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NON-COHEN ORACLE C.C.C.

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Abstract. The oracle c.c.c. is closely related to Cohen forcing. During an iteration we can "omit a type"; i.e. preserve "the intersection of a given family of Borel sets of reals is empty" provided that Cohen forcing satisfies it. We generalize this to other cases. In Section 1 we replace Cohen by "nicely" definable c.c.c., do the parallel of the oracle c.c.c. and end with a criterion for extracting a subforcing (not a complete subforcing, $\lessdot$!) of a given nicely one and satisfying the oracle.

0. Introduction

This answers a question from [2, Chapter IV] (the chapter dealing with the oracle c.c.c.) asking to replace Cohen by e.g. random. Later we will deal with the parallel for oracle proper and for the case $\bar{\varphi}_\alpha$ is a (definition of a) nep forcing. An application will appear in a work with T. Bartoszynski.

How do we use this framework? We start with a universe satisfying $\diamondsuit_{\aleph_1}$ and probably $2^{\aleph_1} = \aleph_2$ and choose $\langle S^*_i \colon i < \omega_2\rangle$, $S^*_i \subseteq S^* \subseteq \omega_1$ such that $S^*_i/\mathcal{D}_{\omega_1}$ is strictly increasing and for every $i < \omega_2$, $\diamondsuit_{S^*_{i+1}\setminus S^*_i}$ holds and for simplicity $S^*_i \subseteq S^*_{i+1}$ where $\mathcal{D}_{\omega_1}$ is the club filter on ω_1. We choose by

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induction on $i < \omega_2$, a c.c.c. forcing $\mathbb{P}_i$ of cardinality $\aleph_1$, a sequence $\bar{M}^i = \langle M^i_\alpha \colon \alpha \in S^*_i \rangle$ of countable models $\subseteq (\mathcal{H}(\aleph_1), \in)$ of some version of ZFC, without loss of generality transitive and a 1-commitment mainly connected to a $\mathbb{P}_i$-name $\underset{\sim}{\nu}^i_\alpha$ which is, e.g. random over M^i_α (and the commitment is that if, $j > i$, $\mathbb{P}'_j \cong \mathbb{P}_j/\mathbb{P}_i$ is represented such that it has set of elements $\subseteq \omega_1$, $G \subseteq \mathbb{P}'_j$ is generic over $\mathbf{V}^{\mathbb{P}_i}$, then for a club of $\alpha \in S_i$, $\underset{\sim}{\nu}^i_\alpha \in \mathbf{V}^{\mathbb{P}_i}$ is random also over $M^i_\alpha[G]$ which naturally is $M^i_\alpha[G \cap \alpha]$). They are increasing in the relevant sense and the work at limit stages is done by the general claims here. In stage i, by bookkeeping we are given a task connected with a $\mathbb{P}_i$-name $\underset{\sim}{X}_i$ we have some freedom in choosing $\mathbb{P}_{i+1}$, usually $\mathbb{P}_{i+1} = \mathbb{P}_i * \underset{\sim}{\mathbb{Q}}_i$. So, working in $\mathbf{V}^{\mathbb{P}_i}$, $\mathbb{Q}_i$ has to satisfy a 0-commitment on S^*_i, and we like it to satisfy that task, usually connected with $X_i \subseteq \mathbb{R}^{V[\mathbb{P}_i]}$, say $X_i = \underset{\sim}{X}_i[G_{\mathbb{P}_i}]$. We essentially have to choose $\bar{M}^{i+1}$ such that $\bar{M}^{i+1} \restriction S^*_i = \bar{M}^i$ but we have freedom (in addition to choosing $\underset{\sim}{\mathbb{Q}}_i$) to choose $\langle \bar{M}^{i+1}_\alpha \colon \alpha \in S^*_{i+1} \backslash S^*_i \rangle$ and a 0-commitment on $S^*_{i+1} \backslash S^*_i$. Also the reals generic for the chosen forcing notion (for $\alpha \in S^*_i!$) as well as M^{i+1}_α for $\alpha \in S^*_{i+1} \backslash S^*_i$ can be chosen considering X_i. E.g. M^{i+1}_α can be the Mostowski Collapse of some $M \prec (\mathcal{H}(\aleph_2), \in)$ to which $\mathbb{P}_i, \bar{M}^i$ and $\underset{\sim}{X}_i$ belong.

Really this corresponds to the omitting type as in [1, XI]. This was originally part of [4], particularly close to faking.

1. Non-Cohen oracle c.c.c.

Hypothesis 1.1.

(*a*) We assume CH, moreover $\diamondsuit_{S^*}$ where $S^* \subseteq \{\delta < \omega_1 \colon \delta \text{ limit}\}$ is stationary.

Definition/Notation 1.2. 1) $\bar{M}$ denotes an oracle, i.e., a sequence of the form $\langle M_\delta \colon \delta \in S \rangle$, M_δ a transitive countable model of ZFC^-_* satisfying $\delta \subseteq M_\delta$ and $S \subseteq S^*$ is stationary satisfying: for every $X \subseteq \omega_1$, the set $\{\delta \in S^* \colon X \cap \delta \in M_\delta\}$ is stationary.

2) $\mathcal{D}$ denotes a normal filter on ω_1 usually extending $\mathcal{D}_{\bar{M}}$ which is defined in 1.3(1) below (of course, the default value is $\mathcal{D}_{\bar{M}}$, see 1.4(1)).

3) For a countable forcing $\mathbb{P}$, a wide $\mathbb{P}$-name is a Borel function giving for every directed $G \subseteq \mathbb{P}$ an object (so if $\mathbb{P} \leq_{\text{ic}} \mathbb{P}'$ then any wide $\mathbb{P}$-name is still a wide $\mathbb{P}'$-name hence a $\mathbb{P}'$-name).

We first give the old definitions from [3, IV]

Definition 1.3. 1) $\mathcal{D}_{\bar{M}}$ is

$$\{X \subseteq \omega_1 \colon \text{for some } Y \subseteq \omega_1 \text{ we have: } \delta \in S_{\bar{M}} \cap X \Rightarrow Y \cap \delta \in M_\delta\}.$$

2) A forcing notion $\mathbb{P}$ of cardinality $\leq \aleph_1$ satisfies the $(\bar{M}, \mathcal{D})$-c.c. <u>if</u> for some (equivalently any) one to one $f \colon \mathbb{P} \to \omega_1$ the set:

$$\Big\{\delta \in S_{\bar{M}} \colon \text{if } X \in M_\delta \text{ and } \{y \in \mathbb{P} \colon f(y) < \delta \text{ and } f(y) \in X\} \\ \text{is predense in } \mathbb{P} \restriction \{y \in \mathbb{P} \colon f(y) < \delta\} \text{ \underline{then} } X \text{ is predense in } \mathbb{P}\Big\}$$

belongs to $\mathcal{D}$ and $\mathbb{P}$ has minimal element $\emptyset_{\mathbb{P}}$.

3) If $\mathcal{D} = \mathcal{D}_{\bar{M}}$ we may write "$\bar{M}$-c.c.". Recall that $\mathcal{D}^+ = \{A \subseteq \omega_1 \colon \omega_1 \backslash A \notin \mathcal{D}\}$.

4) Let $\bar{M}^1 \leq \bar{M}^2$ if $\bar{M}^\ell = \langle M^\ell_\delta \colon \delta \in S_\ell \rangle$ and $\{\delta \colon \delta \in S_1 \backslash S_2 \text{ or } \delta \in S_1 \cap S_2 \text{ and } M^1_\delta \neq M^2_\delta\}$ is not stationary; let $\bar{M}^1 \leq_{\mathcal{D}} \bar{M}^2$ be defined similarly (i.e. the set is $= \emptyset \bmod \mathcal{D}$).

5) A forcing notion $\mathbb{P}$ satisfies the $(\bar{M}, \mathcal{D})$-c.c. <u>if</u>: $|\mathbb{P}| \leq \aleph_0$ or for every $X \subseteq \mathbb{P}$ of cardinality $\leq \aleph_1$ there is $\mathbb{P}_1 \lessdot \mathbb{P}$ of cardinality $\aleph_1$ which includes X and satisfies the $(\bar{M}, \mathcal{D})$-c.c.

Fact 1.4. 1) $\mathcal{D}_{\bar{M}}$ is a normal filter on ω_1.

2) The $\bar{M}$-c.c. implies the c.c.c., and if $\mathcal{D}_{\bar{M}} \subseteq \mathcal{D}$ (or just there is a normal filter $\mathcal{D}' \supseteq \mathcal{D}_{\bar{M}} \cup \mathcal{D}$) <u>then</u> the $(\bar{M}, \mathcal{D})$-c.c.c. implies the c.c.c. and if $\mathcal{D}_2 \supseteq \mathcal{D}_1 \supseteq \mathcal{D}_{\bar{M}}$ are normal filters, <u>then</u> the $(\bar{M}, \mathcal{D}_1)$-c.c. implies the $(\bar{M}, \mathcal{D}_2)$-c.c.

3) We can find $\langle S^*_\zeta \colon \zeta < \omega_2 \rangle$ such that $S^*_\zeta \subseteq S^*$, $\zeta < \xi \Rightarrow S^*_\zeta \subseteq S^*_\xi \bmod \mathcal{D}_{\bar{M}}$, $S^*_\zeta \subseteq S^*_{\zeta+1}$ and $S^*_{\zeta+1} \backslash S^*_\zeta \in \mathcal{D}^+_{\bar{M}}$, moreover $S^*_{\zeta+1} \backslash S^*_\zeta$ is countable.

