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Source: *The Journal of Symbolic Logic*, Vol. 63, No. 2 (Jun., 1998), pp. 427-438
Published by: Association for Symbolic Logic
Stable URL: http://www.jstor.org/stable/2586841
Accessed: 02-11-2015 12:55 UTC

THE JOURNAL OF SYMBOLIC LOGIC
Volume 63, Number 2, June 1998

# DOP AND FCP IN GENERIC STRUCTURES

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**§1. Context.** We work throughout in a finite relational language $L$. This paper is built on [2] and [3]. We repeat some of the basic notions and results from these papers for the convenience of the reader but familiarity with the setup in the first few sections of [3] is needed to read this paper. Spencer and Shelah [6] constructed for each irrational $\alpha$ between 0 and 1 the theory $T^\alpha$ as the almost sure theory of random graphs with edge probability $n^{-\alpha}$. In [2] we proved that this was the same theory as the theory $T_\alpha$ built by constructing a generic model in [3]. In this paper we explore some of the more subtle model theoretic properties of this theory. We show that $T^\alpha$ has the dimensional order property and does not have the finite cover property.

We work in the framework of [3] so probability theory is not needed in this paper. This choice allows us to consider a wider class of theories than just the $T_\alpha$. The basic facts cited from [3] were due to Hrushovski [4]; a full bibliography is in [3]. For general background in stability theory see [1] or [5].

We work at three levels of generality. The first is given by an axiomatic framework in Context 1.10. Section 2 is carried out in this generality. The main family of examples for this context is described in Example 1.3. Sections 3 and 4 depend on a function $\delta$ assigning a real number to each finite $L$-structure as in these examples. Some of the constructions in Section 3 (labeled at the time) use heavily the restriction of the class of examples to graphs. The first author acknowledges useful discussions on this paper with Sergei Starchenko.

**1.1. Notation.** Let $\mathbf{K}_0$ be a class of finite structures closed under substructure and isomorphism and containing the empty structure. Let $\overline{\mathbf{K}}_0$ be the universal class determined by $\mathbf{K}_0$.

**1.2. Notation.** Let $B \cap C = A$. The *free amalgam* of $B$ and $C$ over $A$, denoted $B \bigotimes_A C$, is the structure with universe $BC$ but no relations not in $B$ or $C$.

We write $A \subseteq_\omega B$ to mean $A$ is a finite subset of $B$. A structure $A$ is called *discrete* if there are no relations among the elements of $A$. Let $\delta : \mathbf{K}_0 \mapsto \Re^+$ (the nonnegative reals) be an arbitrary function with $\delta(\emptyset) = 0$. Extend $\delta$ to $d : \overline{\mathbf{K}}_0 \times \mathbf{K}_0 \mapsto \Re^+$ by for each $N \in \overline{\mathbf{K}}_0$,

$$d(N, A) = \inf\{\delta(B) : A \subseteq B \subseteq_\omega N\}.$$

Received April 19, 1996; revised July 11, 1996.
The first author was partially supported by NSF grant 9308768.
This is paper 567 in Shelah's bibliography. Both authors thank Rutgers University and the Binational Science Foundation for partial support of this research.


0022-4812/98/6302-0007/$2.20

We usually write $d(N, A)$ as $d_N(A)$. We only use this definition when $\delta$ is defined on every finite subset of $N$. We will omit the subscript $N$ if it is clear from context.

For $g = \delta$ or $d_N$ and finite $A, B$, we define *relative dimension* by $g(A/B) = g(AB) - g(B)$. For infinite $B$ and finite $A$, $d(A/B) = \inf\{d(A/B_0) : B_0 \subset_\omega B\}$. This definition is justified in e.g., Section 3 of [3]. For any finite sequence $\overline{a} \in N$, $d_N(\overline{a})$ is the same as $d_N(A)$ where $\overline{a}$ enumerates $A$.

Consider a finite structure $B$ for a finite relational language $L$. We assume that each relation of $L$ holds of a tuple $\overline{a}$ only if the elements $\overline{a}$ are distinct and if $R(\overline{a})$ holds, $R(\overline{a}')$ holds for any permutation $\overline{a}'$ of $\overline{a}$.

$R(B)$ denotes the collection of subsets $B_0 = \{b_1, \dots b_n\}$ of $B$ such that for some (any) ordering $\overline{b}$ of $B_0$, $B \models R(\overline{b})$ for some relation symbol $R$ of $L$; $e(B) = |R(B)|$. Let $A$, $B$, $C$ be disjoint sets. We write $R(A, B)$ for the collection of subsets from $AB$ that satisfy some relation of $L$ (counting with multiplicity if a set satisfies more than one relation) and contain at least one member of $A$ and one of $B$. Write $e(A, B)$ for $|R(A, B)|$. Similarly, we write $R(A, B, C)$ for the collection of subsets from $ABC$ that satisfy some relation of $L$ and contain at least one member of $A$ and one of $C$. Write $e(A, B, C)$ for $|R(A, B, C)|$.

**1.3. Example.** The most important examples arise by defining $\delta$ as follows. In the last section of [3] we enumerated several other examples to which this axiomatization applies. Let

$$\delta_{\beta,\alpha}(A) = \beta|A| - \alpha e(A).$$

We may write $\delta_\alpha$ for $\delta_{1,\alpha}$. The class $\mathbf{K}_\alpha$ is the collection of finite $L$-structures $A$ such that for any $A' \subseteq A$, $\delta_\alpha(A') \geq 0$. We denote by $T_\alpha$ the theory of the generic model of $\mathbf{K}_\alpha$.

