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MAIN GAP FOR LOCALLY SATURATED ELEMENTARY SUBMODELS OF A HOMOGENEOUS STRUCTURE

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Abstract. We prove a main gap theorem for locally saturated submodels of a homogeneous structure. We also study the number of locally saturated models, which are not elementarily embeddable into each other.

Hard experience has indicated that before we speak on this particular paper, we should say something on classification theory for nonelementary classes and of the specific context chosen here. Classification theory for first order theories is so established now that many tend to forget that there are other possibilities. There are some good reasons to consider these other possibilities: first, it is better to understand a more general context, we like to classify more; second, concerning applications many classes arising in 'nature' are not first order; third, understanding more general contexts may shed light on the first order one.

Of course, we may suspect that applying to a wider context will leave us with less content, but only trying will teach us if there are enough interesting things to discover.

In any case, 'not first order' does not define our family of classes of models. We are in particular interested in generalizing the main gap theorem for $\aleph_\varepsilon$-saturated models (see more below). Tending to the general case, we may consider replacing the first order theory by an $L_{\kappa^+,\omega}$-sentence ψ. Fixing the vocabulary, the notion of elementary submodel is with respect to this logic (all formulas with finitely many free variables) or at least with respect to a fragment, a family of formulas of $L_{\kappa^+,\omega}$ of cardinality $\leq \kappa$ closed under subformulas and including ψ. We may even consider abstract such classes discarding the logic altogether and working with 'algebraic' properties of the class of models. For such an approach see [37], [38], [39], Makkai and Shelah [25], and [40],[31] (both on universal classes), Grossberg and Hart [6], Hart and Shelah [14], Kolman and Shelah [21], [29], [28], [27], Shelah and Villaveces [43], [42] and Villaveces [44]. (See [28] on history and earlier works.) See also the closely related Grossberg and Shelah [11],[13], [12], Grossberg [5] and Baldwin and Shelah [2], [3], [4]. Naturally much of the work is on categoricity (as was the early

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history of the first order case). Anyhow in those cases even a very weak form of compactness may fail: 'compactness of types', see below.

To explain this we have to say first what we mean by 'the elements a and b realize the same type over the set B in the model $\mathscr{A}$'. If for simplicity we assume that for our class of models and our notion of elementary submodel $\prec$, there is a monster model $\boldsymbol{M}$, then 'the elements a and b realize the same type over the set B in the model $\mathscr{A}$' means that there is an automorphism of $\boldsymbol{M}$ which maps a to b and is the identity on B. (Without the monster we should say that this occurs in some extension of $\mathscr{A}$. If we have amalgamation this works nicely.) Now 'failure of compactness for types' means that for some model $\mathscr{A}$ in our class, elements a and b (or finite sequences) from $\mathscr{A}$ and a subset B of $\mathscr{A}$, the type of the elements a and b over the set B in the model $\mathscr{A}$ is not determined by their restrictions to finite subsets of B; i.e., for every finite subset A of B, there is an automorphism of $\boldsymbol{M}$ which is the identity on A and maps a to b but for $A = B$ there is no such automorphism. (Another way to point out the difficulty is that for an increasing sequence of sets or even models $(\mathscr{A}_\alpha : \alpha < \delta)$, in the appropriate sense, if $p_\alpha \in S(\mathscr{A}_\alpha)$ is increasing with α, do we have a limit type i.e., does $\bigcup_{\alpha<\delta} p_\alpha$ exist? This means: is there $p \in S(\bigcup_{\alpha<\delta} \mathscr{A}_\alpha)$ such that for every $\alpha < \delta$ we have $p \restriction \mathscr{A}_\alpha = p_\alpha$ and is it unique?) So the assumption that such a failure does not occur is quite reasonable.

Assuming that we have a class of models of $\psi \in L_{\lambda^+,\omega}$ with amalgamation and the joint embedding property and with 'compactness of types', we can prove the existence of a monster model $\boldsymbol{M}$ of cardinality κ, which is not saturated, but is 'κ-homogeneous for sequences'. So our class of models $\boldsymbol{K} = \boldsymbol{K}_M$ is the class of elementary submodels of $\boldsymbol{M}$ of cardinality $< \kappa$. Here 'κ-homogeneous for sequences' means that, if f is a partial map from $\boldsymbol{M}$ to $\boldsymbol{M}$ which preserves the satisfaction of first order formulas and has cardinality $< \kappa$, then it can be extended to an automorphism of $\boldsymbol{M}$. This gives a situation where we cannot use compactness for *arbitrary* sets of formulas, but types, defined as usually, behave 'normally'. Note that $\boldsymbol{M}$ is determined up to isomorphism by its cardinality κ and its finite diagram:

$D(\boldsymbol{M}) = \{t(a, \emptyset) : a \text{ a finite sequence from } \boldsymbol{M}\}$.

Classification theory in this context (i.e., using the family of elementary submodels of a κ-homogeneous for sequences monster model $\boldsymbol{M}$ as the class of models and the usual notion of elementary submodel) was started in [32], (and [33] , called there context IV, see page 250, particularly Theorem 1.13) and continued in [16], [17], [19], [18] and [10]. This is the context chosen here. Note that some attention was given to some special cases of it: [33] deal mainly with the following two related cases: In the first $\boldsymbol{M}$ is the universal homogeneous model for κ under usual embeddings for the class of models of T, a first order theory with amalgamation and the joint embedding property. Then we can restrict ourselves to existentially closed models. This is called context II there. The second is the class of existentially closed models of a first order theory with the joint embedding property, again under usual embeddings. This is called context III there. Lately Hrushovski [15] has dealt with context II: he shows that for it, some hopeful properties of non forking fail for simple such classes (on simple first order theories see e.g., [7] and [20]). See also [26].

Another simpler context is when instead stable we generalize $\aleph_0$−stable, such investigation have been carried by Grossberg and Lessmann [9], [8], Lessmann [23],[24].

By [41], a major result in classification theory is the main gap theorem for the class of models of a first order countable T. This essentially gives an understanding of the function counting the number of models of the class in a cardinality up to isomorphism. Weaker but still very important one is the main gap for the class of $\aleph_\varepsilon$-saturated models of a first order theory. This is proved in [35], [36], and the tenth chapter of [41] is dedicated for representing it (or see the tenth chapter of the book [22] and Part D of Baldwin's book [1]). Recall that in the first order case, a model $\mathscr{A}$ is called $\aleph_\varepsilon$-saturated if for every finite $A \subseteq \mathscr{A}$ and element $a \in \boldsymbol{M}$, there is an element $b \in \mathscr{A}$, which is equivalent with a for every equivalence relations with finitely many equivalence classes and definable by a first order formula with parameters form A.

Our aim here is to prove a parallel of this theorem in our context (see [6] and [9] for other main gap results for nonelementary classes). Note that for this we have to choose what is the right parallel of $\aleph_\varepsilon$-saturation. Why was the case of $\aleph_\varepsilon$-saturated models more accessible to analysis? It has enough saturation to make the existence of primary models work on the one hand, but not too much so that the class of such models is closed under union of increasing elementary chains. We find here a similar notion. For making it preserved by the union of increasing chains, it only says that 'for every finite subset A of $\mathscr{A}$ we have $\mathscr{B} \prec \mathscr{A}$ such that ...'. In order to have relevant primary models, we need to have something like the following property of $\aleph_\varepsilon$-saturated models: Let B be a subset of $\boldsymbol{M}$, A a finite subset of B and $p \in S(B)$ be such that $p \restriction A$ does not have a forking extension $q \in S(C)$ over a bigger finite subset C of B including A. Now if $\mathscr{A}$ is an $\aleph_\varepsilon$-saturated model including A, then p is realized in $\mathscr{A}$. This motivate our choice.

This work continues in particular [19]. Naturally, parallels to 'regular types', 'decomposition theorems' etc. play an important part.

Throughout this paper we let $\boldsymbol{M}$ be our monster model. As in [19], we assume that $\boldsymbol{M}$ is homogeneous and that $|\boldsymbol{M}|$ is strongly inaccessible. This can be done without loss of generality.

By a, b, etc. we mean finite sequences of elements of $\boldsymbol{M}$. Subsets of $\boldsymbol{M}$ of power $< |\boldsymbol{M}|$ are denoted by A, B, etc. and we write $\mathscr{A}$, $\mathscr{B}$, etc. for elementary submodels of $\boldsymbol{M}$ of power $< |\boldsymbol{M}|$.

We assume that the reader is familiar with [19] and we use its notions and results freely. Especially, we use the notion of independence defined in [19]. It is similar to non-forking. In fact, if $\boldsymbol{M}$ is saturated, then it is the same as non-forking.

The difference is that in our situation, the independence notion does not have all the properties of non-forking in the full strength. In [40], a related notion has been studied. We also assume that the reader knows the basic methods of using the non-forking calculus.