4) If $\bar{M}^1 \leq \bar{M}^2$ and the forcing notion $\mathbb{P}_2$ satisfies the $(\bar{M}^2, \mathcal{D})$-c.c. and $\mathbb{P}_1 \lessdot \mathbb{P}_2$, <u>then</u> $\mathbb{P}_1$ satisfies the $(\bar{M}^1, \mathcal{D})$-c.c.

Proof. See [3, IV], but for the reader's convenience we prove part (4). Without loss of generality $\mathbb{P}_2$ has cardinality $\aleph_1$ and even set of elements ω_1. As $\mathbb{P}_1 \lessdot \mathbb{P}_2$ there is a function $f \colon \mathbb{P}_2 \to \mathbb{P}_1$ such that

$(*)_1$ $q \in \mathbb{P}_2 \wedge f(q) \leq_{\mathbb{P}_1} p \in \mathbb{P}_1 \Rightarrow p, q$ are compatible in $\mathbb{P}_2$.

Let $g \colon \mathbb{P}_2 \times \mathbb{P}_2 \to \mathbb{P}_2$ be such that

$(*)_2$ if $p, q \in \mathbb{P}_2$ are compatible then $g(p,q)$ is a common upper bound and $p, q \in \mathbb{P}_1 \Rightarrow g(p,q) \in \mathbb{P}_1$.

So there is a club E of ω_1 which is closed under f, g so

$(*)_3$ if $\delta \in E \cap S$, $\mathcal{I} \subseteq \mathbb{P}_1 \cap \delta$ is predense in $\mathbb{P}_1 \restriction \delta$ <u>then</u> $\mathcal{I}$ is predense in $\mathbb{P}_2 \restriction \delta$.

[Why? If $q \in \mathbb{P}_2 \cap \delta$ then $f(q) \in \mathbb{P}_1 \cap \delta$ so by the assumption on $\mathcal{I}$, $f(q)$ is compatible with some $r_1 \in \mathcal{I} \in \mathbb{P}_1 \cap \delta$, so there is $r_2 \in \mathbb{P}_1 \restriction \delta$ above q and r_2. By the definition of f the conditions r_2, q are compatible in $\mathbb{P}_2$ hence $g(r_2, q)$ is a common upper bound of them in $\mathbb{P}_2 \restriction \delta$.] $\square_{1.4}$

Remark 1.5. 1) Note that below when $\mathbb{P}^{\bar{\varphi}_\alpha}$ is the older case ([3, IV]) we just preserve every predense set, so in M_α (in the based obeyed) the forcing is countable.

2) We may forget to mention this case as it is by now easy.

Definition 1.6. 1) We say $\mathcal{Y} = (S, \Phi, \underset{\sim}{\bar{\eta}}, \bar{\nu}) = (S^{\mathcal{Y}}, \Phi^{\mathcal{Y}}, \underset{\sim}{\bar{\eta}}^{\mathcal{Y}}, \bar{\nu}^{\mathcal{Y}})$ is a 0-<u>commitment</u> for $\bar{M}$ <u>if</u> for some $E \in \mathcal{D}_{\bar{M}}$:

- (a) $S \subseteq S^*$, $S \in \mathcal{D}^+_{\bar{M}}$
- (b) $\bar{\eta} = \langle \underset{\sim}{\eta}_\alpha : \alpha \in S \rangle$, $\Phi = \langle \bar{\varphi}_\alpha : \alpha \in S \rangle$ and if $\alpha \in S \cap E$ then $\bar{\varphi}_\alpha \in M_\alpha$ and $M_\alpha \models$ "$\bar{\varphi}_\alpha$ is an absolute definition of a c.c.c. forcing notion called $\mathbb{Q}_\alpha = \mathbb{Q}^{\bar{\varphi}_\alpha}$ with generic real $\underset{\sim}{\eta}_\alpha$"; note, absolute here means that forcing extensions of M_α, preserve predensity of countable sets (in the sense of M_α) order and incompatibility
- (c) $\bar{\nu} = \langle \nu_\alpha : \alpha \in S \rangle$ where $\nu_\alpha \in {}^\omega\omega$ and for every $\alpha \in S \cap E$ the real ν_α is $(\mathbb{Q}_\alpha, \underset{\sim}{\eta}_\alpha)$-generic over M_α.

We ignore $\bar{M}$ if clear from the context. We can replace $\bar{M}$ by $(\bar{M}, \mathcal{D})$ if above $E \in \mathcal{D}$, $S \in \mathcal{D}^+$.

1A) A forcing notion $\mathbb{P}$ of cardinality $\leq \aleph_1$ satisfies the 0-commitment $\mathcal{Y} = (S^{\mathcal{Y}}, \Phi^{\mathcal{Y}}, \underset{\sim}{\bar{\eta}}^{\mathcal{Y}}, \bar{\nu}^{\mathcal{Y}})$ for an $\aleph_1$-oracle $\bar{M}$ (we may suppress) <u>if</u>: $\mathbb{P}$ is a forcing notion and for any one-to-one mapping $h : \mathbb{P} \to \omega_1$ for some $E \in \mathcal{D}_{\bar{M}}$ we have

- (d) if $\alpha \in S \cap E$ then $\Vdash_{\mathbb{P}}$ "the real ν_α is a $(\mathbb{Q}_\alpha, \underset{\sim}{\eta}_\alpha)$-generic real over $M_\alpha[\alpha \cap h'' \underset{\sim}{G}_{\mathbb{P}}]$"
- (e) [the old case]: if $\alpha \in S \cap E$ even when $\mathbb{Q}_\alpha$ is a singleton (hence $\nu_\alpha \in M_\alpha$, a degenerated case), <u>then</u> every predense subset $\mathcal{I}$ of $\{p \in \mathbb{P} : h(p) < \alpha\}$ for which $\{h(p) : p \in \mathcal{I}\} \in M_\alpha$ is a predense subset of $\mathbb{P}$.

2) Let $\mathbb{P} \in \mathcal{H}(\aleph_2)$ be an $\bar{M}$-c.c. forcing notion. We say that $\mathcal{Y} = (S, \underset{\sim}{\bar{\Phi}}, \underset{\sim}{\bar{\eta}}, \underset{\sim}{\bar{\nu}}) = (S^{\mathcal{Y}}, \underset{\sim}{\Phi}^{\mathcal{Y}}, \underset{\sim}{\bar{\eta}}^{\mathcal{Y}}, \underset{\sim}{\bar{\nu}}^{\mathcal{Y}})$ is a 1-commitment on $\mathbb{P}$ for $\bar{M}$ if: for any $\bar{N}$ satisfying $(*)_1$ below, the clauses (a)–(d) of $(*)_2$ below hold

$(*)_1$ $\bar{N} = \langle N_\alpha : \alpha < \omega_1 \rangle$ is increasing continuous, $N_\alpha \prec (\mathcal{H}(\aleph_2), \in)$ is countable, $\bar{N} \restriction (\alpha + 1) \in N_{\alpha+1}$ and $\{\bar{M}, \mathbb{P}\} \subseteq \bigcup_{\alpha < \omega_1} N_\alpha$

$(*)_2$ (a) $S \subseteq \mathrm{Dom}(\bar{M}) \subseteq S^*$, $S \in \mathcal{D}^+_{\bar{M}}$

(b) $\bar{\eta} = \langle \underset{\sim}{\eta}_\alpha : \alpha \in S \rangle$, $\Phi = \langle \underset{\sim}{\bar{\varphi}}_\alpha : \alpha \in S \rangle$ so $(\underset{\sim}{\bar{\varphi}}_\alpha, \underset{\sim}{\eta}_\alpha)$ is a $\mathbb{P}$-name of a pair as in 1.6(1)(a), both are hereditarily countable over $\mathbb{P}$

(c) $\bar{\nu} = \langle \underset{\sim}{\nu}_\alpha : \alpha \in S \rangle$ and $\underset{\sim}{\nu}_\alpha$ a $\mathbb{P}$-name of a real given by countably many conditions

(d) the set of the $\alpha \in S$ satisfying the following belongs to $(\mathcal{D}_{\bar{M}} + S)^+$: $\underset{\sim}{\bar{\varphi}}_\alpha \in M_\alpha$, Mos $\mathrm{Col}_{N_\alpha}(N_\alpha) \in M_\alpha$, and letting

$$\mathbb{P}'_\alpha = \text{Mos } \mathrm{Col}_{N_\alpha}(\mathbb{P}^{N_\alpha}) \in M_\alpha$$

we have $M_\alpha \models$ "$\underset{\sim}{\bar{\varphi}}_\alpha$ is a wide $\mathbb{P}'_\alpha$-name of an absolute definition of a c.c.c. forcing with generic real $\underset{\sim}{\eta}_\alpha$" and $\Vdash_{\mathbb{P}}$ "the real $\underset{\sim}{\nu}_\alpha$ is a $(\mathbb{Q}_{\underset{\sim}{\bar{\varphi}}^\alpha}, \underset{\sim}{\eta}_\alpha)$-generic real over $M_\alpha[\underset{\sim}{G}_{\mathbb{P}}]$".

For simplicity the reader may concentrate on the case $\langle (\bar{\varphi}_\alpha, \underset{\sim}{\eta}_\alpha) : \alpha \in S \rangle \in \mathbf{V}$.

3) Let

$$IS = \{(\mathbb{P}, \mathcal{Y}, \bar{M}) : \mathbb{P} \in \mathcal{H}(\aleph_2) \text{ is an } \bar{M}\text{-c.c. forcing notion and } \mathcal{Y} \text{ is a 1-commitment on } \mathbb{P}\}.$$

We shall omit $\bar{M}$ if clear from the context. We can replace $\bar{M}$ by $(\bar{M}, \mathcal{D})$-naturally and write $\mathrm{IS}_{\mathcal{D}}$, but the claims are the same.

