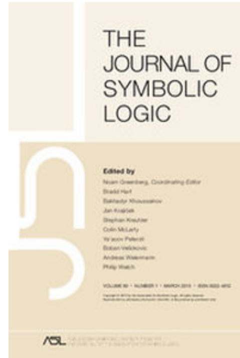

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FORCING CONSTRUCTIONS FOR UNCOUNTABLY CHROMATIC GRAPHS

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§0. Introduction. In this paper we solve some of Pál Erdős's favorite problems on uncountably chromatic graphs. Generalizing a finite graph theory result of Tutte, Erdős and R. Rado showed that for every infinite cardinal κ there exists a triangle-free, κ-chromatic graph of size κ. For $\kappa = \aleph_0$, Erdős established the existence of $\aleph_0$-chromatic graphs excluding even $C_4, C_5, \ldots, C_n$, i.e. circuits up to a given length. For $\kappa > \aleph_0$ the situation is different. As shown by Erdős and A. Hajnal, a graph is necessarily countably chromatic if it omits any finite bipartite graph. We can, however, exclude any finite list of nonbipartite graphs (this obviously reduces to excluding finitely many odd circuits). They posed an even stronger conjecture, namely, that similar examples must occur in every uncountably chromatic graph. To be specific, they conjectured that for every infinite κ, every κ-chromatic graph contains a κ-chromatic triangle-free subgraph. Here we show that this may not be true for $\kappa = \aleph_1$, i.e. we exhibit a model where it is false. We must emphasize that the conjecture is probably false already in ZFC, but we have been unable to show this.

A. Hajnal and A. Máté proved that under $V = L$ there is, on ω_1, an uncountably chromatic graph which is small in another sense: every countable ordinal $< \omega_1$ is joined to at most a converging ω-sequence among the smaller ordinals. They also showed that these graphs are countably chromatic if $\mathrm{MA}_{\aleph_1}$ holds. The first author showed that, under diamond, even a triangle-free such graph exists, but was unable to exclude longer odd circuits. Here we do construct these objects, first by forcing (this is the easiest), then from $\diamond$, and, finally, using the previous constructions, we construct slightly weaker examples in ZFC alone. Of course, the full strength of the Hajnal-Máté notion of being small is lost, but we still can omit the complete countable bipartite graph (we call it $K(\aleph_0, \aleph_0)$). It was Hajnal who first produced, under CH, an uncountably chromatic graph omitting both the triangle $K(3)$ and $K(\aleph_0, \aleph_0)$. To guarantee uncountable chromatic number, he used a different, stronger idea: his graph witnessed $\omega_1^2 \not\to (\omega_1^2, 3)^2$. We are going to show the rather surprising fact that it is impossible to reach our target via a similar construction, as $\omega_1^2 \to (\omega_1^2, C_5)^2$. Notice that it is not known if $\omega_1^2 \to (\omega_1^2, 3)^2$ is consistent.

Theorems 1 and 5 were proved by S. Shelah, the other results by P. Komjáth.

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§1. Uncountably chromatic graphs with no triangle-free uncountably chromatic subgraphs. An old problem of P. Erdős and A. Hajnal asked if for every $\kappa \geq \aleph_0$ every κ-chromatic graph contains a κ-chromatic triangle-free subgraph. See [1]–[6]. This was shown to hold for $\kappa = \aleph_0$ by V. Rödl [14]. Here we show that for $\kappa = \aleph_1$ the answer is consistently not; there may exist a graph X with size and chromatic number $\aleph_1$ such that every $Y \subseteq X$ which does not contain a $K(\aleph_0)$ is countably chromatic. ($K(\alpha)$ denotes the complete graph/ordered graph on α whenever α is a cardinal/ordinal.) We notice that every uncountably chromatic graph obviously contains an uncountably chromatic subgraph not embedding a $K(\aleph_1)$. We also show that there may exist a $K(4)$-free uncountably chromatic graph with only countably chromatic triangle-free subgraphs.

THEOREM 1. *It is consistent that $2^{\aleph_0} = \aleph_2$ and there exists a graph X on ω_1 with* $\mathrm{Chr}(X) = \aleph_1$ *and every $Y \subseteq X$ with $K(\omega + 1) \not\subseteq Y$ satisfies* $\mathrm{Chr}(Y) \leq \aleph_0$.

PROOF. We are going to build an iterated forcing construction. Assume V is a model of ZFC + CH. Our notion of forcing will be a finite-support iteration of length ω_2, $P = P_{\omega_2}$. As usual, it is enough to specify Q_α such that $P_{\alpha+1} = P_\alpha * Q_\alpha$. Let Q_0 be the partial order which adds a generic graph X on ω_1 with finite conditions, and for $1 \leq \alpha < \omega_2$, if $Y_\alpha \in V^{P_\alpha}$ is a subgraph of X not containing a $K(\omega + 1)$, then we let Q_α be the partial order for adding an ω-coloring of Y_α by finite conditions. As usual, if P_{ω_2} will have the ccc, we can eventually treat every $Y \subseteq X$ not containing a $K(\omega + 1)$.

We now give the exact definitions.

$\langle s, g\rangle \in Q_0$ iff $s \in [\omega_1]^{<\omega}$ and g is a graph on s, i.e. $g \subseteq [s]^2$.

$\langle s', g'\rangle \leq \langle s, g\rangle$ iff $s' \supseteq s$ and $g = g' \cap [s]^2$.

Let X denote the generic graph added by Q_0. In V^{P_α}, if $Y_\alpha \subseteq X$ is a $K(\omega + 1)$-free graph, we let $f \in Q_\alpha$ iff $\mathrm{Dom}(f) \in [\omega_1]^{<\omega}$, $\mathrm{Rng}(f) \subseteq \omega$, and for $x, y \in \mathrm{Dom}(f)$, if $\{x, y\} \in Y_\alpha$ then $f(x) \neq f(y)$. We write $f' \leq f$ just in case f' extends f.