Let $A \subseteq \boldsymbol{M}$ and p be a type over A. We say that p is $\boldsymbol{M}$-consistent if it is realized in $\boldsymbol{M}$. We say that $\boldsymbol{M}$ is stable if there is $\lambda < |\boldsymbol{M}|$ such that for all $A \subseteq \boldsymbol{M}$ of power $\leq \lambda$, the number of complete $\boldsymbol{M}$-consistent types over A is $\leq \lambda$. Here we have a general rule: Mostly the notions used in this paper are got from their usual definition from stability theory ([41]) by replacing 'consistent' by '$\boldsymbol{M}$-consistent' and/or by replacing non-forking by the independence notion from [19]. E.g., $F^{\boldsymbol{M}}_\kappa$-saturation is got from F^s_κ-saturation by this rule (see Definition 0.1 (i)). Like this one, several of these concepts appeared already in [32] (but with different notation and in a slightly different context). The main exception to this rule is the notion of strong type. Instead of the usual strong types we use Lascar strong types. In fact, we do not talk about strong types over A but equivalence classes in the minimal equivalence relation $E^m_{min,A}$ (over A and between sequences of length m).

Notice that $\boldsymbol{M}$ may be stable while $Th(\boldsymbol{M})$ is unstable.

DEFINITION 0.1. (i) Suppose $\boldsymbol{M}$ is stable. We say that $\mathscr{A}$ is s-saturated if it is $F^{\boldsymbol{M}}_{\lambda(\boldsymbol{M})}$-saturated i.e., for all $A \subseteq \mathscr{A}$ of power $< \lambda(\boldsymbol{M})$ and a there is $b \in \mathscr{A}$ such that $t(b, A) = t(a, A)$.

(ii) We say that $\mathscr{A}$ is locally $F^{\boldsymbol{M}}_\kappa$-saturated if for all finite $A \subseteq \mathscr{A}$ there is $F^{\boldsymbol{M}}_\kappa$-saturated $\mathscr{B}$ such that $A \subseteq \mathscr{B} \subseteq \mathscr{A}$. If $\boldsymbol{M}$ is stable, then we say that $\mathscr{A}$ is e-saturated if it is locally $F^{\boldsymbol{M}}_{\lambda(\boldsymbol{M})}$-saturated.

(iii) Suppose $\boldsymbol{M}$ is stable. We say that $\mathscr{A}$ is strongly $F^{\boldsymbol{M}}_\kappa$-saturated if for all $A \subseteq \mathscr{A}$ of power $< \kappa$ and a there is $b \in \mathscr{A}$ such that $b\ E^m_{min,A}\ a$. By a-saturated we mean strongly $F^{\boldsymbol{M}}_{\kappa(\boldsymbol{M})}$-saturated.

LEMMA 0.2. *(i) Every $F^{\boldsymbol{M}}_\kappa$-saturated model is locally $F^{\boldsymbol{M}}_\kappa$-saturated and so (assuming $\boldsymbol{M}$ is stable) every s-saturated model is e-saturated.*

(ii) Suppose $\boldsymbol{M}$ is stable. Then every e-saturated model is strongly $F^{\boldsymbol{M}}_\omega$-saturated.

(iii) Suppose $\boldsymbol{M}$ is superstable and $\kappa \geq \lambda(\boldsymbol{M})$. Then every locally $F^{\boldsymbol{M}}_\kappa$-saturated model is $F^{\boldsymbol{M}}_\kappa$-saturated, in particular every e-saturated model is s-saturated.

PROOF. (i) is trivial and (ii) is immediate since by [19] Lemma 1.9 (iv), every $F^{\boldsymbol{M}}_{\lambda(\boldsymbol{M})}$-saturated model is strongly $F^{\boldsymbol{M}}_{\lambda(\boldsymbol{M})}$-saturated. So we prove (iii): Assume $\mathscr{A}$ is locally $F^{\boldsymbol{M}}_\kappa$-saturated. Notice that by (ii), $\mathscr{A}$ is a-saturated. Let $A \subseteq \mathscr{A}$ be of power $< \kappa$ and a arbitrary. We show that there is $b \in \mathscr{A}$ such that $t(b, A) = t(a, A)$. Clearly we may assume that $a \cap \mathscr{A} = \emptyset$.

Choose finite $B \subseteq \mathscr{A}$ so that $a \downarrow_B \mathscr{A}$. Since $\mathscr{A}$ is locally $F^{\boldsymbol{M}}_\kappa$-saturated, we can find $F^{\boldsymbol{M}}_\kappa$-saturated $\mathscr{B}$ such that $B \subseteq \mathscr{B} \subseteq \mathscr{A}$. Since by [19] Lemma 1.9 (iii) $\mathscr{B}$ is strongly $F^{\boldsymbol{M}}_\kappa$-saturated, we can find $a_i \in \mathscr{B}$, $i < \kappa$, such that $a_i\ E^m_{min,B}\ a$ and $a_i \downarrow_B \cup_{j<i} a_j$. Let $I = \{a_i |\ i < \kappa\}$. For all $i < \kappa(\boldsymbol{M})$, choose b_i so that $t(b_i, \mathscr{A}) = t(a, \mathscr{A})$ and $b_i \downarrow_{\mathscr{A}} \cup_{j<i} b_j$. Let $J = \{b_i |\ i < \kappa(\boldsymbol{M})\}$. By [19] Corollaries 3.5 (iv) and 3.11, $I \cup J$ is indiscernible over B. So

$$Av(I, A) = Av(J, A) = t(a, A).$$

Since $|A| < \kappa$ and $\kappa(\boldsymbol{M}) = \omega$, we can find $C \subseteq B \cup I$ of power $< \kappa$ such that for all $c \in A$, $t(c, B \cup I)$ does not split strongly over C. Let $b \in I$ $(\subseteq \mathscr{B} \subseteq \mathscr{A})$ be such that $b \cap C = \emptyset$. Then clearly $t(b, A) = Av(I, A) = t(a, A)$. ⊣

We prove a main gap theorem for e-saturated submodels of $\boldsymbol{M}$. To some extend, the proofs are similar to the related proofs in the case of complete first-order theories.

§1. Regular types. In (the end of) the next section, regular types are needed. In this section we prove the basic properties and the existence of regular types. In this section we assume that $\boldsymbol{M}$ is stable.

DEFINITION 1.1. (i) We say that a stationary pair (p, A) is regular if the following holds: if $C \supseteq dom(p)$, $a \models p$ and $a \not\downarrow_A C$, then (p, A) is orthogonal to $t(a, C)$.

(ii) Assume $\mathscr{A}$ is s-saturated and $p \in S(\mathscr{A})$. We say that p is regular, if there are $A \subseteq B \subseteq \mathscr{A}$ such that p does not split strongly over A, $(p \restriction B, A)$ is a regular stationary pair and $|B| < \kappa(\boldsymbol{M})$.

LEMMA 1.2. *Assume $\mathscr{A}$ is s-saturated and $p \in S(\mathscr{A})$ is regular, not orthogonal to $t(a, \mathscr{A})$ and $\mathscr{B}$ is s-primary over $\mathscr{A} \cup a$. Then there is $b \in \mathscr{B}$ such that $t(b, \mathscr{A}) = p$.*

PROOF. Assume not. Let $A \subseteq B \subseteq \mathscr{A}$ be as in Definition 1.1 (ii). For all $i < \kappa(\boldsymbol{M})$ choose $\mathscr{A}_i$ as follows:

(i) $\mathscr{A}_0 = \mathscr{A}$,
(ii) if i is limit, then $\mathscr{A}_i \subseteq \mathscr{B}$ is s-primary over $\cup_{j<i} \mathscr{A}_j$,
(iii) if $i = j + 1$ and there is $b_j \in \mathscr{B}$ such that $t(b_j, B) = p \restriction B$ and $a \not\downarrow_{\mathscr{A}_j} b_j$, then $\mathscr{A}_i \subseteq \mathscr{B}$ is s-primary over $\mathscr{A}_j \cup b_j$, if such b_j does not exist then we let $\mathscr{A}_i = \mathscr{A}_j$.

Clearly there is $i < \kappa(\boldsymbol{M})$ such that $\mathscr{A}_i = \mathscr{A}_{i+1}$. Let i^* be the least such ordinal. Then

$$(*) \qquad t(a, \mathscr{A}_{i^*}) \text{ is orthogonal to } p.$$

Let $\mathscr{A}^*$ be s-primary over $\mathscr{A}_{i^*} \cup a$.

CLAIM. Assume $b \models p$. Then $p \vdash t(b, \mathscr{A}^*)$.

