(5) We say a forcing notion Q is ε-strategically complete if for the following game, $\otimes_Q^\varepsilon$, player I has a winning strategy.

Playing. A play lasts ε moves. In the ζth move:

Player I. If $\zeta \neq 0$ he chooses $q_\zeta \in Q$ such that $(\forall \xi < \zeta)\ p_\xi \leq q_\zeta$, if $\zeta = 0$ let $q_\zeta = \emptyset_Q$.

Player II. He chooses $p_\zeta \in Q$ such that $q_\zeta \leq p_\zeta$.

Outcome. In the end Player I wins provided that he always has a legal move.

(6) We say Q is $(< \mu)$-strategically complete if for each $\varepsilon < \mu$ it is ε-strategically complete.

Remark 8.

(1) In this paper, in the case $\mu = \aleph_0$ we can use the Knaster condition instead of $*_\mu^\varepsilon$.

(2) We use below $*^{\varepsilon}_{\mu}$ and not $**^{\varepsilon}_{\mu}$ but $**^{\varepsilon}_{\mu}$ could serve as well.

(3) We may consider omitting the strategic completeness (a weak version of it is hidden in player I winning $*^{\varepsilon}_{D}[Q]$), but no present use.

DEFINITION 9.

(1) Let $\bar{F}^{\ell} = \langle F^{\ell}_{\zeta} : \zeta < \varepsilon \rangle$ be a strategy for player I in the game $*^{\varepsilon}_{D}[Q]$ for $\ell = 1$, 2. We say $\bar{F}^1 \leq \bar{F}^2$ equivalently, $\bar{F}^2$ is above $\bar{F}^1$ if any play

$$\langle (\bar{q}^{\zeta}, f_{\zeta}, \bar{p}^{\zeta}) : \zeta < \varepsilon \rangle$$

in which player I uses the strategy $\bar{F}^2$ (that is letting

$$(\langle q'_i : i < \mu^+ \rangle, f) = F_{\zeta}(\langle \bar{p}^{\xi} : \xi < \zeta \rangle)$$

we have $i < \mu^+ \Longrightarrow q'_i \leq q^{\zeta}_i$ and for some $E \in D$, $i \in E \wedge j \in E \wedge f(i) = f(j) \Longrightarrow f_{\zeta}(i) = f_{\zeta}(i)$) is also a play in which player I uses the strategy $\bar{F}^1$.

(2) Let $\alpha^* < \beta^* \leq \mu$, **St** be a winning strategy for player I in the game $\otimes^{\beta^*}_{Q}$. We say $\langle \bar{F}^{\alpha} : \alpha < \alpha^* \rangle$ is an increasing sequence of strategies of player I in $*^{\varepsilon}_{D}[Q]$ obeying **St** if:

(a) $\bar{F}^{\alpha}$ is a winning strategy of player I in $*^{\varepsilon}_{D}[Q]$

(b) for $\alpha < \beta < \alpha^*$, $\bar{F}^{\beta}$ is above $\bar{F}^{\alpha}$

(c) if $\langle (\bar{q}^{\zeta}, f_{\zeta}, \bar{p}^{\zeta}) : \zeta < \varepsilon \rangle$ is a play of $*^{\varepsilon}_{D}[Q]$, Player I uses his strategy $\bar{F}^{\beta}$, then for any $i < \mu^+$, letting $F^{\alpha}(\langle \bar{p}^{\xi} : \xi < \zeta \rangle) = (\bar{q}^{\alpha,\zeta}, f'_{\alpha,\zeta})$ we have:

$$Q \models \mathbf{St}(\langle q^{\alpha,\zeta}_i : \alpha < \beta \rangle) \leq q^{\alpha,\zeta}_i.$$

(3) Similarly to (1), (2) for the game $\otimes^{\varepsilon}_{Q}$ (instead $*^{\varepsilon}_{D}[Q]$), omitting **St** and clause (c) in (2).

OBSERVATION 10.

(1) Assume Q is μ-complete. If $\delta < \mu$ and $\langle \bar{F}^{\alpha} : \alpha < \delta \rangle$ is an increasing sequence of winning strategies of player I in $*^{\varepsilon}_{D}[Q]$, then some winning strategy $\bar{F}^{\delta}$ of player I in $*^{\varepsilon}_{D}[Q]$ is above every $\bar{F}^{\alpha}(\alpha < \delta)$.

(2) Assume $\beta^* \leq \mu$ and Q is β^*-strategically complete with a winning strategy **St**. If $\beta < \beta^*$ and $\langle \bar{F}^{\alpha} : \alpha < \beta \rangle$ is an increasing sequence of winning strategies of player I in $*^{\varepsilon}_{D}[Q]$ obeying **St**, then for some $\bar{F}^{\beta}$, $\langle \bar{F}^{\alpha} : \alpha < \beta + 1 \rangle$ is an increasing sequence of winning strategies of player I in $*^{\varepsilon}_{D}[Q]$ obeying **St**.

(3) Similarly with $\otimes^{\varepsilon}_{Q}$ instead of $*^{\varepsilon}_{Q}[D]$.

PROOF. Straight. ⊣

DEFINITION 11. Assume P, R are forcing notions, $P \subseteq R$, $P \lessdot R$.

(1) We say $\restriction$ is a restriction operation for the pair (P, R) (or $(P, R, \restriction)$) is a strong restriction triple if (P, R are as above, of course, and) for every member $r \in R$, $r \restriction P \in P$ is defined such that:

(a) $r \restriction P \leq r$,

(b) if $r \restriction P \leq p \in P$ then r, p are compatible in R in fact have a least upper bound,

(c) $r^1 \leq r^2 \Longrightarrow r^1 \restriction P \leq r^2 \restriction P$,

(d) if $p \in P$ then $p \restriction P = p$ and $\emptyset_R \restriction P = \emptyset_P$

(so this is a strong, explicit way to say $P \lessdot R$).

(1′) We say weak restriction triple if we omit in clause (b) the "have a least upper bound".

(2) We say "$(P, R, \restriction)$ is ε-strategically complete" if

(α) $\restriction$ is a restriction operation for the pair (P, R).

(β) P is ε-strategically complete.

(γ) if $\mathbf{St}_1$ is a winning strategy for player I in the game $\otimes^\varepsilon_P$, then in the game $\otimes^\varepsilon = \otimes^\varepsilon[P, R, \restriction; \mathbf{St}_1]$ the first player has a winning strategy $\mathbf{St}_2$.

Playing. A play of $\otimes^\varepsilon$ is a play $\langle (p_\zeta, q_\zeta) : \zeta < \varepsilon \rangle$ of $\otimes^\varepsilon_R$ but

(α) $\langle (q_\zeta \restriction P, q_\zeta \restriction P) : \zeta < \varepsilon \rangle$ is a play of the game $\otimes^\varepsilon_P$ in which the first player uses the strategy $\mathbf{St}_1$ (see 7 (2)!).

Outcome. If condition $(\beta)_\zeta$ below fails in stage ζ for some $\zeta < \varepsilon$ then the first player loses immediately, and if not, then he wins.

$(\beta)_\zeta$ for every $\zeta < \varepsilon$, if $\mathbf{St}_1$ dictate to player 1 in the play $\langle (q_\xi \restriction P, p_\xi \restriction P) : \xi < \zeta \rangle$ to choose $q'_\zeta \in P$ and $p \in P$ is above $q'_\zeta \in P$ then $\{p\} \cup \{ q_\xi : \xi < \zeta \}$ has an upper bound. (Read second sentence in 7 (2)).

(2′) We say $(P, R, \restriction)$ is $(< \varepsilon)$-strategically complete if it is ζ-strategically complete for every $\zeta < \varepsilon$.

(3) Let "$(P, R, \restriction)$ satisfy $*^\varepsilon_\mu$" mean (for this and in other definitions many times $\restriction$ will be understood from context hence omitted):

(α) $\restriction$ is a restriction operation for the pair (P, R)

(β) P satisfies $*^\varepsilon_\mu$

(γ) If $\mathbf{St}_1$ is a winning strategy for player I in the game $*^\varepsilon_\mu[P]$ then in the following game called $*^\varepsilon_\mu[P, R, \restriction; \mathbf{St}_1]$ the first player has a winning strategy $\mathbf{St}_2$.

Playing. As before in $*^\varepsilon_\mu[R]$, but

$$\langle\langle q^\zeta_i \restriction P : i < \mu^+\rangle, \langle p^\zeta_i \restriction P : i < \mu^+\rangle, f_\zeta : \zeta < \varepsilon\rangle$$

is required to be a play of $*^\varepsilon_\mu[P]$ in which first player uses the strategy $\mathbf{St}_1$ (see the second sentence of 7 (2)).

We also demand that if $\{ p^\zeta_j : j < i \} \subseteq P$, then $q^\zeta_i \in P$; (seem technical, but help in iterations).

Outcome. Player I wins provided that for each $i < \mu^+$ and limit $\zeta < \varepsilon$ the sequence $\langle (q^\xi_i, p^\xi_i) : \xi < \zeta \rangle$ satisfies clause $(\beta)_\zeta$ above and:

($*$) for some club E of μ^+ if $i < j$ are from E, $\mathrm{cf}(i) = \mathrm{cf}(j) = \mu$,

$$\bigwedge_{\zeta<\varepsilon} f_\zeta(i) = f_\zeta(j)$$

and $r \in P$ is a $\leq_P$-upper bound of $\{ p^\zeta_i \restriction P : \zeta < \varepsilon \} \cup \{ p^\zeta_j \restriction P : \zeta < \varepsilon \}$, then[1] $\{r\} \cup \{ p^\zeta_i : \zeta < \varepsilon \} \cup \{ p^\zeta_j : \zeta < \varepsilon \}$ has an upper bound in R.

[1]Could let some strategy determine r, no need at present.

In this case we say that $\mathbf{St}_2$ projects to $\mathbf{St}_1$ or is above $\mathbf{St}_1$. If we omit the demand on the outcome (so maybe $\mathbf{St}_2$ is not necessarily a winning strategy of player I in $*^\varepsilon_\mu[R]$), we say $\mathbf{St}_2$ weakly projects to $\mathbf{St}_1$.

NOTE. Naturally in $\mathbf{St}_2$ the functions f_ζ code more information than $\mathbf{St}_1$, so we may use a function g to decode the "older" part.

(3′) The game $*^\varepsilon_D[P, R, \restriction]$ and "$(P, R, \restriction)$ satisfies $*^\varepsilon_D$" are defined naturally and similarly projections of strategies. Similarly concerning part (4).

(4) We say $(P, R, \restriction)$ satisfies strongly $*^\varepsilon_\mu$ if (so when $\restriction$ is clear from context, it is omitted; not used):

(α) $\restriction$ is a restriction operation for the pair (P, R)

(β) P satisfies $*^\varepsilon_\mu$

(γ) the first player has a winning strategy in the game $*^\varepsilon_\mu[P, R, \restriction]$ where

PLAYING. Just like a play of $*^\varepsilon_\mu[R]$, except that

$\oplus$ in addition, for every limit ordinal $\zeta < \varepsilon$, in the ζth move first the second player is allowed to choose $\langle r^\zeta_i : i < \mu^+ \rangle$ such that:

$$r^\zeta_i \in P$$

is an upper bound of $\{ p^\xi_i \restriction P : \xi < \zeta \}$ and the first player choosing q^ζ_i has to satisfy also $(\forall^D i)\ (r^\zeta_i \leq q^\zeta_i)$.

OUTCOME. Player I wins if $(*)$ from part (3) holds or

$(*)^-$ in the play $\langle \langle p^\zeta_i \restriction P : i < \mu^+ \rangle, \langle q^\zeta_i \restriction P : i < \mu^+ \rangle : \zeta < \varepsilon \rangle$ of $*^\varepsilon_\mu[P]$ the first player loses, (note concerning the outcome, then now in $(*)$ in part (3), the existence of r is not (even essentially) guaranteed); so possibly for some $\zeta < \varepsilon$ player I has no legal move.

(5) If $\restriction_\ell$ is a restriction operation for $(P_\ell, P_{\ell+1})$ for $\ell = 1, 2$, $\restriction = \restriction_1 \circ \restriction_2$, then "a strategy $\mathbf{St}$ of first player in $*^\varepsilon_\mu[P_1, P_3]$ project to one for $*^\varepsilon_\mu[P_1, P_2]$" is defined naturally.

REMARK 12. We may restrict ourselves to a suitable family of strategies $\mathbf{St}_1$ (to work in the iteration this family has to be suitably closed).

CLAIM 13.

