Result.Math. 54 (2009), 53–64

1422-6383/010053-12, *published online* July 20, 2009
DOI 10.1007/s00025-009-0382-0

Results in Mathematics

$\aleph_n$-Free Modules with Trivial Duals

Rüdiger Göbel and Saharon Shelah

Abstract. In the first part of this paper we introduce a simplified version of a new Black Box from Shelah [11] which can be used to construct complicated $\aleph_n$-free abelian groups for any natural number $n \in \mathbb{N}$. In the second part we apply this prediction principle to derive for many commutative rings R the existence of $\aleph_n$-free R-modules M with trivial dual $M^* = 0$, where $M^* = \mathrm{Hom}(M, R)$. The minimal size of the $\aleph_n$-free abelian groups constructed below is $\beth_n$, and this lower bound is also necessary as can be seen immediately if we apply GCH.

Mathematics Subject Classification (2000). Primary 13C05, 13C10, 13C13, 20K20, 20K25, 20K30; Secondary 03E05, 03E35.

Keywords. Prediction principles, almost free abelian groups, dual groups.

1. Introduction

The existence of almost free R-modules M over countable principal ideal domains (but not fields) with trivial dual $M^* := \mathrm{Hom}(M, R) = 0$ is a well-known fact by results using strong prediction principles like diamonds in $V = L$. Only note that any (non-trivial) R-module with endomorphism ring $\mathrm{End}\, M = R$ is such an example. For every regular cardinal $\kappa > |R|$ (which is not weakly compact) we can find (strongly) κ-free R-modules M of size κ with $\mathrm{End}\, M = R$. But from the singular compactness theorem follows, that such modules M do not exist for singular cardinals, see e.g. [4] or [8]. Thus we want to get rid of additional set theoretic restrictions and work exclusively with ZFC:

If we restrict to $\aleph_1$-free R-modules (meaning that all countable submodules are free) and do not care about the size of M, then we have an abundance of such modules and those of minimal cardinal $2^{\aleph_0}$ can be constructed by applications of

---

This is GbSh 920 in the second author's list of publications.
The collaboration was supported by an NSF-Grant DMS-0100794 and by the project No. I-963-98.6/2007 of the German-Israeli Foundation for Scientific Research & Development and the Minerva Foundation.

the (ordinary) Black Box, see for example [8]. However, it is much harder to find examples like this of size $\aleph_1$ (recall that $\aleph_1$ may be much smaller than $2^{\aleph_0}$ and hence we do not have that many possible extension of a module to eliminate unwanted homomorphisms). A first example of an $\aleph_1$-free R-module M of size $\aleph_1$ with trivial dual was given in Eda [3]. Some years later we improved this result showing the existence of such modules with endomorphism ring R, see [2,7] or [8]. If we want to replace $\aleph_1$ by $\aleph_2$ or any higher cardinal, then we necessarily encounter additional set theoretic restriction, see [6]. If we require only the existence of indecomposable abelian groups, then their κ-freeness is restricted to small cardinals; see [10]. Thus, in order to construct $\aleph_n$-free R-modules M with trivial dual we must relax the restriction on the size of M. Clearly (note that GCH is not excluded, in which case $\aleph_n = \beth_n$), the size of M must be at least $\beth_n$. These cardinals are defined inductively as $\beth_0 = 2^{\aleph_0}$ and $\beth_{n+1} = 2^{\beth_n}$; see Jech [9]. Using a recent refined Black Box from Shelah [11] which takes care of additional freeness of the module, we can give a reasonably short proof of the existence of $\aleph_n$-free R-modules M with trivial dual $M^* = 0$ of cardinality $|M| = \beth_n$ for any natural number n; see Theorem 4.3 and Corollary 4.4. It remains an open question if we can go any further and pass $\aleph_\omega$ or possibly replace $M^* = 0$ by $\operatorname{End} M = R$. In this context it is also worthwhile to recall (from [10]) that there are models of ZFC in which $\aleph_{\omega^2+1}$-free implies $\aleph_{\omega^2+2}$-free.

We would like to thank Daniel Herden for several very useful suggestions and some corrections.

2. The combinatorial Black Box for $\overline{\lambda}$

The new Black Box depends on a finite sequence of cardinals satisfying some cardinal conditions. Thus let $k_* < \omega$ and $\overline{\lambda} = \langle \lambda_1, \ldots, \lambda_{k_*} \rangle$ be a sequence of cardinals such that for $\chi_l := \lambda_l^{\aleph_0}$ $(l \leq k_*)$ the following $\blacksquare$-*conditions* holds.

$$\chi_{l+1}^{\chi_l} = \chi_{l+1} \ (l < k_*)\,. \tag{2.1}$$

We will also say that $\overline{\lambda}$ is a $\blacksquare$-sequence and note that $\chi_l = \chi_l^{\aleph_0} < \chi_{l+1}$.

Condition (2.1) is used to enumerate all maps which we want to predict before constructing the modules. If λ is any cardinal, then we can define inductively a $\overline{\lambda}$-sequence: Let $\chi_1 = \lambda^{\aleph_0}$ and if λ_l is defined for $l < k_*$, then choose a suitable $\lambda_{l+1} > \lambda_l$ with (2.1), e.g. put $\lambda_{l+1} = \chi_l^{\chi_l}$. The sequence $\langle \beth_1, \ldots, \beth_{k_*} \rangle$ is an example of such a $\blacksquare$-sequence.

If λ is a cardinal, then ${}^{\omega\uparrow}\lambda$ will denote all *order preserving* maps $\eta : \omega \to \lambda$ (which we also call *infinite branches*) on λ, while ${}^{\omega\uparrow>}\lambda$ denotes the family of all order preserving *finite branches* $\eta : n \to \lambda$ on λ, where the natural number n, λ and ω (the first infinite ordinal) are considered as sets, e.g. $n = \{0, \ldots, n-1\}$, thus the finite branch η has length n.