PROOF. Since p is not realized in $\mathscr{B}$, for all $i < i^*$, $b_i \not\downarrow_A \mathscr{A}_i$ and so, since p is regular, for all $i < i^*$, p is orthogonal to $t(b_i, \mathscr{A}_j)$. By induction on $i \leq i^*$ it is easy to see that $p \vdash t(b, \mathscr{A}_i)$. And so by (*) above, $p \vdash t(b, \mathscr{A}^*)$. ⊣

By Claim, p is orthogonal to $t(a, \mathscr{A})$, a contradiction. ⊣

COROLLARY 1.3. *Assume $\mathscr{A}_i$, $i < 3$, are s-saturated, $p_i \in S(\mathscr{A}_i)$ and p_1 is regular. If p_0 is not orthogonal to p_1 and p_1 is not orthogonal to p_2, then p_0 is not orthogonal to p_2.*

PROOF. Immediate by Lemma 1.2 and [19] Lemma 5.4 (iii). ⊣

LEMMA 1.4. *Assume that $\mathscr{A}$ is s-saturated, $a \not\downarrow_{\mathscr{A}} b$ and $t(b, \mathscr{A})$ is regular. Then $a \rhd_{\mathscr{A}} b$.*

PROOF. Let $\lambda = (\lambda(\boldsymbol{M}))^+$. Clearly we may assume that $\mathscr{A}$ is $F^{\boldsymbol{M}}_\lambda$-saturated. For a contradiction, assume that there is c such that $c \downarrow_{\mathscr{A}} a$ and $c \not\downarrow_{\mathscr{A}} b$. Choose $A \subseteq B \subseteq \mathscr{B} \subseteq \mathscr{A}$ such that

(i) $(t(b, B), A)$ is a regular stationary pair and $b \downarrow_A \mathscr{A}$,
(ii) $|B| < \kappa(\boldsymbol{M})$ and $|\mathscr{B}| = \lambda(\boldsymbol{M})$,
(iii) $\mathscr{B}$ is s-saturated and $a \cup b \cup c \downarrow_{\mathscr{B}} \mathscr{A}$.

Then $b \not\downarrow_{\mathscr{B}} a$, $b \not\downarrow_{\mathscr{B}} c$ ([19] Lemma 3.8 (iv)) and $a \downarrow_{\mathscr{B}} c$. Let $\mathscr{A}^*$ be $F^{\boldsymbol{M}}_\lambda$-primary over $\mathscr{A} \cup a$ and $\mathscr{C} \subseteq \mathscr{A}^*$ s-primary over $\mathscr{B} \cup a$. Without loss of generality we may assume that $b \cup c \downarrow_{\mathscr{C}} \mathscr{A}$.

For all $i < \kappa(\boldsymbol{M})$, choose $b_i \in \mathscr{A}^*$ such that $t(b_i, \mathscr{C} \cup \bigcup_{j<i} b_j) = t(b, \mathscr{C} \cup \bigcup_{j<i} b_j)$. Let $I = \{b_i |\, i < \kappa(\boldsymbol{M})\}$. Then $I \cup \{b\}$ is indiscernible over $\mathscr{C}$. Since $b \not\downarrow_{\mathscr{B}} \mathscr{C}$, it is easy to see that $I \cup \{b\}$ is not $\mathscr{B}$-independent. So we can choose finite $J \subseteq I$ such that

$$(*) \qquad J \cup \{b\} \text{ is not } \mathscr{B}\text{-independent.}$$

If J is chosen so that $|J|$ is minimal, then J is $\mathscr{B}$-independent.

Let $\mathscr{D}$ be s-primary over $\mathscr{B} \cup c$. By (iii) and the choice of c, $c \downarrow_B \mathscr{A}^*$. Then $J \downarrow_{\mathscr{B}} \mathscr{D}$ and so J is $\mathscr{D}$-independent. Since p is regular and $b \not\downarrow_{\mathscr{B}} \mathscr{D}$, $J \downarrow_{\mathscr{D}} b$ and so $J \downarrow_{\mathscr{B}} b$. Clearly this contradicts (*) above. ⊣

Assume $\mathscr{A}$ is s-saturated and $a \notin \mathscr{A}$. We write $Dp(a, \mathscr{A}) > 0$ if there is s-primary model $\mathscr{B}$ over $\mathscr{A} \cup a$ and $b \notin \mathscr{B}$ such that $t(b, \mathscr{B})$ is orthogonal to $\mathscr{A}$.

LEMMA 1.5. *Assume that $\boldsymbol{M}$ is superstable without $(\lambda(\boldsymbol{M}))^+$-dop. Let $\mathscr{A}$ be s-saturated, I be $\mathscr{A}$-independent and $a \not\downarrow_{\mathscr{A}} I$. If $t(a, \mathscr{A})$ is regular and $Dp(a, \mathscr{A}) > 0$, then there is $b \in I$ such that $a \not\downarrow_{\mathscr{A}} b$. And so by Lemma 1.4, $a \downarrow_{\mathscr{A}} \cup(I - \{b\})$.*

PROOF. Assume not. Clearly we may assume that $|\mathscr{A}| = \lambda(\boldsymbol{M})$. Choose a_i, $\mathscr{A}_i$ and $\mathscr{C}_i$, $i < \alpha^*$, so that

(i) $a \downarrow_{\mathscr{A}} a_i$,
(ii) $\mathscr{A}_i$ is s-primary over $\mathscr{A} \cup a_i$,
(iii) $\{a_i |\, i < \alpha^*\}$ is $\mathscr{A}$-independent,
(iv) $\mathscr{C}_0 = \mathscr{A}_0$ and $\mathscr{C}_{i+1}$ is s-primary over $\mathscr{C}_i \cup \mathscr{A}_{i+1}$,
(v) $a \not\downarrow_{\mathscr{C}_i} \mathscr{A}_{i+1}$,
(vi) $(a_i)_{i<\alpha^*}$ is a maximal sequence satisfying (i)-(v) above and $\alpha^* > 1$.

Since $\boldsymbol{M}$ is superstable, $\alpha^* < \omega$. Let n be such that $\alpha^* = n + 1$. Let $\lambda = (\lambda(\boldsymbol{M}))^+$ and $\mathscr{B}$ be $F^{\boldsymbol{M}}_{\lambda}$-saturated model such that $\mathscr{A} \subseteq \mathscr{B}$ and $\mathscr{B} \downarrow_{\mathscr{A}} \mathscr{C}_n$. Let $\mathscr{B}_i$ be $F^{\boldsymbol{M}}_{\lambda}$-primary over $\mathscr{B} \cup \mathscr{A}_i$ and $\mathscr{D}$ $F^{\boldsymbol{M}}_{\lambda}$-primary over $\cup_{i \le n} \mathscr{B}_i$. It is easy to see that $\mathscr{C}_n$ is s-primary over $\cup_{i \le n} \mathscr{A}_i$ and so we may choose $\mathscr{D}$ so that $\mathscr{C}_n \subseteq \mathscr{D}$. Choose $a' \in \mathscr{D}$ so that $t(a', \mathscr{C}_n) = t(a, \mathscr{C}_n)$. Let $\mathscr{A}'$ be s-primary over $\mathscr{A} \cup a'$.

CLAIM 1. $\mathscr{A}' \downarrow_{\mathscr{A}} \mathscr{B}$.

PROOF. Immediate by Lemma 1.4. ⊣

CLAIM 2. For all $i \le n$, $\mathscr{A}' \downarrow_{\mathscr{A}} \mathscr{B}_i$.

PROOF. Clearly it is enough to show that $a' \downarrow_{\mathscr{A}} \mathscr{B} \cup \mathscr{A}_i$. Let $I = \{j \le n |\, j \neq i\}$. By Claim 1 and (vi) above,

$$(*) \qquad a' \downarrow_{\mathscr{C}_n} \mathscr{B}.$$

By the choice of $\mathscr{B}$, $\cup_{j \in I} \mathscr{A}_j \downarrow_{\mathscr{A}_i} \mathscr{B}$ and so $\mathscr{C}_n \downarrow_{\mathscr{A}_i} \mathscr{B}$. With (*) above, this implies that $a' \downarrow_{\mathscr{A}_i} \mathscr{B}$. Since $a' \downarrow_{\mathscr{A}} \mathscr{A}_i$, $a' \downarrow_{\mathscr{A}} \mathscr{B} \cup \mathscr{A}_i$. ⊣

Since $Dp(a, \mathscr{A}) > 0$, there is $b \notin \mathscr{A}'$ such that $t(b, \mathscr{A}')$ is orthogonal to $\mathscr{A}$ and $b \downarrow_{\mathscr{A}'} \mathscr{D}$. By Claim 2 and [19] Corollary 4.8, $t(b, \mathscr{D})$ is orthogonal to $\mathscr{B}_i$ for all $i \le n$. This contradicts the following claim:

CLAIM 3 ($\boldsymbol{M}$ is superstable without λ-dop.). Assume $\mathscr{B}$, $\mathscr{B}_i$, $i \le n < \omega$, and $\mathscr{D}$ are $F^{\boldsymbol{M}}_{\lambda}$-saturated, for all $i \le n$, $\mathscr{B} \subseteq \mathscr{B}_i$, $(\mathscr{B}_i)_{i \le n}$ is $\mathscr{B}$-independent and $\mathscr{D}$ is $F^{\boldsymbol{M}}_{\lambda}$-primary over $\cup_{i \le n} \mathscr{B}_i$. If $b \notin \mathscr{D}$, then there is $i \le n$ such that $t(b, \mathscr{D})$ is not orthogonal to $\mathscr{B}_i$.