4) For $(\mathbb{P}^\ell, \mathcal{Y}^\ell, \bar{M}^\ell) \in IS\, (\ell = 1, 2)$ let $(\mathbb{P}^1, \mathcal{Y}^1, \bar{M}^1) \le^* (\mathbb{P}^2, \mathcal{Y}^2, \bar{M}^2)$ means $\bar{M}^1 \le \bar{M}^2$, $\mathbb{P}^1 \lessdot \mathbb{P}^2$ and for some $E \in \mathcal{D}_{\bar{M}^1}$ we have

$$S^{\mathcal{Y}_1} \cap E \subseteq S^{\mathcal{Y}_2} \cap E,\ \underset{\sim}{\Phi}^{\mathcal{Y}^1} \restriction (S^{\mathcal{Y}_1} \cap E) = \underset{\sim}{\Phi}^{\mathcal{Y}^2} \restriction (S^{\mathcal{Y}} \cap E),$$

$$\underset{\sim}{\bar{\eta}}^{\mathcal{Y}^1} \restriction (S^{\mathcal{Y}^1} \cap E) = \underset{\sim}{\bar{\eta}}^{\mathcal{Y}^2} \restriction (S^{\mathcal{Y}^1} \cap E) \text{ and } \underset{\sim}{\bar{\nu}}^{\mathcal{Y}^1} \restriction (S^{\mathcal{Y}^1} \cap E) = \underset{\sim}{\bar{\nu}}^{\mathcal{Y}^2} \restriction (S^{\mathcal{Y}^1} \cap E).$$

We call E a witness to $(\mathbb{P}^1, \mathcal{Y}^1, \bar{M}^1) \le^* (\mathbb{P}^2, \mathcal{Y}^2, \bar{M}^2)$.

We point out the connection between 0-commitment and 1-commitment.

Fact 1.7. 1) If $\mathcal{Y}$ is a 1-commitment on $\mathbb{P}$ and $\mathbb{P}$ an $\bar{M}$-c.c. forcing notion of cardinality $\le \aleph_1$, <u>then</u> $\Vdash_{\mathbb{P}}$ "$\mathcal{Y}[\underset{\sim}{G}_{\mathbb{P}}] = (S^{\mathcal{Y}}, \underset{\sim}{\Phi}^{\mathcal{Y}}[\underset{\sim}{G}_{\mathbb{P}}], \underset{\sim}{\bar{\eta}}^{\mathcal{Y}}[\underset{\sim}{G}_{\mathbb{P}}], \underset{\sim}{\bar{\nu}}[\underset{\sim}{G}_{\mathbb{P}}])$ is a 0-commitment" so we call it $\mathcal{Y}[\underset{\sim}{G}_{\mathbb{P}}]$. Note $\underset{\sim}{\eta}_\delta[\underset{\sim}{G}_{\mathbb{P}}]$ is still a name.

2) If $\mathbb{P} = \{\emptyset\}$ (the trivial forcing) then: $\mathcal{Y}$ is a 1-commitment on $\mathbb{P}$ <u>iff</u> $\mathcal{Y}$ is a 0-commitment.

3) If $\langle \bar{M}^i : i < \zeta \rangle$ is $\le$-increasing, $\zeta < \omega_2$ and $\mathrm{Dom}(\bar{M}^i) \backslash S$ is not stationary for $i < \zeta$, <u>then</u> there is $\bar{M}$, $\mathrm{Dom}(\bar{M}) = S$ such that $i < \zeta \Rightarrow \bar{M}^i \le \bar{M}$.

4) Increasing $\bar{M}$ preserves everything.

5) If a forcing notion $\mathbb{P}$ satisfies the 0-commitment $\mathcal{Y}$ for the $\aleph_1$-oracle $\bar{M}$ and $S' = \{\delta \in S^{\mathcal{Y}} : \mathbb{Q}_\delta^{\bar{\varphi}_\alpha[\mathcal{Y}]}$ is a singleton (i.e., δ of the old case for $\mathcal{Y}$)$\}$ and $S' \ne \emptyset \bmod D_{\bar{M}}$ <u>then</u>

(a) $\mathbb{P}$ satisfies the $\bar{M}$-c.c.

(b) if $S'' \subseteq S'$ and $S'' \neq \emptyset \bmod D_{\bar{M}}$ and $M_\delta \models$ "$X_\delta \subseteq {}^\omega 2$ is not meagre" for every $\delta \in S''$ then $\bigcup_{\delta \in S''} X_\delta$ is not meagre in $\mathbf{V}[\mathbb{P}]$.

As a warm-up (see [4] for more)

Claim 1.8. 1) *Assume*

(a) *M is a countable transitive model of ZFC$^-$, $M \models$ "$\mathbb{P}_1$ is a countable forcing notion"*

(b) *$M \models$ "φ is an absolute definition of c.c.c. forcing notion $\mathbb{Q}^\varphi$ with generic $\underset{\sim}{\eta}\colon \alpha_1 \to \alpha_2$" and $\alpha_1, \alpha_2 < \omega_1$*

(c) *ν is $(M, \mathbb{Q}^\varphi)$-generic sequence, i.e., there is $G \subseteq (\mathbb{Q}^\varphi)^M$ generic over M such that $\nu = \underset{\sim}{\eta}[G]$.*

Then we can find a countable $\mathbb{P}_2$ such that

(α) *$\mathbb{P}_1 \subseteq_{ic} \mathbb{P}_2$ and every $\mathcal{J} \in M$ which is predense in $\mathbb{P}_1$ is predense in $\mathbb{P}_2$*

(β) *$\Vdash_{\mathbb{P}_2}$ "ν is $(M', \mathbb{Q}^\varphi)$ -generic sequence where $M' = M[G_{\mathbb{P}_2} \cap \mathbb{P}_1]$".*

2) *Similarly for φ defining a nep forcing.*

Proof. 1) In M we can define $\mathbb{P}^+ = \mathbb{P}_1 * (\underset{\sim}{\mathbb{Q}}^\varphi)^{M[G_{\mathbb{P}_1}]}$, now as $\mathbb{Q}^\varphi$ is absolutely c.c.c., we know that $q \mapsto (\emptyset, q)$ is a complete embedding of $(\mathbb{Q}^\varphi)^M$ into $\mathbb{P}^+$. So if $G_* \subseteq (\mathbb{Q}^\varphi)^M$ is generic over M such that $\nu = \underset{\sim}{\eta}[G]$ then let $\mathbb{P}_2^* = \{(p, \underset{\sim}{q}) \in \mathbb{P}_1 * (\underset{\sim}{\mathbb{Q}}^\varphi)^{M[G_{\mathbb{P}_1}]}\colon (p, \underset{\sim}{q})$ is compatible with $(\emptyset, q')$ for every $q' \in G_*\}$. Now check.

2) See [4]. $\square_{1.8}$

Crucial Claim 1.9. *In IS, any $\leq^*$-increasing ω-chain has an upper bound.*

Remark. 1) The ω-limit is the crucial one not the ω_1-limit? Actually for ω_1-limit we take the union and we preserve what we need by using the square (and having done something toward it in earlier limits or stages of cofinality $\aleph_0$).

2) When is the union not an upper bound? If, e.g., for each $\alpha \in S' \subseteq S^{\mathcal{Y}}$ the forcing note $\varphi_\alpha^{\mathcal{Y}}$ is random real forcing we have in particular to preserve $\{\nu_\alpha\colon \alpha \in S'\}$ is non-null, but the union normally adds a Cohen.

Proof. So assume $(\mathbb{P}^n, \mathcal{Y}^n, \bar{M}^n) \in IS$ and $(\mathbb{P}^n, \mathcal{Y}^n, \bar{M}^n) \leq^* (\mathbb{P}^{n+1}, \mathcal{Y}^{n+1}, \bar{M}^{n+1})$ for $n < \omega$, let $\bar{M}$ be such that $\bar{M} \geq \bar{M}^n$ for each n; so let $E_n \in \mathcal{D}_{\bar{M}}$ witness both. For simplicity assume that above any $p \in \mathbb{P}^n$ there are two incompatible elements, and $0 \in \mathbb{P}^0$ is minimal in all $\mathbb{P}^n$, i.e. is

$\emptyset_{\mathbb{P}_n}$. Without loss of generality the set of elements of $\mathbb{P}^n$ is $\subseteq \omega_1$ and $\omega_1 \setminus \bigcup_{n<\omega} \mathbb{P}^n$ has cardinality $\aleph_1$ and let X^* be such that $\bigcup_{n<\omega} \mathbb{P}_n \subseteq X^* \subseteq \omega_1$ and $|X^* \setminus \bigcup_{n<\omega} \mathbb{P}^n| = \aleph_1$; this notation helps in a future use, also there we replace ω by a (countable) ordinal of cofinality $\aleph_0$. We can define functions F_n, $F_{n,m}$, $F_{n,m,\ell}$ (when $n < m < \omega, \ell < \omega$) such that

$(a)_n$ if $p, q \in \mathbb{P}_n$ are compatible then $F_n(p,q) \in \mathbb{P}_n$ is a common upper bound

$(b)_{n,m}$ if $n < m$ and $p \in \mathbb{P}_m$, then $\langle F_{n,m,\ell}(p) \colon \ell < \omega \rangle$ is a maximal antichain of $\mathbb{P}_n$, such that for each ℓ:
<u>either</u> $p, F_{n,m,\ell}(p)$ are incompatible (in $\mathbb{P}_m$)
<u>or</u> p is compatible in $\mathbb{P}_m$ with every $q \in \mathbb{P}_n$ which is above $F_{n,m,\ell}(p)$

$(c)_{n,m}$ if $n < m$, $p \in \mathbb{P}_m$, $q \in \mathbb{P}_n$ then $p \leq F_{n,m}(p,q) \in \mathbb{P}_n$ and if there is r such that $q \leq_{\mathbb{P}_n} r$ and r, p are incompatible in $\mathbb{P}_m$ then $p, F_{n,m}(p,q)$ are incompatible in $\mathbb{P}_m$.