**1.4. Axioms.** Let $N$ be in $\overline{\mathbf{K}}_0$ and let $A,\ B,\ C \in \mathbf{K}_0$ be substructures of $N$.

1. If $A$, $B$, and $C$ are disjoint then $\delta(C/A) \geq \delta(C/AB)$.
2. For every $n$ there is an $\varepsilon_n > 0$ such that if $|A| < n$ and $\delta(A/B) < 0$ then $\delta(A/B) \leq -\varepsilon_n$.
3. There is a real number $\varepsilon$ independent of $N, A, B, C$ such that if $A, B, C$ are disjoint subsets of a model $N$ and $\delta(A/B) - \delta(A/BC) < \varepsilon$ then $R(A, B, C) = \emptyset$ and $\delta(A/B) = \delta(A/BC)$.
4. For each $A \in \mathbf{K}_0$, and each $A' \subseteq A, \delta(A') \geq 0$.

We call a function $d = d_N$ derived from $\delta$ satisfying Axioms 1.4 a *dimension function*.

**1.5. Lemma.** *If $\delta$ is a dimension function satisfying the properties of Axiom* 1.4 *and $\leq_s$ (read* strong submodel*) is defined by $A \leq_s N$ if $d_N(A) = d_A(A)$, then $\leq_s$ satisfies the following propositions. Let $M, N, N' \in \overline{\mathbf{K}}_0$.*

**A1**. *$M \leq_s M$.*

**A2**. *If $M \leq_s N$ then $M \subseteq N$.*

**A3**. *$M \leq_s N' \leq_s N$ implies $M \leq_s N'$.*

**A4**. *If $M \leq_s N$, $N' \subseteq N$ then $M \bigcap N' \leq_s N$.*

**A5**. *For all $M \in \overline{\mathbf{K}}_0$, $\emptyset \leq_s M$.*

We need to analyze extensions which are far from being strong.

**1.6. Definition.** For $A, B \in S(\mathbf{K}_0)$, $A \leq_i B$ if $A \subseteq B$ but there is no $A'$ properly contained in $B$ with $A \subseteq A' \leq_s B$. If $A \leq_i B$, we say $B$ is an *intrinsic* extension of $A$.

**1.7. Definition.** The intrinsic closure of $A$ in $M$, $\mathrm{icl}_M(A)$ is the union of $B$ with $A \subseteq B \subseteq M$ and $A \leq_i B$. When $M$ is clear from context, we write $\overline{A}$ for $\mathrm{icl}_M(A)$. The intrinsic closure can be more finely analyzed as follows.

1. For any $M \in \mathbf{K}$, any $m \in \omega$, and any $A \subseteq M$,
$$\mathrm{icl}^m_M(A) = \bigcup\{B : A \leq_i B \subseteq M \& |B - A| < m\}.$$
2.
$$\mathrm{icl}_M(A) = \bigcup_m \mathrm{icl}^m_M(A).$$
3. $M$ has *finite closures* if for each finite $A \subseteq M$, $\mathrm{icl}(A)$ is finite.
4. $\mathbf{K}$ has *finite closures* if each $M \in \mathbf{K}$ has finite closures.

Using **A4**, note that the intrinsic closure of $A$ in $M$ is the intersection of the strong substructures of $M$ which contain $A$. Thus, when finite, $\mathrm{icl}_M(A) \in \mathbf{K}_0$ and is a strong substructure of $M$. Moreover, a countable $M$ has finite closures if and only if $M$ can be written as an increasing union of finite strong substructures.

**1.8. Definition.** The countable model $M \in \overline{\mathbf{K}}_0$ is $(\mathbf{K}_0, \leq_s)$-*generic* if

1. If $A \leq_s M, A \leq_s B \in \mathbf{K}_0$, then there exists $B' \leq_s M$ such that $B \cong_A B'$,
2. $M$ has finite closures.

**1.9. Fact.** If $(\mathbf{K}_0, \leq_s)$ satisfies the properties of Lemma 1.5 and the amalgamation property with respect to $\leq_s$ then there is a countable $\mathbf{K}_0$-generic model.

**1.10. Context.** Henceforth, $(\mathbf{K}_0, \leq_s)$ is class of finite structures closed under isomorphism and substructure with $\leq_s$ induced by a function $\delta$ obeying Axioms 1.4. Moreover, we assume $(\mathbf{K}_0, \leq_s)$ satisfies the amalgamation property and $\mathbf{K}$ is the class of models of the theory of the generic model $M$ of $(\mathbf{K}_0, \leq_s)$. $\mathcal{M}$ is a large saturated model of $T = \mathrm{Th}(M)$. In the absence of other specification, the dimension function $d$ is the function induced on $\mathcal{M}$ by $\delta$ and we work with substructures of $\mathcal{M}$.

**§2. Independence and orthogonality.** As indicated in Context 1.10, the following definitions take place in a suitably saturated model elementarily equivalent to the generic. We work in that context throughout this section.

**2.1. Definition.** We say the finite sets $A$ and $B$ are $d$-independent over $C$ and write

1. $A \downarrow^d_C B$ if
   (a) $d(A/C) = d(A/CB)$.
   (b) $\overline{AC} \cap \overline{BC} \subseteq \overline{C}$.
2. We say the (arbitrary) sets $A$ and $B$ are $d$-independent over $C$ and write $A \downarrow^d_C B$ if for every finite $A' \subseteq A$ and $B' \subseteq B$, $A' \downarrow^d_C B'$.