PROOF. We prove this by induction on n. The case $n = 0$ is trivial and the case $n = 1$ follows immediately from λ-ndop. So assume $n > 1$.

Let $\mathscr{B}'$ be F^M_λ-primary over $\cup_{i<n}\mathscr{B}_i$ and $\mathscr{D}'$ be F^M_λ-primary over $\mathscr{B}' \cup \mathscr{B}_n$. By [19] Lemma 5.4 (ii), $\mathscr{B}' \cup \mathscr{B}_n$ is F^M_λ-constructible over $\cup_{i\leq n}\mathscr{B}_i$ and so $\mathscr{D}'$ is F^M_λ-primary over $\cup_{i\leq n}\mathscr{B}_i$. By the uniqueness of F^M_λ-primary sets, we can choose $\mathscr{B}'$ and $\mathscr{D}'$ so that $\mathscr{D}' = \mathscr{D}$. Clearly we may assume that $t(b, \mathscr{D})$ is orthogonal to $\mathscr{B}_n$.

By [19] Lemma 5.11, choose $b' \notin \mathscr{D}$ so that $t(b', \mathscr{D})$ is a c-type ([19] Definition 5.10) and $b \rhd_{\mathscr{D}} b'$. Since $t(b, \mathscr{D})$ is orthogonal to $\mathscr{B}_n$, so is $t(b', \mathscr{D})$. Then by λ-ndop, $t(b', \mathscr{D})$ is not orthogonal to $\mathscr{B}'$. Since $t(b', \mathscr{D})$ is a c-type, there is $b'' \notin \mathscr{D}$ such that $b'' \downarrow_{\mathscr{B}'} \mathscr{D}$ and $b' \rhd_{\mathscr{D}} b''$. By the induction assumption, there is $i < n$, such that $t(b'', \mathscr{B}')$ is not orthogonal to $\mathscr{B}_i$. Then $t(b'', \mathscr{D})$ is not orthogonal to $\mathscr{B}_i$ and because $b \rhd_{\mathscr{D}} b''$, also $t(b, \mathscr{D})$ is not orthogonal to $\mathscr{B}_i$. ⊣

⊣

LEMMA 1.6. *Assume that M is superstable, $\mathscr{A} \subseteq \mathscr{B}$ are s-saturated and $\mathscr{A} \neq \mathscr{B}$. Then there is a singleton $a \in \mathscr{B} - \mathscr{A}$ such that $t(a, \mathscr{A})$ is regular.*

PROOF. As in the case of superstable theories (see e.g., [1] XII Exercise 2.4). ⊣

§2. Superstable with ndop.

Throughout this section we assume that M is superstable and does not have $\lambda(M)$-dop. If P is a tree and $t \in P$ is not the root, then by t^- we mean the immediate predecessor of t.

DEFINITION 2.1. (i) We say that $(P, f, g) = ((P, <), f, g)$ is an s-free tree of the (s-saturated) model $\mathscr{A}$ if the following holds:

(1) $(P, <)$ is a tree without branches of length $> \omega$, $f : (P - \{r\}) \to \mathscr{A}$ and $g : P \to P(\mathscr{A})$, where $r \in P$ is the root of P and $P(\mathscr{A})$ is the power set of $\mathscr{A}$,

(2) $g(r)$ is an s-primary model (over $\emptyset$ i.e., saturated model of power $\lambda(M)$),

(3) if t is not the root and $u^- = t$ then $t(f(u), g(t))$ is orthogonal to $g(t^-)$,

(4) if $t = u^-$ then $g(u)$ is s-primary over $g(t) \cup f(u)$,

(5) Assume $T, V \subseteq P$ and $u \in P$ are such that

(a) for all $t \in T$, t is comparable with u,

(b) T is downwards closed.

(c) if $v \in V$ then for all t such that $v \geq t > u$, $t \notin T$.

Then
$$\bigcup_{t\in T} g(t) \downarrow_{g(u)} \bigcup_{v\in V} g(v).$$

(ii) We say that (P, f, g) is an s-decomposition of $\mathscr{A}$ if it is a maximal s-free tree of $\mathscr{A}$.

(iii) We say that (P, f, g) is an s-free tree, if it is an s-free tree of some $\mathscr{A}$.

Notice that by Lemma 0.2 (iii) it is easy to see, that every e-saturated model has an s-decomposition.

THEOREM 2.2 (M superstable without $\lambda(M)$-dop). *Assume $\mathscr{A}$ is e-saturated and (P, f, g) is an s-decomposition of $\mathscr{A}$. If $\mathscr{B} \subseteq \mathscr{A}$ is s-primary over $\cup_{t\in P} g(t)$, then $\mathscr{B} = \mathscr{A}$.*

PROOF. Immediate by Lemma 0.2 (iii) and [19] Theorem 5.13. ⊣

COROLLARY 2.3. *Suppose $\mathscr{A}$ and $\mathscr{B}$ are e-saturated. If (P, f, g) is a decomposition of both $\mathscr{A}$ and $\mathscr{B}$, then $\mathscr{A}$ and $\mathscr{B}$ are isomorphic over $\cup_{t \in P} g(t)$.*

PROOF. By Theorem 2.2, $\mathscr{A}$ is s-primary over $\cup_{t \in P} g(t)$. By [19] Theorem 5.3 (ii), there is an embedding $f : \mathscr{A} \to \mathscr{B}$ such that $f \restriction \cup_{t \in P} g(t) = id_{\cup_{t \in P} g(t)}$. By Theorem 2.2, $rng(f) = \mathscr{B}$. ⊣

We say that an s-free tree (P, f, g) is regular if the following holds: if $t, u \in P$ are such that u is an immediate successor of t, then $t(f(u), g(t))$ is regular. We say that (P, f, g) is a regular s-decomposition of e-saturated $\mathscr{A}$, if it an s-decomposition of $\mathscr{A}$ and a regular s-free tree.

LEMMA 2.4. *Every e-saturated model $\mathscr{A}$ has a regular s-decomposition.*

PROOF. For this, it is enough to show that every maximal regular s-free tree of $\mathscr{A}$ is a maximal s-free tree of $\mathscr{A}$. But this follows immediately from Lemma 1.6. ⊣

DEFINITION 2.5. (i) We say that $\boldsymbol{M}$ is shallow if every branch in every regular s-free tree is finite. If $\boldsymbol{M}$ is not shallow, then we say that $\boldsymbol{M}$ is deep.

(ii) If $P = (P, <)$ is a tree without infinite branches, then by $Dp(P)$ we mean the depth of P.

(iii) Assume that $\boldsymbol{M}$ is shallow. We define the depth of $\boldsymbol{M}$ to be

$$\sup\{Dp(P) + 1 \mid (P, f, g) \text{ is a regular } s\text{-free tree}\}.$$

LEMMA 2.6. *Assume that $\boldsymbol{M}$ is shallow. Then the depth of $\boldsymbol{M}$ is $< \lambda(\boldsymbol{M})^+$.*

PROOF. Choose a minimal regular s-free tree (P, f, g) so that the following holds: for all $t \in P$ and $p \in S(g(t))$, if (*) below holds, then there is an immediate successor $u \in P$ of t such that $t(f(u), g(t)) = p$.

(*) p is regular and if t has an immediate predecessor t^-, then p is orthogonal to $g(t^-)$.

Clearly $Dp(P) < \lambda(\boldsymbol{M})^+$.

CLAIM. If (P', f', g') is a regular s-free tree, then there is an order-preserving function $h : P' \to P$.

PROOF. By induction on $height(t)$, $t \in P'$, we define $h(t)$ so that

(i) if u is an immediate predecessor of t, then $h(u)$ is an immediate predecessor of $h(t)$,

(ii) there is an elementary function $h_t : g'(t) \to g(h(t))$ such that if u is an immediate predecessor of t, then $h_t(g'(u)) \subseteq g(h(u))$ and $h_t(g'(t)) \downarrow_{h_t(g'(u))} g(h(u))$.

If $height(t) = 0$, then we let $h(t)$ be the root of P. Then h_t exists because $g'(t)$ and $g(h(t))$ are $F^M_{\lambda(M)}$-saturated models of power $\lambda(\boldsymbol{M})$ and thus isomorphic. Assume then, that $height(t) > 0$. Let u be the immediate predecessor of t and let h_u be the function given by the induction assumption. Then there is $h(t) \in P$ such that it is an immediate successor of $h(u)$, $t(f(h(t)), h_u(g'(u))) = h_u(t(f'(t), g'(u)))$ and $f(h(t)) \downarrow_{h_u(g'(u))} g(h(u))$. This is because the free extension of $h_u(t(f'(t), g'(u)))$ is clearly regular and if u^- is an immediate predecessor of u, then by (ii) of the induction assumption and [19] Corollary 4.8 the free extension of $h_u(t(f'(t), g'(u)))$ is orthogonal to $g(h(u^-))$. We need to define h_t.