For the sake of generality we first consider any sequence $\overline{\lambda} = \langle \lambda_1, \ldots, \lambda_{k_*} \rangle$ of cardinals such that $\lambda_l^{\aleph_0} = \chi_l$ $(l \leq k_*)$ (which later will be strengthened to a

■-sequence). Moreover, we associate with $\overline{\lambda}$ two sets Λ and Λ_*. Thus let

$$\Lambda = {}^{\omega\uparrow}\lambda_1 \times \cdots \times {}^{\omega\uparrow}\lambda_{k_*}\,.$$

For the second set we replace the m-th (and only the m-th) coordinate ${}^{\omega\uparrow}\lambda_m$ by the finite branches ${}^{\omega\uparrow>}\lambda_m$, thus

$$\Lambda_m = {}^{\omega\uparrow}\lambda_1 \times \cdots \times {}^{\omega\uparrow>}\lambda_m \times \cdots \times {}^{\omega\uparrow}\lambda_{k_*} \quad \text{for} \quad m \le k_* \quad \text{and let} \quad \Lambda_* = \bigcup_{m\le k_*} \Lambda_m\,. \tag{2.2}$$

The elements of Λ, Λ_* will be written as sequences $\overline{\eta} = (\eta_1, \ldots, \eta_{k_*})$ with $\eta_l \in {}^{\omega\uparrow}\lambda$ or $\eta_l \in {}^{\omega\uparrow>}\lambda$, respectively. Using these $\overline{\eta}$s as support of elements of the module will make enough room for linear independence which will then give $\aleph_n$-freeness.

With each member of Λ we can associate a subset of Λ_*:

Definition 2.1. If $\overline{\eta} = (\eta_1, \ldots, \eta_{k_*}) \in \Lambda$ and $m \le k_*, n < \omega$, then let $\overline{\eta}\upharpoonleft\langle m, n\rangle$ be the following element in Λ_m (thus in Λ_*)

$$\big(\overline{\eta}\upharpoonleft\langle m,n\rangle\big)_l = \begin{cases} \eta_l & \text{if} \quad m \ne l \le k_* \\ \eta_m \upharpoonright n & \text{if} \quad l = m\,. \end{cases}$$

We associate with $\overline{\eta}$ its *support* $[\overline{\eta}] = \{\overline{\eta}\upharpoonleft\langle m,n\rangle \mid m \le k_*, n < \omega\}$ which is a countable subset of Λ_*. Similarly, for $m \le k_*$ also let $[\overline{\eta}\upharpoonleft m] = \{\overline{\eta}\upharpoonleft\langle m,n\rangle \mid n < \omega\} \subseteq [\overline{\eta}]$.

Definition 2.2. Let $\overline{C} = \langle C_1, \ldots, C_{k_*}\rangle$ be a sequence of sets C_m satisfying $|\,C_m\,| \le \chi_m$ for all $m \le k_*$. We let $C = \bigcup_{m\le k_*} C_m$ and define a *set-trap (for $\Lambda, \overline{C}$)* as a map $\varphi_{\overline{\eta}} : [\overline{\eta}] \to C$ with a label $\overline{\eta} \in \Lambda$.

The following lemma will be used for the inductive proof of our next theorem.

Lemma 2.3. *Let λ be an infinite cardinal, $\chi = \lambda^{\aleph_0}$ and $\mathfrak{P}$ a set of size $|\,\mathfrak{P}\,| = \chi$. Then there is a sequence $\langle \Phi_\eta \mid \eta \in {}^{\omega\uparrow}\lambda\rangle$ such that*

(a) $\Phi_\eta = \langle \Phi_{\eta n} \mid n < \omega\rangle$, *with* $\Phi_{\eta n} \in \mathfrak{P}$,
(b) *If $\overline{f} = \{f_\nu \mid f_\nu \in \mathfrak{P}, \nu \in {}^{\omega\uparrow>}\lambda\}$, $\alpha \in \lambda$ and $\rho \in {}^{\omega\uparrow>}\lambda$, then there is $\eta \in {}^{\omega\uparrow}\lambda$ such that $0\eta = \alpha$, $\rho \subset \eta$ and $\Phi_{\eta n} = f_{\eta\upharpoonright n}$ for all $n < \omega$.*

Proof. Since $|\,\mathfrak{P}\,| = \chi = \lambda^{\aleph_0} = |\,{}^{\omega}\lambda\,|$, we can fix an embedding

$$\pi : \mathfrak{P} \hookrightarrow {}^{\omega}\lambda\,.$$

And since $|\,{}^{\omega>}\lambda\,| = \lambda$ there is also a list ${}^{\omega>}\lambda = \langle \mu_\alpha \mid \alpha < \lambda\rangle$ with enough repetitions for each $\eta \in^{\omega>} \lambda$:

$$\{\alpha < \lambda \mid \mu_\alpha = \eta\} \subseteq \lambda \quad \text{is unbounded}\,.$$

Moreover we define for each $n < \omega$ a coding map

$$\pi_n : {}^{n}\mathfrak{P} \longrightarrow {}^{n^2}\lambda \subseteq {}^{\omega>}\lambda$$
$$\overline{\varphi} = \langle \varphi_0, \ldots, \varphi_{n-1}\rangle \mapsto \overline{\varphi}\pi_n = (\varphi_0\pi\upharpoonright n)^\wedge \ldots {}^\wedge(\varphi_{n-1}\pi\upharpoonright n)\,.$$

Finally let $X \subseteq {}^{\omega\uparrow}\lambda$ be the collection of all order preserving maps $\eta : \omega \longrightarrow \lambda$ such that the following holds.

$$\exists\ \overline{\varphi} = \langle \varphi_i \mid i < \omega \rangle \in {}^{\omega}\mathfrak{P} \quad \text{and} \quad \exists\ \kappa < \omega$$
$$\text{with} \quad (\overline{\varphi} \restriction n)\pi_n = \mu_{n\eta} \quad \text{for all} \quad n > k\,. \tag{2.3}$$

By definition of π_n it follows that $\overline{\varphi}$ is uniquely determined by (2.3). (Just note that, $\mu_{n\eta}$ determines $\varphi_m \pi \restriction n$ for all $n > k, m$.)

We now prove the two statements of the lemma. For (a) we consider any $\eta \in {}^{\omega\uparrow}\lambda$. If $\eta \notin X$, then we can choose arbitrary members $\Phi_{\eta n} \in \mathfrak{P}$, and if $\eta \in X$, then choose the uniquely determined sequence $\overline{\varphi}$ from (2.3) and let $\Phi_{\eta n} = \varphi_n$, so $\Phi_\eta = \overline{\varphi}$.

For (b) we consider some $\overline{f} = \{f_\nu \mid f_\nu \in \mathfrak{P}, \nu \in {}^{\omega\uparrow>}\lambda\}$ and $\rho \in {}^{\omega\uparrow>}\lambda$. In this case we must define an extension $\eta = \langle \alpha_n \mid n < \omega \rangle \in {}^{\omega\uparrow}\lambda$ of ρ. Thus put $0\eta = \alpha$, $\alpha_n = n\rho$ for $n < \lg(\rho)$. And if $n \geq \lg(\rho)$, then using the above and that the list of μ_αs is unbounded, we can choose inductively $\alpha_n > \alpha_{n-1}$ with $\langle f_{\eta \restriction m} \mid m < n \rangle \pi_n = \mu_{\alpha_n}$.