The compatibility of the two definitions is shown, e.g., in Section 3 of [3]. The following is well known (cf. 3.31 of [3]).

**2.2. Lemma.** *Suppose $A, B$ and $C = A \cap B$ are closed and $A \downarrow^d_C B$. Then $AB$ is closed, i.e., $\overline{AB} = \overline{A} \cup \overline{B}$.*

The equivalence of $d$-independence and stability theoretic independence was first proved in this generality in [3] but the basic setup comes from [4].

**2.3. Fact.** Suppose $T$ satisfies Context 1.10. If $C$ is intrinsically closed then for any $A$ and $B$, $A \downarrow_C B$ if and only if $A \downarrow^d_C B$.

We give a different proof that is not as involved with the intricacies of amalgamation in the case without finite closures as the one in [3].

Suppose for contradiction that $R(A, C, B) \neq \emptyset$. Then for $\varepsilon$ chosen according to Axiom 1.4, $\delta(A/B) - \delta(A/BC) > \varepsilon$. Now, construct a nonforking sequence $\langle A_i, B_i \rangle$ in $\mathrm{tp}(AB/C)$. Since $A$ is not in the algebraic closure of $BC$, no $A_j$ is in the algebraic closure of the union of $B_i$ for $i < j$. We will use this fact to show that the types $p_i = \mathrm{tp}(A_i/CB_i)$ are $n$-contradictory for some $n$. If not, for each $n$ there is an $A^*$ which is common solution for, say $p_1, \dots, p_n$. Fix $n$ such that $n \cdot \varepsilon > \delta(A/C)$. But $\delta(A^*/B_1, \dots B_n) \leq \delta(A/C) - n \cdot \alpha$ so this implies $A \subseteq \mathrm{acl}(CB_1 \dots, B_n)$ and this contradiction yields the result. The extension property for nonforking types and uniqueness suffice to deduce the converse from $d$-dependence implies forking dependence so we finish as in Lemma 3.35 of [3].

We extend our notion of dimension to a global real-valued rank on types.

**2.4. Definition.** Let $p \in S(A)$. Define $d(p)$ as $d(\overline{a}/A)$ for some (any) $\overline{a}$ realizing $p$.

**2.5. Definition.** Let $p_1, p_2 \in S(A)$.

1. $p_1$ and $p_2$ are disjoint if for any $\overline{a}_1, \overline{a}_2$ realizing $p_1$, $p_2$, $\mathrm{icl}(A\overline{a}_1) \cap \mathrm{icl}(A\overline{a}_2) \subseteq \mathrm{icl}(A)$.
2. $p_1 \in S(A)$ and $p_2 \in S(B)$ are disjoint if any pair of nonforking extensions of $p_1$ and $p_2$ to $AB$ are disjoint.

**2.6. Lemma.** *Let $A \subset B$, $p \in S(B)$ and $p|A = q$ and suppose $A$ is intrinsically closed.*

1. *If $d(p) < d(q)$ then $p$ forks over $A$.*
2. *$q$ is stationary.*

Proof. (1) follows immediately from Fact 2.3; (2) is also proved in [3] (Lemma 3.38).

**2.7. Lemma.** *Let $A$ be intrinsically closed, $p_1, p_2 \in S(A)$. If $p_1$ and $p_2$ are disjoint and $d(p_1) = 0$ then $p_1$ and $p_2$ are orthogonal.*

Proof. If not, there exist sequences $a_1 \dots a_k$ and $b_1 \dots b_m$ of realizations of $p_1$ and $p_2$ respectively, which are independent over $A$, such that $\overline{a} \not\downarrow_A \overline{b}$. Since $d(p_1) = 0$, $d(\overline{a}/A) = 0$ and $\mathrm{icl}(A\overline{a}) \cap \mathrm{icl}(A\overline{b}) \not\subseteq A$. By Lemma 2.2, intrinsic closure is a trivial dependence relation. Since the $a_i$ and the $b_j$ are independent, this implies that for some $i, j$, $\mathrm{icl}(Aa_i) \cap \mathrm{icl}(Ab_j) \not\subseteq A$. But this contradicts the disjointness of $p_1$ and $p_2$ and we finish.

The *dimensional order property* (DOP) and *dimensional discontinuity property* DIDIP are defined in [5]. Either of these conditions implies $T$ has many models in uncountable powers. $T$ has the eventually non-isolated dimensional order property (eni-dop) if some type witnessing the dimensional order property is not isolated. This condition implies that $T$ has the maximal number of countable models. Since $T_\alpha$ is not small for irrational $\alpha$, this is not new information. However, the eni-dop seems to be a much more intrinsic feature of the construction than the smallness. (For precise definition see e.g., [1].)

**2.8. Theorem.** *Let* $\mathbf{K}_0$ *be a class satisfying Context* 1.10. *Let* $T$ *be the theory of the generic model for* $(\mathbf{K}_0, \leq_s)$. *Suppose further that there is a pair of independent points* $B = \{x, y\}$ *and a nonalgebraic type* $p$ *with* $d(p/B) = 0$ *but* $d(p/x) > 0$ *and* $d(p/y) > 0$.