(1) *If the forcing notion P satisfies $*^\varepsilon_\mu$ then P satisfies the μ^+-c. c.*

2) *If P satisfies $*^\varepsilon_\mu$ and R is the trivial forcing $\{\emptyset_P\}$ then the pair (R, P) satisfies $*^\varepsilon_\mu$ where $\restriction$ is defined by $p \restriction R = \emptyset$.*

(3) *If $(P, R, \restriction)$ satisfies $*^\varepsilon_\mu$ then P and R satisfy $*^\varepsilon_\mu$.*

(4) *If triples $(P_0, P_1, \restriction_0)$, $(P_1, P_2, \restriction_1)$ satisfy $*^\varepsilon_\mu$ then $(P_0, P_2, \restriction_0 \circ \restriction_1)$ satisfies $*^\varepsilon_\mu$.*

(5) *If P satisfies $*^\varepsilon_\mu$ and $\Vdash_P$ "$\underset{\sim}{Q}$ satisfies $*^\varepsilon_\mu$" then $P * \underset{\sim}{Q}$ satisfies $*^\varepsilon_\mu$, moreover the pair $(P, P * \underset{\sim}{Q})$ (with the natural $\restriction$) satisfies $*^\varepsilon_\mu$.*

PROOF. Should be clear. ⊣

REMARK 14.

(1) If D is a normal filter on μ^+ to which $\{ \delta < \mu^+ : \mathrm{cf}(\delta) = \mu \}$ belongs, then in 13 we can replace $*^\varepsilon_\mu$ by $*^\varepsilon_D$ (of course, in part (5), D in V^P means the normal filter it generates).

Similarly for the claim below.

(2) Assume that in the game of choosing $A_i \in D^+$ for $i < \varepsilon$ (or $i < \mu$), with player I choosing A_{2i}, player II choosing A_{2i+1}, A_i decreasing, player II loses if and only if he sometime has no legal move; player I has a strategy guaranteeing that he has legal moves. (If κ in measurable V in $V^{\mathrm{Levy}(\mu<\kappa)}$ this holds for some D by [1].) In fact assume more generally that $\mathscr{P}$ is a partial order and $\mathscr{F} : \mathscr{P} \to \{ A : A \subseteq \mu^+ \}$ is decreasing:

$$\mathscr{P} \models x \leq y \Longrightarrow \mathscr{F}(y) \subseteq \mathscr{F}(x)$$

and $\mathscr{E}$ is a function with domain $\mathscr{P}$ where $\mathscr{E}(x)$ is a non-empty subset of $[\mathscr{F}(x)]^2$ and

$$\mathscr{P} \models x \leq y \Longrightarrow \mathscr{E}(y) \subseteq \mathscr{E}(y)$$

and if $x \in \mathscr{P}$, E is a club of μ^+ and f be a pressing down function from μ^+ to μ^+ then for some y satisfying $x \leq y$ we have $f \restriction \{ \sup(E \cap \alpha) : \alpha \in \mathscr{F}(y) \}$ is constant (above $\mathscr{P} = (D^+, \supseteq)$, $\mathscr{F}$ is the identity $\mathscr{E}(x) = [\mathscr{F}(x)]^2$ and we say that a forcing notion Q satisfies $*^{\varepsilon}_{\mathscr{P},\mathscr{F},\mathscr{E}}$ if in the following game $*^{\varepsilon}_{\mathscr{P},\mathscr{F},\mathscr{E}}[Q]$, the first player has a winning strategy.

A play lasts ε moves, in the ζth move player I chooses $x_\zeta \in \mathscr{P}$ such that

$$\xi < \zeta \Longrightarrow y_\xi \leq_{\mathscr{P}} x_\zeta$$

and if $\zeta > 0$ also $\langle q^\zeta_i : i \in \mathscr{F}(x_\zeta) \rangle$ such that

$$\xi < \zeta \text{ and } i \in \mathscr{F}(x_\zeta) \Longrightarrow p^\xi_i \leq q^\zeta_i$$

and player II chooses $y_\zeta \in \mathscr{P}$ such that $x_\zeta \leq y_\zeta$ and $\langle p_i : i \in \mathscr{F}(y_\zeta) \rangle$ such that

$$\zeta > 0 \wedge i \in \mathscr{F}(y_\zeta) \Longrightarrow q^\zeta_i \leq_Q p^\zeta_i.$$

OUTCOME. Player I wins a play if

(α) for every limit $\zeta < \varepsilon$ he has a legal move (this depends on having upper bounds in $\mathscr{P}$ and in Q)

(β) for every $\{i, j\} \in \bigcap_{\zeta<\varepsilon} \mathscr{E}(x_\zeta)$, in Q there is an upper bound to

$$\{ p^\zeta_i : \zeta < \varepsilon \} \cup \{ p^\zeta_j : \zeta < \varepsilon \}.$$

The natural generalizations of the relevant lemmas works for this notion.

(3) We can systematically use the weak restriction triples, and/or use the strong version of $*^{\varepsilon}_{\mu}$ for triples in this paper.

CLAIM 15.

(1) *If the forcing notions P_1, P_2 are equivalent then P_1 satisfies $*^{\varepsilon}_{\mu}$ if and only if P_2 satisfies $*^{\varepsilon}_{\mu}$.*

(2) *Suppose $\restriction$ is a restriction operation for (P_1, P_2), B_ℓ the complete Boolean algebra corresponding to P_ℓ (so $B_1 \lessdot B_2$) and $\restriction'$ is the projection from B_2 to B_1 and $P'_\ell = (B_\ell \setminus \{0\}, \geq)$ then*

(a) *$(P'_1, P'_2, \restriction')$ is a restriction triple and*

(b) *$(P_1, P_2, \restriction)$ satisfies $*^{\varepsilon}_{\mu}$ if and only if $(P'_1, P'_2, \restriction')$ satisfies $*^{\varepsilon}_{\mu}$.*

(2′) *In part* (2) *it is enough to assume that* $\restriction$ *is a weak restriction operation.*

(3) *If a forcing notion* Q *satisfies* $*^{\varepsilon}_{\mu}$ *then player I has a winning strategy in the play even if we demand from him*:

$$\bigwedge_{\xi<\zeta}[p^{\xi}_i = \emptyset_Q \Longrightarrow q^{\zeta}_i = \emptyset_Q]$$

for each $i < \mu^+$.

(4) *Similarly for* $(P, R, \restriction)$ *satisfying* $*^{\varepsilon}_{\mu}$ *demanding*

$$\bigwedge_{\xi<\zeta}[p^{\xi}_i = \emptyset_R \Longrightarrow q^{\zeta}_i = \emptyset_R] \quad \text{and} \quad \bigwedge_{\xi<\zeta}[p^{\xi}_i \in P \Longrightarrow q^{\zeta}_i \in P].$$

CONVENTION 16. Strategies are as in 15 (3), (4).

DEFINITION/CLAIM 17. Assume for $\ell = 1, 2$ that $(P, R_\ell, \restriction_\ell)$ is a restriction triple, $(P, R_\ell, \restriction_\ell)$ satisfies $*^{\varepsilon}_{\mu}$, and we let

$$R = \{ (p, r_1, r_2) : p \in P,\ r_1 \in R_1,\ r_2 \in R_2,$$
$$P \models \text{“}r_1\restriction P \le p\text{” and } P \models \text{“}r_2\restriction P \le p\text{”} \}$$

identifying $r_1 \in R_1$ with $(r_1\restriction P, r_1, \emptyset_{R_2})$, and identifying $r_2 \in R_2$ with $(r_2\restriction P, \emptyset_{R_1}, r_2)$.

Under the quasi order

$$(p, r_1, r_2) \le (p'_1, r'_1, r'_2) \quad \text{if and only if} \quad p \le_P p' \text{ and}$$
$$\mathrm{lub}_{R_1}\{p, r_1\} \le_{R_1} \mathrm{lub}_{R_1}\{p', r'_1\} \text{ and } \mathrm{lub}_{R_2}\{p, r_2\} \le_{R_2} \mathrm{lub}_{R_2}\{p', r'_2\}.$$

Then $R_\ell \lessdot R$ (for $\ell = 1, 2$) and $(R_\ell, R, \restriction'_\ell)$ is a restriction triple and it satisfies $*^{\varepsilon}_{\mu}$, where $(p, r_1, r_2)\restriction'_\ell R_\ell =$ the least upper bound of p, r_ℓ in R_ℓ (see clause (b) of Definition 11 (1)). Moreover if for $\ell = 1, 2$ we have $\mathbf{St}_\ell$ is a winning strategy for player I in the game $*^{\varepsilon}_{\mu}$ for R_ℓ projecting to $\mathbf{St}_0$, a winning strategy for player I in the game $*^{\varepsilon}_{\mu}$ for P, then player I has a winning strategy in the game $*^{\varepsilon}_{\mu}$ for R which project to $\mathbf{St}_\ell$ for $\ell = 1, 2$.

DEFINITION/LEMMA 18. Let $\mu = \mu^{<\mu} < \kappa = \mathrm{cf}(\kappa) \le \lambda \le \chi$. (Usually fixed hence suppressed in the notation.) We define and prove the following by induction on (the ordinal) α:

(1) [Definition]. Let $\mathscr{K}^\alpha = \mathscr{K}^\alpha_{\mu,\kappa,\lambda,\chi}$ be the family of sequences

$$\bar{Q} = \langle P_\beta, \underset{\sim}{Q}_\beta, a_\beta : \beta < \alpha \rangle$$

such that:

- (a) $\langle P_\beta, \underset{\sim}{Q}_\beta : \beta < \alpha \rangle$ is a $(< \mu)$-support iteration (so $P_\alpha = \mathrm{Lim}_\mu\, \bar{Q}$ denotes the natural limit)
- (b) $a_\beta \subseteq \beta$, $|a_\beta| < \kappa$, $[\gamma \in a_\beta \Longrightarrow a_\gamma \subseteq a_\beta]$
- (c) $\underset{\sim}{Q}_\beta$ is $(< \mu)$-strategically complete, has cardinality $< \lambda$ and is a $P^*_{a_\beta}$-name (see parts 18 (2) (b) and 18 (5) (b) below).

(1′) [Definition]. $\bar{Q}$ is called standard if: for every $\beta < \lg(\bar{Q})$ each element of $\underset{\sim}{Q}_\beta$ is from V, even from $\mathscr{H}(\chi)$, and the order is a fixed quasi order from V such that any chain of length $< \mu$ which has an upper bound has a least upper bound and for any sequence $\bar{x} = \langle x_i : i < \delta < \mu \rangle$, for some y we have $p \Vdash_{P_\alpha}$

"if $\bar{x}$ is $\leq_{Q_\alpha}$-increasing then y is its lub" (we can use less), but note that the set of elements is not necessarily from V.

(2) [Definition]. For $\bar{Q}$ as above:

(a) $a \subseteq \alpha$ is called $\bar{Q}$-closed if $[\beta \in a \Longrightarrow a_\beta \subseteq a]$; we also call it $\langle a_\beta : \beta < \alpha \rangle$-closed and let $\bar{a}^{\bar{Q}} = \langle a_\beta : \beta < \alpha \rangle$

(b) for a $\bar{Q}$-closed subset a of α we let

$$P_a = \{ p \in P_\alpha : \mathrm{Dom}(p) \subseteq a \text{ and for each } \beta \in \mathrm{Dom}(p) \text{ we have: } p(\beta) \text{ is a } P_{a\cap\beta}\text{-name (i.e., involving only } G_{P_\beta} \cap P_{a\cap\beta} \text{ so necessarily } Q_\wedge \in V[G_{P_\beta} \cap P_{a\cap\beta}]) \}$$

$$P^*_a = \{ p \in P_\alpha : \mathrm{Dom}(p) \subseteq a \text{ and for each } \beta \in \mathrm{Dom}(p) \text{ we have: } p(\beta) \text{ is a } P^*_{a_\beta}\text{-name and: if } \bar{Q} \text{ is standard, then } p(\beta) \text{ is from } V \text{ not just a name} \}.$$

On both P_a and P^*_a, the order is inherited from P_α. Note that P^*_a is defined by induction on $\sup(a)$.

(3) [Lemma]. For $\bar{Q}$ as above, $\beta < \alpha$

(a) $\bar{Q} \restriction \beta \in \mathscr{K}^\beta$ and is standard if $\bar{Q}$ is

(b) if $a \subseteq \beta$ then: a is $\bar{Q}$-closed if and only if a is $(\bar{Q}\restriction\beta)$-closed

(c) if $a \subseteq \alpha$ is $\bar{Q}$-closed, then so is $a \cap \beta$, in fact β is $\bar{Q}$-closed and the intersection of a family of $\bar{Q}$-closed subsets of α is $\bar{Q}$-closed.