Notice that by trivial density arguments the generic coloring added by Q_α will color Y_α by countably many colors.

DEFINITION 1. For $p \in P_\alpha$, $\mathrm{supp}(p) = \{\beta < \alpha : p \mid \beta \nVdash p(\beta) = \emptyset\}$.

Notice that $\mathrm{supp}(p)$ is a finite subset of α.

DEFINITION 2. For $\alpha < \omega_2$, set $D_\alpha = \{p \in P_\alpha$: there is a function t with $\mathrm{Dom}(t) = \mathrm{supp}(p)$ such that for every $\beta < \alpha$, $p \mid \beta \Vdash p(\beta) = t(\beta)$, and $p(0) = \langle s, g\rangle$ with $s \supseteq \mathrm{Dom}(p(\beta))$ for every $\beta \neq 0\}$.

LEMMA 1. *For every $\alpha < \omega_2$, D_α is dense in P_α.*

PROOF. By induction on α. The case $\alpha = 1$ is trivial, as $D_1 = P_1$. The limit case is again trivial, as we take direct limits.

Assume $\alpha = \beta + 1$. If $p \in P_\alpha = P_\beta * Q_\beta$, then we first extend p to a p' such that $p' \mid \beta \Vdash p'(\beta) = f$ for a finite function f, and $\mathrm{Dom}(f) \subseteq p'(0)$; then we use the induction hypothesis and extend $p' \mid \beta$ to a $q \in D_\beta$. Now $q^\wedge f \in D_\alpha$ and extends p.

LEMMA 2. *Assume that $\alpha \leq \omega_2$, $p, q \in D_\alpha$, $p(\beta)$ and $q(\beta)$ are compatible for every $\beta < \alpha$, and, if $x \in \mathrm{Dom}(p(\beta)) - \mathrm{Dom}(q(\beta))$, then $x \in s^p - s^q$, where $\langle s^p, g^p\rangle = p(0)$ and $\langle s^q, g^q\rangle = q(0)$, and similarly for p and q interchanged. Then p and q are compatible.*

PROOF. Put $r(0) = \langle s^p \cup s^q, g^p \cup g^q\rangle$ (this is the smallest extension of $p(0)$ and $q(0)$, which are, by assumption, compatible). For $0 < \beta < \alpha$, put $r(\beta) = p(\beta) \cup q(\beta)$.

We show that r is a condition. If $r|\beta$ has already been shown to be a condition, then, as $r|\beta \leq p|\beta$, $q|\beta$, it forces that $p(\beta)$ and $q(\beta)$ both partially color Y_β. We show that this is still true for $p(\beta) \cup q(\beta)$. The only problem is if $x \in \mathrm{Dom}(p(\beta)) - \mathrm{Dom}(q(\beta))$, $y \in \mathrm{Dom}(q(\beta)) - \mathrm{Dom}(p(\beta))$, and $r|\beta$ does not force $\{x, y\} \notin Y_\beta$. But then, by assumption, $x \in s^p - s^q$ and $y \in s^q - s^p$, so $\{x, y\} \notin g^p \cup g^q$, i.e. $r(0)$ forces that $\{x, y\} \notin X$.

LEMMA 3. *For $\alpha \leq \omega_2$, P_α has the ccc.*

PROOF. Assume that $p_\xi \in P_\alpha$ $(\xi < \omega_1)$. We can assume that $p_\xi \in D_\alpha$, $\mathrm{supp}(p_\xi) = a \cup b_\xi$ with $b_\xi \cap b_{\xi'} = \varnothing$ for $\xi < \xi' < \omega_1$, and for $\alpha \in a$, $\alpha > 0$, and $\xi < \omega_1$ the following holds:

$$p_\xi(\alpha) = f_\alpha \cup f_{\alpha,\xi}$$

with $\mathrm{Dom}(f_{\alpha,\xi})$ disjoint from each other. For $\alpha = 0$ we can assume that $p_\xi(0) = \langle s \cup s_\xi, g \cup g_\xi \rangle$. There are $\xi < \xi'$ with $\mathrm{Dom}(f_{\alpha,\xi}) \cap s = \varnothing$ and $\mathrm{Dom}(f_{\alpha,\xi'}) \cap s = \varnothing$ for $\alpha \in a$, $\alpha \neq 0$. By Lemma 2, p_ξ and $p_{\xi'}$ are compatible.

To finish the proof of Theorem 1, we need to show that the chromatic number of X remains $\aleph_1$, even after forcing by P_{ω_2}. We show more, that there is no stationary set which is independent in X. Assume that this latter statement fails, i.e. that for some $p \in P_{\omega_2}$, $p \Vdash S$ is independent in X, where S is a name for a stationary set. Then for stationary many ξ there is a $p_\xi \leq p$ forcing $\xi \in S$.

LEMMA 4. *For every $\xi < \omega_1$ and $p \in P_\alpha$ $(\alpha \leq \omega_2)$, there is a $q \in D_\alpha$, $q \leq p$, such that for every $\beta \in \mathrm{supp}(q)$ there exists an $s \in [\xi]^{<\omega}$ such that $q|\beta$ forces that $s \cup \{\xi\}$ spans a complete graph in Y_β, but not $s \cup \{\zeta, \xi\}$ for $\max(s) < \zeta < \xi$.*

PROOF. By induction on α. The cases $\alpha = 1$ and α limit are again trivial, so we can assume $\alpha = \beta + 1$. Given $p \in P_\alpha$, first extend $p|\beta$ to decide $p(\beta)$ and an $s \in [\xi]^{<\omega}$ as in the lemma for Y_β (such an s exists, as Y_β does not embed a $K(\omega + 1)$); then extend $p|\beta$ by the lemma for $p|\beta$, β (using the induction hypothesis).