1. *The theory* $T$ *has the dimensional order property.*
2. *If* $p$ *is not isolated the theory* $T$ *has the eni dimensional order property.*
3. *The theory* $T$ *has the dimensional discontinuity property.*

PROOF. (1) Let $A = \{a, b\}$ where $a$ and $b$ are independent over the empty set. It suffices to show that there is a type $p \in S(A)$ with $d(p) = 0$ and such that if $\overline{c}$ realizes $p$, $\overline{c} \not\downarrow_a b$ and $\overline{c} \not\downarrow_b a$. For then we can construct an independent sequence of points $a_i$ and disjoint copies $p_{i,j}$ over $\{a_i, a_j\}$ which will be pairwise orthogonal by Lemma 2.7. The required type is constructed in Theorem 3.6. (2) follows by the same argument if $p$ is not isolated.

For (3) it suffices to find an independent sequence of sets $B_n$ for $n < \omega$ and $p \in S(B)$ where $B = \bigcup B_n$ such that $p \dashv \bigcup_{n<j} B_n$ for each $j$. Choose $B_n$ and $C_n$ as described at the beginning of the proof of Theorem 3.6. Let $B$ be the union for $n < \omega$ of $B_n = \{x_n, y_n\}$ with no relations on $B$. For each $n$, let $f_n$ map $c_n$ to $c$, $x$ to $x_n$ and $y$ to $y_n$. Then $B \cup \{c\}$ is as required. That is, $d(t(c/B)) = 0$ but $d(t(c/\bigcup_{n<m} B_n)) > 0$.

**§3. Constructing types of $d$-rank 0.** We construct a nonalgebraic type $p$ over a two element set with $d(p) = 0$.

**3.1. Context.** We work with a class $\mathbf{K}_0$ of finite structures as in Example 1.3. Thus, $(\mathbf{K}_0, \leq_s)$ witnesses Context 1.10. Recall that $\mathbf{K}$ is the class of models of the theory of the generic $M$, $\mathscr{M}$ is a saturated model of this theory, and $S(\mathbf{K})$ is the universal class it determines.

Finally, the $\alpha$ parameterizing the dimension function may be rational or irrational. This distinction affects only the question of whether the type with rank 0 is isolated and we discuss that when it arises.

**3.2. Definition.** $(\mathbf{K}_0, \leq_s)$ has the *full amalgamation property* if $B \cap C = A$ and $A \leq_s B$ imply $B \bigotimes_A C \in \mathbf{K}_0$ and $C \leq_s B \bigotimes_A C$.

It is easy to check (Section 4 of [3]) that if $(\mathbf{K}_0, \leq_s)$ is closed under free amalgamation then it has full amalgamation.

**3.3. Assumption.** $(\mathbf{K}_0, \leq_s)$ has the *full amalgamation property*.

**3.4. Examples.** Each of the following classes is closed under free amalgamation.

1. The class $(\mathbf{K}_\alpha, \leq_s)$ of all finite $L$-structures $A$ with $\delta_{1,\alpha}(A)$ hereditarily positive. The resulting theory is $\omega$-stable if $\alpha$ is rational and stable if $\alpha$ is irrational.
2. The class yielding the stable $\aleph_0$-categorical pseudoplane of [4].

The main aim of this section is to establish the following result which leads easily by Theorem 2.8 to showing the theory of the generic model $\mathscr{M}$ has DOP and DIDIP.

**3.5. Definition.** We say $C$ is a *primitive* extension of $B$ if $B \leq_s C$ but there is no $B'$ properly between $B$ and $C$ with $B' \leq_s C$.

**3.6. Theorem.** *There exists a triple* $\{x, y, c\} \in \mathscr{M}$ *such that* $B = \{x, y\}$ *is an independent pair over* $\emptyset$ *and* $d(c/xy) = 0$ *but* $d(c/x) > 0$, $d(c/y) > 0$ *and* $c \notin \mathrm{acl}(x, y)$.

Proof. Fix a discrete structure $B$ with universe $\{x, y\}$. We will construct a family $\langle (C_n, x_n, y_n, c_n) : n < \omega \rangle$ of structures in $\mathbf{K}_0$ which satisfy the following conditions. Let $B_n = \{x_n, y_n\}$. The inequalities in the following discussion automatically become strict inequalities if $\alpha$ is irrational.

1. $0 \leq \delta(C_n/B_n) < 1/n$.
2. $(x_n, y_n, c_n)$ is a discrete substructure of $C_n$.
3. $C_n$ is a primitive extension of $B_n$.

Now map each $B_n$ to $B$ and amalgamate the images of the $C_n$ disjointly over $B$. Then identify all the $c_n$ as $c$ to form a structure $A$. Without loss of generality we can assume $A$ is strongly embedded in $\mathscr{M}$. Thus, $\mathrm{icl}_{\mathscr{M}}(cB) = A$. Then $d(c/B) = 0$ but $d(c/x)$ and $d(c/y)$ are both at least one. Thus $c \not\downarrow_x xy$ and $c \not\downarrow_y xy$. Since $\delta(C_n/B_n) \geq 0$, for every $n$, $c \notin \mathrm{acl}(B)$.

**3.7. Remark.** If $\alpha$ is irrational, all the $C_n$ are necessary and $\mathrm{tp}(c/xy)$ is nonprincipal. If $\alpha$ is rational, for some $n$, $\delta(C_n/B_n) = 0$. (We expand on this remark after Observation 3.9.) The type is principal but still not algebraic since in this context there are infinitely many copies (in a generic) of a primitive extension with relative dimension 0.

The construction of the $C_n$ follows a rather tortured path. We first need to consider structures with negative dimension over $B$.

**3.8. Definition.** Let $\mathscr{A} = \mathscr{A}_\alpha$ be the class of structures of the form $(A, a, b, e)$ which satisfy the following conditions. Let $B$ be the structure with universe $\{a, b\}$ and no relations.