Since $g'(t)$ is s-primary over $g(u) \cup f(t)$, it is s-primitive over $g(u) \cup f(t)$. So there is h_t such that $h_t \restriction g'(u) = h_u$ (and so $h_t(g'(u)) \subseteq g(h(u))$), $h_t(f'(t)) = f(h(t))$ and $h_t(g'(t)) \subseteq g(h(t))$. By the choice of $h(t)$ and [19] Lemma 5.4 (i), $h_t(g'(t)) \downarrow_{h_t(g'(u))} g(h(u))$. $\dashv$

By Claim, if (P', f', g') is a regular s-free tree, then $Dp(P') \leq Dp(P) < \lambda(\boldsymbol{M})^+$. $\dashv$

By $|L|$ we mean the number of L-formulas modulo the equivalence relation $\models \forall x(\phi(x) \leftrightarrow \psi(x))$.

THEOREM 2.7. *Assume that $\boldsymbol{M}$ is shallow. Then the depth of $\boldsymbol{M}$ is $< (|S(\emptyset)|^{\omega})^+$ and so it is $< (2^{|L|})^+$.*

PROOF. By Lemma 2.6, we may assume that $\lambda(\boldsymbol{M}) > \omega$. Choose a minimal regular s-free tree (P, f, g) so that if $t \in P$ and $p \in S(g(t))$ is regular such that if t has an immediate predecessor t^-, then p is orthogonal to $g(t^-)$, then there is an immediate successor $u \in P$ of t and an automorphism h of $g(t)$ such that such that $t(f(u), g(t)) = h(p)$.

CLAIM 1. $Dp(P) < (|S(\emptyset)|^{\omega})^+$.

PROOF. Clearly it is enough to show that for all $t \in P$ the number of immediate successors of t is at most $|S(\emptyset)|^{\omega}$. As in the proof of Lemma 0.2, for all $p \in S(g(t))$, there is a countable indiscernible $I \subseteq g(t)$ such that $Av(I, g(t)) = p$. Also if $t(I, \emptyset) = t(I', \emptyset)$, then there is an automorphism h of $g(t)$ such that $h(I) = I'$ (remember that $g(t)$ is an $F^{\boldsymbol{M}}_{|g(t)|}$-saturated model of power $\lambda(\boldsymbol{M}) > \omega$). So the number of immediate successors of t is at most

$$|\{t(I, \emptyset) \mid I \subseteq g(t) \text{ countable indiscernible}\}|.$$

Clearly this is at most $|S(\emptyset)|^{\omega}$. $\dashv$

CLAIM 2. If (P', f', g') is a regular s-free tree, then there is an order-preserving function $h : P' \to P$.

PROOF. By induction on $height(t)$, $t \in P'$, we define $h(t)$ so that

(i) if u is an immediate predecessor of t, then $h(u)$ is an immediate predecessor of $h(t)$,

(ii) there is an elementary function $h_t : g'(t) \to g(h(t))$ such that if u is an immediate predecessor of t, then $h_t(g'(u)) \subseteq g(h(u))$ and $h_t(g'(t)) \downarrow_{h_t(g'(u))} g(h(u))$.

The case $height(t) = 0$ is as in the proof of Lemma 2.6. So assume that $height(t) > 0$. Let u be the immediate predecessor of t and h_u the isomorphism given by the induction assumption. As in the proof of Lemma 2.6, we can find $h(t) \in P$ and an automorphism h^* of $g(h(u))$ such that $h(t)$ is an immediate successor of $h(u)$, $t(f(h(t)), (h^* \circ h_u)(g'(u))) = (h^* \circ h_u)(t(f'(t), g'(u)))$ and $f(h(t)) \downarrow_{(h^* \circ h_u)(g'(u))} g(h(u))$. Now we can proceed as in the proof of Lemma 2.6 ($h^* \circ h_u$ in place of h_u). $\dashv$

As in the proof of Lemma 2.6, Claim 1 and 2 imply that the depth of $\boldsymbol{M}$ is $< (|S(\emptyset)|^{\omega})^+$. $\dashv$

THEOREM 2.8. *Assume that $\boldsymbol{M}$ is shallow and γ^* is the depth of $\boldsymbol{M}$. Then the number of non-isomorphic e-saturated models of power $\aleph_\alpha$ is at most $\beth_{\gamma^*}(|\alpha| + \lambda(\boldsymbol{M}))$.*

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PROOF. By Corollary 2.3, it is enough to count the number of 'non-isomorphic' regular s-free trees (P, f, g) of power $\aleph_\alpha$. This is an easy induction on $Dp(P)$, see the related results in [41]. $\dashv$

THEOREM 2.9. *Assume that $\boldsymbol{M}$ is shallow and γ^* is the depth of $\boldsymbol{M}$. Let $\kappa = \beth_{\gamma^*}(\lambda(\boldsymbol{M}))^+$. If $\mathscr{A}_i$, $i < \kappa$, are e-saturated models, then there are $i < j < \kappa$ such that $\mathscr{A}_i$ is elementarily embeddable into $\mathscr{A}_j$.*

PROOF. By Corollary 2.3, this question can be reduced to the question of 'embeddability' of labelled trees. So this follows immediately from [41] X Theorem 5.16C. $\dashv$

A cardinal κ is called beautiful if $\kappa = \omega$ or for all $\xi < \kappa$, $\kappa \overset{w}{\rightarrow} (\omega)^{<\omega}_{\xi}$, see [34] Definition 2.3.

THEOREM 2.10. *($\boldsymbol{M}$ is superstable without $\lambda(\boldsymbol{M})$-dop but not necessarily shallow.) Assume that there is a beautiful cardinal $> \lambda(\boldsymbol{M})$. Let κ^* be the least such cardinal. If $\mathscr{A}_i$, $i < \kappa^*$, are e-saturated models, then there are $i < j < \kappa^*$ such that $\mathscr{A}_i$ is elementarily embeddable into $\mathscr{A}_j$.*

PROOF. Again by Corollary 2.3, this follows immediately from [34] Theorems 5.8 and 2.10. $\dashv$

If $(P, <)$ is a tree without branches of length $\geq \omega$ and $t \in P$, then by $Dp(t, P)$ we mean the depth of t in P.

THEOREM 2.11. *Assume that $\boldsymbol{M}$ is superstable, deep, does not have $\lambda(\boldsymbol{M})$-dop and $(\lambda(\boldsymbol{M}))^+$-dop and $\lambda > \lambda(\boldsymbol{M})$. Then there are s-saturated (and so e-saturated) models $\mathscr{A}_i$, $i < 2^\lambda$, of power λ such that for all $i < j < 2^\lambda$, $\mathscr{A}_i \not\cong \mathscr{A}_j$.*

REMARK. In the next section we show that $\boldsymbol{M}$ has many e-saturated models if $\boldsymbol{M}$ is superstable and has $\lambda(\boldsymbol{M})$-dop. Similarly we can show that $\boldsymbol{M}$ has many e-saturated models if $\boldsymbol{M}$ is superstable and has $(\lambda(\boldsymbol{M}))^+$-dop.

PROOF. Assume $X_i \subseteq \lambda$, $i < 2$, are such that $X_0 \neq X_1$ and $|X_i| = \lambda$. Choose regular s-free trees (P_i, f_i, g_i), $i < 2$, so that

(i) P_i does not have branches of length $\geq \omega$ but for all $t \in P_i$, if t is not the root, then $Dp(f(t), g(t^-)) > 0$ (see just before Lemma 1.5),

(ii) for all $\alpha \in X_i$, there are λ many $t \in P_i$ such that the height of t is one and $Dp(t, P_i) = \alpha$ and if $Dp(t, P_i) = \beta$ and the height of t is one, then $\beta \in X_i$,

(iii) for all $t \in P_i$, if $Dp(t, P_i) = \alpha$ and $\beta < \alpha$, then
$$|\{u \in P_i \mid u^- = t \text{ and } Dp(u, P_i) \geq \beta\}| = \lambda,$$

(iv) if $t, u \in P_i$ are not the root and $t^- = u^-$, then
$$t(f_i(t), g_i(t^-)) = t(f_i(u), g_i(u^-)),$$
we write p_{t^-} for this type.

Let r_i be the root of P_i, Choose finite $A_i \subseteq B_i \subseteq g_i(r_i)$ so that p_{r_i} does not split strongly over A_i and $(p_{r_i} \restriction B_i, A_i)$ is a regular stationary pair. Then we require also

(v) $B_0 = B_1$ $(=B)$, $A_0 = A_1$ $(=A)$ and $p_{r_0} \restriction B = p_{r_1} \restriction B$.

Let $\mathscr{A}_i$, $i < 2$, be s-primary over $\cup_{t \in P_i} g_i(t)$. We show that there is no isomorphism $F : \mathscr{A}_0 \to \mathscr{A}_1$ such that $F \restriction B = id_B$. Clearly this is enough (since $\lambda^{<\omega} < 2^\lambda$, 'naming' finite number of elements does not change the number of models and since M is λ-stable, $|\mathscr{A}_i| = \lambda$). For a contradiction we assume that F exists. Clearly we may assume that $F = id_{\mathscr{A}_0}$, this simplifies the notation.