To finish the proof that X will remain uncountably chromatic, apply Lemma 4 for the p_ξ and ξ given before the formulation of Lemma 4. This gives $q_\xi \leq p_\xi$ for stationary many ξ. By applying the pressing-down lemma, we can assume that $\mathrm{supp}(q_\xi) = a \cup b_\xi$ with pairwise disjoint b_ξ's, and for $\alpha \in a$ the graphs/functions $q_\xi(\alpha)$ are compatible. Moreover, we can assume that the set $s_{\alpha,\xi}$ given by Lemma 4 depends only on α (and not on ξ), $s_{\alpha,\xi} = s_\alpha$. We show that $r \leq q_\xi$, q_ζ, where $\zeta < \xi < \omega_1$, $r(\alpha) = q_\xi(\alpha) \cup q_\zeta(\alpha)$ for $\alpha \in a \cup b_\xi \cup b_\zeta - \{0\}$, and $r(0)$ even adds the edge $\{\zeta, \xi\}$. We need to show that r is a condition, i.e. for every $\alpha \in a - \{0\}$, $r|\alpha \Vdash \{\zeta, \xi\} \notin Y_\alpha$. But $s|\alpha$ already forces that $s_\alpha \cup \{\zeta\}$ and $s_\alpha \cup \{\xi\}$ both form a complete graph while $s_\alpha \cup \{\zeta, \xi\}$ does not. Once all this is done, r forces that ζ and ξ are in S and they are joined, a contradiction.

We next show that a graph with the properties of our main result may exist under CH.

THEOREM 2. *It is consistent that CH holds and there exists an uncountably chromatic graph X on ω_1 such that every triangle-free $Y \subseteq X$ is countably chromatic.*

PROOF. Let V model ZFC + GCH, and let $S \subseteq \omega_1$ be a stationary, costationary set of limit ordinals. We are going to build a countable-support iterated forcing of length ω_2, $\langle P_\alpha, Q_\alpha : \alpha < \omega_2 \rangle$.

Sh:303

Let Q_0 be the partial order which adds a graph $X \subseteq [\omega_1]^2$ with the following properties:

(1.1) for $\alpha \in S$, $\{\beta < \alpha : \{\beta, \alpha\} \in X\}$ is either finite, or cofinal in α with order-type ω,

(1.2) $$\text{if } \beta < \alpha,\ \alpha \notin S, \text{ then } \{\beta, \alpha\} \notin X,$$

with countable conditions. That is, $q \in Q_0$ iff $q = \langle \alpha, s \rangle$ with $\alpha < \omega_1$, $s \subseteq [\alpha]^2$ with (1.1), (1.2), and $\langle \alpha', s' \rangle \leq \langle \alpha, s \rangle$ iff $\alpha' \geq \alpha$ and $s = s' \cap [\alpha]^2$. Graphs with similar properties will be investigated in §2.

For $1 \leq \alpha \leq \omega_2$ let Y_α be a triangle-free subgraph of X, living in V^{P_α}. Let Q_α be the generic ω-coloring with countable conditions, i.e.

(1.3) $$q \in Q_\alpha \text{ iff } q: \gamma \to \omega \text{ is a good coloring of } Y_\alpha \text{ and } \gamma \notin S.$$

Obviously, Q_0 is ω_1-closed and all the Q_α are S-proper; therefore, by [14, Chapter V], $P = P_{\omega_2}$ does not add reals, nor does it collapse cardinals. Clearly, all the Y_α will be $\aleph_0$-chromatic.

We are going to show that X will be $\aleph_1$-chromatic in the final model. Assume that $1 \Vdash f: \omega_1 \to \omega$ is a good coloring. Let λ be a large enough regular cardinal, and let N be a countable elementary submodel of $\langle H(\lambda); \in, P, \text{supp}, \text{f}, \ldots \rangle$, with $\alpha = N \cap \omega_1 \in S$.

By S-properness, there is a generic sequence G of P, N. Let $p = \bigcup G$. p is not yet a condition, but we show that there exists an extension of p which is a condition, and forces that α can get no color under f. Enumerate $N \cap \omega_2$ as $\{\gamma_0, \gamma_1, \ldots\}$. Below we will sometimes write Y_i instead of Y_{γ_i}. p determines X, Y_i $(i < \omega)$, and f up to α, and also a good coloring g_i for $Y_i | \alpha$. Let α_i converge to α. We build an inductive construction on i. In the ith step, we select $y_i < \alpha$, $c(i,j) < \omega$ $(j < i)$, or do nothing. Assume that in the ith step

(1.4) $$\begin{gathered}\text{there is a } y \text{ with } \alpha_i < y < \alpha,\ f(y) = i, \text{ and for every } j < i,\\ \text{if } g_j(y) = c(t,j) \text{ for some } t < j, \text{ then } \{y, y_t\} \in Y_j.\end{gathered}$$

If this condition holds, choose as y_i any such y; then choose colors $c(i,j)$ with $g_j(y_t) < c(i,j)$ for $t \leq i$. If (1.4) does not hold, we do nothing.

Extend p to p' by adding all the $\{y_i, \alpha\}$ to $p(0)$. First we show that p' can be extended to a condition. By induction on $\beta \in N \cap \omega_2$ (i.e. in increasing order), we define a condition $q | \beta$ which decides the edges between $\{\alpha\}$ and α. If it forces that no edge of $Y_i(\beta = \gamma_i)$ goes between $g_i^{-1}(0)$ and α, let $g_i(\alpha) = 0$. If $g_i(y_j) = 0$ and $\{y_j, \alpha\} \in Y_\beta$, then every $t \geq j$ with $g_i(y_t) = c(j,i)$ will satisfy $\{y_t, y_j\} \in Y_i$; so, as Y_i is triangle-free, $\{y_t, \alpha\} \notin Y_i$, and we can define $g_i(\alpha) = c(j,i)$. Then keep further extending q.