Let E be a club of ω_1, such that $\delta \in E \Rightarrow \delta$ is closed under $F_n, F_{n,m}, F_{n,m,\ell}$ and

$$\mathrm{otp}(X^* \cap \delta \setminus \bigcup_{n<\omega} \mathbb{P}^n) = \delta.$$

We would like to define a forcing notion $\mathbb{P}^\omega$ with universe X^*, and 1-commitment $\mathcal{Y}^\omega$, and functions $F_\omega, F_{n,\omega,\ell}$ satisfying the natural requirements. First, let

$$E_\omega = \bigcap_{n<\omega} E_n \cap E, S^{\mathcal{Y}^\omega} = \bigcup_{n<\omega} S^{\mathcal{Y}^n} \cap E_\omega,$$

and for $\alpha \in S^{\mathcal{Y}^\omega}$ the triple $(\underset{\sim}{\varphi}^{\mathcal{Y}^\omega}_\alpha, \underset{\sim}{\eta}^{\mathcal{Y}^\omega}_\alpha, \underset{\sim}{\nu}^{\mathcal{Y}^\omega}_\alpha)$ is $(\underset{\sim}{\bar{\varphi}}^{\mathcal{Y}^{n(\alpha)}}_\alpha, \underset{\sim}{\eta}^{\mathcal{Y}^{n(\alpha)}}_\alpha, \underset{\sim}{\nu}^{\mathcal{Y}^{n(\alpha)}}_\alpha)$ where $n(\alpha) = \operatorname{Min}\{n \colon \alpha \in S^{\mathcal{Y}^n}\}$.

Defining $\mathbb{P}_\omega, F_\omega, F_{n,\omega,\ell}$ is harder, so we first define AP, a set of approximations to it. A member t of AP has the form $(\delta^t, \mathbb{P}^t, F^t_\omega, F^t_{n,\omega,\ell}, \Gamma^t)_{\ell<\omega}$ satisfying

(α) $\delta^t \in E_\omega$

(β) $\mathbb{P}^t$ is a forcing notion with set of elements $\subseteq X^* \cap \delta^t$ and $\supseteq \delta^t \cap \bigcup_n \mathbb{P}_n$ and $0 \leq_{\mathbb{P}^t} p$ for every $p \in \mathbb{P}^t$

(γ) $\mathbb{P}^t \restriction (\mathbb{P}^n \cap \delta^t) = \mathbb{P}^n \restriction (\mathbb{P}^n \cap \delta^t)$

(δ) if $p, q \in \mathbb{P}^t$ are compatible in $\mathbb{P}^t$ then $F^t_\omega(p,q)$ is such an upper bound

(ε) if $p \in \mathbb{P}^t$, $n < \omega$ then $\langle F^t_{n,\omega,\ell}(p) \colon \ell < \omega \rangle$ is a maximal antichain of $\mathbb{P}_n$, the members are $< \delta^t$, and for each ℓ, <u>either</u> $p, F^t_{n,\omega,\ell}(p)$ are incompatible in $\mathbb{P}^t$ <u>or</u> $(\forall q \in \mathbb{P}_n \cap \delta)$ ($\mathbb{P}_n \models$ "$F_{n,\omega,\ell}(p) \leq q$" $\Rightarrow p, q$ are compatible in $\mathbb{P}^t$) and for at least one ℓ the second case occurs

(ζ) if $p \in \mathbb{P}^t \cap \mathbb{P}_m \setminus \bigcup_{\ell<m} \mathbb{P}_\ell$ then $F^t_{n,\omega,\ell}(p) = F_{n,m,\ell}(p)$

(η) Γ^t is a sequence $\langle \bar p^t_\zeta : \zeta < \zeta^t \rangle$, $\zeta^t < \omega_1$ and $\bar p^t_\zeta$ is a sequence of length ω of members of $\mathbb{P}^t$ which form a maximal antichain (of $\mathbb{P}^t$)

(θ) if $p \in \mathbb{P}^t$ and $n < m < \omega$ and $r \in \mathbb{P}_n \cap \delta^t$ and $[r \le r' \in \mathbb{P}_n \cap \delta^t \Rightarrow r', p$ are compatible in $\mathbb{P}^t]$, then the set $\{F^t_{m,\omega,\ell}(p) : \ell < \omega$ and p is compatible with $F^t_{m,\omega,\ell}(p)$ in $\mathbb{P}^t\}$ satisfies: if $r \le q \in \mathbb{P}_n$ then in $\mathbb{P}_m, q$ is compatible with some member of this set

(ι) if $\zeta < \zeta^t$ and $n < \omega$ then:
$\{F^t_{n,\omega,\ell}(p^t_{\zeta,k}) : k < \omega,\ \ell < \omega$ and $p^t_{\zeta,k}, F^t_{n,\omega,\ell}(p^t_{\zeta,k})$ are compatible in $\mathbb{P}^t\}$ is a predense subset of $\mathbb{P}_n$. Note that trivially this subset is predense in $\mathbb{P}_n \cap \delta^t$; similarly in clause ($\kappa$)

Moreover,

(κ) if $p^* \in \mathbb{P}^t$ and $n < \omega$ and $\zeta < \zeta^t$ then

$$\begin{aligned}\mathcal{I}^t_{\zeta,n,p^*} =: \{r' \in \mathbb{P}_n \cap \delta^t : (i)\ & r', p^* \text{ incompatible in } \mathbb{P}^t \text{ \underline{or}} \\ (ii)\ & \text{for some } k < \omega \text{ and } p' \text{ we have} \\ & (\forall r'')\ [r' \le r'' \in \mathbb{P}_n \cap \delta^t \rightarrow \{r'', p'\} \\ & \text{has an upper bound in } \mathbb{P}^t] \text{ and} \\ & p^* \le_{\mathbb{P}^t} p',\ p^t_{\zeta,k} \le_{\mathbb{P}^t} p'\}.\end{aligned}$$

is predense in $\mathbb{P}_n$.

* * *

We define the (natural) partial order $\le^*$ on AP: for $t, s \in AP$ as follows; we let $t \le^* s$ iff:

(i) $\delta^t \le \delta^s$

(ii) $\mathbb{P}^t \subseteq \mathbb{P}^s$

(iii) $F^t_\omega \subseteq F^s_\omega$

(iv) $F^t_{n,\omega,\ell} \subseteq F^s_{n,\omega,\ell}$

(v) Γ^t is an initial segment of Γ^s.

Fact A. $AP \ne \emptyset$.

Proof. Easy: choose $\delta \in E$, let

$$\mathbb{P}^t = \Big(\bigcup_{n<\omega} \mathbb{P}^n\Big) \restriction \delta, F_\omega(p,q) = F_{n(p,q)}(p,q)$$

where $n(p,q) =$ Min$\{n : p \in \mathbb{P}_n$ and $q \in \mathbb{P}_n\}$.

For $n < \omega$, $p \in \mathbb{P}^t \cap \delta$ let $\langle F^t_{n,\omega,\ell}(p) \colon \ell < \omega \rangle$ be $\langle F_{n,m,\ell}(p) \colon \ell < \omega \rangle$ for the first $m \geq n$ such that $p \in \mathbb{P}_m$.

Lastly, $\Gamma =$ empty sequence.

Fact B. If $t \in AP$ and $\delta^t < \delta \in E$, then there is s satisfying $t \leq^* s \in AP$ with $\delta^s \geq \delta, \zeta^s = \zeta^t$.

Proof. Without loss of generality t, $\langle \mathbb{P}^n \restriction \delta \colon n < \omega \rangle$, $X^* \cap \delta$ belongs to M_δ and $\delta \in E \cap \bigcap_{n<\omega} E_n$ and $X^* \cap \delta \setminus \bigcup_{n<\omega} \mathbb{P}_n \setminus \delta^t$ is infinite and even has order type δ. [Why? As $\emptyset \notin \mathcal{D}_{\bar{M}}$ and we can increase δ.] So (for the last phrase see the proof of 1.4(4))

- $(*)$ any $\mathcal{J} \in M_\delta$ which is a predense subset of $\mathbb{P}_n \restriction \delta$ is a predense subset of $\mathbb{P}_n$ and $n < m \Rightarrow \mathbb{P}_n \restriction \delta \lessdot \mathbb{P}_m \restriction \delta$.