1. $A \in \mathbf{K}_0$.
2. $\{a, b, e\}$ is the universe of a discrete substructure of $A$.
3. For each $A'$ with $B \subseteq A'$ and $A'$ properly contained in $A$, $\delta(A') > \delta(A)$.
4. $-1 < \delta(A/B) \leq 0$.

**3.9. Observation.**

1. The choice of $\delta$ as $\delta_\alpha$ makes $\mathscr{A}$ depend on $\alpha$.
2. If the last three conditions are satisfied, the first is as well.
3. The last condition implies that $\delta(A/a) > 0$ and $\delta(A/b) > 0$.

We first show that the set

$$X = X_\alpha = \{\beta : \beta = \delta(A/\{a, b\}) \text{ for some } (A, a, b, e) \in \mathscr{A}\}$$

is not bounded away from zero. If $\alpha$ is irrational, $0 \notin X$ so $X$ is infinite. If $\alpha = p/q$ is rational, every element of $X$ has the form $(mq - np)/q$ so there cannot be an infinite sequence of members of $X$ tending to 0. That is, there will be an $A$ with $\delta(A/B) = 0$. As indicated $X$ depends on $\alpha$ (through $\delta = \delta_\alpha$ and $\mathscr{A} = \mathscr{A}_\alpha$.) But the bulk of the proof is uniform in $\alpha$, so to enhance readability we keep track of $\alpha$ only for that part of the proof where the dependence is not uniform.

**3.10. Construction.** There are two elementary steps in the construction. It is easy to check that if the constituent models described here are in $\mathbf{K}_0$, then so is the result.

1. If $\delta(A/B) = \beta$ and $\beta \in X$, and $A^*$ is the free amalgam over $B$ of $k$ copies of $A$, then $\delta(A^*/B) = k\beta$.
2. Let $(A_1, a_1, b_1, c_1)$ and $(A_2, a_2, b_2, c_2)$ be in $\mathscr{A}$. Let $A^*$ be formed by identifying $b_1$ and $a_2$ and freely amalgamating over that point.

**3.11. Lemma.** *If $\beta > -1/k$ and $\beta \in X$ then $k\beta \in X$.*

Proof. Use Construction 3.10 (1).

It is straightforward to determine the following properties of the second construction.

**3.12. Lemma.** *Suppose $\delta(A_1/\{a_1, b_1\}) = \beta_1$, $\delta(A_2/\{a_2, b_2\}) = \beta_2$ and $\beta_1, \beta_2 \in X$. Let $A^*$ be formed as in Construction* 3.10 (2).

1. $\delta(A^*/\{a_1, b_2\}) = \beta_1 + \beta_2 + 1$.
2. *If $-2 < \beta_1 + \beta_2 \leq -1$ then $\beta_1 + \beta_2 + 1 \in X$ and $\langle A^*, a_1, b_2, c_1\rangle \in \mathscr{A}$.*
3. *If $-1 \leq \beta_1 + \beta_2 < -1 + 1/n$ then*
   (a) $0 \leq \delta(A^*/\{a_1, b_2\}) < 1/n$.
   (b) $\delta(A^*/a_1) \geq 1$ *and* $\delta(A^*/b_2) \geq 1$.

Proof. The key observations for (1) and thus (2) and (3a) is that for any $B \subseteq A_1 \subseteq A^*$,

$$\delta(A'/\{a_1, b_2\}) = \delta(A' \cap A_1/\{a_1, b_1\}) + \delta(A' \cap A_2/\{a_2, b_2\}) + 1.$$

For (3b) we need the further remark:

$$\delta(A'/a_1) = \delta(A'/b_2) = \delta(A'/\{a_1, b_2\}) + 1.$$

**3.13. Lemma.** *If $L$ contains a single binary relation and $\mathbf{K}_0 = \mathbf{K}_\alpha$, then $X$ is not empty.*

Proof. It suffices to show that each $\mathscr{A}_\alpha$ is nonempty for $0 < \alpha \leq 1$. The construction is somewhat ad hoc and proceeds by a number of cases depending on $\alpha$. Thus to establish Lemma 3.13 we will use the notations $\mathscr{A}_\alpha, \delta_\alpha$. These constructions are very specific to graphs. The second author has an alternative argument which avoids the dependence on $\alpha$. However, it passes through hypergraphs and has its own computational complexities.

**3.14. Case 1.** $3/4 < \alpha < 1$: Let $A_1$ be the structure obtained by adding to $\{a, b, e\}$ two points $b_1, b_2$ such that $b_1$ is connected to $a$ and $e$ while $b_2$ is connected to $b$ and $e$. Then

$$-1 < \delta_\alpha(A_1/B) = 3 - 4\alpha < 0$$

for the indicated $\alpha$ and $(A_1, a, b, e) \in \mathscr{A}_\alpha$.

**3.15. Case 2.** $2/3 \leq \alpha < 4/5$: Let $A_2$ be the structure obtained by adding to $\{a, b, e\}$ two points $b_1, b_2$ such that $b_1$ is connected to $a$, $b$, and $e$ while $b_2$ is connected to $b$ and $e$. Then

$$-1 < \delta_\alpha(A_2/B) = 3 - 5\alpha < 0$$

for the indicated $\alpha$ and $(A_2, a, b, e) \in \mathscr{A}_\alpha$.