Once q is given, extend it to an r with $r \Vdash f(\alpha) = i$. Then r forces that there is a y (i.e. $y = \alpha$) with $\gamma_i < y$, $f(y) = i$, and, if $g_j(y) = c(t,j)$, then $\{y, y_t\} \in Y_j$ for $j, t < i$. But, as N is an elementary submodel, p forces this, i.e. an appropriate y_i has been chosen, so $f(y_i) = f(\alpha) = i$ and they are joined in X, a contradiction.

In our example the strongest possible property X may possess is that it does not contain a $K(4)$. Notice that by an easy argument this already gives the following theorem of J. Folkman: for every $n < \omega$, there exists a $K(4)$-free (finite) graph such that, coloring its edges by n colors, there always exists a monocolored triangle [6].

THEOREM 3. *It is consistent that* $2^{\aleph_0} = \aleph_2$ *and there exists a* $K(4)$*-free graph* X *on* ω_1 *with* $\mathrm{Chr}(X) = \aleph_1$, *such that every* $Y \subseteq X$ *with* $K(3) \not\subset Y$ *has* $\mathrm{Chr}(Y) \leq \aleph_0$.

PROOF. We slightly modify the forcing construction of Theorem 1. We let $q \in Q_0$ iff $q = \langle s, h, g \rangle$, where $s \in [\omega_1]^{<\omega}$, h is a regressive function on $s - \{0\}$, g is a $K(4)$-free graph on s, and if $\beta < h(\alpha)$, then $\{\beta, \alpha\} \notin g$. We write $\langle s', h', g' \rangle \leq \langle s, h, g \rangle$ iff $s' \supseteq s$, $h' \supseteq h$, and $g = [s]^2 \cap g'$.

Let X denote the generic graph. Obviously, X does not contain a $K(4)$. Now construct the partial orders P_α and Q_α for $\alpha < \omega_2$ as in Theorem 1. The proof that P_{ω_2} is ccc is the same as there. We have to show that $\mathrm{Chr}(X) = \aleph_1$ in $V^{P_{\omega_2}}$.

If A and B are graphs, let $A \to (B)^2_2$ denote that however the edges of A are colored, there always exists a monocolored copy of B. The generalization of the above-mentioned result of Folkman given by Nešetřil and Rödl says that for every finite $K(4)$-free graph B there is a finite $K(4)$-free A such that $A \to (B)^2_2$. (See [12].)

Assume that $p \Vdash f\colon \omega_1 \to \omega$ is a good coloring of X. Let $\mathfrak{M}$ be an elementary submodel of $\langle H(\lambda); \in, P, p, \Vdash, \mathrm{f}, \mathrm{supp}, \ldots \rangle$ of size $\aleph_1$, containing ω_1, where λ is a large enough regular cardinal. If the underlying set of $\mathfrak{M}$ is M, add F to $\mathfrak{M}$, where $F\colon \omega_1 \to M$ is a bijection. Now form an increasing, continuous sequence $\langle M_\alpha\colon \alpha < \omega_1 \rangle$ of countable elementary submodels of $\langle \mathfrak{M}, F \rangle$. Put $\delta_\alpha = M_\alpha \cap \omega_1$. If $p(0) = \langle s, h, g \rangle$, extend p to a q with $q(0) = \langle s \cup \{\delta_0\}, h^\frown \langle \delta_\omega, \delta_0 \rangle, g \rangle$, and $q(\beta) = p(\beta)$ for $\beta \neq 0$. Next, extend q to a p_ω with $p_\omega \Vdash f(\delta_\omega) = i$.

Put $S = h^{-1}(\mathrm{supp}(p_\omega)) \cap \delta_\omega$. If $|S| = n + 1$, choose $T_0 = K(2)$ and $T_{k+1} \to (T_k, K(3))^2_2$ with T_{k+1} not containing a $K(4)$. (This is possible by the above-mentioned result of Nešetřil and Rödl.) Let T_n be a graph on N vertices, $N < \omega$.

Put $y_t = \delta_{2N-t}$ and $x_t = \delta_t$ for $0 \leq t \leq N - 1$. Let $p_t \leq p$ be a condition with $p_t \Vdash f(y_t) = i$, such that

$$p_t(0) = \langle s_t, h_t, g_t \rangle \leq \langle s \cup \{y_t\}, h^\frown \langle y_t, x_t \rangle, g \rangle$$

with $g_t \cap [x_t]^2 = g_\omega \cap [x_t]^2$, $s_t \cap y_t \subseteq x_{t+1}$, and

$$F^{-1}(\mathrm{supp}(p_t)) \cap x_t = S.$$

This is possible, as we are working with elementary submodels. Put $F''S = \{\alpha_0, \ldots, \alpha_n\}_<$, with $\alpha_0 = 0$. Notice that the sets $\mathrm{supp}(p_t) - \{\alpha_0, \ldots, \alpha_n\}$ are disjoint.

We are going to show by induction on $k \leq n$ that there exists a condition $r_k \in P_{\alpha_k}$ such that $r_k(0)$ spans a T_{n-k} on an appropriate $H_k \subseteq \{y_0, \ldots, y_{N-1}\}$,

$$r_k \leq p_t | \alpha_k \quad \text{for } y_t \in H_k,$$

and if y_t and $y_{t'}$ are joined in $r_k(0)$, then

$$r_k \Vdash \{y_t, y_{t'}\} \in Y_{\alpha_k}.$$

Now for $k = n$ this gives the contradiction.

For $k = 0$, we have to fuse the conditions $p_t(0)$ with T_n on $\{y_0, \ldots, y_{N-1}\}$. An easy check gives that no $K(4)$ is given, so this is a condition.