Finally we check statement (b). Using (2.3) it will follow that the sequence η belongs to X:

If $\overline{\varphi} = \langle f_{\eta \restriction i} \mid i < \omega \rangle \in {}^{\omega}\mathfrak{P}$ and $k = \lg(\rho)$, then we have

$$(\overline{\varphi} \restriction n)\pi_n = \langle f_{\eta \restriction m} \mid m < n \rangle \pi_n = \mu_{\alpha_n} = \mu_{n\eta} \quad \text{for all} \quad n > k$$

and $\Phi_{n\eta} = \varphi_n = f_{\eta \restriction n}$ for all $n < \omega$ is immediate. □

The $\overline{\lambda}$-Black Box 2.4. *Let $\langle \lambda_1, \ldots, \lambda_{k_*} \rangle$ be a ■-sequence satisfying* (2.1), Λ, Λ_* *as above and C as in Definition* 2.2. *Then there is a family of set-traps $\langle \varphi_{\overline{\eta}} \mid \overline{\eta} \in \Lambda \rangle$ satisfying the following*

PREDICTION PRINCIPLE: *If $\varphi : \Lambda_* \to C$ is any map with the trap-condition $\Lambda_m \varphi \subseteq C_m$ $(m \leq k_*)$ and $\alpha \in \lambda_{k_*}$, then for some $\overline{\eta} \in \Lambda$ there is a set-trap $\varphi_{\overline{\eta}}$ with $\varphi_{\overline{\eta}} \subseteq \varphi$ and $0\eta_{k_*} = \alpha$.*

Proof. The proof of Theorem 2.4 will follow by induction on k_*. Temporarily we will attach parameters k_* to the above symbols like $\Lambda^{k_*}, \overline{C}^{k_*}, \varphi^{k_*}_{\overline{\eta}}, \overline{\eta}^{k_*}, \ldots$.

The first step is $k_* = 1$. In this case the claim is a special case of Lemma 2.3. Indeed, we have $\Lambda^{k_*} = {}^{\omega\uparrow}\lambda_{k_*}$ and $\Lambda^{k_*}_* = {}^{\omega\uparrow>}\lambda_{k_*}$ and $\overline{\eta}\upharpoonleft\langle m, n \rangle = \eta_{k_*} \restriction n$ holds. We put $\mathfrak{P} = C^{k_*} = C^{k_*}_{k_*}$ and note that $|\mathfrak{P}| \leq \chi_{k_*} = \chi^{\aleph_0}_{k_*}$. The trap functions $\varphi_{\overline{\eta}}$ are defined by

$$\big(\overline{\eta}\upharpoonleft\langle m, n \rangle\big)\varphi_{\overline{\eta}} = \Phi_{\eta_{k_*} n}$$

and with $f_\nu = \nu\varphi$ condition (b) of Lemma 2.3 reads as

$$\big(\overline{\eta}\upharpoonleft\langle m, n \rangle\big)\varphi_{\overline{\eta}} = \Phi_{\eta_{k_*} n} = f_{\eta_{k_*} \restriction n} = \big(\overline{\eta}\upharpoonleft\langle m, n \rangle\big)\varphi$$

and the prediction principle in Theorem 2.4 is clear.

The induction step $k_* = k + 1$:

Suppose that Theorem 2.4 is shown for k. We must find a family of traps $\{\varphi^{k_*}_{\overline{\eta}} \mid \overline{\eta} \in \Lambda^{k_*}\}$ for $\Lambda^{k_*}, \overline{C}^{k_*}$ and verify the prediction principle in Theorem 2.4. By induction hypothesis there is such a family $\{\varphi^{k}_{\overline{\eta}} \mid \overline{\eta} \in \Lambda^{k}\}$ for $\Lambda^{k}, \overline{C}^{k}$.

Let $\chi_{k_*} = \chi$, $\lambda_{k_*} = \lambda$ and recall that $\chi = \lambda^{\aleph_0}$. Moreover, by assumption $\left| C^{k_*} \right| \leq \chi$. We now consider $\mathfrak{P} = \operatorname{map}(\Lambda^k, C^{k_*}_{k_*})$ which has size $|\mathfrak{P}| = |C^{k_*}_{k_*}|^{|\Lambda^k|} \leq \chi^{\chi_k} = \chi$ by condition (2.1) of the $\blacksquare$-sequence.

If $\overline{\eta} \in \Lambda^{k_*}$, then let $\varphi^{k_*}_{\overline{\eta}} : [\overline{\eta}] \to C^{k_*}$ be the following map in (2.4). Recall that $\overline{\eta} = \langle \eta_1, \ldots, \eta_{k_*} \rangle$ and thus $\eta_{k_*} \in {}^{\omega\uparrow}\lambda$ and $\Phi_{\eta_{k_*} n} \in \operatorname{map}(\Lambda^k, C^{k_*}_{k_*})$ by Lemma 2.3. If $\overline{\eta}' = \langle \eta_1, \ldots, \eta_k \rangle \in \Lambda^k$, then $\overline{\eta}' \Phi_{\eta_{k_*} n}$ is a well-defined element of $C^{k_*}_{k_*}$ and $\varphi^{k}_{\overline{\eta}'}$ is given by induction hypothesis. We can now define

$$\big(\overline{\eta} \upharpoonleft \langle m, n \rangle\big) \varphi^{k_*}_{\overline{\eta}} = \begin{cases} \overline{\eta}' \Phi_{\eta_{k_*} n} & \text{if} \quad m = k_* \\ \big(\overline{\eta}' \upharpoonleft \langle m, n \rangle\big) \varphi^{k}_{\overline{\eta}'} & \text{if} \quad m < k_* \,. \end{cases} \tag{2.4}$$

In order to show the prediction principle we consider an arbitrary map $\varphi : \Lambda^{k_*}_* \to C^{k_*}$ satisfying the trap-condition $\Lambda_m \varphi \subseteq C_m$ for all $m \leq k_*$. We want to find $\overline{\eta} \in \Lambda^k$ and $\mu \in {}^{\omega\uparrow}\lambda$ such that $\overline{\eta}^* = \overline{\eta}^\wedge \langle \mu \rangle \in \Lambda^{k_*}$ satisfies $\varphi \restriction [\overline{\eta}^*] = \varphi^{k_*}_{\overline{\eta}^*}$ and $0\eta^*_{k_*} = \alpha$ (which is the claim of Theorem 2.4).