**3.16. Case 3.** $0 < \alpha < 2/3$: Let $A_{n,k}$ be the structure obtained by adding to $\{a, b, e\}$ both $n$ points $a_1, \ldots, a_n$ such that each $a_i$ is connected to $a$, $b$, and $e$ and $k$ points $b_1, \ldots, b_k$ such that each $b_i$ is connected to all the $a_i$.

Then $\delta_\alpha(A_{n,k}/B) = n + k + 1 - (nk + 3n)\alpha$. We say $\alpha$ is *acceptable* for $n$ and $k$ if the following inequality is satisfied.

$$\ell_{n,k} = \frac{n+k+1}{nk+3n} < \alpha < \frac{n+k+2}{nk+3n} = u_{n,k}.$$

To show that if $\alpha$ is acceptable for $n$ and $k$, then $(A_{n,k}, a, b, e) \in \mathscr{A}_\alpha$ we need several claims.

**3.17. Claim 1.** For each $k$,

1. $u_{n+1,k} > \ell_{n,k}$,
2. $\ell_{n+1,k} < \ell_{n,k}$,
3. $\lim_{n\to\infty} \ell_{n,k} = 1/(k+3)$.

Claim 1 is established by routine computations.

**3.18. Claim 2.** For every $\alpha$ that is acceptable for $n$ and $k$, if $B \subseteq A' \subseteq A_{n,k}$, $\delta_\alpha(A'/B) \geq \delta_\alpha(A_{n,k}/B)$.

To see this, note that any such $A'$, for some $m \leq n$ and $\ell \leq k$, either $A'$ has the form $A_{m,\ell}$ or the form $B_{m,\ell}$, where $B_{m,\ell}$ is the structure obtained by omitting the element $e$ from $A_{m,\ell}$. Now note that if $\delta_\alpha(B_{m,\ell}/B) < 0$ then $\delta_\alpha(B_{m,\ell}/B) \geq \delta_\alpha(B_{m+1,\ell}/B)$ and $\delta_\alpha(B_{m,\ell}/B) \geq \delta_\alpha(B_{m,\ell+1}/B)$. The same assertion holds when $A_{m,\ell}$ is substituted for $B_{m,\ell}$. Finally, $\delta_\alpha(B_{n,k}/B) \geq \delta_\alpha(A_{n,k}/B)$. These three observations yield the second claim.

From these two claims we see that for each $\alpha$, there is a pair $n, k$ with $A_{n,k} \in \mathscr{A}_\alpha$. The remainder of the argument does not depend on $\alpha$ so we return to the use of the notation $X$ and $\mathscr{A}$.

**3.19. Lemma.** *For every $n$ there is an element $\beta$ of $X$ with $\beta > -1/n$.*

Proof. If not, fix the least $n$ such that all elements of $X$ are at most $-1/(n+1)$ and fix $\beta_0 \in X$ with $-1/n < \beta_0 \leq -1/(n+1)$. (If $\beta_0 = -1/(n+1)$, $\beta_1 = 0$ and we finish.) Define by induction $\beta_{\ell+1} = (n+1)\beta_\ell + 1$. Combining the two elementary steps we see that each $\beta_\ell \in X$. Let $\beta'_\ell$ be the distance between $-1/n$ and $\beta_\ell$. That is, $\beta'_\ell = |-1/n - \beta_\ell| = 1/n + \beta_\ell$. Now $\beta_\ell \leq -1/(n+1)$ if and only if $\beta'_\ell \leq 1/(n)(n+1)$.

But

$$\beta'_{\ell+1} = 1/n + (n+1)\beta_\ell + 1 = (n+1)\beta'_\ell.$$

So

$$\beta'_\ell = (n+1)^\ell \beta'_0.$$

As $\beta'_0 > 0$, for sufficiently large $\ell$, $\beta'_\ell > 1/(n)(n+1)$ so $\beta_\ell > -1/(n+1)$ as required.

With a few more applications of our fundamental constructions, we can find the $C_n$ needed for Theorem 3.6.

By applying Construction 3.10 (1) and Lemma 3.19 for any $n$, and $i = 1, 2$ we can find $(A_1^n, x_1^n, y_1^n, c_1^n)$ and $(A_2^n, x_2^n, y_n^n, c_2^n)$ containing $B_i^n = \{x_i^n, y_i^n\}$ such that $\{x_i^n, y_i^n, c_i^n\}$ is discrete and $\delta(A_i^n/B_i^n) = \beta_i^n$ with $-1 < \beta_1^n + \beta_2^n < -1 + 1/n$.

To construct $A_1^n$, choose using Lemma 3.19 a $(D^n, x_1^n, y_1^n, c_1^n) \in \mathscr{A}$ with $-1/n < \delta(D^n/B_1^n) \leq 0$. Take an appropriate number, $k$, of copies of $D^n$ over $B_1^n$ and apply Construction 3.10 (1) to form $A_1^n$ with

$$-1 < k\delta(D^n/B_1^n) = \delta(A_1^n/B_1^n) = \beta_1^n < -1 + 1/n$$

and choose $c_1^n \in A_1^n$ so that $(x_1^n, y_1^n, c_1^n)$ is discrete. By Lemma 3.19 again choose $(A_2^n, x_2^n, y_2^n, c_2^n) \in \mathscr{A}$ with

$$-(\beta_1^n + 1)/2 < \delta(A_2^n/B_2^n) = \beta_2^n < 0.$$

Now apply Construction 3.10 (2) to $(A_1^n, x_1^n, y_1^n, c_1^n)$ and $(A_2^n, x_2^n, y_n^n, c_2^n)$ to form $(C_n, x_n, y_n, c_n)$ where $x_n = x_1^n$, $y_n = y_2^n$, and $c_n = c_1^n$. Denote $\{x_n, y_n\}$ by $B_n$. Then $0 < \delta(C_n/B_n) = 1 + \beta_1^n + \beta_2^n < 1/n$. Each $C_n$ contains a discrete set $\{x_n, y_n, c_n\}$ and the third property of the $C_n$ follows using the second part of Lemma 3.12. This completes the construction of the type of $d$-rank 0.