(4) [Lemma]. For $\bar{Q}$ as above, and $\beta < \alpha$,

(a) $P_\beta \lessdot P_\alpha$, moreover, if $p \in P_\alpha$, $p\restriction\beta \leq q \in P_\beta$ then $(p\restriction(\alpha \setminus \beta)) \cup q \in P_\alpha$ is a least upper bound of p, q

(b) P_α / P_β is $(< \mu)$-strategically complete (hence does not add new sequences of length $< \mu$ of old elements).

(5) [Lemma]. For $\bar{Q}$ as above

(a) P^*_α is a dense subset of P_α

(b) if a is $\bar{Q}$-closed then $P_a \lessdot P_\alpha$ and P^*_a is a dense subset of P_a.

(c) if a is $\bar{Q}$-closed, $p \in P_\alpha$, $p\restriction a \leq q \in P_a$ then $(p\restriction(\alpha \setminus a)) \cup q$ belongs to P_α and is a least upper bound of p, q in P_α

(d) if a is $\bar{Q}$-closed, then $\bar{Q}\restriction a \in \mathscr{K}^{\mathrm{otp}(a)}$ (up to renaming of indexes)

(e) if $a \subseteq b \subseteq \lg(\bar{Q})$ are $\bar{Q}$-closed, then $(P^*_a, P^*_b, \restriction)$ is a restriction triple (where $p\restriction P^*_b = p\restriction a$)

6) [Lemma]. The sequence $\bar{Q} = \langle P_\beta, \underset{\sim}{Q}_\beta, a_\beta : \beta < \alpha \rangle$ belongs to $\mathscr{K}^\alpha$ if α is a limit ordinal and

$$\bigwedge_{\gamma<\alpha} \bar{Q}\restriction\gamma \in \mathscr{K}^\gamma.$$

(7) [Lemma]. The sequence $\bar{Q} = \langle P_\beta, \underset{\sim}{Q}_\beta, a_\beta : \beta < \alpha \rangle$ belongs to $\mathscr{K}^\alpha$ if $\alpha = \gamma + 1$, $a_\gamma \subseteq \gamma$ is a $(\bar{Q}\restriction\gamma)$-closed set of cardinality $< \kappa$, $\underset{\sim}{Q}_\gamma$ is a $P^*_{a_\gamma}$-name of a $(< \mu)$-strategically complete forcing notion of cardinality $< \lambda$.

8) [Definition]. $\mathscr{K}^{<\alpha} = \bigcup_{\beta<\alpha} \mathscr{K}^{\beta}$.

PROOF. Straightforward. ⊣

DEFINITION 19. Let $\mu = \mu^{<\mu} < \kappa = \mathrm{cf}(\kappa) \leq \lambda \leq \chi$ (usually fixed hence suppressed in the notation) and ε a limit ordinal $< \mu$. We define the following by induction on (the ordinal) α:

(1) We let $\mathscr{K}^{\varepsilon,\alpha} = \mathscr{K}^{\varepsilon,\alpha}_{\mu,\kappa,\lambda,\chi}$ be the family of sequences

$$\bar{Q} = \langle P_\beta, \underset{\sim}{Q}_\beta, a_\beta, I_\beta : \beta < \alpha \rangle$$

such that:

(α) $\langle P_\beta, \underset{\sim}{Q}_\beta, a_\beta : \beta < \alpha \rangle \in \mathscr{K}^\alpha$

(β) I_β is a family of $\bar{Q}$-closed (see part (2) below, it is not what was defined in 18 (2) (a)) subsets of a_β, closed under finite unions, increasing unions of length $< \mu$ and such that $\emptyset \in I_\beta$

(γ) each a_β is $(\bar{Q}\restriction\beta)$-closed (see part (2) below, this is not as in 18)

(δ) if $b \in I_\beta$ then the pair $(P^*_b, P^*_{a_\beta\cup\{\beta\}})$ satisfies $*^\varepsilon_\mu$, of course for the natural restriction operation.

(2) For $\bar{Q} \in \mathscr{K}^{\varepsilon,\alpha}$ (even satisfying just 19 (1) (α) and (β)) we say that a set a is $\bar{Q}$-closed in b (or is $\langle a_\beta, I_\beta : \beta < \alpha \rangle$-closed in b) if $a \subseteq b \subseteq \alpha$, $[\beta \in a \Longrightarrow a_\beta \subseteq a]$ and $[\beta \in b \setminus a \Longrightarrow a \cap a_\beta \in I_\beta]$. If we omit "in b" we mean $b = \alpha$.

(3) (a) $\bar{Q}$ is simple if for all $\beta < \alpha$

$$I_\beta = \{ b \subseteq a_\beta : b \text{ is } \bar{a}^{\bar{Q}\restriction\beta}\text{-closed and for every } \gamma \in a_\beta \cup \{\beta\},$$
$$\text{if } \mathrm{cf}(\gamma) = \mu^+ \text{ and } \gamma = \sup(\gamma \cap b), \text{ then } \gamma \in b \}.$$

(b) $\bar{Q}^- = \langle P_\beta, \underset{\sim}{Q}_\beta, a_\beta : \beta < \alpha \rangle$, $\bar{a}^{\bar{Q}} = \langle a_\beta : \beta < \alpha \rangle$, and $\bar{I}^Q = \langle I_\beta : \beta < \alpha \rangle$

(c) $\bar{Q}$ is standard if $\bar{Q}^-$ is standard

(d) $\mathscr{K}^{\varepsilon,<\alpha} = \bigcup_{\beta<\alpha} \mathscr{K}^{\varepsilon,\beta}$.

CLAIM 20. *Let $\bar{Q} \in \mathscr{K}^{\varepsilon,\alpha}$.*

(1) *If $\beta < \alpha$ then $\bar{Q}\restriction\beta =: \langle P_\gamma, \underset{\sim}{Q}_\gamma, a_\gamma, I_\gamma : \gamma < \beta \rangle$ belongs to $\mathscr{K}^{\varepsilon,\beta}$; moreover, if $b \subseteq \alpha$ is $\bar{a}^{\bar{Q}}$-closed then $\bar{Q}\restriction b \in \mathscr{K}^{\varepsilon,\mathrm{otp}(b)}$ (up to renaming of index sets) understanding $I^{\bar{Q}\restriction b}_\beta = I^{\bar{Q}}_\beta\restriction b$.*

(2) *If $a \subseteq b \subseteq \beta \leq \alpha$ and a is $\bar{Q}$-closed in b then: a is $(\bar{Q}\restriction\beta)$-closed in b.*

(3) *If $\beta < \alpha$, $a \subseteq \alpha$ is $\bar{Q}$-closed and $\gamma \in \alpha \setminus \beta \Longrightarrow a \cap a_\gamma \cap \beta \in I_\gamma$, then $a \cap \beta$ is $\bar{Q}$-closed.*

(4) *If $\bar{Q}$ is simple, $\beta < \alpha$, $a \subseteq \alpha$ is $\bar{Q}$-closed and $\mathrm{cf}(\beta) \neq \mu^+ \vee (\forall\gamma \in \alpha \setminus \beta)$ $(a_\gamma \cap a \cap \beta$ is bounded in $\beta)$, then $a \cap \beta$ is $\bar{Q}$-closed.*

(5) *The family of $\bar{Q}$-closed $a \subseteq \alpha$ is closed under increasing union of length $< \mu$ and $\emptyset$ belongs to it and α is $\bar{Q}$-closed.*

(6) *If a, b are $\bar{Q}$-closed, then so is $a \cup b$.*

(7) *If $a \subseteq b \subseteq c \subseteq \lg(\bar{Q})$, a is $\bar{Q}$-closed in c, then a is $\bar{Q}$-closed in b.*

(8) *If $\beta \leq \alpha$, $a \subseteq b \subseteq \alpha$, a is $\bar{Q}$-closed in b, then $a \cap \beta$ is $(Q\restriction\beta)$-closed in $b \cap \beta$.*

PROOF. Straight. ⊣

REMARK 21. Simple $\bar{Q}$ is what we shall use.

LEMMA 22. *Assume $\bar{Q} \in \mathscr{K}^{\varepsilon,\alpha}$ and a, b are $\bar{Q}^-$-closed subsets of α and a is a $\bar{Q}$-closed subset of b $(\subseteq \alpha)$ and $\bar{Q}$ is simple or at least*

$$(*) \qquad a \in \{a_\beta\} \cup I_\beta \wedge \gamma < \beta < \alpha \Longrightarrow a \cap (\gamma+1) \in I_\beta.$$

(Hence $\gamma < \beta < \alpha$ and $\mathrm{cf}(\gamma) < \mu \Longrightarrow a_\beta \cap \gamma \in I_\beta$.) *Then the pair (P^*_a, P^*_b) satisfies $*^\varepsilon_\mu$.*

PROOF. We can assume by 20 (1) that $b = \alpha$. By induction on α we shall show that for all $\bar{Q}$-closed subsets a of α the pair (P^*_a, P^*_α) satisfies $*^\varepsilon_\mu$ (see Definition 11 (3)) and this is proved first when $a = \emptyset$ and then when $a \neq \emptyset$. So we fix a strategy $\mathbf{St}_a$ for the first player in $*^\varepsilon_\mu[P^*_a]$; why does it exist? If $a = \emptyset$, trivially, if $a \neq \emptyset$ by the way the proof is arranged we know the conclusion for $(a', b') = (\emptyset, a)$, and as $\mathrm{otp}(a) \leq \alpha$ clearly $\mathbf{St}_a$ exists. Next we shall choose a strategy for the first player in the game $*^\varepsilon_\mu[P^*_a, P^*_\alpha, \mathbf{St}_a]$, where at stage $\zeta < \varepsilon$ the first player chooses $\{ q^\zeta_\xi : \xi < \mu^+ \}$, a regressive function f_ζ from μ^+ to μ^+ and the second player replies with suitable $\{ p^\zeta_\xi : \xi < \mu^+ \}$.

For simplicity the reader may assume that the $\underset{\sim}{Q}_\beta$ are μ-complete (which is the case used; otherwise we have to use the $(< \mu)$-strategic completeness (and remember 7 (2) second sentence).

CASE 1. $\alpha = \beta + 1$, $\beta \in a$.

So $a_\beta \subseteq a$, now $a \cap \beta$ is $(\bar{Q} \restriction \beta)$-closed (by 20 (2)) hence by the induction hypothesis $(P^*_{a\cap\beta}, P^*_\beta)$ satisfies $*^\varepsilon_\mu$. Apply 17 with $P^*_{a\cap\beta}$, P^*_β, P^*_a here standing for P, R_1, R_2 there and we get that (R_2, R) satisfies $*^\varepsilon_\mu$, which (translating) is the desired conclusion.

CASE 2. $\alpha = \beta + 1$, $\beta \notin a$.

We know that $a \cap a_\beta \in I_\beta$. If $a = \emptyset$ use 17, so assume $a \neq \emptyset$.

By Definition 19 (1) (δ) we know that $(P^*_{a\cap a_\beta}, P^*_{a_\beta \cup\{\beta\}})$ satisfies $*^\varepsilon_\mu$. By 17 we get that $(P^*_a, P^*_{a_\beta\cup\{\beta\}\cup a})$ satisfies $*^\varepsilon_\mu$. Now $a' =: a_\beta \cup \{\beta\} \cup a$ is $\bar{Q}$-closed by 20 (6) and $\beta \in a'$ so by Case 1 we have: $(P^*_{a'}, P^*_\alpha)$ satisfies $*^\varepsilon_\mu$. Together by 13 (4) we have: (P^*_a, P^*_α) satisfies $*^\varepsilon_\mu$.

CASE 3. α a limit ordinal, $\mathrm{cf}(\alpha) \leq \mu$.

Here we use 15 (3).