First we search for μ and define for each $\nu \in {}^{\omega\uparrow>}\lambda$ a map $f_\nu : \Lambda^k \to C^{k_*}_{k_*}$ from $\mathfrak{P}$ depending on φ. If $\overline{\eta} \in \Lambda^k$, then $\overline{\eta}^\wedge \langle \nu \rangle \in \Lambda^{k_*}_*$, thus

$$\overline{\eta} f_\nu := \big(\overline{\eta}^\wedge \langle \nu \rangle\big) \varphi \tag{2.5}$$

is well-defined. By the Lemma 2.3 we find $\mu \in {}^{\omega\uparrow}\lambda$ such that

$$0\mu = \alpha \quad \text{and} \quad f_{\mu \restriction n} = \Phi_{\mu n} : \Lambda^k \to C^{k_*}_{k_*} \quad \text{for all} \quad n \in \omega \,.$$

By Lemma 2.3(b), (2.4) and (2.5) we have for any $\overline{\eta}$ and $\overline{\eta}^* = \overline{\eta}^\wedge \langle \mu \rangle$ that

$$\big(\overline{\eta}^* \upharpoonleft \langle k_*, n \rangle\big) \varphi^{k_*}_{\overline{\eta}^*} = \overline{\eta} \Phi_{\mu n} = \overline{\eta} f_{\mu \restriction n} = \big(\overline{\eta}^\wedge \langle \mu \restriction n \rangle\big) \varphi = \big(\overline{\eta}^* \upharpoonleft \langle k_*, n \rangle\big) \varphi$$

and $0\mu = \alpha$ which is the prediction as required for $m = k_*$.

Now we consider the case when $m < k_*$ and define a map $\varphi' : \Lambda^k_* \to C^k$ depending on φ and μ. If $\overline{\eta}' \in \Lambda^k_*$, then $\overline{\eta}'^\wedge \langle \mu \rangle \in \Lambda^{k_*}_*$ (because $\mu \in {}^{\omega\uparrow}\lambda$), thus

$$\overline{\eta}' \varphi' := \big(\overline{\eta}'^\wedge \langle \mu \rangle\big) \varphi$$

is well-defined and by induction hypothesis on the traps $\varphi^k_{\overline{\eta}'}$ there is some $\overline{\eta} \in \Lambda^k$ such that

$$\big(\overline{\eta} \upharpoonleft \langle m, n \rangle\big) \varphi^k_{\overline{\eta}} = \big(\overline{\eta} \upharpoonleft \langle m, n \rangle\big) \varphi' \quad \text{for} \quad m \leq k \quad \text{and} \quad n < \omega \,.$$

Now let $\overline{\eta}^* = \overline{\eta}^\wedge \langle \mu \rangle \in \Lambda^{k_*}$. By the last displayed equation and (2.4) we have for $m < k_*$ that

$$\big(\overline{\eta}^* \upharpoonleft \langle m, n \rangle\big) \varphi^{k_*}_{\overline{\eta}^*} = \big(\overline{\eta} \upharpoonleft \langle m, n \rangle\big) \varphi^{k}_{\overline{\eta}} = \big(\overline{\eta} \upharpoonleft \langle m, n \rangle\big) \varphi' = \big(\overline{\eta}^* \upharpoonleft \langle m, n \rangle\big) \varphi \,.$$

Thus $\varphi^{k_*}_{\overline{\eta}^*}$ predicts φ with $0\eta^*_{k_*} = 0\mu = \alpha$ as suggested above. $\square$

Definition 2.5. Let $F : \Lambda \to \Lambda_*$ be a given map. A subset $\Omega \subseteq \Lambda$ is *free* (with respect to F) if there is an enumeration $\langle \overline{\eta}^\alpha \mid \alpha < \alpha_* \rangle$ of Ω (we write $\Omega_\alpha = \{\overline{\eta}^\beta \mid \beta < \alpha\}$) and there are $\ell_\alpha \leq k_*, n_\alpha < \omega$ $(\alpha < \alpha_*)$ such that for $\alpha < \alpha_*$ and $n_\alpha \leq n$

$$\overline{\eta}^\alpha \restriction \langle \ell_\alpha, n\rangle \notin \left\{\overline{\eta}^\beta \restriction \langle \ell_\alpha, n\rangle \mid \beta < \alpha\right\} \cup \Omega_\alpha F \,.$$

Moreover, Ω is κ-free (with respect to F) for some cardinal κ if the above holds for all subsets of Ω of cardinality $< \kappa$.

This is to say, that every newly chosen element $\overline{\eta}^\alpha$ picks up some unused element from Λ_* in its support. Note that the enumeration of Ω in Definition 2.5 does not permit repetitions. We want to show the following

Freeness-Proposition 2.6. *With the notions from Theorem* 2.4 *and Definition* 2.5 *the set* Λ *is* $\aleph_{k_*}$*-free with respect to any function* $F : \Lambda \to \Lambda_*$*. For any* $k < k_*$*,* $\Omega \subseteq \Lambda$ *of cardinality* $|\,\Omega\,| \leq \aleph_k$ *and* $\langle u_{\overline{\eta}} \subseteq \{1, \dots, k_*\} \mid |\,u_{\overline{\eta}}\,| > k, \overline{\eta} \in \Omega\rangle$ *we can find an enumeration* $\langle \overline{\eta}^\alpha \mid \alpha < \aleph_k\rangle$ *of* Ω*,* $\ell_\alpha \in u_{\overline{\eta}^\alpha}$ *and* $n_\alpha < \omega$ $(\alpha < \aleph_k)$ *such that*

$$\overline{\eta}^\alpha \restriction \langle \ell_\alpha, n\rangle \notin \left\{\overline{\eta}^\beta \restriction \langle \ell_\alpha, n\rangle \mid \beta < \alpha\right\} \cup \Omega_\alpha F \quad \textit{for all} \quad n \geq n_\alpha \,.$$

Proof. The proof follows by induction on k. We begin with $k = 0$, hence we may assume that $|\,\Omega\,| = \aleph_0$. Let $\Omega = \{\overline{\eta}^\alpha \mid \alpha < \omega\}$ be an enumeration without repetitions. From $0 = k < |\,\overline{u}_{\overline{\eta}}\,|$ follows $\overline{u}_{\overline{\eta}} \neq \emptyset$ and we can choose any $\ell_\alpha \in u_{\overline{\eta}^\alpha}$ for all $\alpha < \omega$. To be definite we may choose $\ell_\alpha = \min u_{\overline{\eta}^\alpha}$. If $\alpha \neq \beta < \omega$, then $\overline{\eta}^\alpha \neq \overline{\eta}^\beta$ and there is $n_{\alpha,\beta} \in \omega$ such that $\overline{\eta}^\alpha \restriction \langle \ell_\alpha, n\rangle \neq \overline{\eta}^\beta \restriction \langle \ell_\alpha, n\rangle$ for all $n \geq n_{\alpha\beta}$. Since $\Omega_\alpha F$ is finite, we may enlarge $n_{\alpha,\beta}$, if necessary, such that $\overline{\eta}^\alpha \restriction \langle \ell_\alpha, n\rangle \notin \Omega_\alpha F$ for all $n \geq n_{\alpha,\beta}$. If $n_\alpha = \max_{\beta<\alpha} n_{\alpha,\beta}$, then $\overline{\eta}^\alpha \restriction \langle \ell_\alpha, n\rangle \notin \{\overline{\eta}^\beta \restriction \langle \ell_\alpha, n\rangle \mid \beta < \alpha\} \cup \Omega_\alpha F$ for all $n \geq n_\alpha$. Hence case $k = 0$ is settled and we let $k' = k + 1$ and assume that the proposition holds for k.