We let P_i^* be the set of those $t \in P_i$, which are not leaves. For all $t \in P_0^*$, we let $G(t) \in P_1^*$ be (some node) such that p_t is not orthogonal to $p_{G(t)}$ (if exists).

Claim. G is an one-to-one function from P_0^* onto P_1^*.

Proof. Since for all $t \in P_0^*$, $|\{u \in P_0 | \, u^- = t\}| = \lambda > \lambda(M)$, the existence of $G(t)$ follows easily. Since for all $u, u' \in P_1^*$, $u \neq u'$, p_u is orthogonal to $p_{u'}$, $G(t)$ is unique by Corollary 1.3. But then by symmetry, claim follows. ⊣

We prove a contradiction (with (i) above) by constructing a strictly increasing sequence $(t_j)_{j<\omega}$ of elements of P_0^*. We construct also a strictly increasing sequence $(u_j)_{j<\omega}$ of elements of P_1, sets I_j^i, $i < 2$, and models $\mathscr{B}_j$ so that

(1) $Dp(u_j, P_1) < Dp(t_j, P_0)$ and for all $t \geq t_j$, $G(t) \geq u_j$,
(2) $I_j^i \subseteq P_i$ is downwards closed, non-empty and of power $\leq \lambda(M)$ and $I_j^i \subseteq I_{j+1}^i$,
(3) $t_j \in I_{j+1}^0$ and $G(t_j) \in I_{j+1}^1$,
(4) $\mathscr{B}_j$ is s-primary over $\cup_{t \in I_j^0} g_0(t)$ and over $\cup_{u \in I_j^1} g_1(u)$ and $\mathscr{B}_j \subseteq \mathscr{B}_{j+1}$.

We do this by induction on $j < \omega$.

$j = 0$: Choose I_0^0, I_0^1 and $\mathscr{B}_0$ so that (2) and (4) above are satisfied (if $\mathscr{B}' \subseteq \mathscr{B}_0$ is s-primary over $\cup_{t \in I} g(t)$, $I \subseteq P_0$, then by Theorem 2.2 and [19] Lemma 5.4 (ii), $\mathscr{B}_0$ is s-primary over $\mathscr{B}' \cup \bigcup_{t \in P_0} g(t)$). Let $t_0 \in P_0$ be such that $t_0 \notin I_0^0$ and $(t_0)^- = r_0$. Then

$$(*) \qquad f_0(t_0) \downarrow_A \mathscr{B}_0.$$

By Lemma 1.5, there is $u_0 \in P_1 - I_1^1$ such that $f_1(u_0) \not\downarrow_{\mathscr{B}_0} f_0(t_0)$ and $(u_0)^- \in I_0^1$. By Lemma 1.4,

$$f_0(t_0) \downarrow_{\mathscr{B}_0} \cup \{g_1(u) | \, u \not\geq u_0\}.$$

So u_0 is unique and the latter half of (1) holds. By (*), $(u_0)^- = r_1$ and so since $X_0 \neq X_1$ we can choose t_0 so that $Dp(u_0, P_1) \neq Dp(t_0, P_0)$. By symmetry, we may assume that $Dp(u_0, P_1) < Dp(t_0, P_0)$. Finally, this implies that $t_0 \in P_0^*$.

$j = k + 1$: Essentially, just repeat the argument above. ⊣

§3. Superstable with dop or unstable.

We start by making changes to a result from [17]. Our conclusion is weaker but so are the cardinal assumptions.

Theorem 3.1. *Assume M is superstable with $\lambda(M)$-dop, $\kappa > (\lambda_r(M))^+$ is regular and $\xi > (\kappa^+)^{(\lambda_r(M)^+)}$. Then there are F_κ^M-saturated (and so e-saturated) models $\mathscr{A}_i$, $i < 2^\xi$, of power ξ such that for all $i \neq j$, $\mathscr{A}_i$ is not isomorphic to $\mathscr{A}_j$.*

Proof. Let $\lambda = (\lambda_r(M))^+$. We write $p \in F_\lambda^M(A)$ if $p \restriction A$ F_λ^M-isolates p. By [19] Corollary 6.5, M has λ-sdop (see [19]) and so by [Hy] Corollary 2.3, there are F_λ^M-saturated models A_i of cardinality λ, $i < 3$, and an indiscernible sequence I over $A_1 \cup A_2$ of power κ^+ such that

(i) $A_0 \subseteq A_1 \cap A_2$, $A_1 \downarrow_{A_0} A_2$,
(ii) there is $D \subseteq A_1 \cup A_2$ of power $< \lambda$ with the following property: if C_i, $i < 3$, are such that $C_0 \downarrow_{A_0} A_1 \cup A_2$ and for $i \in \{1, 2\}$ and all $c_i \in C_i$, there is

$D_i \subseteq A_i \cup C_0$ of power $< \lambda$ such that $t(c_i, A_1 \cup A_2 \cup C_0 \cup C_{3-i}) \in F^M_\lambda(D_i)$, then

$$t(I, A_1 \cup A_2 \cup C_0 \cup C_1 \cup C_2) \in F^M_\lambda(D).$$

Let η be a linear ordering. We define an F^M_κ-saturated model $\mathscr{A}_\eta$ as follows. For all $i \in \eta$ we choose B_i and C_i so that $t(B_i, A_0) = t(A_1, A_0)$, $t(C_i, A_0) = t(A_2, A_0)$ and $\{B_i \mid i \in \eta\} \cup \{C_i \mid i \in \eta\}$ is independent over A_0. For all $i, j \in \eta$ we choose I_{ij} so that $t(I_{ij} \cup B_i \cup C_j, A_0) = t(I \cup A_1 \cup A_2, A_0)$. Then we let $\mathscr{A}_\eta$ be F^M_κ-primary over $\bigcup\{B_i \mid i \in \eta\} \cup \bigcup\{C_i \mid i \in \eta\} \cup \bigcup\{I_{ij} \mid (i,j) \in \eta^2,\ i < j\}$.

We let $\psi(x, y)$, $x = x_1 \frown x_2, y = y_1 \frown y_2$, $length(x_1) = length(x_2) = length(y_1) = length(y_2) = \lambda$, be a formula (in some language), which says that there is J such that $t(J \cup x_1 \cup y_2, \emptyset) = t(I \cup A_1 \cup A_2, \emptyset)$. Then by [Hy] Lemma 2.5, for all $i, j \in \eta$, $i < j$ iff $\mathscr{A}_\eta \models \psi(B_i \cup C_i, B_j \cup C_j)$.

Let τ be a similarity type and χ a cardinal. We say that ψ is a $PC(L_{\chi,\omega}(\tau))$-formula if ψ is of the form $(\exists f_i)_{i<\alpha}\phi$, where $\alpha < \chi$, f_i are new function symbols and ϕ is a $L_{\chi,\omega}$-formula of similarity type $\tau \cup \{f_i \mid i < \alpha\}$. In $L_{\chi,\omega}$-formulas any number of free variables may appear. We write $PC(L_{\chi,\omega})$ for $PC(L_{\chi,\omega}(\tau))$ if τ is the similarity type of M.

Claim. (i) There is a $PC(L_{\kappa^+,\omega})$-sentence ψ_0 such that for all models $\mathscr{C}$ of power $< |M|$, $\mathscr{C} \models \psi_0$ iff $\mathscr{C}$ is (isomorphic to) an F^M_κ-saturated elementary submodel of M.

(ii) There is a $PC(L_{\kappa^{++},\omega})$-formula $\psi_1(x, y)$ such that for all linear orderings η and $i, j \in \eta$, $\mathscr{A}_\eta \models \psi_1(B_i \cup C_i, B_j \cup C_j)$ iff $i < j$.

(iii) There is a $PC(L_{\kappa^+,\omega})$-formula $\psi_2(x, y)$ such that for all linear orderings η and $i, j \in \eta$, $\mathscr{A}_\eta \models \psi_2(B_i \cup C_i, B_j \cup C_j)$ iff $i \geq j$.

(iv) $\{\psi_0, \psi_1(c, d), \psi_2(c, d)\}$ is inconsistent, where c and d are sequences of new constant symbols.

Proof. (i) To say that $\mathscr{C}$ is an elementary submodel of M, it is enough to say that for all $n < \omega$, $\mathscr{C}$ does not realize n-types over $\emptyset$, which are not realized in M. This can be expressed by an $L_{\kappa^+,\omega}$-sentence. To say that $\mathscr{C}$ is F^M_κ-saturated, by Lemma 0.2 (iii), it is enough to say that $\mathscr{C}$ is locally F^M_κ-saturated. This can be expressed by a $PC(L_{\kappa^+,\omega})$-sentence.

(ii) Clearly ψ is equivalent to a $PC(L_{\kappa^{++},\omega})$-formula.