We can find an increasing continuous sequence $\langle \gamma_\Upsilon : \Upsilon < \mathrm{cf}(\alpha) \rangle$ of ordinals $< \alpha$ with limit α, $\gamma_0 = 0$ and $\gamma_{\Upsilon+1}$ a successor ordinal. Note that $(a \cap \gamma_{\Upsilon+1}) \cup \gamma_\Upsilon$ is $(\bar{Q} \restriction \gamma_{\Upsilon+1})$-closed as $[\gamma_\Upsilon$ limit $\Longrightarrow \Upsilon$ limit and $\mathrm{cf}(\Upsilon) < \mu]$ moreover $a \cup \gamma_\Upsilon$ is $\bar{Q}$-closed. We define by induction on $\Upsilon \leq \mathrm{cf}(\alpha)$ a strategy $\mathbf{St}^*_\Upsilon$ of player I in the game $*^\varepsilon_\mu[P^*_a, P^*_{a\cup\gamma_\Upsilon}]$ such that for $\Upsilon_1 < \Upsilon$ we have that $\mathbf{St}^*_\Upsilon$ projects to $\mathbf{St}^*_{\Upsilon_1}$ (see Definition 11 (4)) and $\mathbf{St}^*_0$ is $\mathbf{St}_a$.

If we do not assume that all the $\underset{\sim}{Q}_\beta$ are μ-complete, then we demand that, moreover, they satisfy:

$\boxtimes$ if $\langle \langle q^\zeta_i : i < \mu^+ \rangle, f_\zeta, \langle p^\zeta_i : i < \mu^+ \rangle : \zeta < \varepsilon \rangle$ is a play of $*^\varepsilon_\mu[P^*_a, P^*_{a\cup\gamma_\Upsilon}, \mathbf{St}_a]$, then for any ordinal β, looking at $\langle q^\zeta_i(\beta), p^\zeta_i(\beta) : \zeta < \varepsilon \rangle$ letting

$$\zeta(\beta, \emptyset) = \min\{ \zeta : q^\zeta_i(*\beta) \neq \emptyset_Q \}$$

if $\zeta \in [\zeta(\beta,0),\zeta(\beta,1))$ and $q_i^\zeta \restriction \beta$ forces that $\langle q_i^\xi(\beta) : \xi \in [\zeta,(\beta,0),\zeta] \rangle$ is increasing, then $q_i^\zeta \restriction \beta$ forces that some $\langle q'_\xi, p'_\xi : \xi < \zeta - \zeta(\beta,0)+1 \rangle$ is a play of $\otimes^\varepsilon_{\underset{\sim}{Q}_\beta}$ in which player I uses a fix winning strategy (as in 7 (2)!) and $p'_0 = q_i^{\zeta(\beta,0)}(\beta)$, (remember q'_0 not chosen) and

$$0 < \xi < \zeta - \zeta(\beta,0)+1 \Longrightarrow q'_\xi = q_i^{\zeta(\beta,0)+\xi}(\beta)$$

and

$$0 < \xi < \zeta - \zeta(\beta,0) \Longrightarrow p'_\xi = p_i^\xi(\beta).$$

This, of course, puts on us a burden also in successor γ just to increase the condition.

The inductive step is done by 17, the limit stage is straight (using ⊠ to show we can).

CASE 4. α limit ordinal, $\mathrm{cf}(\alpha) > \mu^+$.

During the play, player I in the ζth move also chooses an ordinal γ_ζ, γ_ζ increases continuously with ζ, $\gamma_0 = 0$ as follows:

$$\gamma_{\zeta+1} = \min\{\gamma < \alpha : (\forall i < \mu^+)(\forall \xi \leq \zeta)\,(p_i^\xi, q_i^\xi \in P_\gamma) \text{ and } \gamma \text{ is a successor ordinal}\}$$

and he will make $q_i^\zeta \in P_{\gamma_\zeta}$, and the rest is as in Case 3.

CASE 5. $\mathrm{cf}(\alpha) = \mu^+$.

Let $\langle \gamma_\Upsilon : \Upsilon < \mu^+ \rangle$ be increasing continuously with limit α, $\gamma_0 = 0$, $\mathrm{cf}(\gamma_\Upsilon) \leq \mu$, $\gamma_{\Upsilon+1}$ a successor ordinal and we imitate Case 4, separating to different plays according to the value of

$$j_i^\zeta = \min\{j < i : \text{for each } \xi < \zeta \text{ we have } p_i^\xi \restriction \gamma_i \in P_{\gamma_j} \text{ and } q_i^\xi \restriction \gamma_i \in P_{\gamma_j}\}. \qquad \dashv$$

CLAIM 23. *Assume*

(a) $\bar{Q} = \langle P_\alpha, \underset{\sim}{Q}_\alpha, a_\alpha, I_\alpha : \alpha < \delta \rangle$
(b) *δ a limit ordinal*
(c) *for every $\alpha < \delta$ we have $\bar{Q} \restriction \alpha \in \mathscr{K}^{\varepsilon,\alpha}$.*

Then $\bar{Q} \in \mathscr{K}^{\varepsilon,\delta}$.

PROOF. Check. ⊣

CLAIM 24. *Assume*

(a) $\bar{Q} \in \mathscr{K}^{\varepsilon,\alpha}$
(b) *$a_\alpha \subseteq \alpha$ is $\bar{Q}$-closed, $|a_\alpha| < \kappa$*
(c) *$I_\alpha \subseteq \{b \subseteq a_\alpha : b$ is $\bar{Q}$-closed$\}$*
(d) *I_α is closed under finite unions, I_α is closed under increasing unions of length $< \mu$ and $\emptyset \in I_\alpha$*
(e) *$\underset{\sim}{Q}_\alpha$ is a $P^*_{a_\alpha}$-name of a forcing notion of cardinality $< \lambda$*
(f) *if $b \in I_\alpha$ then $(P_b, P^*_{a_\alpha} * \underset{\sim}{Q}_\alpha)$ satisfies $*^\varepsilon_\mu$*
(g) $P_\alpha = \mathrm{Lim}_\mu\, \bar{Q}$.

Then $\bar{Q}{}^\frown\langle P_\alpha, \underset{\sim}{Q}_\alpha, a_\alpha, I_\alpha\rangle$ belongs to $\mathscr{K}^{\varepsilon,\alpha+1}$.

PROOF. Check. ⊣

THEOREM 25. *Suppose $\mu = \mu^{<\mu} < \kappa = \lambda < \chi$ and χ is measurable.*

(1) *For some forcing notion P of cardinality χ, μ-complete not collapsing cardinalities not changing cofinalities we have*:

$$\Vdash_P \text{“}2^\mu = \chi \text{ and for every } \sigma < \mu \text{ and } \theta < \kappa \text{ we have } \chi \to [\theta]^2_{\sigma,2}\text{”}$$

(*and for a fixed ε we can add the Axiom*: *if Q is a μ-complete forcing notion of cardinality $< \kappa$ satisfying $*^\varepsilon_\mu$ and $\mathscr{I}_\alpha \subseteq Q$ dense for $\alpha < \alpha^* < \chi$ then some directed $G \subseteq Q$ is not disjoint to any $\mathscr{I}_\alpha$*).

(2) *We can replace "μ-complete" by "$(< \mu)$-strategically complete"* (*in the demand on P and, in the axiom, on Q*).

REMARK 26. We can add "P satisfies $*^\varepsilon_\mu$" if the appropriate squared diamond holds which is true in reasonable inner models.

PROOF. We concentrate on part (2). If we would like to do part (1), we should just demand all the $\underset{\sim}{Q}_i$ are μ-complete.

STAGE A. Fix a limit ordinal $\varepsilon < \mu$ and let

$$\mathscr{K}^\alpha_* = \{\, \bar{Q} \in \mathscr{K}^{\varepsilon,\alpha} : \bar{Q} \text{ is simple and standard} \,\},$$
$$\mathscr{K}_* = \bigcup_{\alpha<\chi} \mathscr{K}^\alpha_*.$$

(Note: $\bar{Q}$-closed will mean as in 19 (3) (a), 19 (2).) As the $\bar{Q}$'s are simple we shall not write the I's. By preliminary forcing without loss of generality "χ measurable" is preserved by forcing with $({}^{\chi>}2, \trianglelefteq)$ (= adding a Cohen subset of χ), see Laver [2]. Let us define a forcing notion R:

$$R = \{\, \bar{Q} : \bar{Q} \in \mathscr{K}^\alpha_* \text{ for some } \alpha < \chi \text{ and } \bar{Q} \in \mathscr{H}(\chi) \,\}$$

ordered by: $\bar{Q}^1 \le \bar{Q}^2$ if and only if $\bar{Q}^1 = \bar{Q}^2 \restriction \lg(\bar{Q}^1)$.

As R is equivalent to $({}^{\chi>}2, \trianglelefteq)$ we know that in V^R, χ is still measurable. Let $\bar{Q}^\chi = \langle P_\beta, \underset{\sim}{Q}_\beta, a_\beta : \beta < \chi \rangle$ be $\bigcup G_R$ and P_χ be the limit so $P^* = P^*_\chi \subseteq P_\chi$ is a dense subset, those are R-names. Now $R * \underset{\sim}{P}^*$ is the forcing P we have promised. The non-obvious point is $\Vdash_{R * \underset{\sim}{P}^*_\chi}$ "$\chi \to [\theta]^2_{\sigma,2}$" (where $\theta < \kappa$, $\sigma < \mu$). So suppose $(r^*, \underset{\sim}{p}^*) \in R * \underset{\sim}{P}^*_\chi$ and $(r^*, \underset{\sim}{p}^*) \Vdash$ "the colouring $\underset{\sim}{\tau} : [\chi]^2 \to \sigma$ is a counterexample". Let $\chi_1 = (2^\chi)^+$. Let $G_R \subseteq R$ be generic over V, $r^* \in G_R$. By [7], but the meaning is explained below in V^R we can find an end extension strong $(\chi_1, \chi, \chi, 2^{\kappa+\lambda+2^\mu}, (\kappa + \lambda + 2^\mu)^+, \omega)$-system $\bar{M} = \langle M_s : s \in [B]^{<\aleph_0} \rangle$ such that $M_s \prec (\mathscr{H}(\chi_1)^{V[G_R]}, \mathscr{H}(\chi_1), \in)$, for $x = \{\chi, G_R, p^*, \underset{\sim}{\tau}\}$, (i.e., $x \in \bigcap_s M_s$ and $B \in [\chi]^\chi$). We do not define this as for helping to prove the next theorem (27) we assume less, in $V[G_R]$:

$(*)_0$ $\bar{M} = \langle M_s : s \in [B]^{<(1+n^*)} \rangle$ is an end extension $(\chi_1, \chi, \chi, 2^{\kappa+\lambda+2^\mu}, (\kappa + \lambda + 2^\mu)^+, n^*)$-system for x, for some $2 \le n^* \le \omega$.

where $(*)_0$ means, in $V[G_R]$:

$(*)'$ $B \in [\chi]^\chi$ and $M_s \prec (\mathscr{H}(\chi_1)^{V[G_R]}, G_R, \mathscr{H}(\chi_1), \in), x \in \bigcap_s M_s, M_s \cap M_t = M_{s\cap t}$. Furthermore, $\|M_s\| = 2^{\kappa+\lambda+2^\mu}$ and $[M_s]^{\kappa+\lambda+2^\mu} \subseteq M_s$. In addition, for v_1, $v_2 \in [B]^n$, $n < 1 + n^*$ there is f_{v_1,v_2}, the unique isomorphism from M_{v_1} onto M_{v_2}, and:

$$|v_1 \cap \varepsilon_1| = |v_2 \cap \varepsilon_2|,\ \varepsilon_1 \in v_1,\ \varepsilon_2 \in v_2 \Longrightarrow f_{v_1,v_2}(\varepsilon_1) = \varepsilon_2.$$

Finally, $s \lhd t \Longrightarrow M_s \cap \chi \lhd M_t \cap \chi$.

We meanwhile concentrate on case $n^* = 2$.

STAGE B. We assume $(*)^\vee$.