Let $A = \mathbb{P}^t \subseteq X^* \cap \delta^t$, $B = \bigcup_{n<\omega} \mathbb{P}^n \cap \delta$. We define a forcing notion $\mathbb{Q}$, with set of elements $\subseteq A \times B$ identifying $(p, 0)$ with p and $(0, q)$ with q. Now $(p, q) \in A \times B$ belongs to $\mathbb{Q}$ iff: $p = 0$ or $q = 0$ or there are $r \in A \cap B$ and $n = n(p, q)$ such that: $\mathbb{P}_n \models$ "$r \leq q$", and $(\forall r')$ $[r \leq r' \in \mathbb{P}_n \cap \delta^t \rightarrow r'$, p compatible in $\mathbb{P}^t]$; we call such r a witness and n a possible value for $n(p, q)$. The order on $\mathbb{Q}$ is

$$(p, q) \leq (p', q') \Leftrightarrow p \leq_{\mathbb{P}^t} p' \text{ and } \bigvee_n q \leq_{\mathbb{P}_n} q'.$$

Now note

- (α) $\mathbb{Q} \restriction A = \mathbb{P}^t$.
- (β) $\mathbb{Q} \restriction B = \bigcup_{n<\omega} \mathbb{P}_n \restriction \delta$.
- (γ) If $(p, q) \in \mathbb{Q}$, $m = n(p, q)$ and $q \in \mathbb{P}_m \restriction \delta$ and $\mathbb{P}_m \restriction \delta \models$ "$q \leq q'$" and $\mathbb{P}^t \models$ "$p' \leq p$", then $(p', q') \in \mathbb{Q}$ and $\mathbb{Q} \models$ "$(p, q) \leq (p, q')$".
- (δ) If $(p, q) \in \mathbb{Q}$ and $n = n(p, q) \leq m < \omega$, then for some q_1 we have: $(p, q) \leq (p, q_1) \in \mathbb{Q}$ and $n(p, q_1) = m$, or at least m is a possible value for $n(p, q_1)$.

  [Why? Let $r \in \mathbb{P}_{n(p,q)}$ be a witness in particular r is compatible with p in $\mathbb{P}_t$. By clause (θ) of the Definition of AP the set $\mathcal{J} = \{F^t_{m,\omega,\ell}(p) \colon \ell < \omega$ and p is compatible with $F^t_{m,\omega,\ell}(p)$ in $\mathbb{P}^t\}$ is predense above r in $\mathbb{P}_m$. $\mathbb{P}_n \models r \leq q$ hence $\mathbb{P}_m \models r \leq q$ so for some $\ell, F^t_{m,\omega,\ell}(p) \in \mathcal{J}$ is compatible with q in $\mathbb{P}_m$ so there is $q_1 \in \mathbb{P}_m \cap \delta$ such that $\mathbb{P}_m \models q \leq q_1 \wedge F^t_{m,\omega,\ell}(p) \leq q_1$. So $(p, q_1) \in \mathbb{Q}$ as witnessed by m and $r' = F^t_{m,\omega,\ell}(p)$, is as required.]
- (ε) $\mathbb{P}_n \restriction \delta \lessdot \mathbb{Q}$.

  [Why? Let $(p^0, q^0) \in \mathbb{Q}$, of course, we can replace this pair by any larger one, so by clause (δ) above without loss of generality some $m \in$

$[n,\omega)$, is a possible value for $n(p^0,q^0)$ so we have $q^0 \in \mathbb{P}_m \restriction \delta$, hence recalling that $\mathbb{P}_n \restriction \delta \lessdot \mathbb{P}_m \restriction \delta$ there is $q^1 \in \mathbb{P}_n \restriction \delta$ such that:

$$(\forall r \in \mathbb{P}_n)(\mathbb{P}_n \restriction \delta \models q^1 \leq r \Rightarrow r, q^0 \text{ compatible in } \mathbb{P}_m \restriction \delta).$$

Assume $q^1 \leq r \in \mathbb{P}_n \restriction \delta$. So r, q^0 are compatible in $\mathbb{P}_m \restriction \delta$ hence has a common upper bound $q^2 \in \mathbb{P}_m \restriction \delta$.
In particular $q^0 \leq q^2 \in \mathbb{P}_m \restriction \delta$ so by clause (γ) we have $(p^0,q^2) \in \mathbb{Q}$ and $(p^0,q^0) \leq^{\mathbb{Q}} (p^0,q^2)$; also $r = (0,r) \leq (p^0,q^2)$ as $r \leq q^2$ together r, (p^0,q^0) are compatible in $\mathbb{Q}$, so $[q^1 \leq r \in \mathbb{P}_n \restriction n \Rightarrow (p^0,q^0)$, $r = (0,r)$ are compatible in $\mathbb{Q}]$. As $(p^0,q^0) \in \mathbb{Q}$ was arbitrary we are done.]

(ζ) If $p_1, p_2 \in \mathbb{P}^t$ are incompatible in $\mathbb{P}^t$ then they are incompatible in $\mathbb{Q}$.
[Why? Look at the order of $\mathbb{Q}$].

(η) If $\zeta < \zeta^t$ then $\bar{p}^t_\zeta$ is a maximal antichain in $\mathbb{Q}$.
[Why? If not some $(p^*,q^*) \in \mathbb{Q}$ is incompatible in $\mathbb{Q}$ with every $(p^t_{\zeta,k},0)$ for $k < \omega$. Let $n < \omega$ be a possible value of $n(p^*,q^*)$ so $q^* \in \mathbb{P}_n \restriction \delta$ and there is a witness $r^* \leq q^*, r^* \in \mathbb{P}_n \restriction \delta^t$ for $(p^*,q^*) \in \mathbb{Q}$.
By clause (κ) in the definition of $t \in AP$ we know that for some $r \in \mathbb{P}_n \cap \delta^t$ we have:
(i) $r \in \mathcal{I}^t_{\zeta,n,p^*}$
(ii) q^*, r are compatible in $\mathbb{P}_n$.
As q^*, r are compatible and $r^* \leq q^*$ also r^*, r are compatible in $\mathbb{P}_n$ hence in $\mathbb{P}_n \cap \delta^t$, so by the demand on r^*, we have: r, p^* are compatible in $\mathbb{P}^t$. So in clause (κ) of the definition of AP, in the definition of $\mathcal{I}^t_{\zeta,n,p^*}$ for our r subclause (i) fails hence subclause (ii) holds so there are k, p' as in subclause (ii) there. Also let $q^1 \in \mathbb{P}_n \restriction \delta$ be a common upper bound of q^*, r. So r witness that $(p',q^1) \in \mathbb{Q}$ with n a possible value of $n(p',q^1)$. Clearly it is above (p^*,q^*) and above $p^t_{\zeta,k}$ so we are done.]

Let $\delta^s = \delta$. Clearly $\mathbb{Q} \in M_\delta$ and $M_\delta \models$ "$|\mathbb{Q}_\delta| \leq |\delta|$" so, as $X^* \cap \delta \setminus \bigcup_n \mathbb{P}_n$ has order type δ and $\mathbb{P}^t$ is bounded in it, there is $f \in M_\delta$ such that $f \colon \mathbb{Q} \to X^* \cap \delta$ is a one to one (into or even onto), extending $\mathrm{id}_A \cup \mathrm{id}_B$, and define $\mathbb{P}^s$ such that f is an isomorphism from $\mathbb{Q}$ onto $\mathbb{P}^s$. We can define $F^s_\omega, F^s_{n,\omega,\ell}$ $(n, \ell < \omega)$ extending $F^t_\omega, F^s_{n,\omega,\ell}$ as required, e.g., $F^s_{n,\omega,\ell}((p,q)) = F^{\mathbf{s}}_{n,m,\ell}(q)$ for some $m > n$ such that $q \in \mathbb{P}_m$ except when $q = 0$ then $F^s_{n,\omega,\ell}((p,0)) = F^t_{n,\omega,\ell}(p)$. Now it is easy to check clause (θ) of the definition of $s \in AP$, recalling $(*)$ above and clauses $(i), (\kappa)$ holds since the construction is made in M_δ. Lastly, let $\Gamma^s = \Gamma^t$.

Fact C. If $t^n \in AP$ and $t^n \leq^* t^{n+1}$ for $n < \omega$ then there is t such that $n < \omega \Rightarrow t^n \leq^* t \in AP$ and $\delta^t = \bigcup_{n<\omega} \delta^{t^n}$ and $\zeta^t = \bigcup_{n<\omega} \zeta^{t^n}$.
[Why? Just let δ^t, ζ^t be as above, $\mathbb{P}^t = \bigcup_{n<\omega} \mathbb{P}^{t^n}$, $F^t_\omega = \bigcup_{n<\omega} F^{t^n}_n$, $F^t_{m,\omega,\ell} = \bigcup_{n<\omega} F^{t^n}_{m,\omega,\ell}$ and $p^t_{\zeta,k} = p^{t^n}_{\zeta,k}$ for every n large enough. Now check.]

Main Fact D. Assume $t \in AP$, $\delta^t \in E \cap \bigcap_{n<\omega} E_n$, $t \in M_\delta$ and $\delta := \delta^t \in \bigcup_{n<\omega} S^{\mathcal{Y}_n}$. Then there is $s \in AP$ such that $t \leq^* s$ and $\underset{\sim}{\nu}_\delta$ is actually a $\mathbb{P}^s$-name (i.e. all the countably many conditions appearing in its definition belongs to $\bigcup_{n<\omega} \mathbb{P}_m \cap \delta^s \subseteq \mathbb{P}^s$) and:

- $(*)$ if $\mathbb{P}^s \subseteq_{ic} \mathbb{Q}$, and for each $\zeta < \zeta^s$ the sequence $\bar{p}^s_\zeta$ is a maximal antichain of $\mathbb{Q}$, then
  $\Vdash_{\mathbb{Q}}$ "there is $G' \subseteq \mathbb{Q}^{M_\delta[G]}_{\bar{\underset{\sim}{\varphi}}^t_\delta}$ generic over $M_\delta[G]$ such that $(\underset{\sim}{\eta}[\underset{\sim}{G}])[G'] = \underset{\sim}{\nu}_\delta$".

[Why? Chase arrows so similar to the proof of 1.8 (1) (and we use clause (α) there for clause (e) of Definition 1.6 (1A)).]