If the statement is proved for k, we first extend r_k to a condition $r \in P_{\alpha_{k+1}}$ such that $r | (\alpha_k, \alpha_{k+1})$ extends all $p_t | (\alpha_k, \alpha_{k+1})$ for $t \in H_k$ (this is possible, as these conditions have pairwise disjoint support in this interval). Next extend r to a condition s such that for every edge of $X \cap [H_k]^2$, s decides if that edge is in $Y_{\alpha_{k+1}}$ or not. This is

Sh:303

possible, as $Y_{\alpha_{k+1}}$ is in $V^{P_{\alpha_{k+1}}}$. Now we color the edges with two colors according to if they are in $Y_{\alpha_{k+1}}$ or not. By the partition property, there is a monocolored T_{n-k-1}.

Notice the following easy result.

THEOREM 4. *If X is a $K(4)$-free graph with $\mathrm{Chr}(X) > 2^{\aleph_0}$, then there is an uncountably chromatic triangle-free $Y \subseteq X$.*

PROOF. Order the vertices of X. If for some vertex v the graph spanned by $\{y < v: \{y, v\} \in X\}$ is uncountably chromatic, we are done, as this graph is triangle-free. Otherwise, the edges of X can be decomposed into countably many classes, none containing a triangle, i.e., if the theorem fails, X is the union of countably many countably chromatic subgraphs. The fusion of those colorings gives a coloring of X with $2^{\aleph_0}$ colors.

Similarly, one can show e.g. that if κ is strongly compact, $\mathrm{Chr}(X) = \kappa$, and $K(4) \not\subseteq X$, then X contains a triangle-free, κ-chromatic subgraph.

§2. Eliminating short odd circuits. In their paper [9], A. Hajnal and A. Máté proved that under $\diamondsuit^+$ there exists a graph on ω_1 with uncountable chromatic number, such that for every $\alpha < \omega_1$ the set of those ordinals less than α which are joined to α is either finite or of type ω converging to α. In fact, adding a Cohen real also produces such a graph (see [15]). Later, the first author showed that under $\diamondsuit$ a triangle-free graph with the above properties exists [10], and he even succeeded in eliminating some other (ordered) circuits. Here we show that short odd circuits can be avoided. Let C_s denote the circuit on s vertices.

THEOREM 5. *If $n < \omega$, it is consistent that there exists a graph X on ω_1 not containing $C_3, C_5, \ldots, C_{2n+1}$ with $\mathrm{Chr}(X) = \aleph_1$, and with the property that for every $\beta < \alpha < \omega_1$, the set $\{\gamma < \beta: \{\gamma, \alpha\} \in X\}$ is finite.*

PROOF. Assume that $1 \leq n < \omega$. A condition $p \in P$ will be of the form $\langle s, g, h_1, \ldots, h_n\rangle$, where $s \in [\omega_1]^{<\omega}$, $g \subseteq [s]^2$ is a graph not containing $C_3, \ldots, C_{2n+1}$, the functions $h_1, \ldots, h_n$ are defined for $\alpha \in s$, and, for $\alpha \geq \omega$,

$$h_n(\alpha) < h_{n-1}(\alpha) < \cdots < h_1(\alpha) < \alpha,$$

and, finally, if $\beta < \alpha$, $\beta, \alpha \in s$, and there is a path in g of length t between α and β with $t \leq n$, then $\beta \geq h_t(\alpha)$.

Next we define extension. If $\langle s, g, h_1, \ldots, h_n\rangle$ and $\langle s', g', h'_1, \ldots, h'_n\rangle$ are conditions, the latter *extends* the former if $s' \supseteq s$, $g = [s]^2 \cap g'$, $h'_t \supseteq h_t (1 \leq t \leq n)$, and for every $\alpha \in s$, if $\beta < \alpha$, $\{\beta, \alpha\} \in g' - g$, then $\beta > \gamma$ for every $\gamma < \alpha$ with $\{\gamma, \alpha\} \in g$.

If G is a generic subset of P, then $X = \bigcup\{g: \langle s, g, h_1, \ldots, h_n\rangle \in G\}$ is obviously a graph on ω_1 not containing any of the $C_3, \ldots, C_{2n+1}$.

We now show that X possesses the Hajnal-Máté property, i.e. if $\alpha < \omega_1$, the ordinals $< \alpha$ joined to α form either a finite set or an ω-sequence converging to α. Choose $\beta < \alpha$. There is a $p \in G$ with $\beta, \alpha \in s$, where s is the first coordinate of p. This p will force that α is joined to only finitely many vertices smaller than β, as, by the definition of extension, they must be in s.

That P has the ccc can be seen the usual way.

We now show that the chromatic number of X is $\aleph_1$. Assume that p forces that $f: \omega_1 \to \omega$ is a good coloring of X. Assume that λ is a large enough regular cardinal, and

$$\mathfrak{M}_n \prec \mathfrak{M}_{n-1} \prec \cdots \prec \mathfrak{M}_0 \prec \langle H(\lambda); \in, p, P, \mathbf{f}, \ldots\rangle$$

are countable elementary submodels. Put $\delta_i = M_i \cap \omega_1$. If $p = \langle s, g, h_1, \ldots, h_n \rangle$, then obviously $s \subseteq \delta_n$. Extend p to δ_0 by defining $h_t'(\delta_0) = \delta_t$. Then extend further this $\langle s \cup \{\delta_0\}, g, h_1', \ldots, h_n' \rangle$ to a q with $q \Vdash f(\delta_0) = i$. If $q = \langle \bar{s}, \bar{g}, \bar{h}_1, \ldots, \bar{h}_n \rangle$, put $s_t = \bar{s} \cap [\delta_t, \delta_{t-1})$ for $t = 0, \ldots, n$ (with $\delta_{-1} = \omega_1$).