Let

$$C = \{\delta < \chi : \delta = \sup(B \cap \delta) \text{ and } (s \in [B \cap \delta]^n \text{ for some } n < 1 + n^* \Longrightarrow M_s \cap \chi \subseteq \delta)\}.$$

Let $\gamma(*) = \min(B)$. Now for $p \in P^*_\chi \cap M_{\{\gamma(*)\}}$ and $\bar{c} = \langle c_1, c_2 \rangle \in \sigma \times \sigma$ let us define the statement

$(*)^{\bar{c}}_p$ if $p \leq p^0 \in P^* \cap M_{\{\gamma(*)\}}$ then we can find p^1, $p^2 \in P^*_\chi \cap M_{\{\gamma(*)\}}$, $p^0 \leq p^1$, $p^0 \leq p^2$ such that: for $\gamma_1 < \gamma_2$, $\gamma_1 \in B$, $\gamma_2 \in B$, we can find r_1, $r_2 \in P^* \cap M_{\{\gamma_1, \gamma_2\}}$ (so $\mathrm{Dom}(r_\ell) \subseteq M_{\{\gamma_1, \gamma_2\}} \cap \chi$) satisfying for $\ell = 1, 2$:

$$r_\ell \Vdash \text{“}\underset{\sim}{\tau}(\{\gamma_1, \gamma_2\}) = c_\ell\text{”}$$

$$r_\ell \restriction (\chi \cap M_{\{\gamma_\ell\}}) \leq f_{\{\gamma(*)\},\{\gamma_\ell\}}(p^1) \qquad \text{(for strong system: equality)}$$

$$r_\ell \restriction (\chi \cap M_{\{\gamma_{3-\ell}\}}) \leq f_{\{\gamma(*)\},\{\gamma_{3-\ell}\}}(p^2) \qquad \text{(for strong system: equality).}$$

As $|\sigma \times \sigma| < \mu$ and the relevant forcing notions are $(< \mu)$-strategically complete, easily

$$\mathscr{I} = \{ p \in P^* \cap M_{\{\gamma(*)\}} : \text{for some } \bar{c},\ (*)^{\bar{c}}_p \text{ hold} \}$$

is a dense subset of $P^*_\chi \cap M_{\{\gamma(*)\}}$, but this partial forcing satisfies the μ^+-c. c. Hence we can find $\mathscr{I}^* = \{ p_\zeta : \zeta < \mu \} \subseteq \mathscr{I}$, a maximal antichain of $P^*_\chi \cap M_{\{\gamma(*)\}}$ hence of P^*_χ (as ${}^{\mu \geq}(M_{\{\gamma(*)\}})$ is a subset of $M_{\{\gamma(*)\}}$). For $p \in \mathscr{I}^*$ we can choose $c_1(p)$, $c_2(p) \in \sigma$ such that: $(*)^{(c_1(p), c_2(p))}_p$ hold.

STAGE C. As G_R was any subset of R generic over V to which r^* belongs, there are R-names $\underset{\sim}{\gamma}(*)$, $\langle (\underset{\sim}{p}_\xi, \underset{\sim}{c}_1(\underset{\sim}{p}_\xi), \underset{\sim}{c}_2(\underset{\sim}{p}_\xi)) : \xi < \mu \rangle$, $\langle \underset{\sim}{M}_s : s \in [\underset{\sim}{B}]^{<\aleph_0} \rangle$, $\langle \underset{\sim}{f}_{s,t} : (s,t) \in \bigcup_{n < 1+n^*} ([\underset{\sim}{B}]^n \times [\underset{\sim}{B}]^n) \rangle$ forced by r^* to be as above. As R is χ-complete, $\chi > 2^{\kappa + \lambda + 2^\mu}$, without loss of generality r^* forces values $\gamma(*)$, $M_\emptyset$, $M_{\{\gamma(*)\}}$, $\langle (p^*_\zeta, c^*_1(p^*_\zeta), c^*_2(p^*_\zeta)) : \zeta < \mu \rangle$.

We now try to choose by induction on $\zeta \leq \theta + 1$, $\bar{Q}^\zeta$, α^ζ, γ^ζ such that:

(A) (a) $\bar{Q}^\zeta \in R$
 (b) $\bar{Q}^0 = \{r^*\}$
 (c) $\lg(\bar{Q}^\zeta) = \alpha^\zeta$
 (d) $\xi < \zeta \Longrightarrow \bar{Q}^\xi = \bar{Q}^\zeta \restriction \alpha^\xi$
 (e) $\langle \alpha^\zeta : \zeta \leq \theta + 1 \rangle$ is (strictly) increasing continuous
 (f) $\alpha^\zeta < \gamma_\zeta < \alpha^{\zeta+1}$
 (g) $\bar{Q}^{\zeta+1} \Vdash_R$ “$\gamma^\zeta \in \underset{\sim}{B}$”
 (h) $\bar{Q}^{\zeta+1}$ forces ($\Vdash_R$) a value to

$$\langle \underset{\sim}{M}_s \cap V : s \in [\underset{\sim}{B} \cap (\gamma_\zeta + 1)]^{<1+n^*} \rangle$$

which we call $\langle M_s : s \in [B_\zeta]^{<1+n^*} \rangle$.

(B) if $\zeta \leq \theta + 1$, $\mathrm{cf}(\zeta) > \mu$ then:
 (a) $a^{\bar{Q}^{\zeta+1}}_{\alpha_\zeta} = \bigcup \{ \chi \cap M_{\{\xi_1, \xi_2\}} : \{\xi_1, \xi_2\} \in [\{ \gamma_\varepsilon : \varepsilon < \zeta \}]^{<1+n^*} \}$

(b) $I_{\alpha_\zeta}^{\bar{Q}^{\zeta+1}} = \{ b : b$ an initial segment of $a_{\alpha_\zeta}^{\bar{Q}^{\zeta+1}}$ and $\mathrm{cf}(\mathrm{otp}(b)) \neq \mu^+ \}$
[explanation: this satisfies the simplicity demands]

(c) $\underset{\sim}{Q}_{\alpha_\zeta}^{\bar{Q}^{\zeta+1}} = \{ h : h$ a function, $\mathrm{Dom}(\underset{\sim}{h}) \subseteq \mu, |\mathrm{Dom}(h)| < \mu, h(i) \in \underset{\sim}{Q}_{\alpha_\zeta,*}^{\bar{Q}^{\zeta}}$ when defined $\}$ (see (d) below); order $h_1 \leq h_2$ if $i \in \mathrm{Dom}(h_1) \Longrightarrow h_1(i) \subseteq h_2(i)$ where $Q_{\alpha_\zeta,*}^{\bar{Q}^\zeta}$ is defined in clause (d) below
[explanation: the forcing notion in clause (d) adds a subset $\underset{\sim}{u}$ of ζ such that on $\{ \gamma_\beta : \beta \in \underset{\sim}{u} \}$ the colouring $\underset{\sim}{\tau}$ get only two values; the forcing notion from clause (c) makes ζ the union of $\leq \mu$ such sets and this induces a representation of B_ζ as a union of μ sets on each $\underset{\sim}{\tau}$ get at most two colours]

(d) $Q_{\alpha_\zeta,*}^{\bar{Q}^\zeta} = \{ u : u \in [\zeta]^{<\mu}$, and for some $\xi < \mu$ we have: for every $j_1 < j_2$ from u, we can find p^1, p^2, r_1, r_2 such that for $\ell = 1, 2$ we have: $p_\xi^* \leq p^\ell \in M_{\{\gamma(*)\}} \cap P_\chi^*$, $r_\ell \in P_\chi^* \cap M_{\{\gamma_{j_1},\gamma_{j_2}\}}$, $r_\ell \Vdash$ "$\underset{\sim}{\tau}(\{\gamma_{j_1},\gamma_{j_2}\}) = c_\ell^*(p_\xi^*)$", $r_\ell \restriction (\chi \cap M_{\{\gamma_{j_\ell}\}}) \leq f_{\{\gamma(*)\},\{\gamma_{j_1}\}}(p^1)$, $r_\ell \restriction (\chi \cap M_{\{\gamma_{j_{3-\ell}}\}}) \leq f_{\{\gamma(*)\},\{\gamma_{j_{3-\ell}}\}}(p^2)$, and $r_1 \in \underset{\sim}{G}_{P_{\alpha_\zeta}}$ or $r_2 \in \underset{\sim}{G}_{P_{\alpha_\zeta}} \}$.

Stage D. Again we shall use less than obtained for later use.

The point is to verify that we can carry out the induction. Now there is no problem to do this for $\zeta = 0$ and for ζ limit. So we deal with $\zeta + 1$, $\zeta \leq \theta$ and we are assuming that $\bar{Q}^\zeta$ is already defined. If $\mathrm{cf}(\zeta) \leq \mu$ clause (B) is empty and it is easy to satisfy clause (A). So assume $\mathrm{cf}(\zeta) \geq \mu^+$. Now as before clause (A) is easy. The point is to choose $\bar{Q}^{\zeta+1}$ or just $\bar{Q}^{\zeta+1} \restriction (\alpha_\zeta + 1)$ to satisfy clause (B). Now $\underset{\sim}{Q}_{\alpha_\zeta}$ is chosen by clause (B) so $\bar{Q}^{\zeta+1} \restriction (\alpha_\zeta + 1)$ is now fixed.

The point is to prove that the condition concerning $*_\mu^\varepsilon$ from Definition 11 holds as required in Definition 19 (1) (d). From now on we may omit the superscript $\bar{Q}^{\zeta+1}$ or $\bar{Q}^{\zeta+1} \restriction (\alpha_\zeta + 1)$ so $P_{\alpha_\zeta}^* = P_{\alpha_\zeta}^{\bar{Q}^{\zeta+1} \restriction (\alpha_\zeta+1)}$, etc.

That is, we assume $b \in I_{\alpha_\zeta}$ and we will prove that $(P_b^*, P_{a_{\alpha_\zeta} \cup \{\alpha_\zeta\}}^*)$ satisfies $*_\mu^\varepsilon$.

Note

$(*)_1$ if $\bar{Q}^{\xi+1}$ is well defined (or just $\bar{Q}^{\xi+1} \restriction (\alpha_\xi + 1) \in R$) and $\mathrm{cf}(\xi) > \mu$ then ($P_{\alpha_{\xi+1}}$ is well defined and) in $V^{P_{\alpha_{\xi+1}}}$, $\{ \gamma_\Upsilon : \Upsilon < \xi \}$ is well defined and it can be represented as $\bigcup_{i<\mu} \mathscr{U}_i$, such that each $u \in [\mathscr{U}_i]^{<\mu}$ belongs to $Q_{\alpha_\xi,*}^{\bar{Q}^\xi}$

$(*)_2$ if $\zeta(1) < \zeta(2) \leq \zeta$ and $\mathrm{cf}(\zeta(1)), \mathrm{cf}(\zeta(2)) > \mu$ then $Q_{\alpha_{\zeta(1)},*}^{\bar{Q}^{\zeta(1)}} \subseteq Q_{\alpha_{\zeta(2)},*}^{\bar{Q}^{\zeta(2)}}$, also for the compatibility relation

$(*)_3$ the elements of $Q_{\alpha_\zeta,*}^{\bar{Q}^\zeta}$ are from V, in fact are sets of ordinals of cardinality $< \mu$ ordered by $\subseteq$ and the least upper bound of set of cardinality $< \mu$ members is the union (if there is an upper bound), so $Q_{\alpha_\zeta,*}^{\bar{Q}^\zeta}$ is μ-complete

$(*)_4$ $\bar{Q}^\zeta$ is well defined and $\Vdash_{P_{\alpha_\zeta}}$ "for $\xi < \zeta$, if $\mathrm{cf}(\xi) > \mu$ then, $Q_{\alpha_\xi,*}^{\bar{Q}^\xi}$ is the union of μ sets, each set $(< \mu)$-directed and with any two elements having a least upper bound".

Hence

$(*)_5$ if $\mathrm{cf}(\zeta) > \mu^+$, then in $V^{P_{\alpha_\zeta}}$, each subset of $Q_{\alpha_\zeta,*}^{\bar{Q}^{\zeta+1}}$ of cardinality $\leq \mu^+$ is included in the union of μ sets, each directed and with any two elements having a least upper bound.

Note that by the definition of $Q^{\bar{Q}^\zeta}_{\alpha_\zeta,*}$ we have

$(*)_6$ a family of $< \mu$ members of $Q^{\bar{Q}^\zeta}_{\alpha_\zeta,*}$ has a common upper bound if and only if any two of them are compatible, and then the union is a least upper bound of the family.

So if $\mathrm{cf}(\zeta) > \mu^+$, we are done as by $(*)_5 + (*)_6$ we have $\Vdash_{P_\zeta}$ "Q_ζ satisfies $*^\varepsilon_\mu$" and can use 13 (4).