Fact E. If in Main Fact D, $\mathbb{Q}_\delta$ is a singleton (hence $\nu_\delta \in M_\delta$ so the main fact is trivial) then there is $s \in \mathrm{AP}$ such that $t \leq^* s$ and $\bar{p} \in M_\delta$ is an ω-sequence listing a predense subset of $\mathbb{P}^t$ then $\bar{p}$ appears in the sequence Γ^t.
[Why? Easy.]
So we can choose $t_\varepsilon \in AP$ by induction on $\varepsilon < \omega_1$ such that t^ε is $\leq^*$-increasing continuous, $\delta^{t^{\varepsilon+1}} > \delta^{t^\varepsilon}$, and if $t^\varepsilon \in M_{(\delta^{t^\varepsilon})}$, $\delta^{t^\varepsilon} \in \bigcap_{n<\omega} E_n \cap E \cap \bigcup_{n<\omega} S^{\mathcal{Y}_n}$ then $t^{\varepsilon+1}$ is gotten by Fact D. No problem to carry this ($\varepsilon = 0$ by Fact A, $\varepsilon = \varepsilon_1 + 1$ by Fact D if possible and by Fact B if not; lastly, if ε is a limit ordinal, use Fact C).

Now let $\mathbb{P}^\omega = \bigcup_{\varepsilon<\omega_1} \mathbb{P}^{t^\varepsilon}$ and it should be clear how to define $\mathcal{Y}^\omega$; now check the requirements. $\square_{1.9}$

Definition 1.10. Let $\bar{C}^* = \langle C^*_\delta \colon \delta < \omega_2 \text{ a limit ordinal}\rangle$ (and $C^*_\alpha = \emptyset$ otherwise) be a square sequence and $\bar{X}^* = \langle X^*_i \colon i < \omega_1\rangle$ be an increasing sequence of subsets of ω_1,

$$|X^*_i \setminus \bigcup_{j<i} X^*_j| = \aleph_1, X^*_{\omega_1} = \bigcup_{i<\omega_1} X^*_i.$$

We say that $\langle(\mathbb{P}_i, \mathcal{Y}_i, f_i, \bar{M}_i) \colon i < \alpha\rangle$ is a $(\bar{C}^*, \bar{X}^*)$-iteration (we omit $\bar{M}^i$ and write $(\bar{M}, \bar{C}^*, \bar{X}^*)$-iteration if $i < \alpha \Rightarrow \bar{M}^i \leq \bar{M}$ or an $\bar{M}$-iteration when $\bar{C}^*$, $\bar{X}$ are clear from context) if:

- (a) $(\mathbb{P}_i, \mathcal{Y}_i, \bar{M}^i) \in IS$ is $<^*$-increasing and $\mathrm{Dom}(\bar{M}^i) = S^{\mathcal{Y}_i}$
- (b) f_i is a one to one function from $\mathbb{P}_i$ onto $X^*_{\mathrm{otp}(C^*_\alpha)}$, and let $(\mathbb{P}'_i, \mathcal{Y}'_i)$ be such that f_i maps $(\mathbb{P}_i, \mathcal{Y}_i)$ to $(\mathbb{P}'_i, \mathcal{Y}'_i)$
- (c) if $j \in \mathrm{acc}(C_i)$ then $f_j \subseteq f_i$
- (d) if $\mathrm{cf}(i) = \aleph_0$ and $i = \sup \mathrm{acc}(C^*_i)$ then $(\mathbb{P}'_i, \mathcal{Y}'_i)$ is gotten from $\langle(\mathbb{P}'_j, \mathcal{Y}'_j) \colon j \in \mathrm{acc}(C^*_\delta)\rangle$ as in the proof of 1.9 (using $\langle X^*_j \colon j \in \mathrm{acc}(C^*_i)\rangle$,

$X^*_{\text{otp}(C^*_i)}$ instead of $\langle X_n : n < \omega \rangle$, X_ω so $\text{acc}(C^*_i)$ replace ω and we generate $\langle t^i_\alpha : \alpha < \omega_1 \rangle$ and by it define $(\mathbb{P}'_i, \mathcal{Y}'_i)$ hence $(\mathbb{P}_i, \mathcal{Y}_i)$)

(*e*) in clause (d), assume $\delta = \text{otp}(C^*_i)$, $\langle (\mathbb{P}'_j, \mathcal{Y}'_j) \restriction \delta : j \in \text{acc}(C^*_i) \rangle \in M_\delta$ and for $j_1 < j_2$ from $\text{acc}(C^*_i)$ the ordinal δ belongs to the club $\{\alpha < \omega_1 : \alpha$ limit closed under the functions F^{j_1} and F^{j_1,j_2}_ℓ (see clause (f) below)$\}$ and $\delta^{t^{j_1}_\delta} = \delta$. Let $t^i_* \in AP$ be defined by $\delta^i_* = \text{otp}(C^*_i)$, $\mathbb{P}^{t^i_*} = \cup\{\mathbb{P}'_j \restriction \delta : j \in \text{acc}(C^*_i)\}$, $F^{t^i_*}_\omega = \cup\{F^{j_1} \restriction \delta_{i,j_1} \in \text{acc}(C^*_i)\}$, $F^{t^i_*}_{j_1,\omega,\ell} = \cup\{F^{j_1,j_2}_\ell : j_2 \in \text{acc}(C^*_i) \backslash j_1\}$ and let $\Gamma^{t^i_*}$ be empty. If $t^i_* \in M_\delta$ then let t^i_0 be gotten from t^i_* as in Fact D.

(*f*) F^j is a (partial) two-place function from $X^*_{\text{otp}(C^*_j)}$ to itself such that $F^j(p,q)$ is the $<$-first common upper bound of p and q in $\mathbb{P}'_j$ and if $j_1 \in \text{acc}(C^*_{j_2})$ then we have $\langle F^{j_1,j_2}_n(p) : n < \omega \rangle$ is a maximal antichain of $\mathbb{P}'_{j_1}$ satisfying: for each n, either $F^{j_1,j_2}_n(p)$ is incompatible with p in $\mathbb{P}'_{j_2}$ <u>or</u> p is compatible with r in $\mathbb{P}'_{j_2}$ wherever $\mathbb{P}'_{j_1} \models F^{j_1,j_2}_n(p) \le r$.

Claim 1.11 (iteration at limit)**.** 1) *Assume $\langle (\mathbb{P}_i, \mathcal{Y}_i, f_i) : i < \zeta \rangle$ is a $(\bar{M}, \bar{C}^*, \bar{X}^*)$-iteration where $\zeta < \omega_2$ is a limit ordinal.* <u>*Then*</u>

(*a*) *we can find $(\mathbb{P}_\zeta, \mathcal{Y}_\zeta, f_\zeta)$ such that $\langle (\mathbb{P}_i, \mathcal{Y}_i, f_i) : i < \zeta + 1 \rangle$ is an $\bar{M}$-iteration*

(*b*) *if $S \subseteq S^*$, $i < \zeta \Rightarrow S^{\mathcal{Y}_i} \subseteq S \bmod \mathcal{D}_{\bar{M}}$,* <u>*then*</u> *we can demand $S^{\mathcal{Y}_\zeta} = S$.*

Proof. If $\text{cf}(\zeta) = \aleph_0$ we use 1.9 but taking care of clause (e), this just dictates to us how to start the induction there. If $\text{cf}(\zeta) = \aleph_1$, then by the square bookkeeping (see clause (e) in Definition 1.10) our work is done (using $f_\zeta = \cup\{f_\xi : \xi \in \text{acc}(C_\zeta)\}$). $\square_{1.11}$

Claim 1.12. 1) *Assume*

(*a*) *$\mathcal{Y} = (S, \bar{\Phi}, \underset{\sim}{\bar{\eta}}, \underset{\sim}{\bar{\nu}})$ is a 1-commitment on the forcing notion $\mathbb{P} \in \mathcal{H}(\aleph_2)$ for $\bar{M}$*

(*b*) *$G_{\mathbb{P}} \subseteq \mathbb{P}$ is generic over $\mathbf{V}$, $\bar{\nu}^0 = \langle \nu^0_\alpha : \alpha \in S \rangle$ where $\nu^0_\alpha = \underset{\sim}{\nu}_\alpha[G_{\mathbb{P}}]$, $\bar{M}^1 = \bar{M}[G_{\mathbb{P}}] = \langle M_\delta[f''(G_{\mathbb{P}})] : \delta \in S^* \rangle$ for some one to one function f from $\mathbb{P}$ into ω_1*

(*c*) *in $\mathbf{V}[G_{\mathbb{P}}]$, $\mathcal{Y}^1 = (S^1, \bar{\Phi}^1, \underset{\sim}{\bar{\eta}}^1, \bar{\nu}^1)$ is a 0-commitment, $S \subseteq S^1 \bmod \mathcal{D}_{\bar{M}[G_{\mathbb{P}}]}$, $\bar{\Phi}^1 \restriction (S \cap S^1) = \bar{\Phi} \restriction (S \cap S_1)$, $\underset{\sim}{\bar{\eta}}^1 \restriction (S \cap S_1) = \underset{\sim}{\bar{\eta}} \restriction (S \cap S_1)$, $\bar{\nu}^1 \restriction (S \cap S_1) = \bar{\nu}^0 \restriction (S \cap S_1)$ and $(S^1, \bar{\Phi}^1, \underset{\sim}{\bar{\eta}}^1) \in V$*

(*d*) *in $\mathbf{V}[G_{\mathbb{P}}]$, $\mathbb{Q}$ is a forcing notion satisfying the 0-commitment $\mathcal{Y}^1$ for $\bar{M}^1$.*

Then for some $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ and 1-commitment $\mathcal{Y}^2$ we have:

(a) $(\mathbb{P}, \mathcal{Y}) \leq^* (\mathbb{P} * \underset{\sim}{\mathbb{Q}}, \mathcal{Y}^2)$

(b) $S^{\mathcal{Y}^2} = S^1$, $\Phi^{\mathcal{Y}^2} = \bar{\Phi}^1$, $\underset{\sim}{\bar{\eta}}^{\mathcal{Y}_2} = \underset{\sim}{\bar{\eta}}^1$, $\underset{\sim}{\bar{\nu}}[G_{\mathbb{P}}] = \bar{\nu}^1$

(c) $\underset{\sim}{\mathbb{Q}}[G_{\mathbb{P}}] = \mathbb{Q}$.