Using the argument for constructing $A_1^n$, we easily show the following density result.

COROLLARY 3.1. *For any $\gamma, \delta$ with $-1 \leq \gamma < \delta < 0$ there is a $(D, a, b, e) \in \mathscr{A}$ with $\gamma < \delta(D/\{a, b\}) < \delta$.*

The restriction to one-types in the following lemma is solely for ease of presentation.

**3.20. Lemma.** *Suppose $A \subseteq M \models T_\alpha$ is intrinsically closed and $p_1, p_2 \in S_1(A)$ are disjoint. If $0 < d(p_i)$ for $i = 1, 2$ then $p_1 \not\perp p_2$.*

PROOF. Clearly if $p_1$ and $p_2$ are not disjoint or if there is an edge between realizations of the two types, they are not orthogonal. Let $a_1, a_2$ realize $p_1, p_2$ and suppose for contradiction that $p_1$ and $p_2$ are orthogonal and $d(a_1a_2/A) = d(a_1/A) + d(a_2/A) = \beta > 0$. In particular, there is no edge linking $a_1$ and $a_2$. By Lemma 3.25 of [3] there are finite $A_1 \supseteq a_1a_2$ and $A_0 \subset A$ with $\beta \leq \gamma = \delta(A_1/A_0) < \beta + 1$. Lemma 3.1 allows us to choose a finite $B \supseteq \{a_1, a_2\}$ with

$$-1 < \delta(B/\{a_1, a_2\}) < \beta - \gamma < 0.$$

Then $Ba_1a_2$ is in $\mathbf{K}_0$. By full amalgamation we can freely amalgamate $B$ with $AA_1$ over $\{a_1, a_2\}$ inside $\mathscr{M}$. Then $d(a_1a_2/A) \leq \delta(A_1B/A_0)$. Note $\delta(B/A_1A_0) = \delta(B/\{a_1, a_2\}) < \beta - \gamma$. So

$$\delta(A_1B/A_0) = \delta(B/A_1A_0) + \delta(A_1/A_0) < \beta.$$

This contradicts $d(a_1a_2/A) = \beta$ so we conclude $p_1 \not\perp p_2$.

Using the Lemmas 2.7 and 3.1 it is easy to see

COROLLARY 3.2. *In $T_\alpha$,*

1. *For disjoint $p_1, p_2$, $p_1 \perp p_2$ if and only if $d(p_1) = 0$ or $d(p_2) = 0$.*
2. *Every regular type satisfies $d(p) = 0$.*

Our construction yields some further information.

**3.21. Definition.** The type $p \in S(A)$ is *minimal* if $p$ is not algebraic but for any formula $\phi(x, \overline{b})$ either $p \cup \{\phi(x, \overline{b})\}$ or $p \cup \{\neg\phi(x, \overline{b})\}$ is algebraic.

**3.22. Definition.** The type $p \in S(A)$ is $i$-minimal if for every $\overline{a}$ realizing $p$, if $c \in \text{icl}(A\overline{a})$, $\text{icl}(Ac) = \text{icl}(A\overline{a})$.

**3.23. Theorem.** *If $p$ is constructed as in Lemma* 3.6 *then $p$ is minimal and trivial.*

PROOF. If $d(p) = 0$ and $p$ is $i$-minimal then $p$ is minimal. We constructed $p$ so that $d(p) = 0$ but the fact that each $C_n$ is primitive over $B$ and $A$ is intrinsically closed guarantees that $p$ is $i$-minimal and we finish.

Clearly, $d(p) = 0$ does not imply $p$ is minimal. For, if $d(a/A) = d(b/A) = 0$ then $d(ab/A) = 0$ but if, for example, $a$ and $b$ are independent $\text{tp}(ab/A)$ is not minimal.

**§4. The finite cover property.** In this section we show that for classes as described in Example 1.3 with the full amalgamation property, and in particular for $(\mathbf{K}_\alpha, \leq_s)$, the theory of the generic does not have the finite cover property. We rely on the following characterization due to Shelah [5, II.2.4].

**4.1. Fact.** If $T$ is a stable theory with the finite cover property then there is a formula $\phi(\overline{x}, \overline{y}, \overline{z})$ such that

1. For every $\overline{c}$, $\phi(\overline{c}, \overline{y}, \overline{z})$ defines an equivalence relation. We call this relation $\overline{c}$-equivalence.
2. For arbitrarily large $n$, there exists $\overline{c}_n$ such that the equivalence relation defined by $\phi(\overline{c}_n, \overline{y}, \overline{z})$ has exactly $n$ equivalence classes.

Here is some necessary notation.

**4.2. Definition.** Let $A, B$ be finite substructures of $M$ with $A \subseteq B$ then

1. $\chi_M(B/A)$ is the number of distinct copies of $B$ over $A$ in $M$.
2. $\chi_M^*(B/A)$ is the supremum of the cardinalities of maximal families of disjoint (over $A$) copies of $B$ over $A$ in $M$.