So we can assume $\zeta = \Upsilon(*) \leq \theta + 1$ and $\mathrm{cf}(\zeta) = \mathrm{cf}(\alpha_\zeta) = \mu^+$, and let $\langle \Upsilon(i) : i < \mu^+ \rangle$ be increasing continuous with limit ζ and $\mathrm{cf}(\Upsilon(i)) \leq \mu$ for $i < \mu^+$. Let $b \in I_{\alpha_\zeta}$, hence b is a bounded subset of a_ζ. So by the induction hypothesis and 13 (4) without loss of generality

$$b = \bigcup\{ M_{\{\gamma_{\Upsilon_0}, \gamma_{\Upsilon_1}\}} \cap \alpha_\zeta : \Upsilon_0 < \Upsilon_1 < \Upsilon(0) \}.$$

Define $c_0 = b_0 = b$ and for $\Upsilon \in [\Upsilon(0), \Upsilon(*))$ let

$$b_{1.\Upsilon} = b_0 \cup (M_{\{\gamma_\Upsilon\}} \cap \alpha_\zeta) \cup \bigcup_{\Upsilon_1 < \Upsilon(0)} (M_{\{\gamma_{\Upsilon_1}, \gamma_\Upsilon\}} \cap \alpha_\zeta)$$

(the third term could be waived with minor changes),

$$b_1 = b_{1.\Upsilon(0)}, b_2 = b_1 \cup b_{1.\Upsilon(0)+1},$$

$$c_2 = \bigcup\{ b_{1.\Upsilon} : \Upsilon \in [\Upsilon(0), \Upsilon(*)) \}$$

$$c_3 = a_{\alpha_{\Upsilon(*)}} = \bigcup\{ M_{\{\gamma_{\Upsilon_1}, \gamma_{\Upsilon_2}\}} \cap \alpha_{\Upsilon(*)} : \Upsilon_1 < \Upsilon(*), \Upsilon_2 < \Upsilon(*) \}$$

and

$$c_4 = a_{\alpha_{\Upsilon(*)}} \cup \{\alpha_\zeta\}.$$

NOTE. There is no c_1.

All these sets are $\bar{Q}^{\alpha_\zeta}$-closed except c_4. We now choose several winning strategies which exist by the induction hypothesis on ζ.

Let $\mathbf{St}_0$ be a winning strategy of the first player in a game above $*^\varepsilon_\mu[P^*_{b_0}]$. Let $\mathbf{St}_1$ be a winning strategy of the first player in $*^\varepsilon_\mu[P^*_{b_0}, P^*_{b_1}]$ which projects to $\mathbf{St}_0$. For every $\Upsilon \in [\Upsilon(0), \Upsilon(*))$ let $\mathbf{St}_{1.\Upsilon}$ be a winning strategy of the first player in $*^\varepsilon_\mu[P^*_{b_0}, P^*_{b_{1.\Upsilon}}]$ conjugate to $\mathbf{St}_1$ (by $\mathrm{OP}_{b_{1.\Upsilon}, b_1}$).

For $\bar{\Upsilon} = \langle \Upsilon_1, \Upsilon_2 \rangle$, $\Upsilon_1 < \Upsilon_2$, $\{\Upsilon_1, \Upsilon_2\} \subseteq [\Upsilon(0), \Upsilon(*))$ let

$$b_{2.\bar{\Upsilon}} = b_{1.\Upsilon_1} \cup b_{1.\Upsilon_2} \cup (M_{\{\Upsilon_1.\Upsilon_2\}} \cap \alpha_\zeta)$$

and let $\mathbf{St}_{2.\bar{\Upsilon}}$ be a winning strategy in $*^\varepsilon_\mu[P^*_{b_{1.\Upsilon_1} \cup b_{1.\Upsilon_2}}, P^*_{b_{2.\bar{\Upsilon}}}]$ which is above $\mathbf{St}_{1.\Upsilon_1} \times \mathbf{St}_{1.\Upsilon_2}$ (remember that both project to $\mathbf{St}_0$, use 17); also note as long as the second player uses conditions in $P^*_{b_{1.\Upsilon_\ell}}$ then so does the first player (for each $i < \mu^+$ separately).

Also, the first player has a winning strategy in $*^\varepsilon_\mu[P^*_{c_0}, P^*_{c_2}]$ but we want a very special winning strategy $\mathbf{St}_2$: (letting g_2 be a fixed pairing function on μ^+) in a play $\langle \langle p^\xi_i : i < \mu^+ \rangle, \langle q^\xi_i : i < \mu^+ \rangle, f^\xi : \xi < \varepsilon \rangle$ where the first player uses the strategy $\mathbf{St}_2$ we demand that clauses (a)–(d) below holds on $f^{1,\xi}$, $p^{2,\Upsilon,\xi}$, $p^{3,\bar{\zeta},\xi}, \ldots$, see clause (d):

(a) $\langle \langle p^\xi_i \restriction b_0 : i < \mu^+ \rangle, \langle q^\xi_i \restriction b_0 : i < \mu^+ \rangle, f^{1,\xi} : \xi < \varepsilon \rangle$ is a play of $*^\varepsilon_\mu[P^*_{b_0}]$ in which the first player uses the strategy $\mathbf{St}_0$)

(b) for each $\Upsilon \in [\Upsilon(0), \Upsilon(*))$ defining

$$p^{2,\Upsilon,\xi}_i = \begin{cases} p^\xi_i \restriction b_{1,\Upsilon} & \text{if } \Upsilon(i) > \Upsilon \\ p^\xi_i \restriction b_0 & \text{if } \Upsilon(i) \le \Upsilon \end{cases}$$

$$q^{2,\Upsilon,\xi}_i = \begin{cases} q^\xi_i \restriction b_{1,\Upsilon} & \text{if } \Upsilon(i) > \Upsilon \\ q^\xi_i \restriction b_0 & \text{if } \Upsilon(i) \le \Upsilon \end{cases}$$

we have: $\langle \langle p^{2,\Upsilon,\xi}_i : i < \mu^+ \rangle, \langle q^{2,\Upsilon,\xi}_i : i < \mu^+ \rangle, f^{2,\Upsilon,\xi} : \xi < \varepsilon \rangle$ is a play of $*^\varepsilon_\mu[P^*_{b_0}, P^*_{b_{1,\Upsilon}}]$ in which the first player uses the strategy $\mathbf{St}_{1,\Upsilon}$.

(c) For any pair $\bar\zeta = (\zeta_1, \zeta_2)$ of ordinals in $\mu \times \varepsilon$, let

$$\Upsilon(i, \bar\zeta) = \Upsilon_{\bar\zeta}(i) \text{ is the } \zeta_1\text{th member of } \mathrm{Dom}(q^{\zeta_2}_i) \setminus \Upsilon(i)$$
$$p^{3,\bar\zeta,\xi}_i = \mathrm{OP}_{b_{1,\Upsilon(0)}, b_{1,\Upsilon_{\bar\zeta}(i)}}(p^\xi_i \restriction b_{1,\Upsilon_{\bar\zeta}(i)})$$
$$q^{3,\bar\zeta,\xi}_i = \mathrm{OP}_{b_{1,\Upsilon(0)}, b_{1,\Upsilon_{\bar\zeta}(i)}}(q^\xi_i \restriction b_{1,\Upsilon_{\bar\zeta}(i)}),$$

we demand that $\langle \langle p^{3,\bar\zeta,\xi}_i : i < \mu^+ \rangle, \langle q^{3,\bar\zeta,\xi}_i : i < \mu^+ \rangle, f^{3,\bar\zeta,\xi} : \xi < \varepsilon \rangle$ is a play of $*^\varepsilon_\mu[P^*_{b_0}, P^*_{b_{1,\Upsilon(0)}}]$ in which the first player uses the strategy $\mathbf{St}_{1,\Upsilon(0)}$.

So for each $i < \mu$, for $\zeta_1 < \mu$ too large $\Upsilon(i, \bar\zeta)$ is not well defined and we stipulate the forcing conditions are $\emptyset$.

(d) $f^\xi(i)$ codes $f^{1,\xi}(i)$, $\langle f^{2,\Upsilon,\xi}(i) : \Upsilon \in [\Upsilon(0), \Upsilon(*))$ and $(\exists \beta \in b_{1,\Upsilon} \setminus b_0)\ [p^\xi_i(\beta) \ne \emptyset_{\underset{\sim}{Q}_\beta}] \rangle$ and $\langle f^{3,\bar\zeta,\xi}(i) : \bar\zeta \in \mu \times \varepsilon$, and $\Upsilon_{\bar\zeta}(i)$ is well defined$\rangle$ and the information on $p^\xi_i(\alpha_{\Upsilon(*)})$ and it codes

$$\{ \langle j_1, \zeta_1, \zeta_2 \rangle : \beta, \text{ the } \zeta_2\text{th member of } \mathrm{Dom}(p^\xi_i) \text{ satisfies}$$
$$j_1 = \min\{ j : \beta \in \mathrm{Dom}(p^\xi_j) \},$$
$$\text{and } \beta \text{ is the } \zeta_1\text{th member of } \mathrm{Dom}(p^\xi_{j_1}) \}$$

and

$$\{ \langle j, \zeta_1, \zeta_2 \rangle : \text{for some } \Upsilon, \beta, \text{ the } \zeta_1\text{th member of } \mathrm{Dom}(p^\xi_i),$$
$$\text{belongs to } b_{1,\Upsilon} \setminus b_0 \text{ and satisfies}$$
$$j = \min\{ j' : (\mathrm{Dom}(p^\xi_{j'}) \cap (b_{1,\Upsilon} \setminus b_0) \ne \emptyset \}$$
$$\text{and the } \zeta_2\text{th member of } \mathrm{Dom}(p^\xi_j) \text{ belongs to } b_{1,\Upsilon} \setminus b_0 \}$$

(note: for each $\zeta_2 < \varepsilon$, $i < \mu^+$ we have: $\{ \zeta_1 < \mu : \Upsilon_{(\zeta_1,\zeta_2)}(i)$ is well defined$\}$ is a bounded subset of μ).

Check that such $\mathbf{St}_2$ exists, (note that the number of times we have to increase $p_i \restriction b_0$ is $< \mu$).

Clearly $c_2 \subseteq c_3$ are $\bar Q^\zeta$-closed, hence there is a winning strategy $\mathbf{St}_3$ of the first player in $*^\varepsilon_\mu[P^*_{c_2}, P^*_{c_3}]$ above $\mathbf{St}_2$ and such that:

($\boxtimes$) For any $\bar{\Upsilon} = (\Upsilon_1, \Upsilon_2)$ such that $\Upsilon(0) \le \Upsilon_1 < \Upsilon_2 < \Upsilon(*)$, and defining $p^{4,\bar{\Upsilon},\xi} = p_i^\xi \restriction b_{2,\bar{\Upsilon}}$, $q_i^{4,\bar{\Upsilon},\xi} = q_i^\xi \restriction b_{2,\bar{\Upsilon}}$ (can behave similarly in clause (b)), we have: $\langle \langle p_i^{4,\bar{\Upsilon},\xi} : i < \mu^+ \rangle, \langle q_i^{4,\bar{\Upsilon},\xi} : i < \mu^+ \rangle, f^{4,\bar{\Upsilon},\xi} : \xi < \varepsilon \rangle$ is a play of $*_\mu^\varepsilon[P^*_{b_{2,\bar{\Upsilon}}}]$ in which the first player uses the strategy $\mathbf{St}_{2,\bar{\Upsilon}}$.

Lastly, let $\mathbf{St}_4$ be a strategy of the first player in $*_\mu^\varepsilon[P^*_{c_3}, P^*_{c_4}]$ which is weakly project to $\mathbf{St}_3$ and it guarantees:

($*$) if $\langle \langle p_i^\xi : i < \mu^+ \rangle, \langle q_i^\xi : i < \mu^+ \rangle, f_\xi^4 : \xi < \varepsilon \rangle$ is a play of the game in which the first player uses his strategy $\mathbf{St}_4$ then:

(α) $q_i^\xi \restriction a_{\alpha_\Upsilon}$ forces a value to $q_i^\xi(\alpha_{\Upsilon(*)})$

(β) if $\Upsilon_1 \ne \Upsilon_2$ are from (the value forced on) $q_i^\xi(\alpha_{\Upsilon(*)})$ then $q_i^\xi \restriction a_\Upsilon$ is above the relevant parts of witnesses to this.

Clearly $\mathbf{St}_4$ is (essentially) a strategy of the first player in $*_\mu^\varepsilon[P^*_{b_0}, P^*_{c_4}]$ (for the almost $*_\mu^\varepsilon$ case above $\mathbf{St}_0$). All we have to prove is that $\mathbf{St}_4$ is a winning strategy above $\mathbf{St}_0$. So let $\langle \langle p_i^\xi : i < \mu^+ \rangle, \langle q_i^\xi : i < \mu^+ \rangle, f_\xi^4 : \xi < \varepsilon \rangle$ be a play of $*_\mu^\varepsilon[P^*_{b_0}, P^*_{c_4}]$ in which the first player uses the strategy $\mathbf{St}_4$.

