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STRONGLY MEAGER SETS DO NOT FORM AN IDEAL

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A set $X \subseteq \mathbb{R}$ is strongly meager if for every measure zero set H, $X + H \neq \mathbb{R}$. Let $\mathcal{SM}$ denote the collection of strongly meager sets. We show that assuming CH, $\mathcal{SM}$ is not an ideal.

1. Introduction

In 1919, Borel wrote the paper [4] in which he attempted to classify all measure zero subsets of the real line. In this paper he introduced a class of measure zero sets, which are now called strong measure zero sets. In 1970s, Galvin, Mycielski and Solovay found a characterization of strong measure zero sets that was formulated using only the concept of a first category set and of a translation. That allowed, after replacing first category with measure zero, to define a dual notion of a strongly meager set. It was expected that the global properties of both families of sets will be similar. Several results listed below support this expectation. Nevertheless additive properties of both families of sets are different. It is well known that the family of strong measure zero sets forms an ideal, i.e. is closed under finite unions. The result of this paper is that, assuming continuum hypothesis, the collection of strongly meager sets is not closed under finite unions.

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In this paper we work exclusively in the space 2^ω equipped with the standard product measure denoted as μ. Let $\mathcal{N}$ and $\mathcal{M}$ denote the ideal of all μ-measure zero sets, and meager subsets of 2^ω, respectively. For $x,\ y \in 2^\omega$, $x+y \in 2^\omega$ is defined as $(x+y)(n) = x(n)+y(n) \pmod 2$. In particular, $(2^\omega, +)$ is a group and μ is an invariant measure.

Definition 1.1. A set X of real numbers or more generally, a metric space, is strong measure zero if, for each sequence $\{\varepsilon_n : n \in \omega\}$ of positive real numbers there is a sequence $\{X_n : n \in \omega\}$ of subsets of X whose union is X, and for each n the diameter of X_n is less than ε_n.

The family of strong measure zero subsets of 2^ω is denoted by $\mathcal{SN}$.

The following characterization of strong measure zero is the starting point for our considerations.

Theorem 1.1 ([7]). *The following are equivalent*:

(1) $X \in \mathcal{SN}$,
(2) *for every set* $F \in \mathcal{M}$, $X + F \neq 2^\omega$.

This theorem indicates that the notion of strong measure zero should have its category analog. Indeed, we define after Prikry:

Definition 1.2. Suppose that $X \subseteq 2^\omega$. We say that X is strongly meager if for every $H \in \mathcal{N}$, $X + H \neq 2^\omega$. Let $\mathcal{SM}$ denote the collection of strongly meager sets.

Observe that if $z \notin X + F = \{x + f : x \in X, f \in F\}$ then $X \cap (F + z) = \emptyset$. In particular, a strong measure zero set can be covered by a translation of any dense G_δ set, and every strongly meager set can be covered by a translation of any measure one set.

If $X \subseteq 2^\omega$ is a group then the concepts of strong measure zero and strongly meager connect to the classical construction of a nonmeasurable set by Vitali (a selector of $\mathbb{R}/\mathbb{Q}$).

Theorem 1.2 (Reclaw). *Suppose that* $X \subseteq 2^\omega$ *is a dense subgroup of* $(2^\omega, +)$. *Then*

(1) $X \in \mathcal{SM}$ *if and only if every selector from* $2^\omega/X$ *is nonmeasurable.*
(2) $X \in \mathcal{SN}$ *if and only if every selector from* $2^\omega/X$ *does not have the Baire property.*

Proof. The proof below requires the group X to be infinite and the set $2^\omega/X$ to be infinite. A dense group will have these properties.

We will show only (1), the proof of (2) is analogous. Note that if X is a selector from $2^\omega/X$ and X is as above then X is nonmeasurable if and only if X does not have measure zero.

$\rightarrow$ Suppose that $X \in \mathcal{SM}$ and $H \in \mathcal{N}$. Let $x \notin X+H$. It follows that $[x]_X \cap H = \emptyset$, hence no selector is contained in H.

$\leftarrow$ Suppose that $X \notin \mathcal{SM}$ and let $H \in \mathcal{N}$ be such that $X + H = 2^\omega$. For each $x \in 2^\omega$, $[x]_X \cap H \neq \emptyset$. It follows that we can choose a selector contained in H. $\square$

Note that $X \notin \mathcal{SN}$ if there exists a meager set F such that the family $\{F + x : x \in X\}$ covers 2^ω. Instead of the assignment $x \rightsquigarrow F + x$ we can consider a more general mapping $x \rightsquigarrow (H)_x$, where $H \subseteq 2^\omega \times 2^\omega$ is a Borel set such that $(H)_x = \{y : \langle x, y\rangle \in H\} \in \mathcal{M}$ for all $x \in 2^\omega$.

Definition 1.3. $X \in \mathsf{COV}(\mathcal{M})$ if for every Borel set $H \subseteq 2^\omega \times 2^\omega$ such that $(H)_x \in \mathcal{M}$ for all $x \in 2^\omega$,

$$\bigcup_{x\in X} (H)_x \neq 2^\omega .$$

Similarly, $X \in \mathsf{COV}(\mathcal{N})$ if for every Borel set $H \subseteq 2^\omega \times 2^\omega$ such that $(H)_x \in \mathcal{N}$ for all $x \in 2^\omega$,

$$\bigcup_{x\in X} (H)_x \neq 2^\omega .$$

Note that

Lemma 1.1. $\mathsf{COV}(\mathcal{N}) \subseteq \mathcal{SM}$ *and* $\mathsf{COV}(\mathcal{M}) \subseteq \mathcal{SN}$.

Proof. Given $F \in \mathcal{M}$ let $H = \{(x, y) : y \in F + x\}$. It is clear that, $\bigcup_{x\in X}(H)_x = F + X$. $\square$

Families $\mathcal{SN}$ and $\mathcal{SM}$ as well as $\mathsf{COV}(\mathcal{M})$ and $\mathsf{COV}(\mathcal{N})$ are dual to each other and we are interested to what extent the properties of one family are shared by the dual one.

Below we present several results of that kind. The proofs of these results as well as quite a lot of additional material can be found in [3].

Definition 1.4. Let Borel Conjecture (BC) be the assertion that there are no uncountable strong measure zero sets, and Dual Borel Conjecture (DBC) be the assertion that there are no uncountable strongly meager sets.

Sierpinski showed that Borel Conjecture contradicts CH. His proof essentially yields the following:

Theorem 1.3. *Assume* **MA**. *Both* $\mathsf{COV}(\mathcal{M})$ *and* $\mathsf{COV}(\mathcal{N})$ *contain sets of size* $2^{\aleph_0}$. *In particular, both Borel Conjectures are false.*

There are many weaker assumptions than **MA** that contradict BC or DBC. Nevertheless we have the following:

Theorem 1.4 ([8]). *Borel Conjecture is consistent with* ZFC.

Theorem 1.5 ([5]). *Dual Borel Conjecture is consistent with* ZFC.

Definition 1.5. An uncountable set $X \subseteq 2^\omega$ is a Luzin set if $X \cap F$ is countable for $F \in \mathcal{M}$, and is a Sierpinski set if $X \cap G$ is countable for $G \in \mathcal{N}$.

Sierpinski showed that every Luzin set is in $\mathcal{SN}$. In addition we have the following:

Theorem 1.6 ([10]). *Every Luzin set is in* $\mathsf{COV}(\mathcal{M})$.

Theorem 1.7 ([9]). *Every Sierpinski set is in* $\mathsf{COV}(\mathcal{N})$ *(and so in* $\mathcal{SM}$*).*

Results presented above indicate that we have certain degree of symmetry between the notions of strongly meager and strong measure zero. The main objective of this paper is to show that as far as additive properties of both families are concerned it is not the case.

Sierpinski showed that $\mathcal{SN}$ is a σ-ideal. In fact, we have the following:

Theorem 1.8 ([5]). *Assume* **MA**. *Then the additivity of* $\mathcal{SN}$ *is* $2^{\aleph_0}$.

Similarly,

Theorem 1.9 ([2]).

(1) $\mathsf{COV}(\mathcal{M})$ *is a* σ*-ideal.*
(1) *Assume* **MA**. *Then the additivity of* $\mathsf{COV}(\mathcal{M})$ *is* $2^{\aleph_0}$.

Surprisingly the dual results are not true.

Theorem 1.10. *It is consistent that* $\mathsf{COV}(\mathcal{N})$ *is not a* σ*-ideal.*

Proof. It is an immediate consequence of the following theorem of Shelah:

Theorem 1.11 ([12]). *It is consistent that* $\mathsf{cov}(\mathcal{N}) = \aleph_\omega$.

Recall that

$$\mathsf{cov}(\mathcal{N}) = \min\left\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{N} \ \& \ \bigcup \mathcal{A} = 2^\omega\right\}.$$

Suppose that $\mathsf{cov}(\mathcal{N}) = \aleph_\omega$ and let a family $\mathcal{A} \subseteq \mathcal{N}$ witness that. Let $H \subseteq 2^\omega \times 2^\omega$ be an Borel set with null vertical sections and such that

$$\forall G \in \mathcal{N} \, \exists x \in 2^\omega \, G \subseteq (H)_x \, .$$

Such a set can be easily constructed from a universal set.

For each $G \in \mathcal{A}$ choose $x_G \in 2^\omega$ such that $G \subseteq (H)_{x_G}$. It follows that $X = \{x_G : G \in \mathcal{A}\} \notin \mathsf{COV}(\mathcal{N})$. On the other hand, every set of size $< \mathsf{cov}(\mathcal{N})$ belongs to $\mathsf{COV}(\mathcal{N})$ and X is a countable union of such sets. $\square$

The purpose of this paper is to show that

Theorem 1.12. *Assume* CH. *Then* $\mathcal{SM}$ *is not an ideal.*

2. Framework

The proof of Theorem 1.12 occupies the rest of the paper. The construction is motivated by the tools and methods developed in [11]. We should note here that by using the forcing notion defined in this paper we can also show that the statement "$\mathcal{SM}$ is not an ideal" is not equivalent to CH. However, since the main result is of interest outside of set theory, we present a version of the proof that does not contain any metamathematical references.

The structure of the proof is as follows:

- In Sec. 2, we show that in order to show that $\mathcal{SM}$ is not an ideal it suffices to find certain partial ordering $\mathcal{P}$ (Theorem 2.1).
- The definition of $\mathcal{P}$ involves construction of a measure zero set H with some special properties. All results needed to define H are proved in Sec. 3 and H together with other parameters is defined in Sec. 4.
- $\mathcal{P}$ is defined in Sec. 7. The proof that $\mathcal{P}$ has the required properties is a consequence of Theorem 5.3, which is the main result of Sec. 5, and Theorems 6.1 and 6.2, which are proved in Sec. 6.

We will show that in order to prove Theorem 1.12, it is enough to construct a partial ordering satisfying several general conditions. Here is the first of them.

Definition 2.1. Suppose that $(\mathcal{P}, \geq)$ is a partial ordering. We say that $\mathcal{P}$ has the fusion property if there exists a sequence of binary relations $\{\geq_n : n \in \omega\}$ (not necessarily transitive) such that

(1) if $p \geq_n q$ then $p \geq q$,
(2) if $p \geq_{n+1} q$ and $r \geq_{n+1} p$ then $r \geq_n q$,
(3) if $\{p_n : n \in \omega\}$ is a sequence such that $p_{n+1} \geq_{n+1} p_n$ for each n then there exists p_ω such that $p_\omega \geq_n p_n$ for each n.

From now on we will work in 2^ω with the set of rationals defined as

$$\mathbb{Q} = \{x \in 2^\omega : \forall^\infty n \; x(n) = 0\}.$$

Let Perf be the collection of perfect subsets of 2^ω. For $p, q \in \mathsf{Perf}$ let $p \geq q$ if $p \subseteq q$.

We will be interested in subsets of $\mathsf{Perf} \times \mathsf{Perf}$. Elements of $\mathsf{Perf} \times \mathsf{Perf}$ will be denoted by boldface letters and if $\mathbf{p} \in \mathsf{Perf} \times \mathsf{Perf}$ then $\mathbf{p} = (p_1, p_2)$. Moreover, for $\mathbf{p}, \mathbf{q} \in \mathsf{Perf} \times \mathsf{Perf}$, $\mathbf{p} \geq \mathbf{q}$ if $p_1 \subseteq q_1$ and $p_2 \subseteq q_2$.

Theorem 2.1. *Assume* CH, *fix a measure zero set* $H \subseteq 2^\omega$, *and suppose that there exists a family* $\mathcal{P} \subseteq \mathsf{Perf} \times \mathsf{Perf}$ *such that*:

(A0) *$\mathcal{P}$ has the fusion property.*
(A1) *For every $\mathbf{p} \in \mathcal{P}$, $n \in \omega$ and $z \in 2^\omega$ there exists $\mathbf{q} \geq_n \mathbf{p}$ such that $q_1 \subseteq H + z$ or $q_2 \subseteq H + z$.*

(A2) *For every* $\mathbf{p} \in \mathcal{P}$, $n \in \omega$, $X \in [2^\omega]^{\leq \aleph_0}$, $i = 1, 2$ *and* $\mathbf{t} \in \mathsf{Perf}$ *such that* $\mu(\mathbf{t}) > 0$,

$$\mu\big(\{z \in 2^\omega : \exists \mathbf{q} \geq_n \mathbf{p}\, X \cup (q_i + \mathbb{Q}) \subseteq \mathbf{t} + \mathbb{Q} + z\}\big) = 1\,.$$

Then $\mathcal{SM}$ *is not an ideal.*

Proof. We intend to build by induction sets X_1, $X_2 \in \mathcal{SM}$ in such a way that H witnesses that $X_1 \cup X_2$ is not strongly meager, that is, $(X_1 \cup X_2) + H = 2^\omega$. By induction we will define an ω_1-tree of members of $\mathcal{P}$ and then take the selector from the elements of this tree. This is a refinement of the method invented by Todorcevic (see [6]), who used an Aronszajn tree of perfect sets to construct a set of reals with some special properties. More examples can be found in [1]. □

For each $\alpha < \omega_1$, $\mathfrak{T}_\alpha$ will denote the αth level of an Aronszajn tree of elements of $\mathcal{P}$. More precisely, we will define $\mathsf{succ}(\mathbf{p}, \alpha) \subseteq \mathcal{P}$ — the collection of all successors of $\mathbf{p}$ on level α. We will require that:

(1) $\mathfrak{T}_0 = \{2^\omega \times 2^\omega\}$,
(2) $\mathsf{succ}(\mathbf{p}, \alpha)$ is countable (so levels of the tree are countable),
(3) if $\mathbf{q} \in \mathsf{succ}(\mathbf{p}, \alpha)$ then $\mathbf{q} \geq \mathbf{p}$,
(4) if $\mathsf{succ}(\mathbf{p}, \alpha)$ is defined then for each $n \in \omega$ there is $\mathbf{q} \in \mathsf{succ}(\mathbf{p}, \alpha)$ such that $\mathbf{q} \geq_n \mathbf{p}$.

Note that the tree constructed in this way will be an Aronszajn tree since an uncountable branch would produce an uncountable descending sequence of closed sets. For an arbitrary $\mathcal{P}$ with fusion property the conditions above will guarantee that we build an ω_1-tree with countable levels. This suffices for the constructions we are interested in.

Let $\mathfrak{T} = \bigcup_{\alpha<\omega_1} \mathfrak{T}_\alpha$ where $\mathfrak{T}_\alpha = \mathsf{succ}(2^\omega \times 2^\omega, \alpha)$. For each $\mathbf{p} \in \mathfrak{T}_\alpha$ choose $x^1_{\mathbf{p}} \in p_1$ and $x^2_{\mathbf{p}} \in p_2$. We will show that we can arrange this construction in such a way that $X_1 = \{x^1_{\mathbf{p}} : \mathbf{p} \in \mathfrak{T}\}$ and $X_2 = \{x^2_{\mathbf{p}} : \mathbf{p} \in \mathfrak{T}\}$ are the sets we are looking for.

Let $\{(\mathbf{t}_\alpha, i_\alpha) : \alpha < \omega_1\}$ be an enumeration of pairs $(\mathbf{t}, i) \in \mathsf{Perf} \times \{1, 2\}$ such that $\mu(\mathbf{t}) > 0$. Let $\{z_\alpha : \alpha < \omega_1\}$ be an enumeration of 2^ω.

Successor step

Suppose that $\mathfrak{T}_\alpha$ is already constructed. Denote $X^\alpha = \left\{x^1_{\mathbf{p}}, x^2_{\mathbf{p}} : \mathbf{p} \in \bigcup_{\beta\leq\alpha} \mathfrak{T}_\beta\right\}$.