(iii) Let $\{a_k \mid k < \kappa^+\}$ be an enumeration of I and $\mathscr{D} \supseteq I \cup A_1 \cup A_2$ be an s-saturated model. By [19] Lemma 4.3, there are finite $D \subseteq E \subseteq \mathscr{D}$ such that $(Av(I, E), D)$ is a stationary pair and $Av(I, \mathscr{D})$ does not split strongly over D. Let $\{e_k \mid k < n\}$ be an enumeration of E so that for some $n' \leq n$, $D = \{e_k \mid k < n'\}$. Now assume that $J \subseteq \mathscr{D}$ is such that $t(J, A_1 \cup A_2) = t((a_i \mid i < \kappa), A_1 \cup A_2)$, $Av(J, E) = Av(I, E)$ and $Av(J, \mathscr{D})$ does not split strongly over D. Then

$$(*) \qquad J \text{ is not maximal in } \mathscr{D} \text{ over } \mathscr{A}_1 \cup A_2,$$

i.e., there is $b \in \mathscr{D} - J$ such that $J \cup \{b\}$ is indiscernible over $\mathscr{A}_1 \cup \mathscr{A}_2$. For this, let $D^* = J \cup A_1 \cup A_2 \cup \{a_i \mid i < \kappa\} \cup E$. By the pigeonhole principle, there is $j < \kappa^+$ such that $t(a_j, D^*) = Av(I, D^*)$. Then $b = a_j$ is as wanted: For this it is enough to show that $Av(I, D^*) = Av(J, D^*)$. By [19] Lemma 2.4 (ii), it is enough to find K such that $|K| = \kappa(M)$ and both $I \cup K$ and $J \cup K$ are indiscernible. For this choose

c_k, $k < \kappa(\boldsymbol{M})$ so that $t(c_k, E) = Av(I, E)$ and $c_k \downarrow_D I \cup J \cup E \cup \{c_p |\ p < k\}$. By the choice of E and D and [19] Lemma 2.4 (i), $K = \{c_i |\ i < \kappa(\boldsymbol{M})\}$ is as wanted.

By (*), the following formula is as wanted: There are functions f_k, $k < \kappa$, such that for all b_p, $p < n$, if $t(\{b_k |\ k < n\} \cup B_i \cup C_j, \emptyset) = t(E \cup A_1 \cup A_2, \emptyset)$, then $J = \{f_p(b_0, ..., b_{n-1}) |\ p < \kappa\}$ is such that $t(J \cup B_i \cup C_j, \emptyset) = t((a_k |\ k < \kappa) \cup A_1 \cup A_2, \emptyset)$, it is a maximal in $\mathscr{A}_\eta$ over $B_i \cup C_j$, $Av(J, \{b_k |\ k \leq n\})$ is a conjugate of $Av(I, E)$ and $Av(J, \mathscr{A}_\eta)$ does not split strongly over $\{b_k |\ k < n'\}$. Notice that the last requirement depends on $t(\{f_p(b_0, ..., b_{n-1}) |\ p < \kappa(\boldsymbol{M})\} \cup \{b_k |\ k < n\}, \emptyset)$ only.

(iv) Immediate by (*).

$\dashv$

By Claim, the theorem follows from [30] Chapter III Theorem 3.23 (2). $\dashv$

REMARK 3.2. To strengthen the conclusion of Theorem 3.1 to '$\mathscr{A}_i$ is not elementarily embeddable to $\mathscr{A}_j$', we need the parallel of [30] Chapter IV Theorem 3.1 for trees of height $\kappa + 1$ dealing with sequences of length $< \kappa$ (instead of height $\omega + 1$ and finite sequences). For humane reasons this has not been done in [30].

LEMMA 3.3. *Assume that $\boldsymbol{M}$ is unstable. Let $\kappa > |L|$ be a regular cardinal, and $\eta = (\eta, <)$ be a linear ordering. Then there are sequences a_i, $i \in \eta$, a model $\mathscr{A}$ and functions $f_i : \boldsymbol{M}^{n_i} \to \boldsymbol{M}$, $i < 2^{<\kappa}$, such that $n_i < \omega$ and if we write $L^* = L \cup \{f_i |\ i < 2^{<\kappa}\}$ then the following holds:*

(i) *$(a_i)_{i \in \eta}$ is order-indiscernible inside $\mathscr{A}$ in the language L^*,*

(ii) *for all $X \subseteq \eta$, the closure $\mathscr{A}_X$ of $\{a_i |\ i \in X\}$ under the functions of L^* is a locally $F^{\boldsymbol{M}}_\kappa$-saturated model (in the language L) and $\mathscr{A} = \mathscr{A}_\eta$,*

(iii) *there is an L-formula $\phi(x, y)$ such that for all $i, j \in \eta$, $\models \phi(a_i, a_j)$ iff $i < j$.*

PROOF. Define functions $f'_i : \boldsymbol{M}^{n_i} \to \boldsymbol{M}$, $i < 2^{<\kappa}$, so that

(*) the closure of any set under the functions f_i is locally $F^{\boldsymbol{M}}_\kappa$-saturated (in L) and L'-elementary submodel of $(\boldsymbol{M}, f'_i)_{i<2^{<\kappa}}$,where $L' = L \cup \{f'_i |\ i < 2^{<\kappa}\}$.

By Erdös-Rado Theorem and [32] I Lemma 2.10 (1), we can find sequences $(a^k_i)_{i<k}$, $k < \omega$, such that

(1) there is a formula $\phi(x, y)$ such that for all $k < \omega$ and $i, j < k$, $\models \phi(a^k_i, a^k_j)$ iff $i < j$,

(2) $(a^k_i)_{i<k}$ is order-indiscernible in the language L',

(3) the L'-type of $(a^k_i)_{i<k}$ (over $\emptyset$) is the same as the L'-type of $(a^{k+1}_i)_{i<k}$.

Since $\boldsymbol{M}$ is homogeneous, we can find for all $i \in \eta$, a_i so that for all $k < \omega$, if $i_0 < i_1 < ... < i_{k-1}$, then $t((a_{i_j})_{j<k}, \emptyset) = t((a^k_j)_{j<k}, \emptyset)$. Again, since $\boldsymbol{M}$ is homogeneous (use e.g., [19] Lemma 1.1) we can define the functions f_i so that for all $i_0 < i_1 < ... < i_{k-1}$ the following holds:

(**) If $\mathscr{A}_1$ is the closure of $(a_{i_j})_{j<k}$ under the functions f_i and $\mathscr{A}_2$ is the closure of $(a^k_j)_{j<k}$ under the functions f'_i, then there is an L-isomorphism $F : \mathscr{A}_1 \to \mathscr{A}_2$, such that $F(a_{i_j}) = a^k_j$ and for all $a, b \in \mathscr{A}_1$ and $i < 2^{<\kappa}$, $f_i(a) = b$ iff $f'_i(F(a)) = F(b)$.

Let $\mathscr{A} = \mathscr{A}_\eta$, i.e., the closure of $\{a_i |\ i \in \eta\}$ under the functions of L^*. Then it is easy to see that (iii) in the claim is satisfied.

(ii): Assume $X \subseteq \eta$. We show that $\mathscr{A}_X$ is locally F^{M}_{κ}-saturated. For this let $A \subseteq \mathscr{A}_X$ be finite. Then there is $X' \subseteq X$ finite, such that $A \subseteq \mathscr{A}_{X'}$. By (**) above, $\mathscr{A}_{X'}$ is locally F^{M}_{κ}-saturated. So there is F^{M}_{κ}-saturated $\mathscr{B}$ such that $A \subseteq \mathscr{B} \subseteq \mathscr{A}_X$.

(i): By (*) and (**) above it is easy to see that for all finite $X \subseteq \eta$, $\mathscr{A}_X$ is an L^*-elementary submodel of $\mathscr{A}$. By (2), (*) and (**) again, (i) follows. ⊣

THEOREM 3.4. *Assume $\boldsymbol{M}$ is unstable. Let λ and κ be regular cardinals, $\lambda > 2^{<\kappa}$ and $\kappa > |L|$. Then there are locally F^{M}_{κ}-saturated models $\mathscr{A}_i$, $i < 2^{\lambda}$, such that $|\mathscr{A}_i| = \lambda$ and if $i \neq j$, then $\mathscr{A}_i$ is not elementarily embeddable into $\mathscr{A}_j$.*

PROOF. By Lemma 3.3 this follows from [30] Chapter VI Theorem 3.1 (3). Notice that the trees can be coded into linear orderings. ⊣

§4. Strictly stable. Through out this section we assume that $\boldsymbol{M}$ is stable but unsuperstable, and that $\kappa = cf(\kappa) > \lambda_r(\boldsymbol{M})$.

We write $\kappa^{\leq\omega}$ for $\{\eta : \alpha \to \kappa |\ \alpha \leq \omega\}$, $\kappa^{<\omega}$ and $\kappa^{\omega} = \kappa^{=\omega}$ are defined similarly (of course these have also the other meaning, but it will be clear from the context, which one we mean). Let $J \subseteq 2^{\leq\kappa}$. We order $P_{\omega}(J)$ (=the set of all finite subsets of J) by defining $u \leq v$ if for every $\eta \in u$ there is $\xi \in v$ such that η is an initial segment of ξ.