By the definition of the game $*_\mu^\varepsilon[P^*_{b_0}, P^*_{c_4}]$ without loss of generality for some club E_1 of μ^+ (see clause (a)):

$(**)_1$ if $\{i, j\} \subseteq S_\mu^{\mu^+} \cap E_1$ (see 3) and $\bigwedge_{\xi<\varepsilon} f_\xi^4(i) = f_\xi^4(j)$ then
$$\{ p_i^\xi \restriction b_0, p_j^\xi \restriction b_0 : \xi < \varepsilon \}$$
has an upper bound in $P^*_{b_0}$.

By clause (b) in the demands on $\mathbf{St}_{1,\Upsilon}$ for some club E_2 of μ^+ we have:

$(**)_2$ if $\{i, j\} \subseteq S_\mu^{\mu^+} \cap E_2$ and $\Upsilon \in [\Upsilon(0), \Upsilon(*))$ and
$$\bigwedge_{\xi<\varepsilon} [(b_{1,\Upsilon} \setminus b_0) \cap \mathrm{Dom}(p_i^\xi) \ne \emptyset$$
$$\text{and } (b_{1,\Upsilon} \setminus b_0) \cap \mathrm{Dom}(p_j^\xi) \ne \emptyset \Longrightarrow f^{2,\Upsilon,\xi}(i) = f^{2,\Upsilon,\xi}(j)]$$
(which holds if $\bigwedge_{\xi<\varepsilon} f_\xi^4(i) = f_\xi^4(j)$), and r is an upper bound of
$$\{ p_i^\xi \restriction b_0, p_j^\xi \restriction b_0 : \xi < \varepsilon \}$$
then
$$\{ p_i^\xi \restriction b_{1,\Upsilon}, p_j^\xi \restriction b_{1,\Upsilon} : \xi < \varepsilon \} \cup \{r\}$$
has an upper bound in $P^*_{b_{1,\Upsilon}}$.

By clause (c) in the choice of $\mathbf{St}_2$ we know that there is a club E_3 of μ^+ such that:

$(**)_3$ if $\bar{\zeta} \in \mu \times \varepsilon$, $\{i, j\} \subseteq S_\mu^{\mu^+} \cap E_3$ and $\bigwedge_{\xi<\varepsilon} f^{3,\bar{\zeta},\xi}(i) = f^{3,\bar{\zeta},\xi}(j)$ (which holds if $\bigwedge_{\xi<\varepsilon} f_\xi^4(i) = f_\xi^4(j)$) and $r \in P^*_{b_0}$ is an upper bound of
$$\{ p_i^\xi \restriction b_0, q_i^\xi \restriction b_0 : \xi < \zeta \}$$
then
$$\{ p_i^{3,\bar{\zeta},\xi}, p_j^{3,\bar{\zeta},\xi} : \xi < \varepsilon \} \cup \{r\}$$
has an upper bound.

By clause (e) in the demand on $\mathbf{St}_3$, for some club E_4 of μ^+

$(**)_4$ if $\{i, j\} \subseteq S^{\mu^+}_{\mu} \cap E_4$ and $\bigwedge_{\xi<\varepsilon} f^4_\xi(i) = f^4_\xi(j)$ and r is an upper bound of

$$\{p^\xi_i \restriction b_0, p^\xi_j \restriction b_0 : \xi < \varepsilon\}$$

then

$$\{\, p^\xi_i \restriction \Upsilon(i), p^\xi_j \restriction \Upsilon(j) : \xi < \varepsilon \,\} \cup \{r\}$$

has an upper bound in $P^*_{\alpha_\zeta}$ (even $P^*_{\alpha_{\max\{\Upsilon(i),\Upsilon(j)\}}}$).

Last

$(**)_5$ E is a club of μ^+ included in $E_1 \cap E_2 \cap E_3 \cap E_4$ such that:

$$i < j \in E \Longrightarrow \mathrm{Dom}(p^\xi_i \restriction c_3) \cup \mathrm{Dom}(q^\xi_i \restriction c_3) \subseteq \alpha_{\Upsilon(j)}.$$

The rest is as in [6, §2]. ⊣

THEOREM 27. *We can in 25 replace "measurable", by (strongly) Mahlo.*

REMARK 28. It is not straightforward; e.g., we may use the version of squared diamond given in Fact 30 below.

We first prove two claims.

CLAIM 29. *Suppose λ is a strongly inaccessible Mahlo cardinal, $\chi > \lambda > \theta = \theta^{<\sigma}$, $\mathfrak{C}$ an expansion of $(\mathscr{H}(\chi), \in, <^*_\chi)$ by $\leq \theta$ relations. Then for some club E of λ for every inaccessible $\kappa \in E$ we have:*

$(*)_\kappa$ *for every $x \in \mathscr{H}(\kappa)$ there are $B \in [\kappa]^\kappa$ and N_s (for $s \in [B \cup \{\kappa\}]^2$), $N'_{\{i\}}$ (for $i \in B \cup \{\kappa\}$), $N_{\{i\}}$ (for $i \in B$) and $N_\emptyset$ (so $N_{\{\kappa\}}$ is meaningless) such that ($L_{\sigma,\sigma}$ is like the first order logic but with conjunctions and a string of existential quantifiers of any length $< \sigma$):*

(a) *$x \in N_s \prec_{L_{\sigma,\sigma}} \mathfrak{C}$ and $\theta \subseteq N_s$*

(b) *$x \in N'_{\{i\}} \prec_{L_{\sigma,\sigma}} \mathfrak{C}$ and $\theta \subseteq N'_{\{i\}} \subseteq N_{\{i\}}$*

(c) *$s \subseteq B \Longrightarrow N_s \cap \lambda \subseteq \kappa$ and $N'_s \cap \lambda \subseteq \kappa$ (when defined)*

(d) *$N_\emptyset \prec_{L_{\sigma,\sigma}} N_{\{i\}}$ and*

$$\min(N_{\{i\}} \cap \lambda \setminus N_\emptyset) > \sup\left[\bigcup\{\, N_s \cap \lambda : s \subseteq [B \cap i]^{\leq 2} \,\}\right]$$

(e) *for $j < i$, $N_{\{j,i\}}$ is the $L_{\sigma,\sigma}$-Skolem hull of $N_{\{j\}} \cup N'_{\{i\}}$ inside $\mathfrak{C}$*

(f) *for $j < i$, $N_{\{j\}} \cap \lambda$ is an initial segment of $N_{\{j,i\}} \cap \lambda$*

(g) *for $j < i$,*

$$\min(N_{\{j,i\}} \cap \lambda \setminus N_{\{j\}}) > \sup\{\, N_{\{j_1,i_1\}} \cap \lambda : j_1 < i_1 < i \,\}$$

(h) *N_s, N'_s have cardinality θ when defined.*

PROOF. Let $\theta_1 = 2^\theta, \theta_2 = 2^{\theta_1}$. Let $\mathfrak{A}$ and κ be such that:

- κ strongly inaccessible
- $\mathfrak{A} \prec_{L_{\theta_2^+,\theta_2}} \mathfrak{C}$
- $\mathfrak{A}^{<\kappa} \subseteq \mathfrak{A}$
- $\mathfrak{A} \cap \lambda = \kappa$.

(Clearly for some club E of λ, for every strongly inaccessible $\kappa \in E$ there is $\mathfrak{A}$ as above; so it is enough to prove $(*)_\kappa$). Without loss of generality, $\kappa > \theta$ and let $x \in \mathscr{H}(\kappa)$. Next choose $\mathfrak{B}_i \prec_{L_{\theta_2^+,\theta_2^+}} \mathfrak{C}$, increasing continuous in i for $i < \kappa$, $\langle \mathfrak{B}_i : i \leq j \rangle \in \mathfrak{B}_{j+1}$, $\|\mathfrak{B}_j\| < \kappa$, $\mathfrak{B}_i \cap \kappa$ an ordinal and $\{x, \lambda, \theta, \sigma, \kappa, \lambda, \mathfrak{A}\} \in \mathfrak{B}_0$.

Let $\mathfrak{B} = \mathfrak{B}_{\theta^+}$, and let f be a function from $\mathfrak{B}$ into $\mathfrak{A}$, which is an $\prec_{L_{\theta_2^+,\theta_1^+}}$ elementary mapping (for the model $\mathfrak{C}$, $\mathrm{Dom}(f) = \mathfrak{B}$, $\mathrm{Rang}(f) \subseteq \mathfrak{A}$).

Let $N \prec_{L_{\sigma,\sigma}} \mathfrak{C}$ be such that $\{x, \mathfrak{A}, \mathfrak{B}, \langle \mathfrak{B}_i : i \leq \theta^+ \rangle, f, \sigma, \theta, \lambda, \kappa\} \in N$, $\theta+1 \subseteq N$, $\|N\| = \theta$, $N^{<\sigma} \subseteq N$.

Let N^+ be N.

Let $N_\emptyset$ be $N^+ \cap \mathfrak{A} \cap \mathfrak{B}$, as $\|N_\emptyset\| \leq \theta$ we have $N_\emptyset \in \mathfrak{A} \cap \mathfrak{B}$. Let $N_{\{0\}} = N^+ \cap \mathfrak{A}$ so $N_\emptyset = N_{\{0\}} \cap \mathfrak{B}$, and $N_\emptyset \cap \lambda$ $(\subseteq \kappa)$ is an initial segment of $N_{\{0\}} \cap \lambda$ $(\subseteq \kappa)$, let $N'_{\{\kappa\}} = N^+ \cap \mathfrak{B}$ and $N'_{\{0\}} = f(N'_{\{\kappa\}})$, so $N'_{\{0\}} \prec L_{\sigma,\sigma} N_{\{0\}}$. Let $\alpha_0 = f(\kappa)$. Now we choose by induction on $i < \kappa$, α_i, $N'_{\{i\}}$, $N_{\{i\}}$, g_i and $N_{\{i,j\}}$ for $j < i$ such that:

(1) g_i is an $\prec_{L_{\sigma,\sigma}}$-elementary mapping from $N_{\{0\}}$ into $\mathfrak{A}$, $g_0 = \mathrm{id}_{N_{\{0\}}}$
(2) $g_i(\alpha_0) = \alpha_i$
(3) for $j < i$, $N_{\{j,i\}}$ is the $L_{\sigma,\sigma}$-Skolem hull of $N_{\{j\}} \cup N'_{\{i\}}$ (in $\mathfrak{C}$)
(4) $N_{\{i,\kappa\}}$ is the $L_{\sigma,\sigma}$-Skolem hull of $N_{\{i\}} \cup N'_{\{\kappa\}}$
(5) $N_{\{i,\kappa\}}, N_{\{0,\kappa\}}$ are isomorphic, in fact there is an ismorphism from $N_{\{0,\kappa\}}$ onto $N_{\{i,\kappa\}}$ extending $g_i \cup \mathrm{id}_{N'_{\{\kappa\}}}$
(6) for $j < i$ there is an isomorphism from $N_{\{j,i\}}$ onto $N_{\{j,\kappa\}}$ extending
$$\mathrm{id}_{N_{\{j\}}} \cup (f^{-1} \circ g_i^{-1}) \restriction N'_{\{i\}}$$
(7) $N_{\{j\}} \cap \lambda$ is an initial segment of $N_{\{j,i\}} \cap \lambda$ for $j < i$.

This is possible and gives the desired result (by renaming, replace $i < \kappa$ by α_i). $\dashv$

Fact 30. Let χ be strongly inaccessible $(k+1)$-Mahlo, $\kappa < \chi$ is regular. By a forcing with a P which is κ^+-complete of cardinality χ, not collapsing cardinals nor cofinalities nor changing cardinal arithmetic we can get:

$(*)_\chi^{\kappa,k}$ there is $\bar{A} = \langle A_\alpha : \alpha < \chi \rangle$ and $\bar{C} = \langle C_\alpha : \alpha \in S \rangle$ such that:

(a) $S \subseteq \{\delta < \chi : \delta > \kappa$ and $\mathrm{cf}(\delta) \leq \kappa\}$ and $\{\delta \in S : \mathrm{otp}(C_\delta) = \kappa\}$ is a stationary subset of χ
(b) $C_\alpha \subseteq \alpha \cap S$, $[\beta \in C_\alpha \Longrightarrow C_\beta = C_\alpha \cap \beta]$, $\mathrm{otp}(C_\alpha) \leq \kappa$, C_α a closed subset of α and $[\sup(C_\alpha) = \alpha \iff C_\alpha$ has no last element$]$
(c) $A_\alpha \subseteq \alpha$
(d) $\beta \in C_\alpha \Longrightarrow A_\beta = A_\alpha \cap \beta$
(e) $\{\lambda < \chi : \lambda$ inaccessible, and for every $X \subseteq \lambda$ the set we have $\{\alpha < \lambda : \mathrm{otp}(C_\alpha) = \kappa, X \cap \alpha = A_\alpha\}$ is a stationary subset of $\lambda\}$ is not only stationary but is a k-Mahlo subset, moreover we actually get:

(e)$^+$ for every strongly inaccessible $\lambda \in (\theta, \chi)$, $\langle (A_\alpha, C_\alpha) : \alpha \in S \cap \lambda \rangle$ is a club guessing squared diamond, that is clauses (a)–(d) hold with λ, $S \cap \lambda$ and: for every club E of λ and $X \subseteq \lambda$ for some $\delta \in S$ we have $C_\delta \cup \{\delta\} \subseteq E$ and $\mathrm{otp}(C_\delta) = \kappa$ and $\alpha \in C_\delta \cup \{\delta\} \Longrightarrow A_\alpha = X \cap \alpha$.