2) *If for every $G_{\mathbb{P}} \subseteq \mathbb{P}$ generic over $\mathbf{V}$ there are $\mathbb{Q}$ satisfying some ψ_1 and $(S^1, \bar{\Phi}^1, \underset{\sim}{\bar{\eta}}^1, \bar{\nu}) \in \mathbf{V}[G_{\mathbb{P}}]$ as above satisfying some ψ_2, then we can demand*

(d) $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}}[\underset{\sim}{G}_{\mathbb{P}}], \mathcal{Y}^2$ *as above satisfies ψ_1, ψ_2 respectively".*

3) *We may allow $\langle(\bar{\varphi}_\alpha, \underset{\sim}{\bar{\eta}}_\alpha) \colon \alpha \in S^1\rangle$ be a sequence of $\mathbb{P}$-names and even $(\mathbb{P} * \underset{\sim}{\mathbb{Q}})$-names.*

Proof. Straight.

Claim 1.13 (iteration in successor case: increase the commitment)**.** *Assume $\langle(\mathbb{P}_i, \mathcal{Y}_i, f_i) \colon i < \zeta\rangle$ is an $\bar{M}$-iteration and $\zeta = \xi + 1$, $S^{\mathcal{Y}_\xi} \subseteq S \subseteq S^*$, $S \subseteq \mathrm{Dom}\,(\bar{M})$ and $\langle(\underset{\sim}{\varphi}_\alpha, \underset{\sim}{\eta}_\alpha) \colon \alpha \in S \backslash S^{\mathcal{Y}_\xi}\rangle$ is as required in Definition 1.6. Lastly $\underset{\sim}{Z}_\alpha \subseteq {}^\omega 2$ is a $\mathbb{P}_\xi$-name of a positive set for $(\underset{\sim}{\bar{\varphi}}_\alpha, \underset{\sim}{\eta}_\alpha)$ for every such α.*

Then we can find $(\mathbb{P}_\zeta, \mathcal{Y}_\zeta, f_\zeta)$ such that

(i) *$\langle(\mathbb{P}_i, \mathcal{Y}_i, f_i) \colon i < \zeta + 1\rangle$ is an $\bar{M}$-iteration*

(ii) *$\mathbb{P}_\zeta = \mathbb{P}_\xi$, $S^{\mathcal{Y}_\xi} = S$, $(\underset{\sim}{\bar{\varphi}}_\alpha^{\mathcal{Y}_\xi}, \underset{\sim}{\eta}_\alpha^{\mathcal{Y}_\xi}) = (\underset{\sim}{\bar{\varphi}}_\alpha, \underset{\sim}{\eta}_\alpha)$ if $\alpha \in S \backslash S^{\mathcal{Y}_\xi}$.*

Proof. Straight.

Claim 1.14 (iteration at successor: increasing the forcing)**.** *Suppose*

(a) *$(\mathbb{P}, \mathcal{Y}) \in IS$ and the set of elements of $\mathbb{P}$ is X_i (the X_j^*'s as in 1.10)*

(b) *$\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}$-name satisfying, for every $G \subseteq \mathbb{P}$ generic over $\mathbf{V}$, the following:*

  (i) *$\underset{\sim}{\mathbb{Q}}[G]$ is a forcing notion with set of elements beings $X_{i+1} \backslash X_i$*

  (ii) *$\{\delta < \omega_1$: if $\mathbb{P} \restriction \delta \in M_\delta$ and $G \cap \delta$ is a generic subset of $\mathbb{P} \restriction \delta, \underset{\sim}{\mathbb{Q}}[G] \restriction \delta \in M_\delta[G \cap \delta]$ and $\underset{\sim}{\nu}_\delta[G]$ is forced to be generic for $\big((\mathbb{Q}^{\underset{\sim}{\varphi}_\delta[G]})^{M_\delta[G]}, \underset{\sim}{\eta}_\delta[G]\big)\} \in \mathcal{D}_{\bar{M}[G]}$.*

Then we can find $(\mathbb{P}^+, \mathcal{Y}^+)$ such that $(\mathbb{P}, \mathcal{Y}) \leq^* (\mathbb{P}^+, \mathcal{Y}^+) \in IS$ and the $\mathbb{P}$-name $\mathbb{P}^+/\underset{\sim}{G}_{\mathbb{P}}$ is equivalent to $\underset{\sim}{\mathbb{Q}}[\underset{\sim}{G}_{\mathbb{P}}]$.

Proof. Straight.

Conclusion 1.15. Assume $(\bar{C}^*, \bar{X}^*)$ is as in 1.9. Let Φ be a set of definitions of forcing notions with some real parameters, and $\langle S_i^* \colon i < \omega_2\rangle$ is as in 1.4 for $\mathcal{D}_{\bar{M}}$.

We can find $\langle(\mathbb{P}_i, \mathcal{Y}_i, f_i, \bar{M}^i) \colon i < \omega_2\rangle$ such that

(a) it is an $(\bar{C}^*, \bar{X}^*)$-iteration

(b) $\mathbb{P} = \bigcup_{i<\omega_2} \mathbb{P}_i$ is a c.c.c. forcing notion of cardinality $\aleph_2$
(so in $\mathbf{V}^{\mathbb{P}}$, $2^{\aleph_0} \le \aleph_2$) and except in degenerated cases equality holds

(c) $S^{\mathcal{Y}_i} = S^*_i$ from 1.4(3)

(d) if in $\mathbf{V}^{\mathbb{P}_i}$ we have $(\bar{\varphi}, \eta)$ is a case of Φ as in 1.6, moreover $\Vdash_{\mathbb{P}_i}$ "$\{\delta \in S^*_{i+1} \backslash S^*_i \colon \tilde{M}^{i+1}_\delta[f''_i(\underset{\sim}{G}_{\mathbb{P}_i})]$" $\models$ "$(\bar{\varphi}, \underset{\sim}{\eta})$ as required in 1.6"$\} \in \mathcal{D}^+_{\bar{M}^{i+1}}$
(even less with more bookkeeping) and $Z \subseteq ({}^\omega 2)^{\mathbf{V}^{\mathbb{P}}}$ is positive for $(\bar{\varphi}, \eta)$, <u>then</u>
 (α) $\{\delta \in S^{\mathcal{Y}_{i+1}} \backslash S^{\mathcal{Y}_i} \colon (\bar{\varphi}, \underset{\sim}{\eta}_\delta)/G_{\mathbb{P}_i} = (\bar{\varphi}, \underset{\sim}{\eta})$ and $\underset{\sim}{\nu}_\delta[G_{\mathbb{P}_i}] \in Z\} \in \mathcal{D}^+_{\bar{M}}$, in fact the set is forced to include such old set (from $\mathbf{V}$) by this we can get
 (β) for some $j > i$, $\delta \in S^{\mathcal{Y}_{j+1}} \backslash S^{\mathcal{Y}_0}_j \Rightarrow (\bar{\varphi}, \underset{\sim}{\eta}_\delta)/G_{\mathbb{P}_i} = (\bar{\varphi}, \underset{\sim}{\eta})$, $\underset{\sim}{\nu}_\delta[G_{\mathbb{P}_i}] \in Z$

(e) if H is a pregiven function such that for every $i < \omega_2$ and $(\mathbb{P}, \mathcal{Y}, \bar{M})$ satisfying $(\mathbb{P}_i, \mathcal{Y}_i) \le^* (\mathbb{P}, \mathcal{Y}) \in IS$ such that $S^{\mathcal{Y}} = S^*_i$ we have $(\mathbb{P}, \mathcal{Y}) \le^* H(\mathbb{P}, \mathcal{Y}) \in IS$ such that $H(\mathbb{P}, \mathcal{Y}, \bar{M})$ satisfies the demands from (a) + (c) on $(\mathbb{P}_{i+1}, \mathcal{Y}_{i+1}, \bar{M}^{i+1})$, <u>then</u> we can demand $(\exists^{\aleph_2} j)[(\mathbb{P}_{j+1}, \mathcal{Y}_{j+1}) = H(\mathbb{P}_j, \mathcal{Y}_j)]$; moreover, if $S^* \subseteq \omega_2$ is stationary we can demand $\{j \in S \colon (\mathbb{P}_{j+1}, \mathcal{Y}_{j+1}) = H(\mathbb{P}_j, \mathcal{Y}_n)\}$ is stationary.
(Of course, we can promise this for $\aleph_2$ such functions).

Proof. Put together the previous claims. (Concerning clause (e) without loss of generality $\{i < \omega_1 \colon \mathrm{otp}(C^*_i) = 0\}$ is stationary) so in those stages we have no influence of clause (e) of 1.10; anyhow the influence of 1.10(e) is minor.

Discussion 1.16. We discuss here some possible extensions.