We next find a very close twin for q. Let

$$A = s \cup \bigcup \{h_j''(x) : x \in s, j \leq n\},$$
$$A_t = [\delta_t, \delta_{t-1}) \cap A \quad (\text{for } t = 0, \ldots, n+1, \delta_{-1} = \omega_1, \delta_{n+1} = 0).$$

Claim. There is a condition $q' = \langle s', g', h_1', \ldots, h_n' \rangle$ *with its* $A_0', \ldots, A_{n+1}'$ *such that* $\langle s, g, h_j, A_j \rangle$ *and* $\langle s', g', h_j', A_j' \rangle$ *are isomorphic, and*

$$(*) \qquad \max A_{t+1} < \min A_t' \leq \max A_t' < \delta_t$$

for $t = 0, \ldots, n$, *and* $A_{n+1} = A_{n+1}'$. *Moreover,* $q' \Vdash f(\delta_0') = i$, *where* δ_0' *is the twin of* δ_0.

Let τ be the type of q over $A_{n+1} \cup \cdots \cup A_0$.

Proof of the claim. Let $\Phi_0(A_{n+1}, \ldots, A_0)$ be the statement that $q \Vdash f(\delta_0) = i$, where $\delta_0 = \min(A_0)$, and q is the condition built from $A_{n+1}, \ldots, A_0$ by type τ. Obviously, this is true. Now put

$$\Phi_{j+1}(A_{n+1}, \ldots, A_{j+1})$$

to denote that for every $\xi < \omega_1$ there is an $A_i \in [\omega_1 - \xi]^{<\omega}$ with $\Phi_j(A_{n+1}, \ldots, A_j)$ (for $j \leq n$). As $\mathfrak{M}_j$ is an elementary submodel, and $A_{n+1} \cup \cdots \cup A_{j+1} \subseteq M_j \cap \omega_1 \leq \min A_j$, we see by induction that all $\Phi_j(A_{n+1}, \ldots, A_j)$ are true.

Again by induction we can select A_j' for $j = n, \ldots, 0$, such that $\Phi_j(A_{n+1}', \ldots, A_j')$ and $(*)$ both hold (and $A_{n+1}' = A_{n+1}$).

To finish the proof of Theorem 5, let q, q' be as in the claim. If $q = \langle s, g, h_i \rangle$, $q' = \langle s', g', h_i' \rangle$, put $r = \langle s \cup s', g \cup g' \cup \{\delta_0, \delta_0'\}, h_i \cup h_i' \rangle$. We have to show that r is a condition, then r obviously forces that $f(\delta_0) = f(\delta_0') = i$, and they are joined, an obvious contradiction.

First we argue that there is no $C_3, C_5, \ldots$, or C_{2n+1} in $g \cup g' \cup \{\delta_0, \delta_0'\}$. The only possibility is by including the edge $\{\delta_0, \delta_0'\}$, but then the remaining part of the circuit must go through $s \cap \delta_n$ (the common vertices of g, g') but the path between δ_0 and $s \cap \delta_n$ must have length at least $n + 1$, as $h_n(\delta_0) = \delta_n$, and similarly for the path from δ_0' to $s \cap \delta_n$. The total length is therefore, at least $1 + 2(n + 1) = 2n + 3$.

Finally, we have to check that none of the conditions on the functions is violated. Assume that $x \in s$, there is a path of length a from x to δ_0 in g, a path of length b from δ_0' to y' in g', and either $x < h_{a+b+1}'(y')$ or $y' < h_{a+b+1}(x)$. (The other possibilities, i.e. not using edge $\{\delta_0, \delta_0'\}$, or using it twice, are easily ruled out.) Now obviously

$$x \geq h_a(\delta_0) = \delta_a \quad \text{and} \quad y' \geq h_b'(\delta_0') = \delta_b' > \delta_{b+1}$$

(by the claim). As in g there is a path of length a from x to δ_0, we get

$$h_{a+b+1}(x) \leq h_{b+1}(\delta_0) = \delta_{b+1} < y',$$

and as in g' there is a path of length b from y' to δ'_0, we get

$$h'_{a+b+1}(y') \leq h'_{a+1}(\delta'_0) = \delta'_{a+1} < \delta_a \leq x,$$

and we are done.

Let $K(\alpha, \beta)$ denote the complete bipartite graph with bipartition classes of size α and β, respectively. Erdős asked if there may exist an $\aleph_1$-chromatic graph not embedding $K(\aleph_0, \aleph_0)$ or C_5. We show that a slight modification of the construction in Theorem 5 gives such a graph.

THEOREM 6. *Let $n < \omega$. It is consistent that there exists an $\aleph_1$-chromatic graph on ω_1 containing neither a $K(\aleph_0, \aleph_0)$ nor any $C_3, C_5, \ldots, C_{2n+1}$.*

PROOF. Define P as in the proof of Theorem 5, except that we now require that g should not contain a C_4 of the following type: the vertices are $x < y < z < t$, the edges $\{x, y\}$, $\{x, z\}$, $\{y, t\}$, $\{z, t\}$. The whole proof goes through, so X will be an $\aleph_1$-chromatic graph with no $C_3, \ldots, C_{2n+1}$. Assume that $\{x_i, y_i: i < \omega\}$ forms a $K(\aleph_0, \aleph_0)$. We can assume that both $\{x_i: i < \omega\}$ and $\{y_i: i < \omega\}$ form a convergent sequence (otherwise we could take subsequences). They must converge to the same point α, as X has the Hajnal-Máté property. Therefore there are $x_i < y_j < y_k < x_l < \alpha$ and they form a forbidden C_4.

It is worth noticing that A. Hajnal originally gave a triangle-free $\aleph_1$-chromatic graph not containing a $K(\aleph_0, \aleph_0)$ by constructing a graph actually witnessing $\omega_1^2 \not\rightarrow (\omega_1^2, 3)^2$ (under CH: see [7]). This is impossible for the pentagon.

THEOREM 7. $\omega_1^2 \rightarrow (\omega_1^2, C_5)^2$.

PROOF. Assume that we are given a graph on ω_1^2, not containing a C_5. We show that there exists an independent set of type ω_1^2. As usual, we call the set $[\omega_1 \alpha, \omega_1(\alpha + 1))$ the αth column.