For each $\mathbf{p} \in \mathfrak{T}_\alpha$ and $n \in \omega$, let

$$Z^n_{\mathbf{p}} = \{z \in 2^\omega : \exists \mathbf{q} \geq_n \mathbf{p}\; X^\alpha \cup (q_{i_\alpha} + \mathbb{Q}) \subseteq \mathbf{t}_\alpha + \mathbb{Q} + z\}\,.$$

Note that by (A2), each set $Z^n_{\mathbf{p}}$ has measure one. Fix

$$y_\alpha \in \bigcap_{\mathbf{p}\in\mathfrak{T}_\alpha} \bigcap_{n\in\omega} Z^n_{\mathbf{p}}\,.$$

For each $\mathbf{p} \in \mathfrak{T}_\alpha$ choose $\{\mathbf{p}^n : n \in \omega\}$ such that

(1) $\mathbf{p}^n \geq_{n+1} \mathbf{p}$ for each n,
(2) $X^\alpha \cup (p^n_{i_\alpha} + \mathbb{Q}) \subseteq \mathbf{t}_\alpha + \mathbb{Q} + y_\alpha$.

Next apply (A1) to get sets $\{\mathbf{q}^n : n \in \omega\}$ such that for all n,

(1) $\mathbf{q}^n \geq_{n+1} \mathbf{p}^n$,
(2) $q^n_1 \subseteq H + z_\alpha$ or $q^n_2 \subseteq H + z_\alpha$.

Define $\mathsf{succ}(\mathbf{p}, \alpha{+}1) = \{\mathbf{q}^n : n \in \omega\}$. Note that for each $n \in \omega$ there is $\mathbf{q} \in \mathsf{succ}(\mathbf{p}, \alpha)$ such that $\mathbf{q} \geq_n \mathbf{p}$. For completeness, if $\mathbf{p} \in \bigcup_{\beta<\alpha} \mathfrak{T}_\beta$ then put

$$\mathsf{succ}(\mathbf{p}, \alpha+1) = \bigcup \{\mathsf{succ}(\mathbf{q}, \alpha+1) : \mathbf{q} \in \mathsf{succ}(\mathbf{p}, \alpha)\}\,.$$

Limit step

Suppose that α is a limit ordinal and $\mathfrak{T}_\beta$ are already constructed for $\beta < \alpha$. Suppose that $\mathbf{p}_0 \in \mathfrak{T}_{\alpha_0}$, $\alpha_0 < \alpha$. Find an increasing sequence $\{\alpha_n : n \in \omega\}$ with $\sup_n \alpha_n = \alpha$, and for $k \in \omega$, let $\{\mathbf{p}^k_n : n \in \omega\}$ be such that

(1) $\mathbf{p}^k_n \in \mathfrak{T}_{\alpha_n}$,
(2) $\mathbf{p}^k_{n+1} \geq_{n+k+1} \mathbf{p}^k_n$ for each $k,\, n \in \omega$.

Let $\mathbf{p}^k_\omega$ be such that $\mathbf{p}^k_\omega \geq_{n+k} \mathbf{p}^k_n$. Define $\mathsf{succ}(\mathbf{p}_0, \alpha) = \{\mathbf{p}^k_\omega : k \in \omega\}$. This concludes the construction of $\mathfrak{T}$ and X_1, X_2.

Lemma 2.1. *$X_1,\, X_2 \in \mathcal{SM}$.*

Proof. We will show that $X_1 \in \mathcal{SM}$. The proof that $X_2 \in \mathcal{SM}$ is the same.

Let $G \subseteq 2^\omega$ be a measure zero set. Find $\alpha < \omega_1$ such that $G \cap (\mathbf{t}_\alpha + \mathbb{Q}) = \emptyset$ and $i_\alpha = 1$. It follows that,

$$X_1 \subseteq X^\alpha \cup \bigcup_{\mathbf{p} \in \mathfrak{T}_{\alpha+1}} p_1 \subseteq \mathbf{t}_\alpha + \mathbb{Q} + y_\alpha \subseteq (2^\omega \setminus G) + y_\alpha\,.$$

Thus $X_1 + y_\alpha \subseteq 2^\omega \setminus G$ and therefore $y_\alpha \notin X_1 + G$, which finishes the proof. □

Lemma 2.2. *$X_1 \cup X_2 \notin \mathcal{SM}$.*

Proof. Let H be the set used in (A1). We will show that $(X_1 \cup X_2) + H = 2^\omega$. Suppose that $z \in 2^\omega$ and let $\alpha < \omega_1$ be such that $z = z_\alpha$. By our construction, for any $\mathbf{p} \in \mathfrak{T}_{\alpha+1}$, $x^1_{\mathbf{p}} \in z + H$ or $x^2_{\mathbf{p}} \in z + H$. Thus $z \in (X_1 \cup X_2) + H$, which ends the proof.

This shows that the sets X_1, X_2 and H have the required properties. The proof of Theorem 2.1 is finished. □

Therefore the problem of showing that $\mathcal{SM}$ is not an ideal reduces to the construction of an appropriate set $\mathcal{P}$. We will do that in the following sections.

3. Measure Zero Set

In this section, we will develop tools to define a measure zero set H that will be used in the construction of $\mathcal{P}$ and will witness that the union of two strongly meager sets X_1, X_2 defined in the proof of Theorem 2.1 is not strongly meager. The set H will be defined at the end of the next section.

We will need several definitions.

Definition 3.1. Suppose that $I \subseteq \omega$ is a finite set. Let F^I be the collection of all functions $f : \mathsf{dom}(f) \longrightarrow 2$, with $\mathsf{dom}(f) \subseteq 2^I$. For $f \in \mathsf{F}^I$, let $m^0_f = |\{s : f(s) = 0\}|$ and $m^1_f = |\{s : f(s) = 1\}|$.

For a set $B \subseteq 2^I$ let $(B)^1 = 2^I \setminus B$ and $(B)^0 = B$.

We will work in the space $(2^I, +)$ with addition mod 2. For a function $f \in \mathsf{F}^I$ let

$$(B)^f = \bigcap_{s \in \mathsf{dom}(f)} (B + s)^{f(s)} .$$

In addition let $(B)^\emptyset = 2^I$.

For $f \in \mathsf{F}^I$ and $k \in \omega$, let

$$\mathsf{F}^I_{f,k} = \left\{ g \in \mathsf{F}^I : f \subseteq g \ \& \ |\mathsf{dom}(g) \setminus \mathsf{dom}(f)| \leq k \right\} .$$

The set H will be defined using an infinite sequence of finite sets. The following theorem describes how to construct one term of this sequence.

Theorem 3.1. *Suppose that $m \in \omega$ and $0 < \delta < \varepsilon < 1$ are given. There exists $n \in \omega$ such that for every finite set $I \in [\omega]^{>n}$ there exists a set $C \subseteq 2^I$ such that $1 - \varepsilon + \delta \geq |C| \cdot 2^{-|I|} \geq 1 - \varepsilon - \delta$ and for every $f \in \mathsf{F}^I_{\emptyset,m}$,*

$$\left| \frac{|(C)^f|}{|(C)^\emptyset|} - (1-\varepsilon)^{m^0_f} \varepsilon^{m^1_f} \right| < \delta .$$

Note that the theorem says that we can choose C is such a way that for any sequences $s_1, \ldots, s_m \in 2^I$ the sets $s_1 + C, \ldots, s_m + C$ are probabilistically independent with error δ. Thus, we want δ to be much smaller than ε^m. In order to prove this theorem it is enough to verify the following:

Theorem 3.2. *Suppose that $m \in \omega$ and $0 < \delta < \varepsilon < 1$ are given. There exists $n \in \omega$ such that for every finite set $I \in [\omega]^{>n}$ there exists a set $C \subseteq 2^I$ such that $1 - \varepsilon + \delta \geq |C| \cdot 2^{-|I|} \geq 1 - \varepsilon - \delta$ and for every set $X \subseteq 2^I$, $|X| \leq m$*

$$\left| \frac{|\bigcap_{s \in X} (C + s)|}{2^{|I|}} - (1-\varepsilon)^{|X|} \right| < \delta .$$

Proof. Note first that Theorem 3.2 suffices to prove Theorem 3.1. Indeed, if for every $X \in [2^I]^{\leq m}$,

$$\left| \frac{|\bigcap_{s \in X} (C + s)|}{2^{|I|}} - (1-\varepsilon)^{|X|} \right| < \delta$$

then we show by induction on m^1_f that for every $f \in \mathsf{F}^I_{\emptyset,m}$,

$$\left| \frac{|(C)^f|}{|(C)^\emptyset|} - (1-\varepsilon)^{m^0_f} \varepsilon^{m^1_f} \right| < 2^m \delta\,.$$

Fix m, δ and ε, and choose the set $C \subseteq 2^I$ randomly (for the moment I is arbitrary). For each $s \in 2^I$ decisions whether $s \in C$ are made independently with the probability of $s \in C$ equal to $1-\varepsilon$. Thus the set C is a result of a sequence of Bernoulli trials. Note that by the Chebyshev's inequality, the probability that $1-\varepsilon+\delta \geq |C| \cdot 2^{-|I|} \geq 1-\varepsilon-\delta$ approaches 1 as $|I|$ goes to infinity.

Let S_n be the number of successes in n independent Bernoulli trials with probability of success p. We will need the following well–known fact that we will prove here for completeness. □

Theorem 3.3. *For every* $\delta > 0$,

$$P\left(\left|\frac{S_n}{n} - p\right| \geq \delta\right) \leq 2e^{-n\delta^2/4}\,.$$

Proof. We will show that

$$P\left(\frac{S_n}{n} \geq p + \delta\right) \leq e^{-n\delta^2/4}\,.$$

The proof that

$$P\left(\frac{S_n}{n} \leq p - \delta\right) \leq e^{-n\delta^2/4}$$

is the same. Let $q = 1-p$. Then for each $x \geq 0$, we have

$$\begin{aligned}
P\left(\frac{S_n}{n} \geq p + \delta\right) &\leq \sum_{k \geq n(p+\delta)}^{n} \binom{n}{k} p^k q^{n-k} \\
&\leq \sum_{k \geq n(p+\delta)}^{n} e^{-x(n(p+\delta)-k)} \cdot \binom{n}{k} p^k q^{n-k} \\
&\leq e^{-xn\delta} \cdot \sum_{k \geq n(p+\delta)} \binom{n}{k} (pe^{xq})^k (qe^{-xp})^{n-k} \\
&\leq e^{-xn\delta} \cdot \sum_{k=0}^{n} \binom{n}{k} (pe^{xq})^k (qe^{-xp})^{n-k} \\
&= e^{-xn\delta} \left(pe^{xq} + qe^{-xp}\right)^n \\
&\leq e^{-xn\delta} \left(pe^{x^2q^2} + qe^{x^2p^2}\right)^n \\
&\leq e^{-xn\delta} \left(pe^{x^2} + qe^{x^2}\right)^n \\
&= e^{-xn\delta} e^{nx^2} = e^{n(x^2 - \delta x)}\,.
\end{aligned}$$

The inequality $pe^{xq}+qe^{-xp} \leq pe^{x^2q^2}+qe^{x^2p^2}$ follows from the fact that $e^x \leq e^{x^2}+x$, for every x. The expression $e^{n(x^2-\delta x)}$ attains its minimal value at $x=\delta/2$, which yields the desired inequality. □

Consider an arbitrary set $X \subseteq 2^I$. To simplify the notation denote $V = 2^I \setminus C$ and note that $\bigcap_{s\in X}(C+s) = 2^I \setminus (V+X)$. For a point $t \in 2^I$, $t \notin X+V$ is equivalent to $(t+X)\cap V=\emptyset$. Thus the probability that $t \notin X+V$ is equal to $(1-\varepsilon)^{|X|}$.

Let $G(X)$ be a subgroup of $(2^I,+)$ generated by X. Since every element of 2^I has order 2, it follows that $|G(X)| \leq 2^{|X|}$.

Lemma 3.1. *There are sets* $\{U_j : j \leq |G(X)|\}$ *such that*:

(1) $\forall j\ \forall s,t \in U_j\ \big(s \neq t \to s+t \notin G(X)\big)$,
(2) $\forall j \leq |G(X)|\ |U_j| = 2^{|I|}/|G(X)|$,
(3) $\forall i \neq j\ U_i \cap U_j = \emptyset$,
(4) $\bigcup_{j\leq|G(X)|} U_j = 2^I$.

Proof. Choose U_j's to be disjoint selectors from the cosets $2^I/G(X)$. □

Note that if $t_1, t_2 \in U_j$ then the events $t_1 \in X+V$ and $t_2 \in X+V$ are independent since sets t_1+X and t_2+X are disjoint. Consider the sets $X_j = U_j \cap \bigcap_{s\in X}(C+s)$ for $j \leq |G(X)|$. The expected value of the size of this set is $(1-\varepsilon)^{|X|}\cdot 2^{|I|}/|G(X)|$. By Theorem 3.3, for each $j \leq |G(X)|$,

$$P\left(\left|\frac{|X_j|}{2^{|I|}/|G(X)|}-(1-\varepsilon)^{|X|}\right| \geq \delta\right) \leq 2e^{-2^{|I|-2}\delta^2/|G(X)|}.$$

It follows that for every $X \subseteq 2^I$ the probability that

$$(1-\varepsilon)^{|X|}-\delta \leq \frac{\left|\bigcap_{s\in X}(C+s)\right|}{2^{|I|}} \leq (1-\varepsilon)^{|X|}+\delta$$

is at least

$$1-2|G(X)|e^{-2^{|I|-2}\delta^2/|G(X)|} \geq 1-2^{|X|+1}e^{-2^{|I|-|X|-2}\delta^2}.$$

The probability that it happens for every X of size $\leq m$ is at least

$$1-2^{|I|\cdot(m+1)^2}\cdot e^{-2^{|I|-m-2}\delta^2}.$$

If m and δ are fixed then this expression approaches 1 as $|I|$ goes to infinity, since $\lim_{x\to\infty} P(x)e^{-x}=0$ for any polynomial $P(x)$. It follows that for sufficiently large $|I|$ the probability that the "random" set C has the required properties is >0. Thus there exists an actual C with these properties as well.

4. Parameters of the Construction

We will define now all the parameters of the construction. The actual relations [(P1)–(P7) below] between these parameters make sense only in the context of the computations in which they are used, and are tailored to simplify the calculations in the following sections. The reason why we collected these definitions here is that there are many of them and the order in which they are defined is quite important. Nevertheless this section serves only as a reference.

The following notation will be used in the sequel.

Definition 4.1. Suppose that $s : \omega \times \omega \longrightarrow \omega$.

Let $s^{(0)}(i,j) = i$ and $s^{(n+1)}(i,j) = s(s^{(n)}(i,j), j)$. Given $N \in \omega + 1$, $n \in \omega$ and $f \in \omega^N$ let

$$s^{(n)}(f) = \left\{ \left(i, s^{(n)}\left(f(i), i\right) \right) : i < N \right\} .$$

We will write $s(f)$ instead of $s^{(1)}(f)$.

We define real sequences $\{\varepsilon_i, \delta_i, \epsilon_i : i \in \omega\}$, intervals $\{I_i : i \in \omega\}$, sets $\{C_i : i \in \omega\}$ and integers $\{m_i : i \in \omega\}$. In addition we will define functions $\bar{\mathbf{s}}$, $\tilde{\mathbf{s}}$, $\mathbf{s} : \omega \times \omega \longrightarrow \omega$. The sequence $\{\varepsilon_i : i \in \omega\}$ is defined first. We require that

(P1) $0 < \varepsilon_{i+1} < \varepsilon_i$ for $i \in \omega$,
(P2) $\sum_{i \in \omega} \varepsilon_i < 1/2$.

Set $\epsilon_0 = \delta_0 = 1$, $I_0 = C_0 = \emptyset, m_0 = 0$ and $\bar{\mathbf{s}}(n,0) = \tilde{\mathbf{s}}(n,0) = \mathbf{s}(n,0) = 0$ for all $n \in \omega$. Suppose that $\{\delta_i, \epsilon_i, I_i, C_i, m_i : i < N\}$ are defined. Also assume that $\bar{\mathbf{s}}(n,i)$, $\tilde{\mathbf{s}}(n,i)$ and $\mathbf{s}(n,i)$ are defined for $i < N$ and $n \in \omega$.

Put $v_N = \left|\prod_{k<N} 2^{I_k}\right|$, $l_N = \prod_{k<N} v_k$ and define ϵ_N such that

(P3) $0 < v_N \cdot \epsilon_N \leq \varepsilon_N$,
(P4) $2^{l_N+N+2} \cdot \epsilon_N < \epsilon_{N-1}$.

Given ε_N and ϵ_N we will define for $k \in \omega$

$$\bar{\mathbf{s}}(k,N) = \begin{cases} \max\left\{ l : \dfrac{k}{l+1}\epsilon_N^2 \varepsilon_N^l > 4 \right\} & \text{if } k\epsilon_N^2 > 4 \\ 0 & \text{otherwise} . \end{cases}$$

Next let $\tilde{\mathbf{s}}(k,N) = \bar{\mathbf{s}}^{(2u_N)}(k,N)$, where u_N is the smallest integer $\geq \log_2(8/\epsilon_N^2)$. Finally define

$$\mathbf{s}(k,N) = \tilde{\mathbf{s}}^{(2v_N+1)}(k,N) .$$

Note that the functions $\bar{\mathbf{s}}(\cdot,N)$, $\tilde{\mathbf{s}}(\cdot,N)$, and $\mathbf{s}(\cdot,N)$ are nondecreasing and unbounded.

Define

(P5) $m_N = \min\{m : \mathbf{s}^{(N \cdot l_N)}(m, N) > 0\}$,
(P6) $\delta_N = 2^{-N-2} \cdot \varepsilon_N^{m_N}$.