Let $|\,\Omega\,| = \aleph_{k'}$ and choose an $\aleph_{k'}$-filtration $\Omega = \bigcup_{\delta < \aleph_{k'}} \Omega_\delta$ with $\Omega_0 = \emptyset$ and $|\,\Omega_1\,| = \aleph_k$. The crucial idea comes from [11]: We can also assume that this chain is closed, meaning that for any $\delta < \aleph_{k'}$, $\overline{\nu}, \overline{\nu}' \in \Omega_\delta$ and $\overline{\eta} \in \Omega$ with

$$\{\eta_m \mid m \leq k_*\} \subseteq \left\{\nu_m, \nu'_m, (\overline{\nu}F)_m, (\overline{\nu}'F)_m \mid m \leq k_*\right\}$$

follows $\overline{\eta} \in \Omega_\delta$. Thus, if $\overline{\eta} \in \Omega_{\delta+1} \setminus \Omega_\delta$, then the set

$$u^*_{\overline{\eta}} = \left\{\ell \leq k_* \mid \exists n < \omega, \overline{\nu} \in \Omega_\delta \text{ such that } \overline{\eta} \restriction \langle \ell, n\rangle = \overline{\nu} \restriction \langle \ell, n\rangle \text{ or } \overline{\eta} \restriction \langle \ell, n\rangle = \overline{\nu}F\right\}$$

is empty or a singleton. Otherwise there are $n, n' < \omega$ and distinct $\ell, \ell' \leq k_*$ with $\overline{\eta} \restriction \langle \ell, n\rangle \in \{\overline{\nu} \restriction \langle \ell, n\rangle, \overline{\nu}F\}$ and $\overline{\eta} \restriction \langle \ell', n'\rangle \in \{\overline{\nu}' \restriction \langle \ell', n'\rangle, \overline{\nu}'F\}$ for certain $\overline{\nu}, \overline{\nu}' \in \Omega_\delta$. Hence $\{\eta_m \mid m \leq k_*\} \subseteq \{\nu_m, \nu'_m, (\overline{\nu}F)_m, (\overline{\nu}'F)_m \mid m \leq k_*\}$, and the closure property implies the contradiction $\overline{\eta} \in \Omega_\delta$.

If $\delta < \aleph_{k'}$, then let $D_\delta = \Omega_{\delta+1} \setminus \Omega_\delta$ and $u'_{\overline{\eta}} := u_{\overline{\eta}} \setminus u^*_{\overline{\eta}}$ must have size $> k' - 1 = k$. Thus the induction hypothesis applies and we find an enumeration $\overline{\eta}^{\delta\alpha}$ $(\alpha < \aleph_k)$ of D_δ as in the proposition. Finally we put these chains for each $\delta < \aleph_{k'}$ together with the induced ordering to get an enumeration $\langle \overline{\eta}^\alpha \mid \alpha < \aleph_{k'}\rangle$ of Ω satisfying the proposition. □

3. The Black Box for $\aleph_n$-free modules

Let R be a commutative ring with $\mathbb{S}$ a countable multiplicatively closed subset such that the following holds.

(i) The elements of $\mathbb{S}$ are not zero-divisors, i.e. if $s \in \mathbb{S}, r \in R$ and $sr = 0$, then $r = 0$.

(ii) $\bigcap_{s\in\mathbb{S}} sR = 0$.

We also say that R is an $\mathbb{S}$-ring. If (i) holds, then R is $\mathbb{S}$-*torsion-free* and if (ii) holds, then R is $\mathbb{S}$-*reduced*, see [8]. To ease notations we use the letter $\mathbb{S}$ only if we want to emphasize that the argument depends on it. If M is an R-module, then these definitions naturally carry over to M. Finally we enumerate $\mathbb{S} = \{s_n \mid n < \omega\}$, let $s_0 = 1$ and put $q_n = \prod_{i\le n} s_i$, thus $q_{n+1} = q_n s_{n+1}$.

Similar to the Black Box in [1], we first define the basic R-module B, which is

$$B = \bigoplus_{\overline{\eta}\in\Lambda_*} Re_{\overline{\eta}}\,.$$

Definition 3.1. If $U \subset \Lambda_*$, then we get a *canonical summand* $B_U = \bigoplus_{\overline{\eta}\in U} Re_{\overline{\eta}}$ of B, and in particular, let $B_{\overline{\eta}} = B_{[\overline{\eta}]}$ and $B_{\overline{\eta}\restriction m} = B_{[\overline{\eta}\restriction m]}$ be the *canonical summand of* $\overline{\eta}$ and $\overline{\eta}\restriction m$ $(\overline{\eta} \in \Lambda)$, respectively.

We have several R–free summands

$$B_{\overline{\eta}\restriction m} = \bigoplus_{n<\omega} Re_{\overline{\eta}\restriction\langle m,n\rangle} \quad \text{and} \quad B_{\overline{\eta}} = \bigoplus_{m\le k_*} B_{\overline{\eta}\restriction m}\,.$$

The $\mathbb{S}$-topology (generated by the basis sB $(s \in \mathbb{S})$) of neighbourhoods of 0 is Hausdorff on B and (as usual) we can consider the $\mathbb{S}$-completion $\widehat{B}$ of B; see [8] for elementary facts on the elements of $\widehat{B}$. Let $\widetilde{B} = \bigoplus_{\overline{\eta}\in\Lambda_*} \widehat{R}e_{\overline{\eta}}$. Every element $b \in \widehat{B}$ has a natural Λ_*-support $[b]_{\Lambda_*} \subseteq \Lambda_*$ which are those $\overline{\eta} \in \Lambda_*$ which contribute to the sum-representation $b = \sum_{\overline{\eta}\in\Lambda_*} b_{\overline{\eta}} e_{\overline{\eta}}$ with coefficients $0 \ne b_{\overline{\eta}} \in \widehat{R}$. Thus let $[b]_{\Lambda_*} = \{\overline{\eta} \in \Lambda_* \mid b_{\overline{\eta}} \ne 0\}$. Note that $[b]_{\Lambda_*}$ is at most countable and $b \in \widetilde{B}$ iff $[b]_{\Lambda_*}$ is finite. As in the earlier Black Boxes (see [8]) we use conditions on the support (given by the prediction) to select (carefully) elements from $\widehat{B}$ added to B to get the final structure M, such that

$$B \subseteq M \subseteq_* \widehat{B}$$

which is an $\mathbb{S}$-pure submodule of $\widehat{B}$, thus satisfying

$$M \cap s\widehat{B} \subseteq sM$$

for all $s \in \mathbb{S}$. Thus $\mathbb{S}$-topological arguments can be used carelessly switching between these three modules.