As $\omega_1 \rightarrow (\omega_1, C_5)$, we can assume that the columns are independent.

If there exists a vertex x which is joined to $\aleph_1$ vertices y in one column, such that each y is joined to $\aleph_1$ vertices in $\aleph_1$ columns, then we can select $\aleph_1$ vertices $\{y_\alpha: \alpha < \omega_1\}$ and different columns C_α such that y_α is joined to the uncountable $T_\alpha \subseteq C_\alpha$. But then $\bigcup\{T_\alpha: \alpha < \omega_1\}$ is obviously independent.

If there are only countably many columns C such that there exists a vertex joined to uncountably many points of C, then an obvious transfinite construction gives an independent set of type ω_1^2. So we can assume that there are $\aleph_1$ such columns, and, by the previous remark, every vertex is joined to countably many elements of every later column. We can then select a set of type ω_1^2 such that if the ξth element of the αth column is joined to the ζth element of the βth column, and $\alpha < \beta$, then $\xi > \zeta$, i.e. the edges "go down".

If now there exists a vertex which is joined to a set of type ω_1^2 in two steps, we can shrink this set such that the joining vertices are different (as each can be only used countably many times, because the edges go down) and this set must again be independent as there is no C_5.

This means that if there is a vertex which is joined to $\aleph_1$ elements of a column, then these $\aleph_1$ elements can be joined to countably many elements of all but countably many of the columns.

If there are only countably many columns such that there is a vertex joined to $\aleph_1$ elements of that column, we are done, as before. If not, selecting these columns rarely enough, we get a graph of type ω_1^2 such that if C_α and C_β are two columns with $\alpha < \beta$, then C_α is joined to countably many elements of C_β. Removing a countable set from each column, we get an independent set of type ω_1^2.

Next we show how to transform Theorem 5 into a construction from $\diamondsuit$. As the construction is rather involved, we elaborate it for the case of the pentagon.

THEOREM 8. *Under* $\diamondsuit$, *there exists a graph* $G \subseteq [\omega_1]^2$ *with* $\mathrm{Chr}(G) = \aleph_1$, $C_3, C_5 \not\subset G$, *and* G *has the Hajnal-Máté property, i.e. every vertex is joined to either finitely many smaller vertices, or to an* ω*-sequence converging to it.*

PROOF. We use the following observation [11]: $G \subseteq [V]^2$ is C_3, C_5-free if and only if there exists an $F \subseteq [V]^2$ such that

$$\{a,b\}, \{b,c\} \in G \Rightarrow \{a,c\} \in F, \tag{2.1}$$

$$\{a,b\}, \{b,c\} \in F \Rightarrow \{a,c\} \notin G, \tag{2.2}$$

$$F \cap G = \emptyset. \tag{2.3}$$

When building our graphs F and G we are going to specify by a transfinite recursion, for every $\alpha < \omega_1$, $F \cap \alpha \times \{\alpha\}$ and $G \cap \alpha \times \{\alpha\}$, assuming, of course, that $F \cap [\alpha]^2$ and $G \cap [\alpha]^2$ have already been constructed.

By diamond, we can assume that there exist functions $h_0(\alpha)$, $h_1(\alpha)$, $h_2(\alpha)$, and f_α such that for $\omega \leq \alpha < \omega_1$

$$h_0(\alpha) < h_1(\alpha) < h_2(\alpha) < \alpha$$

and for given $\gamma_0 < \gamma_1 < \gamma_2 < \omega_1$ and $f\colon \omega_1 \to \omega$ the set

$$\{\alpha < \omega_1 : h_i(\alpha) = \gamma_i \text{ for } i = 0,1,2, \text{ and } f_\alpha = f \mid \alpha\}$$

is stationary. We are going to build our F and G with the following additional conditions:

(2.4) if $\beta < h_2(\alpha)$, then β is not joined to α in G;

(2.5) if $\beta < h_1(\alpha)$, then β is not joined in two steps to α in G;

(2.6) if $\beta < h_0(\alpha)$, then β is not joined to α in F;

(2.7) $\{\beta < \alpha : \{\beta,\alpha\} \in G\}$ either is finite or converges to α;

(2.8) $\{\beta < h_2(\alpha) : \beta$ is joined to α in two steps in $G\}$ either is finite or converges to $h_2(\alpha)$;

(2.9) $\{\beta < h_1(\alpha) : \beta$ is joined to α in $F\}$ either is finite or converges to $h_1(\alpha)$.

Assume that we have already constructed everything up to the αth step. We do nothing, unless the following is true: α is limit, and there are elementary submodels $\mathfrak{M}_0, \mathfrak{M}_1, \mathfrak{M}_2$ of

$$\langle \alpha; F \cap [\alpha]^2, G \cap [\alpha]^2, h_0 \mid \alpha, h_1 \mid \alpha, h_2 \mid \alpha, \langle f_\beta : \beta < \alpha \rangle, f_\alpha, \ldots \rangle$$

with $M_j = \nu_j = h_j(\alpha)$ for $j = 0,1,2$, each being the union of an ω^2-sequence of elementary submodels. In countably many steps, we are going to select the ω type sets described in (2.7), (2.9), making sure that α can get no color if our models are elementary submodels of $\langle \omega_1; F, G, f\colon \omega_1 \to \omega, \ldots \rangle$. The simplest way of doing this is to join α to a vertex β with $f_\alpha(\beta) = i$ for every $i < \omega$. So assume that at a certain step the finite sets A and B as approximations for the sets in (2.9) and (2.7) have been constructed. Of course, A and B must satisfy

(2.10) if $\beta_1, \beta_2 \in A$, then $\{\beta_1, \beta_2\} \notin G$;

(2.11) if $\beta_1 \in A$, $\beta_2 \in B$, then $\{\beta_1, \beta_2\} \notin F$;

(2.12) if $\beta_1, \beta_2 \in B$, then $\{\beta_1, \beta_2\} \in F$.