Finally use Theorem 3.1 to define I_N and $C_N \subseteq 2^{I_N}$ for $\delta = \delta_N$, $\varepsilon = \varepsilon_N$ and $m = m_N$.

In addition we require that

(P7) I_i are pairwise disjoint.

The set H that will witness that $\mathcal{SM}$ is not an ideal is defined as

$$H = \{x \in 2^\omega : \exists^\infty k \; x{\restriction}I_k \notin C_k\}.$$

Note that

$$\mu(H) \leq \mu\left(\bigcap_n \bigcup_{k>n} \{x \in 2^\omega : x{\restriction}I_k \notin C_k\}\right) \leq \sum_{k>n} \varepsilon_k + \delta_k \stackrel{n\to\infty}{\longrightarrow} 0\,.$$

5. More Combinatorics

This section contains the core of the proof of Theorem 2.1. This is Theorem 5.1 which is in the realm of finite combinatorics and concerns properties of the counting measure on finite product spaces. We will use the following notation:

Definition 5.1. Suppose that $N_0 < N \leq \omega$. Define F^N to be the collection of all sequences $\mathbf{F} = \langle f_i : i < N\rangle$ such that $f_i \in \mathsf{F}^{I_i}$ for $i < N$. For $\mathbf{F} \in \mathsf{F}^N$ and $h \in \omega^N$, let

$$\mathsf{F}^N_{\mathbf{F},h} = \left\{\mathbf{G} \in \mathsf{F}^N : \forall i < N \; \mathbf{G}(i) \in \mathsf{F}^{I_i}_{\mathbf{F}(i),h(i)}\right\}.$$

Similarly,

$$\mathsf{F}^{N_0,N}_{\mathbf{F},h} = \left\{\mathbf{G} \in \mathsf{F}^N_{\mathbf{F},h} : \mathbf{G}{\restriction}N_0 = \mathbf{F}{\restriction}N_0\right\}.$$

We always require that for all $i < N$,

$$\left|\mathsf{dom}\big(\mathbf{F}(i)\big)\right| + h(i) \leq m_i\,.$$

Let $\mathbf{C} = \langle C_i : i < \omega\rangle$ be the sequence of sets defined earlier. For $N_0 < N$ and $\mathbf{F} \in \mathsf{F}^N$ let

$$(\mathbf{C})^{\mathbf{F}}_{N_0} = \prod_{N_0 \leq i < N} (C_i)^{\mathbf{F}(i)} = \left\{s \in 2^{I_{N_0} \cup \ldots \cup I_{N-1}} : \forall i \in [N_0, N) \; s{\restriction}I_i \in (C_i)^{\mathbf{F}(i)}\right\}.$$

We will write $(\mathbf{C})^{\mathbf{F}}$ instead of $(\mathbf{C})^{\mathbf{F}}_0$ and $(C_{N-1})^{\mathbf{F}(N-1)}$ instead of $(\mathbf{C})^{\mathbf{F}}_{N-1}$.

Definition 5.2. Suppose that X is a finite set. A distribution is a function $m : X \longrightarrow \mathbb{R}$ such that

$$0 \leq m(x) \leq \frac{1}{|X|}.$$

Define α_m to be the largest number α such that $m' = \alpha \cdot m$ is a distribution, and put $\overline{m} = \sum_{x \in X} m(x)$ and $\overline{\overline{m}} = \alpha_m \cdot \overline{m}$.

Suppose that a distribution m on X is given and $Y \subseteq X$. Define $m_Y : Y \longrightarrow \mathbb{R}^+$ as

$$m_Y(x) = \frac{|X|}{|Y|} \cdot m(x).$$

Note that

$$\alpha_m = \frac{1}{|X| \cdot \max\{m(x) : x \in X\}}.$$

Observe also that $(m_Y)_Z = m_Z$ if $Z \subseteq Y \subseteq X$.

A prototypical example of a distribution is defined as follows. Suppose that $p \subseteq 2^\omega$ is a closed (or just measurable) set and $n \in \omega$. Let m be defined on 2^n as

$$m(s) = \mu(p \cap [s]) \text{ for } s \in 2^n.$$

Note that $\overline{m} = \mu(p)$.

The following lemmas list some easy observations concerning these notions.

Lemma 5.1. *Suppose that $N \in \omega$, $k^0 + k_0 \leq m_N$, $f \in \mathsf{F}^{I_N}_{\emptyset, k^0}$ and m is a distribution on 2^{I_N}. There exist $f_0, f_1 \in \mathsf{F}^{I_N}_{f, k_0}$ such that $|f_0 \setminus f| = |f_1 \setminus f| = k_0$ and*

$$\overline{m_{(C_N)^{f_0}}} \leq \overline{m_{(C_N)^f}} \leq \overline{m_{(C_N)^{f_1}}}.$$

Proof. For each $x \in 2^{I_N}$ and $h \in \mathsf{F}^{I_N}_{\emptyset, k}$, let $h^0_x = h \cup \{(x, 0)\}$ and $h^1_x = h \cup \{(x, 1)\}$. Note that there is $i \in \{0, 1\}$ such that

$$\overline{m_{(C_N)^{h^i_x}}} \leq \overline{m_{(C_N)^h}} \leq \overline{m_{(C_N)^{h^{1-i}_x}}}.$$

Iteration of this procedure k_0 times will produce the required examples. $\square$

Lemma 5.2. *Suppose that $N_0 \leq N$ are natural numbers, $h^0, h_0 \in \prod_{i<N} m_i$ satisfy $h_0(i) + h^0(i) \leq m_i$ for $i < N$, $\mathbf{F} \in \mathsf{F}^N_{\emptyset, h^0}$ and m is a distribution on $2^{I_0 \cup \ldots \cup I_{N-1}}$. Suppose that for every $\mathbf{G} \in \mathsf{F}^{N_0, N}_{\mathbf{F}, h_0}$, $a \leq \overline{m_{(\mathbf{C})^{\mathbf{G}}}} \leq b$. Let $\mathbf{G}^\star \in \mathsf{F}^{N_0, N}_{\mathbf{F}, h_0}$ be such that $|\mathsf{dom}(\mathbf{G}^\star(i)) \setminus \mathsf{dom}(\mathbf{F}(i))| = h_1(i) < h_0(i)$ for $N_0 \leq i < N$. Then*

$$\forall \mathbf{G} \in \mathsf{F}^{N_0, N}_{\mathbf{G}^\star, h_0 - h_1} \; a \leq \overline{m_{(\mathbf{C})^{\mathbf{G}}}} \leq b.$$

Proof. Since $\mathsf{F}^{N_0, N}_{\mathbf{G}^\star, h_0 - h_1} \subseteq \mathsf{F}^{N_0, N}_{\mathbf{F}, h_0}$, the lemma is obvious. $\square$

The following theorem is a good approximation of the combinatorial result that we require for the proof of Theorem 2.1. The proof of it will give us a slightly stronger but more technical result, Theorem 5.3, which is precisely what we need.

Theorem 5.1. *Suppose that $N_0 < N$ are natural numbers, h^0, $h_0 \in \prod_{i<N} m_i$ satisfy $h_0(i) + h^0(i) \le m_i$ for $i < N$, $\mathbf{F} \in \mathsf{F}^N_{\emptyset,h^0}$ and m is a distribution on $2^{I_0\cup\ldots\cup I_{N-1}}$ such that*

$$\overline{m_{(\mathbf{C})^{\mathbf{F}}}} \ge \frac{2\cdot\sum_{i=N_0}^N \epsilon_i}{\prod_{i=N_0}^N (1-8\epsilon_i)}\,.$$

There exists $\mathbf{F}^\star \in \mathsf{F}^{N_0,N}_{\mathbf{F},h_0-\mathbf{s}(h_0)}$ such that

$$\forall \mathbf{G} \in \mathsf{F}^{N_0,N}_{\mathbf{F}^\star,\mathbf{s}(h_0)}\ \overline{m_{(\mathbf{C})^{\mathbf{G}}}} \ge \overline{m_{(\mathbf{C})^{\mathbf{F}}}} \cdot \prod_{i=N_0}^{N-1} (1-8\epsilon_i)^2 - \sum_{i=N_0}^{N-2} \epsilon_i\,.$$

Remark. It is worth noticing that the complicated formulas appearing in the statement of this theorem are chosen to simplify the inductive proof. Putting them aside, the theorem can be formulated as follows: if $\overline{m_{(\mathbf{C})^{\mathbf{F}}}}$ is sufficiently big (where big means only slightly larger than zero), then there exists $\mathbf{F}^\star \in \mathsf{F}^{N_0,N}_{\mathbf{F},h_0-\mathbf{s}(h_0)}$ such that for all $\mathbf{G} \in \mathsf{F}^{N_0,N}_{\mathbf{F}^\star,\mathbf{s}(h_0)}$ the value of $\frac{\overline{m_{(\mathbf{C})^{\mathbf{G}}}}}{\overline{m_{(\mathbf{C})^{\mathbf{F}}}}}$ cannot be significantly smaller than 1.

The proof of Theorem 5.1 will proceed by induction on $N \ge N_0$, and the following theorem corresponds to the single induction step.

Suppose that $N \in \omega$ is fixed.

Theorem 5.2. *If $k^0 + k_0 \le m_N$, m is a distribution on 2^{I_N} and $f \in \mathsf{F}^{I_N}_{\emptyset,k^0}$ is such that $\overline{\overline{m_{(C_N)^f}}} \ge 2\epsilon_N$ then there exists $f^\star \in \mathsf{F}^{I_N}_{f,k_0-\tilde{\mathbf{s}}(k_0,N)}$ such that*

$$\forall g \in \mathsf{F}^{I_N}_{f^\star,\tilde{\mathbf{s}}(k_0,N)}\ \overline{m_{(C_N)^{f^\star}}}\cdot(1+2\epsilon_N) \ge \overline{m_{(C_N)^g}} \ge \overline{m_{(C_N)^{f^\star}}}\cdot(1-2\epsilon_N)\,,$$

and

$$\forall g \in \mathsf{F}^{I_N}_{f^\star,\tilde{\mathbf{s}}(k_0,N)}\ \overline{m_{(C_N)^g}} \ge \overline{m_{(C_N)^f}}\cdot(1-2\epsilon_N)\,.$$

Proof. We start with the following observation:

Lemma 5.3. *Suppose that $\overline{m_{(C_N)^f}} \ge \epsilon_N$. There exists $\tilde{f} \in \mathsf{F}^{I_N}_{f,k_0-\bar{\mathbf{s}}(k_0,N)}$ such that*

$$\forall g \in \mathsf{F}^{I_N}_{\tilde{f},\bar{\mathbf{s}}(k_0,N)}\ \overline{m_{(C_N)^g}} \ge \overline{m_{(C_N)^f}}\cdot(1-\epsilon_N)\,.$$

Similarly, there exists $\tilde{f} \in \mathsf{F}^{I_N}_{f,k_0-\bar{\mathbf{s}}(k_0,N)}$ such that

$$\forall g \in \mathsf{F}^{I_N}_{\tilde{f},\bar{\mathbf{s}}(k_0,N)}\ \overline{m_{(C_N)^g}} \le \overline{m_{(C_N)^f}}\cdot(1+\epsilon_N)\,.$$

Proof. We will show only the first part, the second part is proved in the same way. If $\bar{\mathbf{s}}(k_0,N) = 0$, then the lemma follows readily from Theorem 5.1. Thus, suppose that $\bar{\mathbf{s}}(k_0,N) > 0$ and let $m_{(C_N)^f}$ be a distribution satisfying the requirements of the lemma.

Construct, by induction, a sequence $\{f_n : n < n^\star\}$ such that

(1) $f_0 = f$,
(2) $f_{n+1} \in \mathsf{F}^{I_N}_{f_n,\bar{\mathbf{s}}(k_0,N)}$,
(3) $\overline{m_{(C_N)^{f_n}}} \geq \overline{m_{(C_N)^f}} \cdot \left(1 + \frac{n}{2}\epsilon_N \varepsilon_N^{\bar{\mathbf{s}}(k_0,N)}\right)$.

First notice that $\bar{\mathbf{s}}(k_0, N)$ was defined in such a way that

$$\begin{aligned}
&\overline{m_{(C_N)^f}} \cdot \left(1 + \frac{1}{2}\left(\frac{k_0}{\bar{\mathbf{s}}(k_0,N)} - 2\right)\epsilon_N \varepsilon_N^{\bar{\mathbf{s}}(k_0,N)}\right) \\
&\quad \geq \epsilon_N \left(1 + \frac{1}{2}\left(\frac{k_0}{\bar{\mathbf{s}}(k_0,N)} - 2\right)\epsilon_N \varepsilon_N^{\bar{\mathbf{s}}(k_0,N)}\right) \\
&\quad \geq \frac{1}{2}\frac{k_0}{\bar{\mathbf{s}}(k_0,N)}\epsilon_N^2 \varepsilon_N^{\bar{\mathbf{s}}(k_0,N)} > 1\,.
\end{aligned}$$

Therefore, after fewer than $\frac{k_0}{\bar{\mathbf{s}}(k_0,N)} - 2$ steps the construction has to terminate (otherwise $\overline{m_{(C_N)^g}} > 1$ for some g, which is impossible).

Suppose that f_n has been constructed.

Case 1. $\forall h \in \mathsf{F}^{I_N}_{f_n,\bar{\mathbf{s}}(k_0,N)}$ $\overline{m_{(C_N)^h}} \geq \overline{m_{(C_N)^f}} \cdot (1 - \epsilon_N)$. In this case put $\tilde{f} = f_n$ and finish the construction. Observe that

$$\begin{aligned}
|\tilde{f}| + \bar{\mathbf{s}}(k_0,N) &\leq k^0 + n^\star \cdot \bar{\mathbf{s}}(k_0,N) + \bar{\mathbf{s}}(k_0,N) \\
&\leq k^0 + \left(\frac{k_0}{\bar{\mathbf{s}}(k_0,N)} - 2\right) \cdot \bar{\mathbf{s}}(k_0,N) + \bar{\mathbf{s}}(k_0,N) \\
&\leq k^0 + k_0 - \bar{\mathbf{s}}(k_0,N) < m_N\,.
\end{aligned}$$

Case 2. $\exists h \in \mathsf{F}^{I_N}_{f_n,\bar{\mathbf{s}}(k_0,N)}$ $\overline{m_{(C_N)^h}} < \overline{m_{(C_N)^f}} \cdot (1 - \epsilon_N)$. Using Lemma 5.1, we can assume that $|h| = |f_n| + \bar{\mathbf{s}}(k_0,N)$.

Consider the partition of $(C_N)^{f_n}$ given by h, i.e.

$$(C_N)^{f_n} = (C_N)^h \cup \left((C_N)^{f_n} \setminus (C_N)^h\right).$$

Note that by considering the worst case we get

$$\begin{aligned}
\frac{|(C_N)^h|}{|(C_N)^{f_n} \setminus (C_N)^h|} &\geq \frac{(1-\varepsilon_N)^{m^0_{f_n}} \varepsilon_N^{m^1_{f_n}} \cdot \varepsilon_N^{\bar{\mathbf{s}}(k_0,N)} - \delta_N}{(1-\varepsilon_N)^{m^0_{f_n}} \varepsilon_N^{m^1_{f_n}} + \delta_N - \left((1-\varepsilon_N)^{m^0_{f_n}} \varepsilon_N^{m^1_{f_n}} \cdot \varepsilon_N^{\bar{\mathbf{s}}(k_0,N)} - \delta_N\right)} \\
&\geq \frac{\varepsilon_N^{\bar{\mathbf{s}}(k_0,N)} - \frac{\delta_N}{(1-\varepsilon_N)^{m^0_{f_n}} \varepsilon_N^{m^1_{f_n}}}}{1 - \varepsilon_N^{\bar{\mathbf{s}}(k_0,N)} + \frac{2\delta_N}{(1-\varepsilon_N)^{m^0_{f_n}} \varepsilon_N^{m^1_{f_n}}}}\,.
\end{aligned}$$

Moreover, since $\delta_N \leq \frac{1}{2}\varepsilon_N^{m_N}$, we have

$$\varepsilon_N^{\bar{\mathbf{s}}(k_0,N)} \geq 2 \cdot \frac{\delta_N}{(1-\varepsilon_N)^{m^0_{f_n}} \cdot \varepsilon_N^{m^1_{f_n}}},$$

and thus

$$\frac{\varepsilon_N^{\bar{\mathbf{s}}(k_0,N)} - \frac{\delta_N}{(1-\varepsilon_N)^{m^0_{f_n}} \varepsilon_N^{m^1_{f_n}}}}{1-\varepsilon_N^{\bar{\mathbf{s}}(k_0,N)} + \frac{2\delta_N}{(1-\varepsilon_N)^{m^0_{f_n}} \varepsilon_N^{m^1_{f_n}}}} \geq \frac{1}{2}\varepsilon_N^{\bar{\mathbf{s}}(k_0,N)}.$$