Proof. This can be obtained, e.g., by iteration with Easton support, in which for each strongly inaccessible $\lambda \in (\kappa, \chi]$ we add $\bar{A}$, $\bar{C}$ satisfying (a)–(d) above, each condition being an initial segment.

More specifically, we define and prove by induction on $\alpha \leq \chi$

(1) [Definition] $P_\gamma = \{ (a, \bar{C}, \bar{A}) :$
  (a) $a \subseteq \gamma \setminus \kappa^+$,
  (b) for every strongly inaccessible $\lambda \in (\kappa, \chi]$ we have $\lambda > \sup(a \cap \lambda)$
  (c) $\bar{C} = \langle C_\gamma : \gamma \in a \rangle$
  (d) $C_\gamma \neq \emptyset \Longrightarrow \mathrm{cf}(\gamma) \leq \kappa$ and $\mathrm{otp}(C_\gamma) \leq \kappa$
  (e) $\beta \in C_\gamma \Longrightarrow \beta \in a$ and $C_\beta = C_\gamma \cap \beta$
  (f) $C_\gamma \neq \emptyset \Longrightarrow C_\gamma$ closed
  (g) $\bar{A} = \langle \underset{\sim}{A}_\gamma : \gamma \in a \rangle$
  (h) $\underset{\sim}{A}_\gamma$ is a P_γ-name of a subset of γ
  (i) $\beta \in C_\gamma \Longrightarrow \Vdash_\gamma$ "$\underset{\sim}{A}_\gamma \cap \beta = \underset{\sim}{A}_\beta \}$
  order $p \leq q$ if and only if $a^p \subseteq a^q$, $\bar{C}^p = \bar{C}^q \restriction a^p$, $\bar{A}^p = \bar{A}^q \restriction a^p$.
(2) [Claim] $\beta < \alpha \Longrightarrow P_\beta \lessdot P_\alpha$.
(3) [Claim] If $p \in P_\alpha$, $\beta < \alpha$, then $p \restriction \beta =: (a^p \cap \beta,\ \bar{C} \restriction (a \cap \beta),\ \bar{A} \restriction (a \cap \beta))$ belongs to P_β and: if $p \restriction \beta \leq q \in P_\beta$ then p, q are compatible in a simple way: $p \,\&\, q$ is a least upper bound of $\{p, q\}$.
(4) [Claim] If λ is strongly inaccessible $\leq \chi$ and $> \kappa$ then $P_\lambda = \bigcup_{\alpha<\lambda} P_\alpha$. If in addition λ is Mahlo, then P_λ satisfies the λ-c.c.

Let $\underset{\sim}{c}_\alpha = c^p_\alpha$, $\underset{\sim}{A}_\alpha = A^p_\alpha$ for every large enough $p \in \underset{\sim}{G}_{P_\chi}$. The point is that for every strongly inaccessible $\lambda \in (\theta, \chi]$, P_χ / P_λ does not add any subset of λ, and so $\langle (\underset{\sim}{C}_i, \underset{\sim}{A}_i[G]) : i < \lambda \rangle$ is as required. ⊣

Conclusion 31. Let $\theta = \theta^{<\sigma} < \lambda$, λ a strongly inaccessible Mahlo cardinal, then for some θ^+-complete, λ-c.c. forcing notion of cardinality λ not collapsing cardinals not changing cofinalities nor changing cardinal arithmetic, in V^P we get:

$(**)^{\theta,2}_\lambda$ there are $\langle (B_\alpha, \bar{M}^\alpha, C_\alpha) : \alpha \in S \rangle$ such that:
  (a) $S \subseteq \{ \delta < \chi : \mathrm{cf}(\delta) \leq \theta \}$ and $\{ \delta \in S : \mathrm{otp}(C_\delta) = \theta \}$ is a stationary subset of χ and even of any strongly inaccessible $\lambda \in (\theta, \chi)$
  (b) $C_\alpha \subseteq \alpha \cap S$, $[\beta \in C_\alpha \Longrightarrow C_\beta = C_\alpha \cap \beta]$, $\mathrm{otp}(C_\alpha) \leq \theta$, C_α a closed subset of α so $[\sup(C_\alpha) = \alpha \iff C_\alpha$ has no last element]
  (c) $B_\alpha \subseteq \alpha$, $\mathrm{otp}(B_\alpha) = \omega \times \mathrm{otp}(C_\alpha)$, $\beta \in C_\alpha \Longrightarrow B_\beta = B_\alpha \cap \beta$
  (d) each $\langle M^\alpha_s : s \in [B_\alpha]^{\leq 2} \rangle$ is as in 29 (and $B_\alpha \subseteq B$) and $\beta \in C_\alpha$ and $s \in [B_\beta]^{\leq 2} \Longrightarrow M^\alpha_s = M^\beta_s$
  (e) diamond property: if $\mathfrak{B}$ is an expansion of $(\mathscr{H}(\chi), \in, <^*_\chi)$ by $\leq \theta$ relations, $B \in [\chi]^\chi$ then for a club E of χ for every strong inaccessible $\lambda \in \mathrm{acc}(E)$ for stationarily many $\delta \in S \cap \lambda$ we have $\mathrm{otp}(C_\delta) = \kappa$, $C_\delta \subseteq E$ and $B_\delta \subseteq B$ and $s \in [B_\delta]^{\leq 2} \Longrightarrow M^\delta_s \prec \mathfrak{B}$.

Proof. By 30 and 29 (alternatively, force this directly: simpler than in 30). ⊣

Remark. In 30 we could force a stronger version.

Proof of 27. We repeat the main proof, the one of Theorem 25, but using the diamond from 30 for $k = 0$. In fact the proof of 25 was written such that it can be read as a proof of 27, mainly in Stage B we can get $(*)$ which is proved using measurability, but use only $(*)'$. ⊣

Combining the above proof and [8] we get

THEOREM 32. *Suppose*

(a) $\mu = \aleph_0$ *or* μ *is Laver indestructible supercompact* (*see* [2]) *or just* μ *as in* [8, §4]
(b) λ *is* n^**-Mahlo,* $\lambda > \theta > \mu$
(c) k_{n^*} *as in* [5] (*see below*).

Then for some μ^+*-c. c. forcing notion* P *of cardinality* λ *we have*:

$$\Vdash_P \text{“}2^\mu = \lambda \to [\theta]^{n^*+1}_{k_{n^*}+1}\text{”},$$

moreover for $\sigma < \mu$,

$$\lambda \to [\theta]^{n^*+1}_{\sigma,k_{n^*}}.$$

REMARK 33. What is k_{n^*}?

CASE 1. $\mu = \aleph_0$.

Define on $[{}^\omega 2]^{n^*}$ an equivalence relation E: if $w_1 = \{\eta_\ell : \ell < n^*\}$, $w_2 = \{\nu_\ell : \ell < n^*\}$ are members of $[{}^w 2]^{n^*}$ both listed in lexicographic increasing order, then $w_1 \mathrel{E} w_2$ if and only if for any $\ell_1 < \ell_2 < n^*$ and $\ell_3 < \ell_4 < n^*$ we have

$$\lg(\eta_{\ell_1} \cap \eta_{\ell_2}) < \lg(\eta_{\ell_3} \cap \eta_{\ell_4}) \iff \lg(\nu_{\ell_1} \cap \nu_{\ell_2}) < \lg(\nu_{\ell_3} \cap \nu_{\ell_4}).$$

Lastly, k_{n^*} is the number of E-equivalence classes.

CASE 2. $\mu > \aleph_0$.

Choose $<_\alpha$ be a well ordering of ${}^\alpha 2$ and let E be the following equivalence relation on $[{}^\mu 2]^{n^*}$: if $w_0 = \{\eta_\ell : \ell < n^*\}$, $w_2 = \{\nu_\ell : \ell < n^*\}$ are members of $[{}^\mu 2]^{n^*}$ both listed in lexicographic increasing order then: $w_1 \mathrel{E} w_2$ if and only if for any $\ell_1 < \ell_2 < n^*$ and $\ell_3 < \ell_4 < n^*$ we have

(a) $\lg(\eta_{\ell_1} \cap \eta_{\ell_2}) < \lg(\eta_{\ell_3} \cap \eta_{\ell_4}) \iff \lg(\nu_{\ell_1} \cap \nu_{\ell_2}) < \lg(\nu_{\ell_3} \cap \nu_{\ell_4})$
(b) $\eta_{\ell_3} \restriction \lg(\eta_{\ell_1} \cap \eta_{\ell_2}) <_{\lg(\eta_{\ell_1} \cap \eta_{\ell_2})} \eta_{\ell_4} \restriction \lg(\eta_{\ell_1} \cap \eta_{\ell_2}) \iff \nu_{\ell_3} \restriction \lg(\nu_{\ell_1} \cap \nu_{\ell_2}) <_{\lg(\nu_{\ell_1} \cap \nu_{\ell_2})} \nu_{\ell_4} \restriction \lg(\nu_{\ell_1} \cap \nu_{\ell_1})$.
(c) $\eta_{\ell_3}(\lg(\eta_{\ell_1} \cap \eta_{\ell_2})) = \nu_{\ell_3}(\lg(\nu_{\ell_1} \cap \nu_{\ell_2}))$

REMARK 34. Of course 22 contains

THEOREM 35. *Assume*

(a) $\mu = \mu^{<\mu}$ *and* D *is a normal filter on* μ^+ *to which the set of ordinals of cofinality* μ *belongs and* ε *is a limit ordinal* $< \mu^+$
(b) $\bar{Q} = \langle P_\alpha, \underset{\sim}{Q}_\beta : \alpha \le \alpha^*, \beta < \alpha^* \rangle$ *is a* $(< \mu)$*-support iteration*
(c) *for each* $\beta < \alpha^*$ *in the universe* V^{P_β} *we have*: *the forcing notion* Q_β *is* $(< \mu)$*-strategically complete satisfying* $*^\varepsilon_D$.

Then *for* $\gamma < \beta < \alpha^*$ *we have*: *in* V^{P_γ} *the forcing notion* P_β / P_γ *satisfies* $*^\varepsilon_D$ *hence satisfies the* μ*-c. c.*

PROOF. For simplicity let D be the club filter on μ^+ plus the set of ordinals of cofinality μ, the proof does not change by this. Let $\kappa = \lambda = \chi$ be regular large enough, e.g., just $> |\alpha^*|$, $|P_{\alpha^*}|$. Let us for $\beta < \alpha$ choose $a_\beta = \beta$. Now (see Definition 18) trivially we have: $\bar{Q}^* = \langle P_\alpha, \underset{\sim}{Q}_\beta, a_\beta : \alpha \le \alpha^*, \beta < \alpha^* \rangle$ belongs to $\mathscr{K}^{\varepsilon,\alpha^*}_{\mu,\kappa,\lambda,\chi}$, (see Definition 19), each $\beta \le \alpha^*$ is $\tilde{\bar{Q}}^*$-closed and $P^*_\beta = P_\beta$.

Next for $\beta < \alpha^*$ we choose $I_\beta = \{\gamma : \gamma < \beta\}$ and we shall prove by induction on $\gamma \le \alpha^*$ that $\bar{Q}^\gamma = \langle P_\alpha, \underset{\sim}{Q}_\beta, a_\beta, I_\beta : \alpha \le \gamma, \beta < \gamma \rangle$ belongs to $K^{\varepsilon,\gamma}_{\mu,\kappa,\lambda,\chi}$. Now for

$\gamma = 0$ there is nothing to do and for γ limit this holds by 23, and for γ successor ordinal this holds by Lemma 24, where clause (f) there is proved by 13 (5) and the induction hypothesis. Having proved this the conclusion holds by 13 (1). $\dashv$

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