**4.3. Definition.** $(A, B)$ is a *minimal pair* if $\delta(B/A) < 0$ and for every $B'$, with $A \subseteq B' \subseteq B, \delta(B/A) < \delta(B'/A)$.

The next result is proved in [3].

**4.4. Fact.** There is a function $t$ taking pairs of integers to integers such that if $A \leq_i B$ then for any $N \in \mathbf{K}$ and any embedding $f$ of $A$ into $N$, $\chi_N(fB/fA) \leq t(|A|, |B|)$.

There is an easy partial converse to this result.

**4.5. Lemma.** *For any $M \in \mathbf{K}_0$, if $\chi_M^*(B/A) > t(|A|, |B|)$ then $A \leq_s B$.*

PROOF. Suppose some $B'$ with $A \subseteq B$ satisfies $A \leq_i B'$. Then there are more than $t(|A|, |B|)$ disjoint copies of $B'$ over $A$ in $M$ contradicting Fact 4.4.

We also need the finer analysis of the intrinsic closure carried out in [2]. In fact, this argument depends on the slightly finer notion of a *semigeneric* which is defined in [2]. The crucial facts from [3] and [2] are the following.

**4.6. Fact.** If $(\mathbf{K}_0, \leq_s)$ satisfies Context 1.10 and has the full amalgamation property then the theory of the generic $T$ satisfies

(1) All models of $T$ are semigeneric.

(2) $T$ is stable. For any formula $\phi(x_1 \dots x_r)$ there is an integer $\ell = \ell_\phi$, such that for any semigeneric $M \in \mathbf{K}$ and any $r$-tuples $\overline{a}$ and $\overline{a}'$ from $M$ if $\text{icl}_M^{\ell_\phi}(\overline{a}) \approx \text{icl}_M^{\ell_\phi}(\overline{a}')$ then $M \models \phi(\overline{a})$ if and only if $M \models \phi(\overline{a}')$.

**4.7. Theorem.** *Let the language L contain only binary relation symbols. If* $(\mathbf{K}_0, \leq)$ *satisfies Context* 1.10 *and has the full amalgamation property then the theory of the generic T does not have the finite cover property.*

PROOF. Suppose not. We know $T$ is stable so there is a formula $\phi$ satisfying the conditions of Fact 4.1. Each model of $T$ is semigeneric. Choose $\ell = \ell_\phi$ as in Fact 4.6 so that the isomorphism type of $\mathrm{icl}^\ell_M(\overline{c}, \overline{a}, \overline{b})$ determines the truth of $\phi(\overline{c}, \overline{a}, \overline{b})$ for any triple of $\overline{c}, \overline{a}, \overline{b}$ of appropriate length. For any $n$ choose $m$ sufficiently large with respect to the maximal cardinality of $\mathrm{icl}^\ell_M(\overline{c}, \overline{a}, \overline{b})$ and $n$ so that applying the pigeonhole principle and Ramsey's theorem we can choose $\overline{c}_m$ so that the $\langle \overline{a}_i : i < n\rangle$ are pairwise $\overline{c}_m$-inequivalent and for $i < n$ letting $A_i = \mathrm{icl}^\ell_M(\overline{c}, \overline{a}_i)$ and $C = \mathrm{icl}^\ell_M(\overline{c})$ the following property $P(C)$ holds.

1. for all $i, j$, $A_i \approx_C A_j$
2. for $i < j$, $A_0 A_1 \approx_C A_i A_j$.

If $n > t(k, |A_0|)!$ for $k < |A_0|$, applying the $\Delta$-system Lemma we can find $\hat{C}$ with $C \subseteq \hat{C} \subseteq A_0$ such that (without loss of generality) the $A_i$ are disjoint over $\hat{C}$. By appropriate choice of $n$, depending only on $|A_0|$, $|C|$, we may assume that $p(\hat{C})$ holds. By Fact 4.5, $\hat{C} \leq_s A_0$. We claim in fact that the structure imposed on $A_0 A_1$ is $A_0 \otimes_{\hat{C}} A_1$. If not, $R(A_0, \hat{C}, A_1)$ is nonempty. Let $E_i$ denote the substructure of $M$ with universe $\bigcup_{j<i} A_j$. By Axiom 1.4 (3) for sufficiently large $k$, $\delta(A_k / E_k) < 0$. There is a minimal pair $(E'_k, A'_k)$ with $E'_k \subseteq E_k$ and $A'_k \subseteq A_k$. But then for each $j > k$ there is a copy $A'_j$ of $A'_k$, contained in $A_j$ and isomorphic to $A'_k$ over $E_k$ (since the language is binary). This contradicts the bound on the number of copies of a minimal pair, Fact 4.4. Thus we establish the claim. But now we have $E_{i+1} \approx E_i \otimes_C A_0$. Since $(\mathbf{K}_0, \leq_s)$ has full amalgamation, this construction can be carried on indefinitely. But the definition of $\ell_\phi$ guarantees that the $\overline{a}_i$ represent distinct $\overline{c}$-equivalence classes and this contradicts the hypothesis that there are only finitely many $\overline{c}$-equivalence classes.

**4.8. Conclusion.** The arguments in the paper are fully worked out only for languages with binary relation symbols. For Section 4, this is just a matter of easing notation; slight modifications of the argument work for any finite relational language. The combinatorial arguments in Section 3 are sufficiently complicated that the proof is the general case is less clear. But it would be quite surprising if the restriction to a binary language is actually necessary.

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