It follows that

$$\begin{aligned}
&\frac{1}{\overline{m_{(C_N)^f}}} \cdot \overline{m_{(C_N)^{f_n} \setminus (C_N)^h}} \\
&\quad\geq \left(1 + \frac{n}{2}\epsilon_N \varepsilon_N^{\bar{\mathbf{s}}(k_0,N)}\right) \cdot \frac{|(C_N)^{f_n}|}{|(C_N)^{f_n} \setminus (C_N)^h|} - (1-\epsilon_N) \cdot \frac{|(C_N)^h|}{|(C_N)^{f_n} \setminus (C_N)^h|} \\
&= \frac{|(C_N)^{f_n}|}{|(C_N)^{f_n} \setminus (C_N)^h|} + \frac{n}{2}\epsilon_N \varepsilon_N^{\bar{\mathbf{s}}(k_0,N)} \cdot \frac{|(C_N)^{f_n}|}{|(C_N)^{f_n} \setminus (C_N)^h|} \\
&\quad - \frac{|(C_N)^h|}{|(C_N)^{f_n} \setminus (C_N)^h|} + \epsilon_N \frac{|(C_N)^h|}{|(C_N)^{f_n} \setminus (C_N)^h|} \\
&= 1 + \frac{n}{2}\epsilon_N \varepsilon_N^{\bar{\mathbf{s}}(k_0,N)} \cdot \frac{|(C_N)^{f_n}|}{|(C_N)^{f_n} \setminus (C_N)^h|} + \epsilon_N \frac{|(C_N)^h|}{|(C_N)^{f_n} \setminus (C_N)^h|} \\
&\quad\geq 1 + \frac{n}{2}\epsilon_N \varepsilon_N^{\bar{\mathbf{s}}(k_0,N)} + \epsilon_N \cdot \frac{\varepsilon_N^{\bar{\mathbf{s}}(k_0,N)} - \frac{\delta_N}{(1-\varepsilon_N)^{m^0_{f_n}} \varepsilon_N^{m^1_{f_n}}}}{1-\varepsilon_N^{\bar{\mathbf{s}}(k_0,N)} + \frac{2\delta_N}{(1-\varepsilon_N)^{m^0_{f_n}} \varepsilon_N^{m^1_{f_n}}}} \\
&\quad\geq 1 + \frac{n}{2}\epsilon_N \varepsilon_N^{\bar{\mathbf{s}}(k_0,N)} + \frac{1}{2}\epsilon_N \varepsilon_N^{\bar{\mathbf{s}}(k_0,N)} \geq 1 + \frac{n+1}{2}\epsilon_N \varepsilon_N^{\bar{\mathbf{s}}(k_0,N)}.
\end{aligned}$$

Let $\{h_1, \dots, h_{2^{\bar{\mathbf{s}}(k_0,N)}}\}$ be the list of all functions in F^{I_N} such that $\mathsf{dom}(h_i) = \mathsf{dom}(h) \setminus \mathsf{dom}(f_n)$. Without loss of generality we can assume $f_n \cup h_1 = h$. The sets $(C_N)^{f_n \cup h_2}, \dots, (C_N)^{f_n \cup h_{2^{\bar{\mathbf{s}}(k_0,N)}}}$ define a partition of the set $(C_N)^{f_n} \setminus (C_N)^h$. Since

$$\overline{m_{(C_N)^{f_n} \setminus (C_N)^h}} \geq \overline{m_{(C_N)^f}} \cdot \left(1 + \frac{n+1}{2}\epsilon_N \varepsilon_N^{\bar{\mathbf{s}}(k_0,N)}\right)$$

it follows that there exists $2 \leq \ell \leq 2^{\bar{\mathbf{s}}(k_0,N)}$ such that

$$\overline{m_{(C_N)^{f_n \cup h_\ell}}} \geq \overline{m_{(C_N)^f}} \cdot \left(1 + \frac{n+1}{2}\epsilon_N \varepsilon_N^{\bar{\mathbf{s}}(k_0,N)}\right).$$

Let $f_{n+1} = f_n \cup h_\ell$. This completes the induction. $\square$

Proof of Theorem 5.2. Suppose that $\overline{\overline{m_{(C_N)^f}}} = a_0 \geq 2\epsilon_N$. Without loss of generality we can assume that $\alpha_{m_{(C_N)^f}} = 1$, that is $\overline{\overline{m_{(C_N)^f}}} = \overline{m_{(C_N)^f}}$. This is because if we succeed in proving the theorem for the distribution $\alpha_{m_{(C_N)^f}} \cdot m_{(C_N)^f}$ then it must be true for $m_{(C_N)^f}$ as well.

Apply Lemma 5.3 to get $f' \in \mathsf{F}^{I_N}_{f,k_0-\bar{\mathbf{s}}(k_0,N)}$ such that

$$\forall g \in \mathsf{F}^{I_N}_{f',\bar{\mathbf{s}}(k_0,N)}\ \overline{m_{(C_N)^g}} \geq a_0(1-\epsilon_N)\,.$$

Let u_N be the smallest integer greater than $\log_2(8/\epsilon_N^2)$ and define by induction sequences $\{f_i, a_i, b_i : i \leq u_N\}$ such that

(1) $b_0 = 1$ and $f_0 = f'$,
(2) $a_i, b_i \in \mathbb{R}$ for $i \leq u_N$,
(3) $|b_i - a_i| \leq 2^{-i}$ for $i \leq u_N$,
(4) $f_{i+1} \in \mathsf{F}^{I_N}_{f_i,\bar{\mathbf{s}}^{(i+1)}(k_0,N)-\bar{\mathbf{s}}^{(i+2)}(k_0,N)}$ for $i < u_N$,
(5) $\forall g \in \mathsf{F}^{I_N}_{f_i,\bar{\mathbf{s}}^{(i+1)}(k_0,N)}\ a_i(1-\epsilon_N) \leq \overline{m_{(C_N)^g}} \leq b_i(1+\epsilon_N)$.

Suppose that a_i, b_i and f_i are defined and let $c = \overline{m_{(C_N)^{f_i}}}$. Observe that $c \geq a_0 \cdot (1-\epsilon_N) > \epsilon_N$.

If $|c - a_i| \leq 2^{-i-1}$ then let $a_{i+1} = a_i$ and $b_{i+1} = c$. Apply Lemma 5.3 to get $f_{i+1} \in \mathsf{F}^{I_N}_{f_i,\bar{\mathbf{s}}^{(i+1)}(k_0,N)-\bar{\mathbf{s}}^{(i+2)}(k_0,N)}$ such that

$$\forall g \in \mathsf{F}^{I_N}_{f_{i+1},\bar{\mathbf{s}}^{(i+2)}(k_0,N)}\ \overline{m_{(C_N)^g}} \leq b_{i+1}(1+\epsilon_N)\,.$$

Otherwise let $a_{i+1} = c$ and $b_{i+1} = b_i$ and let $f_{i+1} \in \mathsf{F}^{I_N}_{h,\bar{\mathbf{s}}^{(i+1)}(k_0,N)-\bar{\mathbf{s}}^{(i+2)}(k_0,N)}$ be such that

$$\forall g \in \mathsf{F}^{I_N}_{f_{i+1},\bar{\mathbf{s}}^{(i+2)}(k_0,N)}\ \overline{m_{(C_N)^g}} \geq a_{i+1}(1-\epsilon_N)\,.$$

Put $f^\star = f_{u_N}$. Note that by the choice of u_N, $|b_{u_N} - a_{u_N}| \leq \epsilon_N^2/8$. In addition, $\bar{\mathbf{s}}^{(u_N+1)}(k_0,N) > \bar{\mathbf{s}}^{(2u_N)}(k_0,N) = \tilde{\mathbf{s}}(k_0,N)$. Since $\overline{m_{(C_N)^{f^\star}}}$ is equal to either a_{u_N} or b_{u_N}, and $a_{u_N} \geq \varepsilon_N$, a simple computation shows that for every $g \in \mathsf{F}^{I_N}_{f^\star,\tilde{\mathbf{s}}(k_0,N)}$,

$$\begin{aligned}\overline{m_{(C_N)^{f^\star}}} \cdot (1-2\epsilon_N) &\leq a_{u_N}(1-\epsilon_N) \leq \overline{m_{(C_N)^g}} \\ &\leq b_{u_N}(1+\epsilon_N) \leq \overline{m_{(C_N)^{f^\star}}} \cdot (1+2\epsilon_N)\,,\end{aligned}$$

and

$$\overline{m_{(C_N)^g}} \geq a_{u_N}(1-\epsilon_N) \geq a_0(1-2\epsilon_N) = \overline{m_{(C_N)^f}} \cdot (1-2\epsilon_N)\,. \qquad \square$$

Before we start proving Theorem 5.1, we need to prove several facts concerning distributions. The following notation will be used in the sequel:

(1) $v_k = \left|2^{I_0 \cup \ldots \cup I_{k-1}}\right|$ for $k \in \omega$.
(2) if $\mathbf{F} \in \mathsf{F}^N$ and $k < N$ then let $w_k(\mathbf{F}) = \left|(\mathbf{C})^{\mathbf{F}\restriction k}\right|$.

Suppose that $\mathbf{F} \in \mathsf{F}^{N+1}$ and m is a distribution on $2^{I_0 \cup \ldots \cup I_N}$.

(1) Let $m^+_{(\mathbf{C})^{\mathbf{F}}}$ be the distribution on $(C_N)^{\mathbf{F}(N)}$ given by

$$m^+_{(\mathbf{C})^{\mathbf{F}}}(s) = \sum \left\{ m_{(\mathbf{C})^{\mathbf{F}}}(t) : s \subseteq t \in (\mathbf{C})^{\mathbf{F}} \right\} \text{ for } s \in (C_N)^{\mathbf{F}(N)} .$$

(2) For $N_0 \leq N$ and $t \in (\mathbf{C})^{\mathbf{F} \restriction N_0}$, let $m^t_{(\mathbf{C})^{\mathbf{F}}}$ be a distribution on $(\mathbf{C})^{\mathbf{F}}_{N_0}$ defined as

$$m^t_{(\mathbf{C})^{\mathbf{F}}}(s) = m_{(\mathbf{C})^{\mathbf{F}}}(t^\frown s) \text{ for } s \in (\mathbf{C})^{\mathbf{F}}_{N_0} .$$

(3) Let $m^-_{(\mathbf{C})^{\mathbf{F}}}$ be the distribution on $(\mathbf{C})^{\mathbf{F} \restriction N}$ defined as

$$m^-_{(\mathbf{C})^{\mathbf{F}}}(t) = \overline{m^t_{(\mathbf{C})^{\mathbf{F}}}} \text{ for } t \in (\mathbf{C})^{\mathbf{F} \restriction N} .$$

Lemma 5.4. *Suppose that $N_0 \leq N$, $\mathbf{F} \in \mathsf{F}^{N+1}$ and $\mathbf{G} \in \mathsf{F}^{N_0, N+1}_{\mathbf{F}, h}$ for some $h \in \omega^\omega$. Then*

$$\left(m^t_{(\mathbf{C})^{\mathbf{F}}}\right)_{(\mathbf{C})^{\mathbf{G}}_{N_0}} = m^t_{(\mathbf{C})^{\mathbf{G}}} .$$

Proof. Fix $t \in (\mathbf{C})^{\mathbf{F} \restriction N_0} = (\mathbf{C})^{\mathbf{G} \restriction N_0}$ and observe that for $s \in (\mathbf{C})^{\mathbf{F}}_{N_0}$,

$$\begin{aligned} \left(m^t_{(\mathbf{C})^{\mathbf{F}}}\right)_{(\mathbf{C})^{\mathbf{F}}_{N_0}}(s) &= \frac{|(\mathbf{C})^{\mathbf{F}}_{N_0}|}{|(\mathbf{C})^{\mathbf{G}}_{N_0}|} \cdot m_{(\mathbf{C})^{\mathbf{F}}}(t^\frown s) \\ &= \frac{|(\mathbf{C})^{\mathbf{F}}_{N_0}|}{|(\mathbf{C})^{\mathbf{G}}_{N_0}|} \cdot \frac{v_{N+1}}{w_{N+1}(\mathbf{F})} m(t^\frown s) \\ &= \frac{v_{N+1}}{w_{N+1}(\mathbf{G})} \cdot m(t^\frown s) \\ &= m^t_{(\mathbf{C})^{\mathbf{G}}}(s) . \end{aligned}$$

□

Lemma 5.5. *Suppose that $\mathbf{F} \in \mathsf{F}^{N+1}$ and $\mathbf{G} \in \mathsf{F}^{N, N+1}_{\mathbf{F}, h}$ for some $h \in \omega^\omega$. Then*

$$(m^+_{(\mathbf{C})^{\mathbf{F}}})_{(C_N)^{\mathbf{G}(N)}} = m^+_{(\mathbf{C})^{\mathbf{G}}} .$$

Proof. Similar to the proof of 5.4. □

Lemma 5.6. *Suppose that $N_0 \leq N$, $\mathbf{F} \in \mathsf{F}^{N+1}$ and $t \in (\mathbf{C})^{\mathbf{F} \restriction N_0}$. Then*

$$\overline{\overline{m^t_{(\mathbf{C})^{\mathbf{F}}}}} \geq w_{N_0}(\mathbf{F}) \cdot \overline{m^t_{(\mathbf{C})^{\mathbf{F}}}} .$$

Proof. Note that

$$w_{N_0}(\mathbf{F}) \cdot m_{(\mathbf{C})^{\mathbf{F}}}(t^\frown s) \leq \frac{w_{N_0}(\mathbf{F})}{w_{N+1}(\mathbf{F})} = \frac{1}{|(\mathbf{C})^{\mathbf{F}}_{N_0}|} .$$

□

The next two lemmas will be crucial in the recursive computations of distributions.

Lemma 5.7. *Suppose that* $N_0 \le N$, $\mathbf{F}$, $\mathbf{G} \in \mathsf{F}^{N+1}$, $\mathbf{F}\restriction[N_0,N] = \mathbf{G}\restriction[N_0,N]$ *and* $t \in (\mathbf{C})^{\mathbf{F}\restriction N_0} \cap (\mathbf{C})^{\mathbf{G}\restriction N_0}$. *Then*

$$\alpha_{m^t_{(\mathbf{C})^{\mathbf{F}}}} \cdot m^t_{(\mathbf{C})^{\mathbf{F}}} = \alpha_{m^t_{(\mathbf{C})^{\mathbf{G}}}} \cdot m^t_{(\mathbf{C})^{\mathbf{G}}} .$$

In particular, if $\mathbf{F}^\star \in \mathsf{F}^{N_0,N+1}_{\mathbf{F},h}$ *for some* $h \in \omega^\omega$ *then*

$$\frac{\overline{m^t_{(\mathbf{C})^{\mathbf{F}\restriction N_0 \frown \mathbf{F}^\star\restriction[N_0,N]}}}}{\overline{m^t_{(\mathbf{C})^{\mathbf{F}}}}} = \frac{\overline{m^t_{(\mathbf{C})^{\mathbf{G}\restriction N_0 \frown \mathbf{F}^\star\restriction[N_0,N]}}}}{\overline{m^t_{(\mathbf{C})^{\mathbf{G}}}}} .$$

Proof. Note that under the assumptions the distributions $m^t_{(\mathbf{C})^{\mathbf{F}}}$ and $m^t_{(\mathbf{C})^{\mathbf{G}}}$ have the same domain and the fraction $\frac{m(t^\frown s)}{m^t_{(\mathbf{C})^{\mathbf{F}}}(s)}$ has the constant value for both $\mathbf{F}$ and $\mathbf{G}$. □

Lemma 5.8. *Suppose that* $\mathbf{F} \in \mathsf{F}^{N+1}$ *and* $\mathbf{G} \in \mathsf{F}^{N+1}_{\mathbf{F},h}$ *for some* $h \in \omega^\omega$. *Then*

$$\overline{m_{(\mathbf{C})^{\mathbf{G}}}} = \sum_{t \in (\mathbf{C})^{\mathbf{G}\restriction N}} m^-_{(\mathbf{C})^{\mathbf{G}\restriction N \frown \mathbf{F}(N)}}(t) \cdot \frac{\overline{m^t_{(\mathbf{C})^{\mathbf{F}\restriction N \frown \mathbf{G}(N)}}}}{\overline{m^t_{(\mathbf{C})^{\mathbf{F}}}}} .$$

Proof. For $t \in (\mathbf{C})^{\mathbf{G}\restriction N}$,

$$\frac{m^-_{(\mathbf{C})^{\mathbf{G}\restriction N \frown \mathbf{F}(N)}}(t)}{\overline{m^t_{(\mathbf{C})^{\mathbf{F}}}}} = \frac{\sum_{s' \in (C_N)^{\mathbf{F}(N)}} \frac{v_{N+1}}{w_{N+1}\left(\mathbf{G}\restriction N^\frown \mathbf{F}(N)\right)} \cdot m(t^\frown s')}{\sum_{s' \in (C_N)^{\mathbf{F}(N)}} \frac{v_{N+1}}{w_{N+1}(\mathbf{F})} \cdot m(t^\frown s')} = \frac{w_N(\mathbf{F})}{w_N(\mathbf{G})} .$$

Therefore

$$\begin{aligned}
&\sum_{t \in (\mathbf{C})^{\mathbf{G}\restriction N}} m^-_{(\mathbf{C})^{\mathbf{G}\restriction N \frown \mathbf{F}(N)}}(t) \cdot \frac{\overline{m^t_{(\mathbf{C})^{\mathbf{F}\restriction N \frown \mathbf{G}(N)}}}}{\overline{m^t_{(\mathbf{C})^{\mathbf{F}}}}} \\
&= \sum_{t \in (\mathbf{C})^{\mathbf{G}\restriction N}} \frac{w_N(\mathbf{F})}{w_N(\mathbf{G})} \cdot \overline{m^t_{(\mathbf{C})^{\mathbf{F}\restriction N \frown \mathbf{G}(N)}}} \\
&= \sum_{t \in (\mathbf{C})^{\mathbf{G}\restriction N}} \frac{w_N(\mathbf{F})}{w_N(\mathbf{G})} \cdot \sum_{t \subseteq s \in (\mathbf{C})^{\mathbf{G}}} \frac{v_{N+1}}{w_{N+1}\left(\mathbf{F}\restriction N^\frown \mathbf{G}(N)\right)} \cdot m(s) \\
&= \sum_{t \in (\mathbf{C})^{\mathbf{G}\restriction N}} \sum_{t \subseteq s \in (\mathbf{C})^{\mathbf{G}}} \frac{v_{N+1}}{w_{N+1}(\mathbf{G})} \cdot m(s) \\
&= \sum_{t \in (\mathbf{C})^{\mathbf{G}\restriction N}} \sum_{t \subseteq s \in (\mathbf{C})^{\mathbf{G}}} m_{(\mathbf{C})^{\mathbf{G}}}(s) \\
&= \overline{m_{(\mathbf{C})^{\mathbf{G}}}} .
\end{aligned}$$

□

We will need one more definition:

Definition 5.3. Suppose that m is a distribution on X and $U \subseteq X$. Let $m_{[U]}$ be the distribution on X defined as

$$m_{[U]}(x) = \begin{cases} m(x) & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } x \in X\,.$$

Now we are ready to prove Theorem 5.1. For technical reasons we will need a somewhat stronger result stated below.