If we cannot extend B with a vertex of color i, then the following is true: there are γ and δ with $\nu_1 < \delta < \nu_2 < \gamma < \alpha$, such that for every β with $\gamma < \beta < \alpha$, $h_2(\gamma) > \delta$, and $f_\alpha(\beta) = i$, *either* there exists a $\xi \in A$ with $\{\xi, \beta\} \in F$, *or* there exists a $\xi \in B$ with $\{\xi, \beta\} \notin F$. This can be formulated as a sentence about the parameters, $\Phi(\nu_0, A, \nu_1, \gamma, \nu_2, B, \delta)$. If we can show that there exist $\nu'_0, A', \nu'_1, \gamma', \nu'_2, B', \delta'$ still satisfying Φ and also

(2.13) $\nu_0 < \nu'_2, A' \subseteq \nu_0 < \gamma' < \nu'_2 < \min B' < \max B' < \nu_1$,

(2.14) γ' is arbitrarily large below ν_1,

(2.15) there is no G-edge between A and B', and

(2.16) there is no F-edge between B' and B,

then we can extend (A, B) to $(A \cup B', B)$ and this extension will guarantee that $f(\alpha) \neq i$.

If no such A', B', etc. can be found, this gives a certain sentence $\Psi(\nu_0, A, \nu_1, \gamma, \nu_2, B, \delta)$.

By the assumption about $\mathfrak{M}_0$, there is an increasing sequence of disjoint sets $\langle A(\tau): \tau < \omega^2 \rangle$ which have the properties of A. The B' of a certain pair (A', B') with (2.13) can have G-edge only to an ω-type subset of these $A(\tau)$'s (by the Hajnal-Máté property). There is therefore an $A(\tau)$ which can be extended with (2.15), so, if $B(\tau)$ is such that $A(\tau)$ and $B(\tau)$ satisfy $\Phi \wedge \Psi$, then there is an F-edge between B' and $B(\tau)$. But this is impossible, as again there is an ω^2 type sequence of appropriate B''s (even with no G-edges to $A(\tau)$), and every element β of $B(\tau)$ is F-joined to only a set of type ω^2 below $h_2(\beta)$, and as $h_2(\beta) \geq h_1(\alpha) = \nu_1$ for every $\beta \in B$, $B(\tau)$ can be selected with these values arbitrarily large below ν_1.

If the construction of F and G is finished, and $f: \omega_1 \to \omega$ is a good coloring of G, then there is an increasing, continuous chain of countable elementary submodels $\langle \mathfrak{M}_\alpha: \alpha < \omega_1 \rangle$ of $\langle \omega_1; F, G, f, \ldots \rangle$. Taking $\delta_\alpha = M_\alpha \cap \omega_1$, we choose an α with $h_j(\alpha) = \delta_{\omega^2 j}$ (for $j = 0, 1, 2$) and $f_\alpha = f | \alpha$, and observe that α can get no color under f.

We can modify this construction along the lines of Theorem 6.

THEOREM 9. *Under $\diamondsuit$ there is an $\aleph_1$-chromatic Hajnal-Máté graph not containing C_3, C_5, or $K(\aleph_0, \aleph_0)$.*

PROOF. To exclude circuits of type $\alpha < \beta < \gamma < \delta$ with edges $\{\alpha, \beta\}$, $\{\beta, \gamma\}$, $\{\gamma, \delta\}$, $\{\alpha, \delta\}$ means that the class F must split into two disjoint subclasses F_1 and F_2 such that if $\alpha < \beta < \gamma$ and $\{\alpha, \beta\}$, $\{\beta, \gamma\} \in G$, then $\{\alpha, \gamma\} \in F_1$, and if $\alpha < \beta < \gamma$ and $\{\alpha, \gamma\}$, $\{\beta, \gamma\} \in G$, then $\{\alpha, \beta\} \in F_2$.

When building our sets A and B, we require that the edges between members of B should be in F_2 (instead of F), and we have the intention that for $\beta \in A$, the edge $\{\beta, \alpha\}$ will be in F_2. The proof is virtually the same.

We can even relax the condition of the theorem by using an idea of F. Galvin.

THEOREM 10. *There exists an uncountably chromatic graph with size* $2^{\aleph_0}$ *which does not contain* C_3, C_5, *or* $K(\aleph_0, \aleph_0)$.

PROOF. In Theorems 8 and 9 we described a method which, given appropriate graphs F and G on α, and a coloring $f: \alpha \to \omega$, extends these graphs to α (assuming α is the union of an $\omega^2 4$ chain of elementary submodels). We are now building F and G on the tree $T = \{f: \alpha \to \omega, \alpha < \omega_1\rangle$, such that only comparable elements will be in F and G. To decide where to join a particular $f: \alpha \to \omega$, we look at the part already constructed on $\{g: g < f\}$, which has order-type α, and think of f as the coloring on this part. We only have to find $h_0(\alpha)$, etc. One way is to handle all possible choices simultaneously, defining countably many disjoint ω-sequences cofinal in α, and joining those points with α; the other possibility is to choose the *smallest* ordinals with those properties (sufficient elementarity). This way we get a graph such that only comparable points are joined, and if $f, g \in T$, the set of those vertices smaller than both and joined to both is finite. This insures, with the method of Theorem 9, that G does not contain a $K(\aleph_0, \aleph_0)$. If $H: T \to \omega$ is a good coloring, define $x_\alpha = H(\langle x_\beta: \beta < \alpha\rangle)$ by recursion on α. Now finish the proof by regarding the branch $\langle x_\alpha: \alpha < \omega_1\rangle$ as in Theorem 8.

The only other problem is that if we have graphs F and G with (2.1), (2.2), and (2.3), and we extend to disjoint sets two different ways: F', G', F'', G'', then the union of them is a subset of graphs with similar properties. This is an easy computation, and is shown in [11].

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