We will now use B, Λ_*, Λ to define the Black Box for $\aleph_n$-free R-modules. As in [1] we will also use the notion of a trap.

Definition 3.2. Let G be any R-module. A *trap* (*for* B, G) is a partial R-homomorphism of B into G with a label $\overline{\eta} \in \Lambda$, say $\varphi_{\overline{\eta}} : \mathrm{Dom}(\varphi_{\overline{\eta}}) \to G$, such that $B_{\overline{\eta}} \subseteq \mathrm{Dom}(\varphi_{\overline{\eta}}) \subseteq B$.

The $\overline{\lambda}$-Black Box 3.3. *Given a* $\blacksquare$*-sequence* $\overline{\lambda} = \langle \lambda_1, \dots, \lambda_{k_*} \rangle$ *with* (2.1) *and an* R*-module* G *of size* $|G| \leq \chi_1$*, let* Λ, Λ_* *be as above. Then there is a family of traps* $\varphi_{\overline{\eta}}$ $(\overline{\eta} \in \Lambda)$ *with the following property:*
The prediction: *If* $\varphi : B \to G$ *is an* R*-homomorphism and* $\alpha \in \lambda_{k_*}$*, then there is* $\overline{\eta} \in \Lambda$ *with* $0\eta_{k_*} = \alpha$ *and* $\varphi_{\overline{\eta}} \subseteq \varphi$.

Proof. The theorem is an immediate consequence of Theorem 2.4. We view the set maps in Theorem 2.4 as the restrictions of the R-homomorphisms in Theorem 3.3 to the canonical R-basis $\{e_{\overline{\nu}} \mid \overline{\nu} \in \Lambda_*\}$ of B. There is a one-to-one correspondence between these maps and thus Theorem 3.3 follows. □

4. The R-modules

Let R be an $\mathbb{S}$-torsion-free and $\mathbb{S}$-reduced commutative ring of size $|R| < 2^{\aleph_0}$, $\chi_{k_*} = \lambda_{k_*}^{\aleph_0} = \lambda_{k_*}$ be as before, $B = \bigoplus_{\overline{\nu} \in \Lambda_*} Re_{\overline{\nu}}$ the R-module freely generated by $\{e_{\overline{\nu}} \mid \overline{\nu} \in \Lambda_*\}$ and

$$\Lambda_* = \bigcup_{\overline{\eta} \in \Lambda} [\overline{\eta}] \quad \text{with} \quad [\overline{\eta}] = \{\overline{\eta}\upharpoonright\langle m, n\rangle \mid m \leq k_*, n < \omega\}.$$

We also choose any bijection

$$\delta : \lambda_{k_*} \longrightarrow \Lambda_* .$$

Thus we can write the basis elements of B in the form $e_{\delta(\alpha)}$ for any $\alpha \in \lambda_{k_*}$.

From [5] follows that the $\mathbb{S}$-adic completion $\widehat{R}$ of R has $2^{\aleph_0}$ algebraically independent elements over R, and in particular $|\widehat{R}| = 2^{\aleph_0}$.

Next we define particular elements in $\widehat{B}$. If $\overline{\eta} \in \Lambda$, then let

$$y_{\overline{\eta}k} = \sum_{n \geq k} \frac{q_n}{q_k} \left(\sum_{m=1}^{k_*} e_{\overline{\eta}\upharpoonright\langle m,n\rangle} + b_{\overline{\eta}n} e_{\delta(0\eta_{k_*})} \right)$$

where $b_{\overline{\eta}n} \in R$. Moreover let $y_{\overline{\eta}} = y_{\overline{\eta}0}$. We will choose $\pi_{\overline{\eta}} \in \widehat{R}$ and write $\pi_{\overline{\eta}} = \sum_{n<\omega} q_n b_{\overline{\eta}n}$ and let $\pi_{\overline{\eta}k} = \sum_{n \geq k} \frac{q_n}{q_k} b_{\overline{\eta}n}$. Thus

$$y_{\overline{\eta}k} = \sum_{n \geq k} \frac{q_n}{q_k} \left(\sum_{m=1}^{k_*} e_{\overline{\eta}\upharpoonright\langle m,n\rangle} \right) + \pi_{\overline{\eta}k} e_{\delta(0\eta_{k_*})}$$

and from

$$s_{k+1} y_{\overline{\eta}k+1} = \sum_{n \geq k+1} \frac{q_n}{q_k} \left(\sum_{m=1}^{k_*} e_{\overline{\eta}\upharpoonright\langle m,n\rangle} + b_{\overline{\eta}n} e_{\delta(0\eta_{k_*})} \right)$$

Sh:920

and $y_{\overline{\eta}k} - s_{k+1}y_{\overline{\eta}k+1} = \sum_{m=1}^{k_*} e_{\overline{\eta}\upharpoonright\langle m,k\rangle} + b_{\overline{\eta}k}e_{\delta(0\eta_{k_*})}$, follows

$$s_{k+1}y_{\overline{\eta}k+1} = y_{\overline{\eta}k} - \sum_{m=1}^{k_*} e_{\overline{\eta}\upharpoonright\langle m,k\rangle} - b_{\overline{\eta}k}e_{\delta(0\eta_{k_*})}\,. \tag{4.1}$$