Theorem 5.3. *Suppose that $N_0 < N$ are natural numbers, $h^0,\ h_0 \in \prod_{i<N} m_i$ satisfy $h_0(i) + h^0(i) \leq m_i$ for $i < N$, $\mathbf{F} \in \mathsf{F}^N_{\emptyset,h^0}$ and m is a distribution on $2^{I_0\cup\ldots\cup I_{N-1}}$ such that*

$$\overline{m_{(\mathbf{C})^{\mathbf{F}}}} \geq \frac{2\sum_{i=N_0}^{N} \epsilon_i}{\prod_{i=N_0}^{N}(1-8\epsilon_i)}\,.$$

There exist $\mathbf{F}^\star \in \mathsf{F}^{N_0,N}_{\mathbf{F},h_0-\mathbf{s}(h_0)}$ and $U^\star \subseteq 2^{I_0\cup\ldots\cup I_{N-1}}$ such that

$$\overline{\left(m_{[U^\star]}\right)_{(\mathbf{C})^{\mathbf{F}^\star}}} \geq \overline{m_{(\mathbf{C})^{\mathbf{F}}}} \cdot \prod_{i=N_0}^{N-1}(1-8\epsilon_i) - \sum_{i=N_0}^{N-2} \epsilon_i\,,$$

and for any $\mathbf{G} \in \mathsf{F}^{N_0,N}_{\mathbf{F}^\star,\mathbf{s}(h_0)}$ and $t \in (\mathbf{C})^{\mathbf{G}\restriction M_0}$, $M_0 \in [N_0, N)$,

$$\overline{\left(m_{[U^\star]}\right)^t_{(\mathbf{C})^{\mathbf{G}}}} \geq \overline{\left(m_{[U^\star]}\right)^t_{(\mathbf{C})^{\mathbf{F}^\star}}} \cdot \prod_{i=M_0}^{N-1}(1-4\epsilon_i)\,.$$

Proof. First notice that Theorem 5.1 follows from Theorem 5.3. If $\mathbf{F}^\star$ and $U^\star$ are as required, then for all $\mathbf{G} \in \mathsf{F}^{N_0,N}_{\mathbf{F}^\star,\mathbf{s}(h_0)}$,

$$\begin{aligned}
\overline{m_{(\mathbf{C})^{\mathbf{G}}}} \geq \overline{\left(m_{[U^\star]}\right)_{(\mathbf{C})^{\mathbf{G}}}} &\geq \sum_{t\in(\mathbf{C})^{\mathbf{G}\restriction N_0}} \overline{\left(m_{[U^\star]}\right)^t_{(\mathbf{C})^{\mathbf{G}}}} \\
&\geq \sum_{t\in(\mathbf{C})^{\mathbf{G}\restriction N_0}} \left(\overline{\left(m_{[U^\star]}\right)^t_{(\mathbf{C})^{\mathbf{F}^\star}}} \cdot \prod_{i=N_0}^{N-1}(1-4\epsilon_i)\right) \\
&\geq \prod_{i=N_0}^{N-1}(1-4\epsilon_i) \cdot \left(\sum_{t\in(\mathbf{C})^{\mathbf{F}^\star\restriction N_0}} \overline{\left(m_{[U^\star]}\right)^t_{(\mathbf{C})^{\mathbf{F}^\star}}}\right) \\
&= \prod_{i=N_0}^{N-1}(1-4\epsilon_i) \cdot \overline{\left(m_{[U^\star]}\right)_{(\mathbf{C})^{\mathbf{F}^\star}}} \\
&\geq \prod_{i=N_0}^{N-1}(1-4\epsilon_i) \cdot \left(\overline{m_{(\mathbf{C})^{\mathbf{F}}}} \cdot \prod_{i=N_0}^{N-1}(1-8\epsilon_i) - \sum_{i=N_0}^{N-2} \epsilon_i\right) \\
&\geq \overline{m_{(\mathbf{C})^{\mathbf{F}}}} \cdot \prod_{i=N_0}^{N-1}(1-8\epsilon_i)^2 - \sum_{i=N_0}^{N-2} \epsilon_i\,.
\end{aligned}$$

We will proceed by induction on N. If $N = N_0$ then the theorem is trivially true. Thus, suppose that the result holds for some $N \geq N_0$ and consider $N+1$. Let $\mathbf{F} \in \mathsf{F}^{N_0,N+1}_{\emptyset,h^0}$ and let m be a distribution on $2^{I_0\cup\ldots\cup I_N}$ such that

$$\overline{m_{(\mathbf{C})^{\mathbf{F}}}} \geq \frac{2\sum_{i=N_0}^{N}\epsilon_i}{\prod_{i=N_0}^{N}(1-\epsilon_i)} .$$

Recall that by Lemma 5.5,

$$\overline{\overline{m^+_{(\mathbf{C})^{\mathbf{F}}}}} \geq \overline{m^+_{(\mathbf{C})^{\mathbf{F}}}} = \overline{m_{(\mathbf{C})^{\mathbf{F}}}} \geq 2\epsilon_N ,$$

and apply Theorem 5.2 with $m = m^+_{(\mathbf{C})^{\mathbf{F}}}$, $k^0 = \left|\mathsf{dom}\big(\mathbf{F}(N)\big)\right|$, $k_0 = h_0(N)$ to get $\tilde{f}_0 \in \mathsf{F}^{I_N}_{\mathbf{F}(N),k_0-\tilde{\mathbf{s}}(k_0,N)}$ such that

$$\forall g \in \mathsf{F}^{I_N}_{\tilde{f}_0,\tilde{\mathbf{s}}(k_0,N)}\ \overline{m^+{}_{(\mathbf{C})^{\mathbf{F}\restriction N\frown g}}} \geq \overline{m_{(\mathbf{C})^{\mathbf{F}}}}\cdot(1-2\epsilon_N) .$$

Let $\{s_i : 1 \leq i \leq w_N(\mathbf{F})\}$ be an enumeration of $(\mathbf{C})^{\mathbf{F}\restriction N}$. By induction, build a sequence $\{\tilde{f}_i : i \leq w_N(\mathbf{F})\}$ such that

(1) $\tilde{f}_i \subseteq \tilde{f}_{i+1}$,
(2) $k_0 - |\mathsf{dom}(\tilde{f}_i)| \geq \tilde{\mathbf{s}}^{(2i+1)}(k_0,N)$,
(3) for every $i \geq 1$, one of the following conditions holds:
  (a) $\forall g \in \mathsf{F}^{I_N}_{\tilde{f}_i,\tilde{\mathbf{s}}^{(2i+1)}(k_0,N)}\ \overline{m^{s_i}{}_{(\mathbf{C})^{\mathbf{F}\restriction N\frown g}}} < \frac{2\epsilon_N}{w_N(\mathbf{F})}$,
  (b) for all $g \in \mathsf{F}^{I_N}_{\tilde{f}_i,\tilde{\mathbf{s}}^{(2i+1)}}$,

$$\overline{m^{s_i}{}_{(\mathbf{C})^{\mathbf{F}\restriction N\frown \tilde{f}_i}}}\cdot(1-2\epsilon_N) \leq \overline{m^{s_i}{}_{(\mathbf{C})^{\mathbf{F}\restriction N\frown g}}} \leq \overline{m^{s_i}{}_{(\mathbf{C})^{\mathbf{F}\restriction N\frown \tilde{f}_i}}}\cdot(1+2\epsilon_N) .$$

Suppose that $\tilde{f}_i$ is given. If

$$\forall g \in \mathsf{F}^{I_N}_{\tilde{f}_i,\tilde{\mathbf{s}}^{(2i+3)}(k_0,N)}\ \overline{m^{s_{i+1}}{}_{(\mathbf{C})^{\mathbf{F}\restriction N\frown g}}} < \frac{2\epsilon_N}{w_N(\mathbf{F})}$$

then put $\tilde{f}_{i+1} = \tilde{f}_i$.

Otherwise, let $\tilde{f}'_{i+1} \in \mathsf{F}^{I_N}_{\tilde{f}_i,\tilde{\mathbf{s}}^{(2i+3)}(k_0,N)}$ be chosen so that

$$\overline{m^{s_{i+1}}{}_{(\mathbf{C})^{\mathbf{F}\restriction N\frown \tilde{f}'_{i+1}}}} \geq \frac{2\epsilon_N}{w_N(\mathbf{F})} .$$

In particular, by Lemma 5.6, $\overline{\overline{m^{s_{i+1}}{}_{(\mathbf{C})^{\mathbf{F}\restriction N\frown \tilde{f}'_{i+1}}}}} \geq 2\epsilon_N$. Let $\tilde{k} = k_0 - |\mathsf{dom}\tilde{f}'_{i+1}|$. By Theorem 5.2, there exist $\tilde{f}_{i+1} \in \mathsf{F}^{I_N}_{\tilde{f}'_{i+1},\tilde{k}-\tilde{\mathbf{s}}(\tilde{k},N)}$ such that for all $g \in \mathsf{F}^{I_N}_{\tilde{f}_{i+1},\tilde{\mathbf{s}}(\tilde{k},N)}$,

$$\overline{m^{s_{i+1}}{}_{(\mathbf{C})^{\mathbf{F}\restriction N\frown \tilde{f}_{i+1}}}}\cdot(1+2\epsilon_N) \geq \overline{m^{s_{i+1}}{}_{(\mathbf{C})^{\mathbf{F}\restriction N\frown g}}} \geq \overline{m^{s_{i+1}}{}_{(\mathbf{C})^{\mathbf{F}\restriction N\frown \tilde{f}_{i+1}}}}\cdot(1-2\epsilon_N) .$$

Note that $\tilde{k} \geq k_0 - |\mathsf{dom}\tilde{f}_i| - \tilde{\mathbf{s}}^{(2i+3)}(k_0,N)$. Using the induction hypothesis we get that $\tilde{k} \geq \tilde{\mathbf{s}}^{(2i+1)}(k_0,N) - \tilde{\mathbf{s}}^{(2i+3)}(k_0,N) \geq \tilde{\mathbf{s}}^{(2i+2)}(k_0,N)$. It follows that $\tilde{\mathbf{s}}(\tilde{k},N) \geq \tilde{\mathbf{s}}^{(2i+3)}(k_0,N)$ and $k_0 - |\mathsf{dom}(\tilde{f}_{i+1})| \geq \tilde{\mathbf{s}}^{(2i+3)}(k_0,N)$, which finishes the induction.

Let $\mathbf{F}^\star(N) = \tilde{f}_{w_N(\mathbf{F})}$. Since $w_N(\mathbf{F}) \leq \left|2^{I_0\cup\ldots\cup I_{N-1}}\right|$ it follows that $\mathbf{s}(k_0,N) \leq \tilde{\mathbf{s}}^{(2w_N(\mathbf{F})+1)}(k_0,N)$. Thus $\mathbf{F}^\star(N) \in \mathsf{F}^{I_N}_{\mathbf{F}(N),h_0(N)-\mathbf{s}(k_0,N)}$.

Observe that $\overline{m^s{}_{(\mathbf{C})^{\mathbf{F}\restriction N}\frown g}} = m^-_{(\mathbf{C})^{\mathbf{F}\restriction N}\frown g}(s)$ for every $s \in (\mathbf{C})^{\mathbf{F}\restriction N}$. In particular, $\overline{m^s{}_{(\mathbf{C})^{\mathbf{F}\restriction N}\frown \mathbf{F}^\star(N)}} = m^-_{(\mathbf{C})^{\mathbf{F}\restriction N}\frown \mathbf{F}^\star(N)}(s)$.

By the construction, for every $g \in \mathsf{F}^{I_N}_{\mathbf{F}^\star(N),\mathbf{s}(k_0,N)}$ and $s \in (\mathbf{C})^{\mathbf{F}\restriction N}$,

$$m^-_{(\mathbf{C})^{\mathbf{F}\restriction N}\frown \mathbf{F}^\star(N)}(s) \cdot \frac{1-2\epsilon_N}{1+2\epsilon_N} \le \overline{m^s{}_{(\mathbf{C})^{\mathbf{F}\restriction N}\frown g}} \le m^-_{(\mathbf{C})^{\mathbf{F}\restriction N}\frown \mathbf{F}^\star(N)}(s) \cdot \frac{1+2\epsilon_N}{1-2\epsilon_N}$$

or otherwise

$$\overline{m^s{}_{(\mathbf{C})^{\mathbf{F}\restriction N}\frown \mathbf{F}^\star(N)}} \le \frac{2\epsilon_N}{w_N(\mathbf{F})} \quad \text{and} \quad \overline{m^s{}_{(\mathbf{C})^{\mathbf{F}\restriction N}\frown g}} \le \frac{2\epsilon_N}{w_N(\mathbf{F})}.$$

Moreover, by the choice of $\tilde{f}_0$, for every $g \in \mathsf{F}^{I_N}_{\mathbf{F}^\star(N),\mathbf{s}(k_0,N)}$,

$$\overline{m^+{}_{(\mathbf{C})^{\mathbf{F}\restriction N}\frown g}} \ge \overline{m_{(\mathbf{C})^{\mathbf{F}}}} \cdot (1-2\epsilon_N).$$

Even though we do not have much control over the values of $m^-_{(\mathbf{C})^{\mathbf{F}\restriction N}\frown \mathbf{F}^\star(N)}(s)$ we can show that many of them are larger than $\frac{2\epsilon_N}{w_N(\mathbf{F})}$. Let

$$U = \left\{ s \in (\mathbf{C})^{\mathbf{F}\restriction N} : m^-_{(\mathbf{C})^{\mathbf{F}\restriction N}\frown \mathbf{F}^\star(N)}(s) \ge \frac{2\epsilon_N}{w_N(\mathbf{F})} \right\}.$$

Note that for every $g \in \mathsf{F}^{I_N}_{\mathbf{F}^\star(N),\mathbf{s}(k_0,N)}$,

$$\begin{aligned}
&(1-2\epsilon_N) \cdot \overline{m_{(\mathbf{C})^{\mathbf{F}}}} \le \overline{m^+{}_{(\mathbf{C})^{\mathbf{F}\restriction N}\frown g}} \\
&\quad = \overline{m_{(\mathbf{C})^{\mathbf{F}\restriction N}\frown g}} = \overline{m^-{}_{(\mathbf{C})^{\mathbf{F}\restriction N}\frown g}} \le \sum_{s\in U} m^-_{(\mathbf{C})^{\mathbf{F}\restriction N}\frown g}(s) \\
&\qquad + \sum_{s\in (\mathbf{C})^{\mathbf{F}\restriction N}\setminus U} m^-_{(\mathbf{C})^{\mathbf{F}\restriction N}\frown g}(s) \le \frac{1+2\epsilon_N}{1-2\epsilon_N} \cdot \sum_{s\in U} m^-_{(\mathbf{C})^{\mathbf{F}\restriction N}\frown \mathbf{F}^\star(N)}(s) \\
&\qquad + w_N(\mathbf{F}) \cdot \frac{2\epsilon_N}{w_N(\mathbf{F})} \cdot \frac{1+2\epsilon_N}{1-2\epsilon_N} \le 2\epsilon_N \cdot \frac{1+2\epsilon_N}{1-2\epsilon_N} \\
&\qquad + \frac{1+2\epsilon_N}{1-2\epsilon_N} \cdot \sum_{s\in U} m^-_{(\mathbf{C})^{\mathbf{F}\restriction N}\frown \mathbf{F}^\star(N)}(s).
\end{aligned}$$