Since $\boldsymbol{M}$ is unsuperstable, by [19] Lemma 5.1, there are a and $F^{M}_{\lambda_r(M)}$-saturated models $\mathscr{A}_i$, $i < \omega$, of power $\lambda_r(\boldsymbol{M})$ such that

(i) if $j < i < \omega$, then $\mathscr{A}_j \subseteq \mathscr{A}_i$,
(ii) for all $i < \omega$, $a \not\downarrow_{\mathscr{A}_i} \mathscr{A}_{i+1}$.

Let $\mathscr{A}_{\omega}$ be an $F^{M}_{\lambda_r(M)}$-primary model over $a \cup \bigcup_{i<\omega} \mathscr{A}_i$. Then for all $\eta \in \kappa^{\leq\omega}$, we can find $\mathscr{A}_{\eta}$ such that

(a) for all $\eta \in \kappa^{\leq\omega}$, there is an automorphism f_{η} of $\boldsymbol{M}$ such that $f_{\eta}(\mathscr{A}_{length(\eta)}) = \mathscr{A}_{\eta}$,
(b) if η is an initial segment of ξ, then $f_{\xi} \restriction \mathscr{A}_{length(\eta)} = f_{\eta} \restriction \mathscr{A}_{length(\eta)}$,
(c) if $\eta \in \kappa^{<\omega}$, $\alpha \in \kappa$ and X is the set of those $\xi \in \kappa^{\leq\omega}$ such that $\eta \frown (\alpha)$ is an initial segment of ξ, then

$$\cup_{\xi\in X}\mathscr{A}_{\xi} \downarrow_{\mathscr{A}_{\eta}} \cup_{\xi\in(\kappa^{\leq\omega}-X)}\mathscr{A}_{\xi}.$$

For all $\eta \in \kappa^{\omega}$, we let $a_{\eta} = f_{\eta}(a)$.

For each $\alpha < \kappa$ of cofinality ω, let $\eta_{\alpha} \in \kappa^{\omega}$ be a strictly increasing sequence such that $\cup_{i<\omega}\eta_{\alpha}(i) = \alpha$. Let $S \subseteq \{\alpha < \kappa |\ cf(\alpha) = \omega\}$. By J_S we mean the set

$$\kappa^{<\omega} \cup \{\eta_{\alpha} |\ \alpha \in S\}.$$

Let $I_S = P_{\omega}(J_S)$.

LEMMA 4.1. *For all $S \subseteq \{\alpha < \kappa |\ cf(\alpha) = \omega\}$, there are sets $\mathscr{A}_u$, $u \in I_S$, such that*

(i) for all $u, v \in I_S$, $u \leq v$ implies $\mathscr{A}_u \subseteq \mathscr{A}_v$,
(ii) for all $u \in I_S$, $\mathscr{A}_u$ is $F^{M}_{\lambda_r(M)}$-primary over $\cup_{\eta\in u} A_{\eta}$,
(iii) if $\alpha \in \kappa - S$, $u \in I_S$ and $v \in P_{\omega}(J_S \cap \alpha^{\leq\omega})$ is maximal such that $v \leq u$, then

$$\mathscr{A}_u \downarrow_{\mathscr{A}_v} \cup_{w\in P_{\omega}(J_S\cap\alpha^{\leq\omega})}\mathscr{A}_w.$$

PROOF. See [18] Lemmas 4 and 7. ⊣

For all $S \subseteq \{\alpha < \kappa | \; cf(\alpha) = \omega\}$, let $\mathscr{A}_S = \cup_{u \in I_S} \mathscr{A}_u$. By Lemma 4.1 (i) and (ii), $\mathscr{A}_S$ is e-saturated and $|\mathscr{A}_S| = \kappa$.

LEMMA 4.2. *There are sets $S_i \subseteq \{\alpha < \kappa | \; cf(\alpha) = \omega\}$, $i < 2^\kappa$, such that if $i \neq j$, then $S_i - S_j$ is stationary.*

PROOF. Let $f_i; \kappa \to \kappa$, $i < 2$, be one to one functions such that $rng(f_0) \cap rng(f_1) = \emptyset$. Let R'_i, $i < 2^\kappa$, be an enumeration of the power set of κ. We define R_i, $i < 2^\kappa$, so that $f_0(\alpha) \in R_i$ iff $\alpha \in R'_i$ and $f_1(\alpha) \in R_i$ iff $\alpha \notin R'_i$. Then clearly, $i \neq j$ implies $R_i - R_j \neq \emptyset$. By [41] Appendix Theorem 1.3 (2), there are pairwise disjoint stationary sets $S'_j \subseteq \{\alpha < \kappa | \; cf(\alpha) = \omega\}$, $j < \kappa$. For $i < 2^\kappa$, we let $S_i = \cup_{j \in R_i} S'_j$. Clearly these are as wanted. ⊣

THEOREM 4.3. *Assume $\boldsymbol{M}$ is stable and unsuperstable and $\kappa = cf(\kappa) > \lambda_r(\boldsymbol{M})$. Then there are e-saturated models $\mathscr{A}_i$, $i < 2^\kappa$, of power κ such that if $i \neq j$, then $\mathscr{A}_i$ is not elementarily embeddable into $\mathscr{A}_j$.*

PROOF. For all $i < 2^\kappa$, let $\mathscr{A}_i = \mathscr{A}_{S_i}$, where the sets S_i are as in Lemma 4.2. Assume $i \neq j$. We show that there are no elementary map $F : \mathscr{A}_i \to \mathscr{A}_j$.

For a contradiction, assume that F exists. For all $\alpha < \kappa$, let $I^\alpha_{S_i}$ be the set of those $u \in I_{S_i}$ such that for all $\eta \in u$, $\sup\{\eta(i) | \; i < length(\eta)\} < \alpha$. Let $\mathscr{A}^\alpha_i = \cup_{u \in I^\alpha_{S_i}} \mathscr{A}_u$. $I^\alpha_{S_j}$ and $\mathscr{A}^\alpha_j$ are defined similarly. We say that α is closed if for all $a \in \mathscr{A}_i$, $a \in \mathscr{A}^\alpha_i$ iff $F(a) \in \mathscr{A}^\alpha_j$. Let C be the set of all closed ordinals and C_{lim} the set of all limit points in C. Then $S^0 = C_{lim} \cap (S_i - S_j)$ is stationary.

For all $\alpha \in S^0$, let $u_\alpha \in I_{S_j}$ be such that $F(a_{\eta_\alpha}) \in \mathscr{A}_{u_\alpha}$. By $g(\alpha)$ we mean the least $\beta \in C$ such that $u_\alpha \downarrow_{\mathscr{A}^\beta_j} \mathscr{A}^\alpha_j$. By Lemma 4.1 (iii) and the fact that $S^0 \cap S_j = \emptyset$, $g(\alpha) < \alpha$. So there is stationary $S^1 \subseteq S^0$ such that $g \restriction S^1$ is constant. Let α^* be this constant value.

Then there is $S^2 \subseteq S^1$ and $n < \omega$ such that $|S^2| = \kappa$ and for all $\beta, \gamma \in S^2$, if $\beta \neq \gamma$, then $\eta_\beta(n) \neq \eta_\gamma(n)$. By choosing n so that it is minimal, we may assume that for all $\beta \in S^2$, $\eta_\beta(n-1) < \alpha^*$. Clearly we may assume that for all $\beta \in S^2$, $\eta_\beta(n) > \alpha^*$.

Then by Lemma 4.1 (iii),

(i) $(F(\mathscr{A}_{\eta_\beta \restriction (n+1)}))_{\beta \in S^2}$ is $F(\mathscr{A}^{\alpha^*}_i)$-independent.

Since $F(a_{\eta_\beta}) \downarrow_{\mathscr{A}^{\alpha^*}_j} F(\mathscr{A}_{\eta_\beta \restriction (n+1)})$ and $F(a_{\eta_\beta}) \not\downarrow_{F(\mathscr{A}^{\alpha^*}_i)} F(\mathscr{A}_{\eta_\beta \restriction (n+1)})$,

(ii) for all $\beta \in S^2$, $F(\mathscr{A}_{\eta_\beta \restriction (n+1)}) \not\downarrow_{F(\mathscr{A}^{\alpha^*}_i)} \mathscr{A}^{\alpha^*}_j$.

Since $\kappa(\boldsymbol{M}) < \kappa$, $|\mathscr{A}^{\alpha^*}_j| < \kappa$ and $|S^2| = \kappa$, (i) and (ii) are contradictory. ⊣

REMARK 4.4. By using [30] Chapter IV Theorem 3.1 (3), it is possible to replace the assumption $\kappa = cf(\kappa) > \lambda_r(\boldsymbol{M})$ by $\kappa > \lambda_r(\boldsymbol{M})$ in Theorem 4.3.

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