We want to define an R-module M with $B \subseteq M \subseteq_* \widehat{B}$ which is $\mathbb{S}$-pure in $\widehat{B}$. Thus M/B is $\mathbb{S}$-torsion-free and $\mathbb{S}$-divisible. It follows that for any non-trivial homomorphism $\sigma : M \to R$ there is $\overline{\nu} \in \Lambda_*$ with $e_{\overline{\nu}}\sigma \neq 0$. If $\overline{\eta} \in \Lambda$, then we will adjoin to B for some suitable $\pi_{\overline{\eta}} \in \widehat{R}$ the element $y_{\overline{\eta}} = \sum_{n<\omega} q_n(\sum_{m=1}^{k_*} e_{\overline{\eta}\upharpoonright\langle m,n\rangle})+ \pi_{\overline{\eta}}e_{\delta(0\eta_{k_*})}$. This will follow with the help of the next

Proposition 4.1. *Let R be an $\mathbb{S}$-torsion-free and $\mathbb{S}$-reduced commutative ring of size $< 2^{\aleph_0}$. Then for any $\overline{\eta} \in \Lambda$ there are $\pi_{\overline{\eta}} \in \widehat{R}$ and*

$$y_{\overline{\eta}} = \sum_{n<\omega} q_n \left(\sum_{m=1}^{k_*} e_{\overline{\eta}\upharpoonright\langle m,n\rangle} \right) + \pi_{\overline{\eta}}e_{\delta(0\eta_{k_*})} \tag{4.2}$$

with no homomorphism $\varphi : \langle B, y_{\overline{\eta}}\rangle_* \longrightarrow R$ such that $\varphi \upharpoonright B_{[\overline{\eta}]} = \varphi_{\overline{\eta}}$ and $e_{\delta(0\eta_{k_*})}\varphi \neq 0$.

Proof. Let $e = e_{\delta(0\eta_{k_*})}$ and choose pairwise distinct elements $\pi_\alpha \in \widehat{R}$ $(\alpha < 2^{\aleph_0})$. Moreover let $y = \sum_{n<\omega} q_n(\sum_{m=1}^{k_*} e_{\overline{\eta}\upharpoonright\langle m,n\rangle})$ and put $y_\alpha = y + \pi_\alpha e$. Suppose that for each $\alpha < 2^{\aleph_0}$ there is a homomorphism $\varphi_\alpha : \langle B, y_\alpha\rangle_* \longrightarrow R$ with $\varphi_\alpha \upharpoonright B_{[\overline{\eta}]} = \varphi_{\overline{\eta}}$ and $e\varphi_\alpha \neq 0$. By a pigeon hole argument there are distinct $\alpha, \beta < 2^{\aleph_0}$ with the same images $y_\alpha\varphi_\alpha = y_\beta\varphi_\beta$ and also $e\varphi_\alpha = e\varphi_\beta =: c \neq 0$. But this implies

$$\begin{aligned} 0 &= y_\alpha\varphi_\alpha - y_\beta\varphi_\beta = (y + \pi_\alpha e)\varphi_\alpha - (y + \pi_\beta e)\varphi_\beta \\ &= y\varphi_{\overline{\eta}} + \pi_\alpha e\varphi_\alpha - y\varphi_{\overline{\eta}} - \pi_\beta e\varphi_\beta = (\pi_\alpha - \pi_\beta)c\,. \end{aligned}$$

And from $\pi_\alpha - \pi_\beta \neq 0$ follows $c = 0$, a contradiction. □

Finally we define the R-module

$$M = \langle B, y_{\overline{\eta}} \mid \overline{\eta} \in \Lambda\rangle_* \subseteq \widehat{B}\,. \tag{4.3}$$

Here we let $y_{\overline{\eta}}$ be as in (4.2) and apply Proposition 4.1.

First we will take care of the freeness of M by applying the set-theoretic version of freeness, i.e. Proposition 2.6. In order to apply our results to rings which are not necessarily PIDs, we more generally say that an R-module M is κ-free if any subset of size $< \kappa$ is contained in a free R-submodule of M.

Freeness-Proposition 4.2. *The module M as defined in* (4.3) *is $\aleph_{k_*}$-free.*

Proof. Besides the Λ_*-support $[g]_{\Lambda_*}$ (discussed at the beginning of the last section) any element g of the module $M = \langle B, y_{\overline{\eta}} \mid \overline{\eta} \in \Lambda\rangle_*$ has a refined natural finite support $[g]$ arriving from the definition (4.3). It consists of all those elements of Λ and Λ_* contributing to g. We observe that g is generated by elements $y_{\overline{\eta}}$ and $e_{\overline{\eta}\upharpoonright\langle m,n\rangle}$ and simply collect the $\overline{\eta}$s and $\overline{\eta}\upharpoonright\langle m,n\rangle$ needed. Clearly $[g]$ is a finite subset of $\Lambda \cup \Lambda_*$. Hence any submodule H of M has a natural support $[H]$ taking the union of supports of its elements and if $|H| < \kappa$ for any cardinal $\kappa > |R|$, then there is a subset $\Omega \subseteq \Lambda$ of size $|\Omega| < \kappa$ such that H is a submodule of the pure R-submodule

$$M_\Omega = \langle e_{\overline{\eta}\upharpoonright\langle m,n\rangle}, e_{\delta(0\eta_{k_*})}, y_{\overline{\eta}} \mid \overline{\eta} \in \Omega, m \leq k_*, n < \omega\rangle_* \subseteq \widehat{B}\,,$$

which also has size $< \kappa$. Thus, in order to show $\aleph_{k_*}$-freeness of M, we only must consider any $\Omega \subseteq \Lambda$ of size $|\Omega| < \aleph_{k_*}$ and show the freeness of the module M_Ω. We may assume that $|\Omega| = \aleph_{k_*-1}$. Let $F : \Lambda \to \Lambda_*$ be the map which assignes to $\overline{\eta} \in \Lambda$ the element $\overline{\eta}F = \delta(0\eta_{k_*}) \in \Lambda_*$.

By Proposition 2.6 we can express the generators of M_Ω of the form

$$M_\Omega = \langle e_{\overline{\eta}^\alpha\upharpoonright\langle m,n\rangle}, e_{\overline{\eta}^\alpha F}, y_{\overline{\eta}^\alpha n} \mid \alpha < \aleph_{k_*-1}, m \leq k_*, n < \omega\rangle$$

and find a sequence of pairs $(\ell_\alpha, n_\alpha) \in (k_* + 1) \times \omega$ such that for $n \geq n_\alpha$

$$\overline{\eta}^\alpha\upharpoonright\langle \ell_\alpha, n\rangle \notin \{\overline{\eta}^\beta\upharpoonright\langle \ell_\alpha, n\rangle \mid \beta < \alpha\} \cup \{\overline{\eta}^\beta F \mid \beta < \alpha\}\,. \tag{4.4}$$