It follows that

$$\sum_{s\in U} m^-_{(\mathbf{C})^{\mathbf{F}\restriction N}\frown \mathbf{F}^\star(N)}(s) \ge \frac{(1-2\epsilon_N)^2}{1+2\epsilon_N} \cdot \overline{m_{(\mathbf{C})^{\mathbf{F}}}} - 2\epsilon_N \ge (1-8\epsilon_N) \cdot \overline{m_{(\mathbf{C})^{\mathbf{F}}}} - 2\epsilon_N.$$

Define distribution $m^\star$ on $2^{I_0\cup\ldots\cup I_{N-1}}$ as

$$m^\star(s) = \begin{cases} m^-_{(\mathbf{C})^{\mathbf{F}\restriction N}\frown \mathbf{F}^\star(N)}(s) & \text{if } m^-_{(\mathbf{C})^{\mathbf{F}\restriction N}\frown \mathbf{F}^\star(N)}(s) \ge \dfrac{2\epsilon_N}{w_N(\mathbf{F})} \\ 0 & \text{otherwise}. \end{cases}$$

Clearly,

$$\overline{m^\star_{(\mathbf{C})^{\mathbf{F}\restriction N}}} = \sum_{s\in U} m^-_{(\mathbf{C})^{\mathbf{F}\restriction N \frown \mathbf{F}^\star(N)}}(s)$$
$$\geq \overline{m_{(\mathbf{C})^{\mathbf{F}}}} \cdot (1-8\epsilon_N) - 2\epsilon_N \geq \frac{2\sum_{i=N_0}^{N-1}\epsilon_i}{\prod_{i=N_0}^{N-1}(1-8\epsilon_i)}\,.$$

Apply the induction hypothesis to $m^\star$, $\mathbf{F}\restriction N$ and $h_0\restriction N$ to obtain $\mathbf{F}^\star\restriction N$ and $V^\star$ as in Theorem 5.3. Let

$$U^\star = \left\{s \in 2^{I_0\cup\ldots\cup I_N} : s\restriction I_0\cup\ldots\cup I_{N-1} \in V^\star \cap U\right\}\,.$$

It remains to check that $\mathbf{F}^\star$ and $U^\star$ have the required properties.

$$\overline{\left(m_{[U^\star]}\right)_{(\mathbf{C})^{\mathbf{F}^\star}}} = \sum_{s\in(\mathbf{C})^{\mathbf{F}^\star\restriction N}} \left(m_{[U^\star]}\right)^-_{(\mathbf{C})^{\mathbf{F}^\star}}$$
$$= \sum_{s\in(\mathbf{C})^{\mathbf{F}^\star\restriction N}} \left(m^\star_{[V^\star]}\right)_{(\mathbf{C})^{\mathbf{F}^\star\restriction N}}(s)$$
$$= \overline{\left(m^\star_{[V^\star]}\right)_{(\mathbf{C})^{\mathbf{F}^\star\restriction N}}} \geq \overline{m^\star_{(\mathbf{C})^{\mathbf{F}\restriction N}}} \cdot \prod_{i=N_0}^{N-1}(1-8\epsilon_i)$$
$$- \sum_{i=N_0}^{N-2}\epsilon_i \geq \left(\overline{m_{(\mathbf{C})^{\mathbf{F}}}}\cdot(1-8\epsilon_N) - 2\epsilon_N\right)\cdot \prod_{i=M_0}^{N-1}(1-8\epsilon_i)$$
$$- \sum_{i=M_0}^{N-2}\epsilon_i \geq \overline{m_{(\mathbf{C})^{\mathbf{F}}}} \cdot \prod_{i=M_0}^{N}(1-8\epsilon_i) - \sum_{i=M_0}^{N-1}\epsilon_i\,,$$

which gives the first condition.

To verify the second condition suppose that $\mathbf{G} \in \mathsf{F}^{N_0,N+1}_{\mathbf{F}^\star,\mathbf{s}(h_0)}$, $M_0 \in [N_0, N]$ and $t \in (\mathbf{C})^{\mathbf{G}\restriction M_0}$. By the inductive hypothesis we have that

$$\overline{\left(m^\star_{[V^\star]}\right)^t_{(\mathbf{C})^{\mathbf{G}\restriction N}}} \geq \overline{\left(m^\star_{[U^\star]}\right)^t_{(\mathbf{C})^{\mathbf{F}^\star\restriction N}}} \cdot \prod_{i=M_0}^{N-1}(1-4\epsilon_i)\,.$$

By Lemmas 5.7 and 5.8,

$$\overline{\left(m_{[U^\star]}\right)^t_{(\mathbf{C})^{\mathbf{G}}}} = \sum_{t\subseteq s\in(\mathbf{C})^{\mathbf{G}\restriction N}} \left(m_{[U^\star]}\right)^-_{(\mathbf{C})^{\mathbf{G}\restriction N\frown\mathbf{F}^\star(N)}}(s)\cdot\frac{\overline{\left(m_{[U^\star]}\right)^s_{(\mathbf{C})^{\mathbf{F}^\star\restriction N\frown\mathbf{G}(N)}}}}{\overline{\left(m_{[U^\star]}\right)^s_{(\mathbf{C})^{\mathbf{F}^\star}}}}$$
$$= \sum_{t\subseteq s\in(\mathbf{C})^{\mathbf{G}\restriction N}} \left(m_{[U^\star]}\right)^-_{(\mathbf{C})^{\mathbf{G}\restriction N\frown\mathbf{F}^\star(N)}}(s)\cdot\frac{\overline{\left(m_{[U^\star]}\right)^s_{(\mathbf{C})^{\mathbf{F}\restriction N\frown\mathbf{G}(N)}}}}{\overline{\left(m_{[U^\star]}\right)^s_{(\mathbf{C})^{\mathbf{F}\restriction N\frown\mathbf{F}^\star(N)}}}}$$
$$= \sum_{t\subseteq s\in(\mathbf{C})^{\mathbf{G}\restriction N}} \left(m_{[U^\star]}\right)^-_{(\mathbf{C})^{\mathbf{G}\restriction N\frown\mathbf{F}^\star(N)}}(t)\cdot\frac{\overline{\left(m_{[U]}\right)^s_{(\mathbf{C})^{\mathbf{F}\restriction N\frown\mathbf{G}(N)}}}}{\overline{\left(m_{[U]}\right)^s_{(\mathbf{C})^{\mathbf{F}\restriction N\frown\mathbf{F}^\star(N)}}}}\,.$$

Now

$$\sum_{t\subseteq s\in(\mathbf{C})^{\mathbf{G}\restriction N}} \left(m_{[U^\star]}\right)^{-}_{(\mathbf{C})^{\mathbf{G}\restriction N\frown \mathbf{F}^\star(N)}}(s)\cdot \frac{\overline{\left(m_{[U]}\right)^{s}_{(\mathbf{C})^{\mathbf{F}\restriction N\frown \mathbf{G}(N)}}}}{\overline{\left(m_{[U]}\right)^{s}_{(\mathbf{C})^{\mathbf{F}\restriction N\frown \mathbf{F}^\star(N)}}}}$$

$$\geq \sum_{t\subseteq s\in(\mathbf{C})^{\mathbf{G}\restriction N}} \left(m^\star_{[U^\star]}\right)^{t}_{(\mathbf{C})^{\mathbf{G}\restriction N}}(s)\cdot\frac{1-2\epsilon_N}{1+2\epsilon_N}$$

$$= \sum_{t\subseteq s\in(\mathbf{C})^{\mathbf{G}\restriction N}} \left(m^\star_{[V^\star]}\right)^{t}_{(\mathbf{C})^{\mathbf{G}\restriction N}}(s)\cdot\frac{1-2\epsilon_N}{1+2\epsilon_N}$$

$$\geq \frac{1-2\epsilon_N}{1+2\epsilon_N}\cdot\overline{\left(m^\star_{[V^\star]}\right)^{t}_{(\mathbf{C})^{\mathbf{F}^\star\restriction N}}}\cdot\prod_{i=M_0}^{N-1}(1-4\epsilon_i)$$

$$\geq \overline{\left(m^\star_{[V^\star]}\right)^{t}_{(\mathbf{C})^{\mathbf{F}^\star\restriction N}}}\cdot\prod_{i=M_0}^{N}(1-4\epsilon_i)$$

$$= \overline{\left(m_{[U^\star]}\right)^{t}_{(\mathbf{C})^{\mathbf{F}^\star}}}\cdot\prod_{i=M_0}^{N}(1-4\epsilon_i)\,,$$

which concludes the proof.

6. Measures and Norms

In this section we will examine the consequences of the combinatorial results proved earlier on measures on 2^ω.

For $U\subseteq 2^I$, $[U]=\{x\in 2^\omega : x\restriction I\in U\}$.

If $p\subseteq 2^{<\omega}$ is a tree, $s\in p$, and $N\in\omega$, then

(1) $[p]$ denotes the set of branches of p,
(2) $p_s=\{t\in p: t\subseteq s \text{ or } s\subseteq t\}$,
(3) $p^N=p\restriction(I_0\cup\ldots\cup I_{N-1})$.

We will identify product with concatenation, i.e. (s,t) with $s^\frown t$, and similarly for infinite products. Most of the time we will also identify p with $[p]$.

Definition 6.1. Let $\mu_{(\mathbf{C})^{\mathbf{F}}}$ be the measure on $(\mathbf{C})^{\mathbf{F}}$ defined as the product of counting measures on the coordinates. In other words, if $s\in 2^{I_k}$ then

$$\mu_{(\mathbf{C})^{\mathbf{F}}}([s])=\begin{cases}\left|(C_k)^{\mathbf{F}(k)}\right|^{-1} & \text{if } s\in (C_k)^{\mathbf{F}(k)}\\ 0 & \text{otherwise}.\end{cases}$$

Given a perfect set $p\in\mathsf{Perf}$,

$$\mu_{(\mathbf{C})^{\mathbf{F}}}(p)=\lim_{N\to\infty}\frac{\left|p^N\cap(\mathbf{C})^{\mathbf{F}\restriction N}\right|}{\left|(\mathbf{C})^{\mathbf{F}\restriction N}\right|}.$$

Note that $\mu_{(\mathbf{C})^{\mathbf{F}}}(p)=\mu_{(\mathbf{C})^{\mathbf{F}}}\left(p\cap(\mathbf{C})^{\mathbf{F}}\right)$.

Definition 6.2. For a function $f \in \omega^\omega$ define $\log_{\mathbf{s}}(f) \in \omega^\omega$ as

$$\log_{\mathbf{s}}(f)(N) = \max\left\{k : \mathbf{s}^{(k \cdot l_N)}(f(N), N) > 0\right\} .$$

For $h_1, h_s \in \omega^\omega$ define $h_1 \simeq h_2$ if $\log_{\mathbf{s}}(h_1) = \log_{\mathbf{s}}(h_2)$. Clearly $\simeq$ is an equivalence relation.

Let $\mathcal{X}$ be the collection of functions $f \in \omega^\omega$ such that

(1) $\lim_{m\to\infty} \log_{\mathbf{s}}(f)(m) = \infty$,
(2) $f = \min\{g : f \simeq g\}$.

For $f \in \omega^\omega$ define functions $\bar{f}, f^- \in \mathcal{X}$ as follows: $\overline{f} = \mathcal{X} \cap \{g : f \simeq g\}$, and

$$f^-(n) = \begin{cases} \min\{k : \log_{\mathbf{s}}(k,n) = \log_{\mathbf{s}}(f(n),n) - 1\} & \text{if } \log_{\mathbf{s}}(f(n),n) > 0 \\ 0 & \text{otherwise} . \end{cases}$$

If $f \in \mathcal{X}$ and $n \in \omega$ let $i_f(n) = \max\{k : \log_{\mathbf{s}}(f)(k) \leq n\}$.

Remark. Note that $\mathcal{X} \neq \emptyset$. By (P5), $\overline{h} \in \mathcal{X}$, where $h(k) = m_k$ for $k \in \omega$. Also, $\lim_{n\to\infty} i_f(n) = \infty$ for $f \in \mathcal{X}$. The purpose of the restriction put on the set $\mathcal{X}$ is to make the mapping $f \mapsto \log_{\mathbf{s}}(f)$ one-to-one. In practice, we will only use the fact that if $\log_{\mathbf{s}}(f)(n) = 0$, then $f(n) = 0$.

Definition 6.3. For a perfect set $p \subseteq 2^\omega$, $\mathbf{F} \in \mathsf{F}^\omega$, $N \in \omega$ and $h \in \mathcal{X}$, define

$$[\![p, \mathbf{F}, h]\!]_N = \inf\left\{\mu_{(\mathbf{C})^{\mathbf{G}}}(p) : \mathbf{G} \in \mathsf{F}^{N,\omega}_{\mathbf{F},h}\right\} .$$

We will write $[\![p, \mathbf{F}, h]\!]$ instead of $[\![p, \mathbf{F}, h]\!]_0$.

The following easy lemma lists some basic properties of these notions.

Lemma 6.1.

(1) *The sequence* $\left\{\frac{|p^N \cap (\mathbf{C})^{\mathbf{F}\restriction N}|}{|(\mathbf{C})^{\mathbf{F}\restriction N}|} : k \in \omega\right\}$ *is monotonically decreasing for every* $p \in \mathsf{Perf}$,
(2) $[\![p, \mathbf{F}_1, h_1]\!]_N \geq [\![p, \mathbf{F}_2, h_2]\!]_N$ *if* $\mathbf{F}_1 \in \mathsf{F}^{N,\omega}_{\mathbf{F}_2, h_2 - h_1}$,
(3) *if* $p_1 \cap p_2 = \emptyset$ *then* $[\![p_1 \cup p_2, \mathbf{F}, h]\!]_N \geq [\![p_1, \mathbf{F}, h]\!]_N + [\![p_2, \mathbf{F}, h]\!]_N$.

Proof. (1) is obvious, and (2) follows from Lemma 5.2(3). Take $\varepsilon > 0$ and let $\mathbf{G} \in \mathsf{F}^{N,\omega}_{\mathbf{F},h}$ be such that

$$[\![p_1 \cup p_2, \mathbf{F}, h]\!]_N + \varepsilon \geq \mu_{(\mathbf{C})^{\mathbf{G}}}(p_1 \cup p_2) .$$

Now

$$[\![p_1 \cup p_2, \mathbf{F}, h]\!]_N + \varepsilon \geq \mu_{(\mathbf{C})^{\mathbf{G}}}(p_1 \cup p_2) \geq \mu_{(\mathbf{C})^{\mathbf{G}}}(p_1 \cup p_2)$$

$$\geq \mu_{(\mathbf{C})^{\mathbf{G}}}(p_1) + \mu_{(\mathbf{C})^{\mathbf{G}}}(p_2) \geq [\![p_1, \mathbf{F}, h]\!]_N + [\![p_2, \mathbf{F}, h]\!]_N .$$

Thus $[\![p_1 \cup p_2, \mathbf{F}, h]\!]_N + \varepsilon \geq [\![p_1, \mathbf{F}, h]\!]_N + [\![p_2, \mathbf{F}, h]\!]_N$ and the inequality follows. □

The following two theorems are the key to the whole construction.

Theorem 6.1. *Suppose that* $\mu_{(\mathbf{C})^{\mathbf{F}}}(p) > 0$, $h \in \mathcal{X}$ *and* $0 < \varepsilon < 1$. *Then there exist* $p^\star \subseteq p$, $h^\star \in \mathcal{X}$, $N_0 \in \omega$ *and* $\mathbf{F}^\star \in \mathsf{F}^{N_0,\omega}_{\mathbf{F},h-h^\star}$ *such that*

$$\mu_{(\mathbf{C})^{\mathbf{F}^\star}}(p^\star) \geq (1-\varepsilon)\cdot \mu_{(\mathbf{C})^{\mathbf{F}}}(p), \qquad [\![p^\star, \mathbf{F}^\star, h^\star]\!] \geq (1-2\varepsilon)\cdot \mu_{(\mathbf{C})^{\mathbf{F}}}(p)$$

and

$$\forall N\ \forall s \in (p^\star)^N\ [\![p^\star_s, \mathbf{F}^\star, h^\star]\!]_N > 0 .$$

Moreover, we can require that $h^\star(N) = \overline{\mathbf{s}(h)}(N) = h^-(N)$ *for* $N \geq N_0$.

Proof. Find $N_0 \in \omega$ such that

(1) $\mu_{(\mathbf{C})^{\mathbf{F}}}(p) > \frac{2\sum_{i=N_0}^\infty \epsilon_i}{\prod_{i=N_0}^\infty (1-8\epsilon_i)}$,
(2) $\prod_{i=N_0}^\infty (1-4\epsilon_i) < \varepsilon$,
(3) $\mu_{(\mathbf{C})^{\mathbf{F}}}(p)\cdot \prod_{i=N_0}^\infty (1-8\epsilon_i) - \sum_{i=N_0}^\infty \epsilon_i \geq (1-\varepsilon)\cdot \mu_{(\mathbf{C})^{\mathbf{F}}}(p)$,
(4) $h(N) > 0$ for $N \geq N_0$.