Claim 1.17. *Assume $\langle S_i \colon i < \omega_2 \rangle$ is a sequence of pairwise almost disjoint stationary subsets of ω_1, each with diamond and $i < j \Rightarrow S_i \subseteq S^+_j \bmod \mathcal{D}_{\omega_1}$, so $S^+_i \subseteq \omega_1$ and $S_i \cap S^+_i = \emptyset$ and $S^+_i / \mathcal{D}_{\omega_1}$ is increasing with i.*

<u>Then</u> in the following game the between the bookkeeper and the forcer, the bookkeeper has a winning strategy.
A Play last ω_2 moves, before the α-th move a sequence $\langle (\mathbb{P}_i, \mathbb{Q}_i, \underset{\sim}{\bar{M}}_i, \mathcal{Y}_i) \colon i < \alpha \rangle$ is defined such that

(a) *$\mathbb{P}_i$ a c.c.c. forcing notion of cardinality $\aleph_1$, say $\subseteq \mathcal{H}_{<\aleph_1}(\aleph_2)$*
(b) *$\mathbb{Q}_i$ is a $\mathbb{P}_i$-name of a forcing notion of cardinality $\le \aleph_1$, say $\subseteq \omega_1$*
(c) *$\mathbb{P}_i$ is $\lessdot$-increasing*

(*d*) $\mathbb{P}_{i+1}, \mathbb{P}_i * \underset{\sim}{\mathbb{Q}}_i$ *are isomorphic over* $\mathbb{P}_i$

(*e*) $\underset{\sim}{\bar{M}}_i$ *is a* $\mathbb{P}_i$*-name of an* S_i*-oracle*

(*f*) $\underset{\sim}{\mathcal{Y}}_i$ *is a* $\mathbb{P}_i$*-name of a* S_i*-commitment.*

In the i-th move:

(*a*) *the bookkeeper chooses* $\mathbb{P}_i$ *and a* $\mathbb{P}_i$*-name* $(\underset{\sim}{\bar{M}}_i^+, \underset{\sim}{\mathcal{Y}}^+)$ *of an* S_i^**-oracle and* 0*-commitment*

(*b*) *the forcer choose* $\underset{\sim}{\mathbb{Q}}_i$ *and* $(\underset{\sim}{\bar{M}}_i, \underset{\sim}{\mathcal{Y}}_i), \mathbb{P}_i$*-names such that* $\underset{\sim}{\mathbb{Q}}_i$ *satisfies* $(\underset{\sim}{M}_i^+, \underset{\sim}{\mathcal{Y}}_i^+)$ *and* $(\underset{\sim}{\bar{M}}_i, \underset{\sim}{\mathcal{Y}}_i)$.

In the end the bookkeeper wins if

$$i < j < \omega_2 \Rightarrow \mathbb{P}_j/\mathbb{P}_i \text{ satisfies } (\underset{\sim}{\bar{M}}_i, \underset{\sim}{\mathcal{Y}}_i).$$

Proof. Similar to earlier proofs.

We give an easy criterion for existence. The following uses more from [4].

Claim 1.18. *Assume*

(*a*) $(\mathbb{P}, \leq, \leq_n)_{n<\omega}$ *is a definition of a forcing notion satisfying condition A of Baumgartner with* $\leq_n$ *as witness and* ZFC_*^- *says this, in a way preserved by suitable forcing*

(*b*) $\mathcal{Y} = (S, \Phi, \underset{\sim}{\bar{\eta}}, \bar{\nu})$ *is a* 0*-commitment so* $\Phi = \langle \varphi_\alpha \colon \alpha \in S \rangle$

(*c*) $\mathbb{P}$ *is absolutely nep such that for each* $\alpha \in S^{\mathcal{Y}}$ *it is* $\leq_n$*-purely* $I_{\mathbb{Q}^{\bar{\varphi}_\alpha}}$*-preserving, i.e.*

 (∗) *if* M *is a* $\mathbb{P}$*-candidate and a* $\mathbb{Q}^{\bar{\varphi}_\alpha}$*-candidate,* $p \in \mathbb{P}^M$*,* $n < \omega$ *and* $q \in (\mathbb{Q}^{\bar{\varphi}_\alpha})^M$ <u>*then*</u> *for some* p'*,* η*,* ν *we have* $p \leq_n p' \in \mathbb{P}$*,* p' *is* $\langle M, \mathbb{P}\rangle$*-generic and* ν *is* $(\mathbb{Q}^{\bar{\varphi}_\alpha}, \underset{\sim}{\eta}_\alpha)$*-generic over* M *satisfying* q *(check def) and* $p' \Vdash_{\mathbb{P}}$ *"*ν *is* $(\mathbb{Q}^{\bar{\varphi}_\alpha}, \underset{\sim}{\eta}_\alpha)$*-generic over* $M\langle \underset{\sim}{G}_{\mathbb{P}} \cap P^M\rangle$*".*

<u>*Then*</u> *there is a c.c.c. forcing notion* $\mathbb{P}' \subseteq \mathbb{P}$ *(not necessarily* $\mathbb{P}' \lessdot \mathbb{P}$*) satisfying the* 0*-commitment* $\mathcal{Y}$ *and* $\Vdash_{\mathbb{P}}$ *"for a club of* $\delta < \omega_1, \varphi(\underset{\sim}{\nu}, \underset{\sim}{\eta}_\delta^*)$*".*

Remark. Why the φ_δ's? We hope it helps, for example in the following; suppose we are given $f \colon \mathbb{R} \to \mathbb{R}$, we like to force $A \subseteq \mathbb{R}$ which is not in $I_{\mathbb{Q}^{\bar{\varphi}_\alpha}}$ and on which the function f is continuous; i.e. to force a continuous $\underset{\sim}{f}^*$ such that $\{\eta \in {}^\omega 2\colon \underset{\sim}{f}^*(\eta) = f(\eta)\} \in (I_{\mathbb{Q}}^{ex})^+$. So not only do we like to find $q \Vdash$ "η_δ is $(\underset{\sim}{\mathbb{Q}}_\delta, \underset{\sim}{\eta}_\delta)$-generic over $M_\delta[\underset{\sim}{G}_{\mathbb{P}}]$" but also $q \Vdash_{\mathbb{P}'}$ "$\underset{\sim}{f}(\eta_\delta) = f(\eta_\delta)$". This is what $\bar{\varphi}$ says. (On $I_{\mathbb{Q}^{\bar{\varphi}_\alpha}}$, $I_{\mathbb{Q}}^{ex}$ see [4].)

Proof. We choose by induction on $\alpha < \omega_1$, a pair $(\mathbb{P}_\alpha, \Gamma_\alpha)$ such that:

(*α*) $\mathbb{P}_\alpha \subseteq \mathbb{P}$ is countable

(*β*) Γ_α is a countable family of predense subsets of $\mathbb{P}_\alpha$

(γ) if $\mathcal{I} \in \Gamma_\alpha$ and $p \in \mathbb{P}_\alpha$ and $n < \omega$ then for some q we have $p \le_n q \in \mathbb{P}_\alpha$ and $\mathcal{I}$ is predense above q in $\mathbb{P}$

(δ) $\mathbb{P}_\alpha$ is increasing continuous in α

(ε) Γ_α is increasing continuous in α.

Case 1: $\alpha = 0$. Trivial.

Case 2: $\alpha = \beta + 1$, β non-limit or $(\mathbb{P}_\beta, \Gamma_\beta) \notin M_\beta$.

Let $(\mathbb{P}_\alpha, \Gamma_\alpha) = (\mathbb{P}_\beta, \Gamma_\beta)$.

Case 3: α limit.

Let $(\mathbb{P}_\alpha, \Gamma_\alpha) = \Big(\bigcup_{\beta<\alpha} \mathbb{P}_\beta, \bigcup_{\beta<\alpha} \Gamma_\beta\Big)$.

Case 4: $\alpha = \delta + 1$ where δ is a limit ordinal and $(\mathbb{P}_\delta, \Gamma_\delta) \in M_\delta$.

We can find $g \subseteq \text{ Levy}(\aleph_0, |\mathbb{P}|)^{M_\delta}$, generic over M_δ such that η^*_δ is still $\mathbb{Q}_\delta$-generic over $M_\delta[g]$ (see [4, §6]).

In $M_\delta[g]$ we define $\mathbb{P}^+_\delta = \{p\colon M_\delta[g] \models p \in \mathbb{P} \text{ and } \mathcal{I} \in \Gamma_\alpha \Rightarrow \mathcal{I} \text{ predense above } p\}$, using the induction hypothesis, as in $M_\delta[g]$ the set Γ_δ is countable, so:

($*$) for every $p \in \mathbb{P}_\delta$ and $n < \omega$ there is $p' \in \mathbb{P}^+_\delta$ such that $\mathbb{P} \models p \le_n p'$.

Again by [4, §6] for every $n < \omega$ and $p \in \mathbb{P}^+_\delta$, there is $q_{p,n} \in \mathbb{P}$ such that $p \le_n q_{p,n} \in \mathbb{P}$, $q_{p,n}$ is $(M_\delta[g], \mathbb{Q})$-generic and $q_{p,n} \Vdash_{\mathbb{P}}$ "ν_δ is a $(\mathbb{Q}_\delta, \underset{\sim}{\eta}_\delta)$-generic real over $M_\delta[g][\underset{\sim}{G}_{\mathbb{P}}]$".

Let $\mathbb{P}_{\delta+1} = \mathbb{P}_\delta \cup \{q_{p,n}\colon p \in \mathbb{P}^+_\delta \text{ and } n < \omega\}$ and $\Gamma_{\delta+1} = \Gamma_\delta \cup \{\mathcal{I}_\delta\}$ where $\mathcal{I}_\delta = \{q_{p,n}\colon p \in \mathbb{P}^+_\delta \text{ and } n < \omega\}$. $\square_{1.18}$

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