Let $M_\alpha = \langle e_{\overline{\eta}^\gamma\upharpoonright\langle m,n\rangle}, e_{\overline{\eta}^\gamma F}, y_{\overline{\eta}^\gamma n} \mid \gamma < \alpha, m \leq k_*, n < \omega\rangle$ for any $\alpha < \aleph_{k_*-1}$; thus

$$\begin{aligned} M_{\alpha+1} &= M_\alpha + \langle e_{\overline{\eta}^\alpha\upharpoonright\langle m,n\rangle}, e_{\overline{\eta}^\alpha F}, y_{\overline{\eta}^\alpha n} \mid m \leq k_*, n < \omega\rangle \\ &= M_\alpha + \langle e_{\overline{\eta}^\alpha\upharpoonright\langle \ell_\alpha,n\rangle} \mid n < n_\alpha\rangle + \langle y_{\overline{\eta}^\alpha n} \mid n \geq n_\alpha\rangle \\ &\quad + \langle e_{\overline{\eta}^\alpha F}, e_{\overline{\eta}^\alpha\upharpoonright\langle m,n\rangle} \mid \ell_\alpha \neq m \leq k_*, n < \omega\rangle\,. \end{aligned}$$

Hence any element in $M_{\alpha+1}/M_\alpha$ can be represented in $M_{\alpha+1}$ modulo M_α of the form

$$\sum_{n \geq n_\alpha} r_n y_{\overline{\eta}^\alpha n} + \sum_{n < n_\alpha} r'_n e_{\overline{\eta}^\alpha\upharpoonright\langle \ell_\alpha,n\rangle} + r e_{\overline{\eta}^\alpha F} + \sum_{n<\omega} \sum_{\ell_\alpha \neq m \leq k_*} r''_{mn} e_{\overline{\eta}^\alpha\upharpoonright\langle m,n\rangle}\,.$$

Moreover, the summands involving the $e_{\overline{\eta}^\alpha\upharpoonright\langle m,n\rangle}$s have disjoint supports. Now condition (4.4) applies recursively. And by the disjointness (identifying $e_{\overline{\eta}^\alpha F}$ with one of the $e_{\overline{\eta}^\alpha\upharpoonright\langle m,n\rangle}$s if possible) it also follows that all coefficients r, r'_n, r''_{mn} must be zero, showing that the set

$$\{e_{\overline{\eta}^\alpha\upharpoonright\langle \ell_\alpha,k\rangle}, e_{\overline{\eta}^\alpha F}, e_{\overline{\eta}^\alpha\upharpoonright\langle m,n\rangle} \mid k < n_\alpha, \ell_\alpha \neq m \leq k_*, n < \omega\} \setminus M_\alpha$$

freely generates $M_{\alpha+1}/M_\alpha$. Thus M_Ω has an ascending chain with only free factors; it follows that M_Ω is free. □

We can finally show the

Theorem 4.3. *Let R be an $\mathbb{S}$-torsion-free and $\mathbb{S}$-reduced commutative ring of size $< 2^{\aleph_0}$. Then for any $\blacksquare$-sequence $\overline{\lambda} = \langle \lambda_1, \ldots, \lambda_{k_*}\rangle$ with (2.1) there exists an $\aleph_{k_*}$-free R-module M of size χ_{k_*} with trivial dual* $\mathrm{Hom}(M,R) = 0$. *In particular, if R is a principal ideal domain but not a field of size $< 2^{\aleph_0}$, then there is an $\aleph_{k_*}$-free R-module M of size χ_{k_*} with trivial dual.*

Proof. If M is the R-module above, then M is $\aleph_{k_*}$-free by Proposition 4.2. Obviously M has size χ_{k_*}. If $\varphi : M \longrightarrow R$ is a non-trivial homomorphism, then there is $\overline{\nu} \in \Lambda_*$ such that for some basis element $e_{\overline{\nu}}\varphi \neq 0$. By the Black Box 3.3 there is $\overline{\eta} \in \Lambda$ with $\delta(0\eta_{k_*}) = \overline{\nu}$ and $\varphi \restriction B_{[\overline{\eta}]} = \varphi_{\overline{\eta}}$. We apply Proposition 4.1 to see that this is a contradiction. Hence $\mathrm{Hom}(M, R) = 0$ follows. □

Corollary 4.4. *If n is a natural number, then we find $\aleph_n$-free abelian groups of size $\beth_n$ with trivial dual.*

References

[1] A. L. S. Corner and R. Göbel, Prescribing endomorphism algebras – A unified treatment, Proc. London Math. Soc. (3) **50** (1985), 447 – 479.

[2] A. L. S. Corner and R. Göbel, Small almost free modules with prescribed topological endomorphism rings, Rendiconti Sem. Mat. Univ. Padova **109** (2003) 217–234.

[3] K. Eda, Cardinality restrictions on preradicals., pp. 277–283 in *Abelian group theory* (Perth, 1987), Contemp. Math. **87**, Amer. Math. Soc., Providence, RI, 1989.

[4] P. Eklof and A. Mekler, *Almost Free Modules, Set-theoretic Methods,* North Holland, 2002.

[5] R. Göbel and W. May, Independence in completions and endomorphism algebras, Forum mathematicum **1** (1989), 215–226.

[6] R. Göbel and S. Shelah, G. C. H. implies the existence of many rigid almost free abelian groups, pp. 253–271 in *Abelian Groups and Modules*, Marcel Dekker New York 1996.

[7] R. Göbel and S. Shelah, Indecomposable almost free modules – the local case, Canadian J. Math. **50** (1998), 719–738.

[8] R. Göbel and J. Trlifaj, *Endomorphism Algebras and Approximations of Modules*, Expositions in Mathematics **41**, Walter de Gruyter Verlag, Berlin (2006).

[9] T. Jech, *Set Theory*, Monographs in Mathematics, Springer, Berlin Heidelberg (2002).

[10] M. Magidor and S. Shelah, When does almost free imply free? (For groups, transversals, etc.), J. Amer. Math. Soc. **7** (1994), 769–830.

[11] S. Shelah, $\aleph_n$-free abelian groups with no non-zero homomorphisms to $\mathbb{Z}$, Cubo – A Mathematical Journal **9** (2007), 59–79.

Rüdiger Göbel
Fachbereich Mathematik
Universität Duisburg Essen
Campus Essen
D-45117 Essen
Germany
e-mail: ruediger.goebel@uni-due.de

Saharon Shelah
Institute of Mathematics
Hebrew University
Einstein Institute of Mathematics
Givat Ram
Jerusalem 91904
Israel
and
Rutgers University
Department of Mathematics
Hill Center
Busch Campus
Piscataway
NJ 08854 8019
USA
e-mail: shelah@math.huji.ac.il

Received: September 12, 2007.
Revised: February 11, 2009.