For $N \in \omega$ let m^N be the distribution on $2^{I_0\cup\ldots\cup I_{N-1}}$ defined as

$$m^N(s) = \begin{cases} 2^{-|\bigcup_{i<N} I_i|} & \text{if } s \in p^N \\ 0 & \text{otherwise}. \end{cases}$$

Note that $\overline{m^N}$ is the counting measure of p^N.

Use Theorem 5.3 to find $\mathbf{F}^\star_N \in \mathsf{F}^{N_0,N}_{\mathbf{F}\restriction N, h\restriction N - \mathbf{s}(h\restriction N)}$ and $U^\star_N \subseteq 2^{I_0\cup\ldots\cup I_{N-1}}$ such that

$$\overline{\left(m^N_{[U^\star_N]}\right)_{(\mathbf{C})^{\mathbf{F}^\star_N}}} = \frac{\left|p^N \cap U^\star_N \cap (\mathbf{C})^{\mathbf{F}^\star_N}\right|}{\left|(\mathbf{C})^{\mathbf{F}^\star_N}\right|} \geq \frac{\left|p^N \cap (\mathbf{C})^{\mathbf{F}\restriction N}\right|}{\left|(\mathbf{C})^{\mathbf{F}\restriction N}\right|}\cdot \prod_{i=N_0}^\infty (1-8\epsilon_i) - \sum_{i=N_0}^\infty \epsilon_i ,$$

and for $M_0 \in [N_0, N)$, $s \in p^N_s \restriction I_0 \cup \ldots \cup I_{M_0-1}$ and $\mathbf{G} \in \mathsf{F}^{M_0,N}_{\mathbf{F}^\star_N, \mathbf{s}(h\restriction N)}$,

$$\frac{\left|p^N_s \cap U^\star_N \cap (\mathbf{C})^{\mathbf{G}}\right|}{\left|(\mathbf{C})^{\mathbf{G}}\right|} \geq \frac{\left|p^N_s \cap U^\star_N \cap (\mathbf{C})^{\mathbf{F}^\star}\right|}{\left|(\mathbf{C})^{\mathbf{F}^\star}\right|}\cdot \prod_{i=M_0}^\infty (1-4\epsilon_i) .$$

By compactness, there exist $\mathbf{F}^\star \in \mathsf{F}^\omega$ and $U^\star \subseteq 2^{<\omega}$ such that

$$\forall N\ \exists M \geq N\ \left(\mathbf{F}^\star\restriction N = \mathbf{F}^\star_M \restriction N\ \&\ (U^\star)^N = (U^\star_M)^N\right).$$

Put $p^\star = p \cap U^\star$ and note that, by Theorem 5.3, for every $N \geq N_0$ there exists $M \geq N$ such that

$$\frac{\left|(p^\star)^N \cap (\mathbf{C})^{\mathbf{F}^\star\restriction N}\right|}{\left|(\mathbf{C})^{\mathbf{F}^\star\restriction N}\right|} = \frac{\left|(p^M \cap U^\star_M)^N \cap (\mathbf{C})^{\mathbf{F}^\star\restriction N}\right|}{\left|(\mathbf{C})^{\mathbf{F}^\star\restriction N}\right|}$$

$$= \frac{\left|(p^M \cap U^\star_M)^N \cap (\mathbf{C})^{\mathbf{F}^\star_M\restriction N}\right|}{\left|(\mathbf{C})^{\mathbf{F}^\star_M\restriction N}\right|} \geq \frac{\left|p^M \cap U^\star_M \cap (\mathbf{C})^{\mathbf{F}^\star_M}\right|}{\left|(\mathbf{C})^{\mathbf{F}^\star_M}\right|}$$

$$\geq \frac{\left|p^M \cap (\mathbf{C})^{\mathbf{F}\restriction M}\right|}{\left|(\mathbf{C})^{\mathbf{F}\restriction M}\right|} \cdot \prod_{i=N_0}^{\infty} (1 - 8\epsilon_i)$$

$$- \sum_{i=N_0}^{\infty} \epsilon_i \geq \mu_{(\mathbf{C})^{\mathbf{F}}}(p) \cdot \prod_{i=N_0}^{\infty} (1 - 8\epsilon_i)$$

$$- \sum_{i=N_0}^{\infty} \epsilon_i \geq (1 - \varepsilon) \cdot \mu_{(\mathbf{C})^{\mathbf{F}}}(p) \,.$$

It follows that

$$\mu_{(\mathbf{C})^{\mathbf{F}^\star}}(p^\star) = \lim_{N\to\infty} \frac{\left|(p^\star)^N \cap (\mathbf{C})^{\mathbf{F}^\star\restriction N}\right|}{\left|(\mathbf{C})^{\mathbf{F}^\star\restriction N}\right|} \geq (1 - \varepsilon) \cdot \mu_{(\mathbf{C})^{\mathbf{F}}}(p) \,.$$

Suppose that $s \in (p^\star)^{M_0}$ for some $M_0 \geq N_0$. As above, for $N \geq M_0$ and $\mathbf{G} \in \mathsf{F}^{M_0,N}_{\mathbf{F}^\star_N,\mathbf{s}(h\restriction N)}$, the inequality

$$\frac{\left|p^N_s \cap U^\star_N \cap (\mathbf{C})^{\mathbf{G}}\right|}{\left|(\mathbf{C})^{\mathbf{G}}\right|} \geq \frac{\left|p^N_s \cap U^\star_N \cap (\mathbf{C})^{\mathbf{F}^\star}\right|}{\left|(\mathbf{C})^{\mathbf{F}^\star}\right|} \cdot \prod_{i=M_0}^{\infty} (1 - 4\epsilon_i),$$

translates to

$$\forall \mathbf{G} \in \mathsf{F}^{M_0,\omega}_{\mathbf{F}^\star,\mathbf{s}(h)} \ \mu_{(\mathbf{C})^{\mathbf{G}}}(p^\star_s) \geq \mu_{(\mathbf{C})^{\mathbf{F}^\star}}(p^\star_s) \cdot \prod_{i=M_0}^{\infty} (1 - 4\epsilon_i)$$

$$\geq \frac{1}{\left|(C_{M_0})^{\mathbf{F}^\star(M_0)}\right|} \cdot \prod_{i=M_0}^{\infty} (1 - 4\epsilon_i) > 0 \,.$$

It follows that if $s \in (p^\star)^{M_0}$, $M_0 \geq N_0$ then for all $\mathbf{G} \in \mathsf{F}^{M_0,\omega}_{\mathbf{F}^\star,\mathbf{s}(h)}$,

$$\mu_{(\mathbf{C})^{\mathbf{G}}}(p^\star_s) \geq (1 - \varepsilon) \cdot \mu_{(\mathbf{C})^{\mathbf{F}^\star}}(p^\star_s) > 0 \,.$$

Define

$$h^\star(N) = \begin{cases} \overline{\mathbf{s}(h)}(N) & \text{if } N \geq N_0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } N \in \omega \,.$$

Suppose that $s \in (p^\star)^N$. If $N \geq N_0$ then the above estimates show that

$$[\![p^\star_s, \mathbf{F}^\star, h^\star]\!]_N \geq (1 - \varepsilon) \cdot \mu_{(\mathbf{C})^{\mathbf{F}^\star}}(p^\star_s) > 0 \,.$$

If $N < N_0$ then by Lemma 6.1(3),

$$[\![p^\star_s, \mathbf{F}^\star, h^\star]\!]_N \geq \sum_{s \subseteq t \in (p^\star)^{N_0}} [\![p^\star_t, \mathbf{F}^\star, h^\star]\!]_N$$

$$= \frac{w_N(\mathbf{F}^\star)}{w_{N_0}(\mathbf{F}^\star)} \cdot \sum_{s \subseteq t \in (p^\star)^{N_0}} [\![p^\star_t, \mathbf{F}^\star, h^\star]\!]_{N_0} > 0 \,.$$

Finally note that for $\mathbf{G} \in \mathsf{F}^{\omega}_{\mathbf{F}^\star, h^\star}$,

$$\mu_{(\mathbf{C})^{\mathbf{G}}}(p^\star) = \sum_{t \in (p^\star)^{N_0}} \mu_{(\mathbf{C})^{\mathbf{G}}}(p^\star_s) \geq \sum_{t \in (p^\star)^{N_0}} \mu_{(\mathbf{C})^{\mathbf{F}^\star}}(p^\star_s) \cdot (1-\varepsilon)$$

$$= (1-\varepsilon) \cdot \mu_{(\mathbf{C})^{\mathbf{F}^\star}}(p^\star) \geq (1-\varepsilon)^2 \cdot \mu_{(\mathbf{C})^{\mathbf{F}}}(p) \geq (1-2\varepsilon) \cdot \mu_{(\mathbf{C})^{\mathbf{F}}}(p) .$$

It follows that

$$[\![p^\star, \mathbf{F}^\star, h^\star]\!] \geq (1-2\varepsilon) \cdot \mu_{(\mathbf{C})^{\mathbf{F}}}(p) . \qquad \square$$

Theorem 6.2. *Suppose that $M_0 \in \omega$, $\varepsilon < 1$ and $\mu_{(\mathbf{C})^{\mathbf{F}}}(A) = 1$. Let $p \subseteq 2^\omega$ and $h \in \mathcal{X}$ be such that*

$$\forall N \ \forall s \in (p)^N \ [\![p, \mathbf{F}, h]\!]_N > 0 .$$

There exist $p^\star$, $h^\star \in \mathcal{X}$ and $\mathbf{F}^\star \in \mathsf{F}^{N_0,\omega}_{\mathbf{F}, h-h^\star}$ such that

(1) $p^\star \subseteq p \cap A$,
(2) $h^\star \restriction M_0 = h \restriction M_0$,
(3) $\forall N \geq M_0 \log_{\mathbf{s}}(h^\star)(N) = \log_{\mathbf{s}}(h)(N) - 1$,
(4) $\forall s \in p^\star \ [\![p^\star_s, \mathbf{F}^\star, h^\star]\!]_N > 0$,
(5) $\forall s \in (p)^{M_0} \ [\![p^\star_s, \mathbf{F}^\star, h^\star]\!]_{M_0} \geq (1-4\varepsilon) \cdot [\![p_s, \mathbf{F}, h]\!]_{M_0}$.

Proof. Let $\alpha = \min\left\{[\![p_s, \mathbf{F}, h]\!]_{M_0} : s \in (p)^{M_0}\right\}$. Fix $\varepsilon > 0$ and for every $s \in (p)^{M_0}$ find $N_0^s \geq M_0$ as in Theorem 6.1. Let $N_0 \geq \max\left\{N_0^s : s \in (p)^{M_0}\right\}$ be such that $\log_{\mathbf{s}}(h)(N_0) > 0$.

Fix an enumeration $\{s_i : 0 < i \leq \ell\}$ of $(p)^{M_0}$, and define sequences $\{\mathbf{F}_i, h_i : i \leq \ell\}$ and $\{p^\star_i : 0 < i \leq \ell\}$ such that

(1) $\mathbf{F}_0 = \mathbf{F}$, $h_0 = h$,
(2) $h_i \in \mathcal{X}$ for $i \leq \ell$,
(3) $p^\star_i \subseteq p_{s_i} \cap A$,
(4) $\mathbf{F}_{i+1} \in \mathsf{F}^{N_0,\omega}_{\mathbf{F}_i, h_i - \mathbf{s}(h_i)}$,
(5) $h_{i+1}(N) = \mathbf{s}(h_i)(N)$ for $N \geq N_0$, $i < \ell$,
(6) $\forall i \leq \ell \ \forall N < N_0 \ h_i(N) = 0$,
(7) $[\![p^\star_i, \mathbf{F}_i, h_i]\!]_{M_0} \geq (1-4\varepsilon) \cdot \mu_{(\mathbf{C})^{\mathbf{F}_i}}(p_{s_i})$,
(8) $\forall N \ \forall s \in (p^\star_i)^N \ [\![p^\star_i, \mathbf{F}_i, h_i]\!]_N > 0$.

Suppose that $\mathbf{F}^\star_i$, $h^\star_i$ are given for some $i < \ell$. Find $q_{i+1} \subseteq p_{s_{i+1}} \cap A$ such that $\mu_{(\mathbf{C})^{\mathbf{F}_i}}(q_{i+1}) \geq (1-\varepsilon)\mu_{(\mathbf{C})^{\mathbf{F}_i}}(p_{s_i})$. Let p_{i+1}, $\mathbf{F}_{i+1}$ and h_{i+1} be obtained by applying 6.1 to q_{i+1}, $\mathbf{F}_i$ and h_i. After ℓ steps we have constructed functions $\mathbf{F}_\ell$, h_ℓ and a set $p^\star = \bigcup_{i \leq \ell} p_i$. Functions $\mathbf{F}_\ell$ and $\overline{h_\ell} = h^-$ will define values of $\mathbf{F}^\star$ and $h^\star$ for $N \geq N_0$.

Define for $N \in \omega$,

$$h^\star(N) = \begin{cases} h(N) & \text{if } N < M_0 \\ h^-(N) & \text{if } M_0 \leq N \end{cases}$$

and $\mathbf{F}^\star(N) = \mathbf{F}_\ell(N)$ for $N \geq N_0$. It remains to define the values of $\mathbf{F}^\star(N)$ for $N < N_0$.

Define $\mathbf{F}^\star{\restriction}N_0$ by the following requirements:

(1) $\mathbf{F}^\star{\restriction}M_0 = \mathbf{F}{\restriction}M_0$,
(2) $\mathbf{F}^\star \in \mathsf{F}^{M_0,\omega}_{\mathbf{F},h-h^-}$,
(3) for $N < N_0$ and $s \in (p^\star)^N$,

$$p^\star_s \cap (\mathbf{C})^{\mathbf{F}^\star{\restriction}N_0} \neq \emptyset \to \left(\forall \mathbf{G} \in \mathsf{F}^{N,N_0}_{\mathbf{F}^\star{\restriction}N_0,h^\star{\restriction}N_0}\ p^\star_s \cap (\mathbf{C})^{\mathbf{G}} \neq \emptyset\right) .$$

More precisely, by induction on $N \in [M_0, N_0)$ define sequences $\left\{\mathbf{F}^N_i : i \leq v_N\right\}$ and $\left\{h^N_i : i \leq v_N\right\}$ such that

(1) $h^{M_0}_0 = h{\restriction}N_0$, $\mathbf{F}^{M_0}_0 = \mathbf{F}{\restriction}N_0$, $\mathbf{F}^{N+1}_0 = \mathbf{F}^N_{v_N}$ and $h^{N+1}_0 = h^N_{v_N}$ for $N \geq M_0$,
(2) $\forall N < N_0\ \forall i \leq v_N\ h^N_i{\restriction}N = h^N_0{\restriction}N$,
(3) $h^N_{i+1} = h^N_0{\restriction}N{}^\frown\mathbf{s}\big(h_i{\restriction}[N, N_0)\big)$ for $i \leq v_N$,
(4) $\mathbf{F}^N_{i+1} \in \mathsf{F}^{N,N_0}_{\mathbf{F}^N_i,h^N_i-h^N_{i+1}}$,
(5) if s is the ith element of $(p)^N$ then exactly one of the following two cases holds:
   (a) $\forall \mathbf{G} \in \mathsf{F}^{N,N_0}_{\mathbf{F}^N_i,h^N_i} (\mathbf{C})^{\mathbf{G}} \cap (p_s)^{N_0} \neq \emptyset$,
   (b) $(\mathbf{C})^{\mathbf{F}^N_i} \cap (p_s)^{N_0} = \emptyset$.

The construction is straightforward. If case 5(a) holds, then we define $\mathbf{F}^N_{i+1} = \mathbf{F}^N_i$, otherwise there exists $\mathbf{G} \in \mathsf{F}^{N,N_0}_{\mathbf{F}^N_i,h^N_i}$ such that $(\mathbf{C})^{\mathbf{G}} \cap (p_s)^{N_0} = \emptyset$, and we put $\mathbf{F}^N_{i+1} = \mathbf{G}$.

Observe that for $N \geq M_0$, $h^\star(N) = h^-(N) = \mathbf{s}^{(l_N)}(h)(N) = h^{N+1}_{v_N}(N)$. Therefore we can carry out this construction provided that $\log_{\mathbf{s}}(h)(N) > 0$. However, by the choice of $\mathcal{X}$, if $\log_{\mathbf{s}}(h)(N) = 0$ then $h(N) = 0$ and the required condition is automatically met.

Finally let

$$\mathbf{F}^\star(N) = \begin{cases} \mathbf{F}(N) & \text{if } N < M_0 \,, \\ \mathbf{F}^N(N) & \text{if } M_0 \leq N < N_0 \,, \\ \mathbf{F}_\ell(N) & \text{if } N \geq N_0 \,. \end{cases}$$

We will show that $p^\star$, $\mathbf{F}^\star$ and $h^\star$ have the required properties. Conditions (1)–(3) of Theorem 6.2 are obvious.

To check (5), consider $s \in (p^\star)^{M_0}$. By the choice of N_0, $p^\star$ and $\mathbf{F}_\ell$ we have

$$[\![p^\star_s, \mathbf{F}^\star, h^\star]\!]_{M_0} \geq \min\left\{\mathbf{G} \in \mathsf{F}^{M_0,N_0}_{\mathbf{F}^\star{\restriction}N_0,h^\star{\restriction}N_0} : \frac{\left|(p^\star_s)^{N_0} \cap (\mathbf{C})^{\mathbf{G}}\right|}{(\mathbf{C})^{\mathbf{G}}|}\right\} \cdot (1 - 4\varepsilon) \,,$$

$$\min\left\{\mathbf{G} \in \mathsf{F}^{M_0,N_0}_{\mathbf{F}^\star{\restriction}N_0,h^\star{\restriction}N_0} : \frac{\left|(p_s)^{N_0} \cap (\mathbf{C})^{\mathbf{G}}\right|}{(\mathbf{C})^{\mathbf{G}}|}\right\} \cdot (1 - 4\varepsilon)$$

$$\geq [\![p_s, \mathbf{F}^\star, h^\star]\!]_{M_0} \cdot (1 - 4\varepsilon) \geq [\![p_s, \mathbf{F}, h]\!]_{M_0} \cdot (1 - 4\varepsilon) \,.$$

To verify (4), we have to show that $[\![p^\star_s, \mathbf{F}^\star, h^\star]\!]_N > 0$ for $s \in (p^\star)^N$. If $N \geq N_0$ it follows from the construction of $\mathbf{F}_\ell$. If $N < N_0$ then

$$[\![p^\star_s, \mathbf{F}^\star, h^\star]\!]_N \geq (1-4\varepsilon) \cdot \min \left\{ \frac{\left|(p^\star_s)^{N_0} \cap (\mathbf{C})^{\mathbf{G}}\right|}{\left|(\mathbf{C})^{\mathbf{G}}\right|} : \mathbf{G} \in \mathsf{F}^{M_0,N_0}_{\mathbf{F}^\star \restriction N_0, h^\star \restriction N_0} \right\}.$$

By the choice of $\mathbf{F}^\star \restriction N_0$, for all $\mathbf{G} \in \mathsf{F}^{M_0,N_0}_{\mathbf{F}^\star \restriction N_0, h^\star \restriction N_0}$,

$$\frac{\left|(p^\star_s)^{N_0} \cap (\mathbf{C})^{\mathbf{G}}\right|}{\left|(\mathbf{C})^{\mathbf{G}}\right|} \neq 0\,.$$

It follows that $[\![p^\star_s, \mathbf{F}^\star, h^\star]\!]_N > 0$. □

7. Definition of $\mathcal{P}$

In this section, we will define a partial order $\mathcal{P}$ having properties (A0)–(A2) from Theorem 2.1. This will conclude the proof of Theorem 1.12.

We start by defining a partial ordering $\mathcal{Q}$ that will be used in the definition of $\mathcal{P}$.

Definition 7.1. Let $\mathcal{Q}$ be the following partial order:
$(p, \mathbf{F}, h) \in \mathcal{Q}$ if

(1) $p \in \mathsf{Perf}$, $\mathbf{F} \in \mathsf{F}^\omega$, $h \in \mathcal{X}$,
(2) $\left|\mathsf{dom}\big(\mathbf{F}(k)\big)\right| + h(k) \leq m_k$ for every k,
(3) $p \subseteq (\mathbf{C})^{\mathbf{F}}$,
(4) $\forall s \in p^N\ [\![p_s, \mathbf{F}, h]\!]_N > 0$.

For $(p^1, \mathbf{F}_1, h_1)$, $(p^2, \mathbf{F}_2, h_2) \in \mathcal{Q}$ define $(p^1, \mathbf{F}_1, h_1) \geq (p^2, \mathbf{F}_2, h_2)$ if

(1) $p^1 \subseteq p^2$,
(2) $\mathbf{F}_1 \in \mathsf{F}^\omega_{\mathbf{F}_2, h_2 - h_1}$.

To see that $\mathcal{Q}$ has the fusion property we define $\geq_n$:

Definition 7.2. For $n > 0$, define $(p^1, \mathbf{F}_1, h_1) \geq_n (p^2, \mathbf{F}_2, h_2)$ if

(1) $(p^1, \mathbf{F}_1, h_1) \geq (p^2, \mathbf{F}_2, h_2)$,
(2) $\forall s \in (p^2)^{n^\star}\, [\![p^1_s, \mathbf{F}_1, h_1]\!]_{n^\star} \geq (1 - 2^{-n-1}) \cdot [\![p^2_s, \mathbf{F}_2, h_2]\!]_{n^\star}$,
(3) $h_1 \restriction n^\star = h_2 \restriction n^\star$,
(4) $\mathbf{F}_1 \restriction n^\star = \mathbf{F}_2 \restriction n^\star$,

where $n^\star = i_{h_1}(n)$.

Note that condition (2) above implies that $(p^1)^{n^\star} = (p^2)^{n^\star}$.

Lemma 7.1. *$\mathcal{Q}$ has the fusion property.*

Proof. Suppose that $\big\{(p^k, \mathbf{F}_k, h_k) : k \in \omega\big\}$ is a sequence of conditions such that $(p^{k+1}, \mathbf{F}_{k+1}, h_{k+1}) \geq_{k+1} (p^k, \mathbf{F}_k, h_k)$ for each k. Let $n^\star(k) = i_{h_{k+1}}(k)$. Note that $\lim_{k\to\infty} n^\star(k) = \infty$. Define

(1) $h = \bigcup_{k\in\omega} h_k \restriction n^\star(k)$,
(2) $\mathbf{F} = \bigcup_{k\in\omega} \mathbf{F}_k \restriction n^\star(k)$,
(3) $p = \bigcup_{k\in\omega} (p^k)^{n^\star(k)}$.

Observe that h, $\mathbf{F}$ and p are well defined.

Suppose that $s \in p^{n^\star(k_0)}$, $\mathbf{G} \in \mathsf{F}^{N,\omega}_{\mathbf{F},h}$ and $k \geq k_0$, and note that

$$\frac{\left|(p_s)^{n^\star(k)} \cap (\mathbf{C})^{\mathbf{G}\restriction n^\star(k)}\right|}{\left|(\mathbf{C})^{\mathbf{G}\restriction n^\star(k)}\right|} = \frac{\left|(p^k_s)^{n^\star(k)} \cap (\mathbf{C})^{\mathbf{G}\restriction n^\star(k)}\right|}{\left|(\mathbf{C})^{\mathbf{G}\restriction n^\star(k)}\right|} \geq [\![p^k_s, \mathbf{F}_k, h_k]\!] .$$

Therefore $\mu_{(\mathbf{C})^{\mathbf{G}}}(p_s) \geq \inf_k [\![p^k_s, \mathbf{F}_k, h_k]\!]$. Hence,

$$[\![p_s, \mathbf{F}, h]\!]_{n^\star(k_0)} \geq [\![p^{k_0}_s, \mathbf{F}_{k_0}, h_{k_0}]\!]_{n^\star(k_0)} \cdot \prod_{k>k_0} \left(1 - \frac{1}{2^{k+1}}\right)$$

$$\geq \left(1 - \frac{1}{2^{k_0+1}}\right) \cdot [\![p^{k_0}_s, \mathbf{F}_{k_0}, h_{k_0}]\!]_{n^\star(k_0)} > 0 .$$

The same computation shows that $(p, \mathbf{F}, h) \geq_k (p^k, \mathbf{F}_k, h_k)$. □

Theorem 7.1. *Suppose that $(p, \mathbf{F}, h) \in \mathcal{Q}$.*

If $q \subseteq p$ and $\mu_{(\mathbf{C})^{\mathbf{F}}}(q) > 0$, then there exist $q^\star \subseteq q$, $\mathbf{F}^\star$ and $h^\star \in \mathcal{X}$ such that $(q^\star, \mathbf{F}^\star, h^\star) \in \mathcal{Q}$ and $(q^\star, \mathbf{F}^\star, h^\star) \geq (p, \mathbf{F}, h)$.

If $n \in \omega$ and $A \subseteq p$ is such that $\mu_{(\mathbf{C})^{\mathbf{F}}}(A) = 1$ then there exist $q^\star \subseteq p \cap A$, $\mathbf{F}^\star$ and $h^\star \in \mathcal{X}$ such that $(q^\star, \mathbf{F}^\star, h^\star) \in \mathcal{Q}$ and $(q^\star, \mathbf{F}^\star, h^\star) \geq_n (p, \mathbf{F}, h)$.

Proof. The first part follows from Theorem 6.1 and the second from Theorem 6.2. □

The following theorem shows that $\mathcal{Q}$ satisfies condition (A2) defined in Sec. 2.

Theorem 7.2. *For every $(p, \mathbf{F}, h) \in \mathcal{Q}$, $n \in \omega$, $X \in [2^\omega]^{\leq\aleph_0}$, and $\mathbf{t} \in \mathsf{Perf}$ such that $\mu(\mathbf{t}) > 0$,*

$$\mu(\{z \in 2^\omega : \exists (q, \mathbf{G}, f) \geq_n (p, \mathbf{F}, h) X \cup (q + \mathbb{Q}) \subseteq \mathbf{t} + \mathbb{Q} + z\}) = 1 .$$

Proof. Suppose that $(p, \mathbf{F}, h) \in \mathcal{Q}$ and $\mathbf{t}$ is a perfect set of positive measure. □

We will need the following observation:

Lemma 7.2.

$$\mu\left(\left\{z \in 2^\omega : \mu_{(\mathbf{C})^{\mathbf{F}}}\big(p \cap (\mathbf{t} + z)\big) > 0\right\}\right) > 0 .$$

Proof. Consider the space $p \times 2^\omega$ equipped with the product measure $(\mu_{(\mathbf{C})^{\mathbf{F}}} \restriction p) \times \mu$. Let $Z = \{(x, z) \in p \times 2^\omega : z \in \mathbf{t} + x\}$. Note that $\mu\big((Z)_x\big) = \mu(\mathbf{t} + x) = \mu(\mathbf{t}) > 0$ for each x. By the Fubini theorem

$$\left\{z : \mu_{(\mathbf{C})^{\mathbf{F}}}\big((Z)^z\big) > 0\right\}$$

has positive measure. But

$$(Z)^z = \{x \in p : z \in \mathbf{t} + x\} = \{x \in p : x \in \mathbf{t} + z\} = p \cap (\mathbf{t} + z)\,. \qquad \square$$

Let $X \subseteq 2^\omega$ be a countable set. Put $Z_X = \{z \in 2^\omega : X \subseteq \mathbf{t} + \mathbb{Q} + z\}$. Note that Z_X has measure one. Thus, without loss of generality, we can assume that $X = \emptyset$.

For each $s \in p$ let

$$Z_s = \left\{z \in 2^\omega : \mu_{(\mathbf{C})^{\mathbf{F}}}\left(p_s \cap (\mathbf{t} + z)\right) > 0\right\}\,.$$

By the lemma, $\mu(Z_s) > 0$ for each s. Let $Z = \bigcap_{s \in p}(Z_s + \mathbb{Q})$. This is the measure one set we are looking for.

Fix $z \in Z$ and $n \in \omega$. Note that $\mu_{(\mathbf{C})^{\mathbf{F}}}(\mathbf{t} + \mathbb{Q} + z) = 1$ and apply Theorem 7.1.

Definition 7.3. Let $\mathcal{P} \subseteq \mathcal{Q} \times \mathcal{Q}$ be the collection of elements $\left((p^1, \mathbf{F}_1, h), (p^2, \mathbf{F}_2, h)\right)$ such that

(1) $\forall k \; \mathsf{dom}\left(\mathbf{F}_1(k)\right) = \mathsf{dom}\left(\mathbf{F}_2(k)\right)$,
(2) $\forall k \; \forall s \in \mathsf{dom}\left(\mathbf{F}_1(k)\right) \; \Big(\mathbf{F}_1(k)(s) = 1 \text{ or } \mathbf{F}_2(k)(s) = 1\Big)$.

For $\left((p^1, \mathbf{F}_1, h_1),\ (q_1, \mathbf{G}_1, h_1)\right),\ \left((p^2, \mathbf{F}_2, h_2),\ (q_2, \mathbf{G}_2, h_2)\right) \in \mathcal{P}$ and $n \in \omega$ define $\left((p^1, \mathbf{F}_1, h_1),\ (q_1, \mathbf{G}_1, h_1)\right) \geq_n \left((p^2, \mathbf{F}_2, h_2),\ (q_2, \mathbf{G}_2, h_2)\right)$ if $(p^1, \mathbf{F}_1, h_1) \geq_n (p^2, \mathbf{F}_2, h_2)$ and $(q_1, \mathbf{G}_1, h_1) \geq_n (q_2, \mathbf{G}_2, h_2)$.

Strictly speaking, the partial order used in the proof of Theorem 2.1 was a subset of $\mathsf{Perf} \times \mathsf{Perf}$ while $\mathcal{P}$ defined above has more complicated structure. Nevertheless it is easy to see that it makes no difference in the proof of Theorem 2.1 as conditions (A1) and (A2) refer only to the first coordinate of $\mathcal{P}$.

Lemma 7.3. *$\mathcal{P}$ has the fusion property.*

Proof. Follows immediately from the definition of $\mathcal{P}$ and Lemma 7.1. $\square$

Next we show that $\mathcal{P}$ satisfies (A1).

Lemma 7.4. *For every $\mathbf{p} \in \mathcal{P}$, $n \in \omega$ and $z \in 2^\omega$ there exists $\mathbf{q} \geq_n \mathbf{p}$ such that $q_1 \subseteq H + z$ or $q_2 \subseteq H + z$.*

Proof. Suppose that $\left((p^1, \mathbf{F}_1, h), (p^2, \mathbf{F}_2, h)\right) \in \mathcal{P}$ and $z \in 2^\omega$.

Case 1. There exist infinitely many k such that $z{\restriction}I_k \in \mathsf{dom}\left(\mathbf{F}_1(k)\right)$.

It follows from the definition of $\mathcal{P}$ that in this case there exists $i \in \{1, 2\}$ and infinitely many k such that $\mathbf{F}_i(k)(z{\restriction}I_k) = 1$. In particular, since $p^i \subseteq (\mathbf{C})^{\mathbf{F}_i}$, for every $x \in p^i$,

$$\exists^\infty k \; x{\restriction}I_k \notin C_k + z{\restriction}I_k.$$

Thus, $p^i \subseteq H + z$.

Case 2. $z \restriction I_k \in \mathsf{dom}\big(\mathbf{F}_1(k)\big)$ for finitely many k.

Let $n^\star = i_h(n)$. Define for $k \in \omega$, and $i = 1, 2$

$$\mathbf{G}_i(k) = \begin{cases} \mathbf{F}_i(k) & \text{if } k \leq n^\star\,, \\ \mathbf{F}_i(k) \cup (z \restriction I_k, 0) & \text{if } k > n^\star\,, \end{cases}$$

$q_i = p^i \cap (\mathbf{C})^{\mathbf{G}_i}$ and

$$f(k) = \begin{cases} h(k) & \text{if } k \leq n^\star\,, \\ \mathbf{s}\big(h(k), k\big) & \text{if } k > n^\star\,. \end{cases}$$

Clearly $\big((q_1, \mathbf{G}_1, f), (q_2, \mathbf{G}_2, f)\big) \geq_n \big((p^1, \mathbf{F}_1, h), (p^2, \mathbf{F}_2, h)\big)$ and the same argument as in the first case shows that it has the required properties. □

Next we show that $\mathcal{P}$ satisfies (A2).

Theorem 7.3. *For every* $\mathbf{p} \in \mathcal{P}$, $n \in \omega$, $X \in [2^\omega]^{\leq \aleph_0}$, $i = 1, 2$ *and* $\mathbf{t} \in \mathsf{Perf}$ *such that* $\mu(\mathbf{t}) > 0$,

$$\mu\Big(\big\{z \in 2^\omega : \exists \mathbf{q} \geq_n \mathbf{p} X \cup (q_i + \mathbb{Q}) \subseteq \mathbf{t} + \mathbb{Q} + z\big\}\Big) = 1\,.$$

Proof. Suppose that $\big((p^1, \mathbf{F}_1, h), (p^2, \mathbf{F}_2, h)\big) \in \mathcal{P}$, $n \in \omega$, $X \subseteq 2^\omega$ is a countable set, and $\mathbf{t}$ is a perfect set of positive measure. Without loss of generality we can assume that $i = 1$. Consider the set

$$Z = \big\{z \in 2^\omega : \exists (q, \mathbf{G}, f) \geq_n (p^1, \mathbf{F}_1, h) X \cup (q + \mathbb{Q}) \subseteq \mathbf{t} + \mathbb{Q} + z\big\}\,.$$

By Theorem 7.2, $\mu(Z) = 1$. Fix $z \in Z$ and let $(p', \mathbf{F}'_1, h') \geq_n (p^1, \mathbf{F}_1, h)$ be such that $p' + \mathbb{Q} \subseteq \mathbf{t} + \mathbb{Q} + z$. Now define $\mathbf{F}'_2$ by putting $\mathbf{F}'_2(s) = 1$ for every $s \in \mathsf{dom}(\mathbf{F}'_1) \setminus \mathsf{dom}(\mathbf{F}_2)$. Clearly, $\big((q, \mathbf{F}'_1, h'), (p^2, \mathbf{F}'_2, h')\big)$ is the condition we are looking for. □

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