ISRAEL JOURNAL OF MATHEMATICS, Vol. 70, No. 1, 1990

UNIVERSAL GRAPHS WITHOUT INSTANCES OF CH: REVISITED

BY

SAHARON SHELAH[†]
EECS and Mathematics Department,
University of Michigan, Ann Arbor, MI 48109, USA;
Institute of Mathematics, The Hebrew University of Jerusalem,
Jerusalem, Israel; and
Mathematics Department, Rutgers University,
New Brunswick, New Jersey, USA

ABSTRACT

We give a complete proof[††] of the consistency of the existence of a universal graph of power λ, where $\kappa = \kappa^{<\kappa} < \lambda = \mathrm{cf}\,\lambda < 2^\kappa$ are arbitrary.

Introduction

The problem of the existence of universal models is very natural and appears in several contexts. Note that the related problem of the existence (for first order T) of saturated models (which is more central for the model theorist, but not so adopted elsewhere) is completely resolved — for a first order T, T has a saturated model in λ iff $\lambda = \lambda^{<\lambda} \geqq |D(T)|$ or T is stable in λ (see [6], VIII, §4 and reforcing there). Also note that when a universal homogeneous, or saturated, or even just special model exists, a universal one exists (the same) (see works of Jonsson and of Morley and Vaught). So if T is first order, $\lambda = \lambda^{<\lambda} > |T|$ or $\lambda > |T|$ strong limit, then T has a universal model in λ. But, of course, this may be rare.

To get non-existence of a universal model in λ is not hard — if we add

---

[†] The author would like to thank the NSF for partially supporting this research, Alice Leonhardt for the beautiful typing, and M. Kojman for proofreading. Publication No. 175A.

[††] The proof in the second section of [9] is flawed.

Received August 14, 1988 and in revised form May 15, 1989

$\aleph_2$-Cohen reals any non-$\aleph_0$-stable countable T has no universal model in $\aleph_1$; if $\lambda = \lambda^{<\lambda}$ and we add μ Cohen subsets of λ, no unstable T has a universal model in any $\chi \in (\lambda, \mu)$ (see representation in Mekler [5]).

In [7] we show that the theory of linear order may have a universal model in $\aleph_1$ though $\aleph_1 < 2^{\aleph_0}$ (using a combination of iteration of proper forcing and oracles). We also show even stronger results (categoricity of a PC class) for a natural unsuperstable theory.

In [9] we show the same for the theory of graphs (the method is related to the one in Abraham, Rubin and Shelah [1]). Here we generalize [9] to higher cardinals.

Meanwhile Mekler [5] generalizes [7] and [9] for a family of universal theories with strong amalgamation properties ($\mathscr{P}^-(3)$, $\mathscr{P}^-(4)$ respectively).

The author found examples of countable theories which never have a universal model in $\aleph_1 < 2^{\aleph_0}$, but are $\aleph_0$-categorical and with amalgamation.[†]

§1.

1.1. THEOREM. *Suppose G.C.H. for simplicity*, $\kappa < \lambda \leqq \mu$, κ *and* λ *are regular and* $\mu^{<\lambda} = \mu$. *Then for some forcing notion* P^*:

(1) P^* *is* κ*-complete and has power* μ.

(2) P^* *does not collapse any* χ, $\kappa < \chi \leqq \mu$.

(3) $\Vdash_{P^*}$ "*there is a universal graph of power* λ".

(4) $\Vdash_{P^*}$ "$2^\kappa = \mu$".

PROOF. By the proof of Baumgartner [2] Theorem 6.1 we easily get:

1.2. PRELIMINARY FORCING. For some forcing notion P, $|P| = \mu$, P is κ-complete and satisfies the $(\lambda)^+$-c.c., does not collapse cardinals and does not change cofinalities, and in $V_a = V^P$ there is a family $\mathscr{U} = \{A_\alpha : \alpha < \mu\}$ of μ subsets of λ such that:

(a) $A \neq B \in \mathscr{U}$ implies $|A \cap B| < \kappa$.

(b) A_α is a stationary subset of $\{\delta < \lambda ; \operatorname{cf} \delta = \kappa\}$.

1.3. GENERAL DESCRIPTION. We shall define a $(<\kappa)$-support iteration of forcing notions satisfying the κ^+-c.c. $\bar{Q} = \langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \mu \rangle$. Q_0 forces a graph $(\lambda, \underset{\sim}{R}_0)$, which shall be a universal graph of power λ. We shall define the $\underset{\sim}{Q}_\alpha$ by induction on α (together with some auxilliary things), and will have to prove that it satisfies the κ^+-c.c.

[†] *Added in proof.* See much more in M. Kojman and S. Shelah, in preparation.

In stage $\alpha > 0$ we will have a P_α-name $\underset{\sim}{R}_\alpha$ so that $\Vdash_{P_\alpha}$ "$(\lambda, \underset{\sim}{R}_\alpha)$ is a graph" and in $V_a^{P_\alpha}$, $\underset{\sim}{Q}_\alpha$ will force an embedding $\underset{\sim}{f}_\alpha$ of the graph $(\lambda, \underset{\sim}{R}_\alpha)$ into the graph $(\lambda, \underset{\sim}{R}_0)$. It is known that we can take care that every P_μ-name of a graph on λ appears as $(\lambda, \underset{\sim}{R}_\alpha)$ for some $\alpha < \mu$.

The problem is, of course, that the various $\underset{\sim}{f}_\alpha$ may give contradicting demands on $(\lambda, \underset{\sim}{R}_0)$. In order to avoid this as much as possible we shall make the $\underset{\sim}{f}_\alpha$'s such that for $\beta < \alpha$ the set (Rang $\underset{\sim}{f}_\alpha$) $\cap$ (Rang $\underset{\sim}{f}_\beta$) has cardinality $< \kappa$. It is reasonable to demand that "Rang $\underset{\sim}{f}_\alpha \subseteq A_\alpha$".

1.4. The Full Inductive Definition. We let

$$Q_0 = \{(w, r) : w \text{ a subset of } \lambda \text{ of power } < \kappa,\ r \text{ a reflexive symmetric two-place relation on } w\}.$$

The order on Q_0 is: $q_1 \leqq q_2$ iff q_1 is a submodel of q_2. Let $\underset{\sim}{R}_0$ be the Q_0-name $\bigcup \underset{\sim}{G}_{Q_0}$ so $\underset{\sim}{R}_0$ is a two-place reflexive symmetric relation on λ.

Now F will be a function such that for each $(<\kappa)$-support iteration (see 1.6) $\bar{Q}^\gamma = \langle P_\alpha, \underset{\sim}{Q}_\beta : \beta < \gamma, \alpha \leqq \gamma \rangle$, $F(\bar{Q}^\gamma)$ is a P_γ-name of a graph (λ, R). Let χ be a large enough regular cardinal.

Now for each $\alpha > 0$, we let $(\lambda, \underset{\sim}{R}_\alpha) = F(\langle P_\beta, \underset{\sim}{Q}_\beta : \beta < \alpha \rangle)$, and we shall define $\langle N_{\alpha,i} : i < \lambda \rangle$, and $\underset{\sim}{Q}_\alpha$.

First let $\langle N_{\alpha,i} : i < \lambda \rangle$ be a sequence of elementary submodels of $(H(\chi)^{V_a}, \in)$ such that for $j < \lambda$:

$$\langle P_\beta, \underset{\sim}{Q}_\beta : \beta < \alpha \rangle,\ (\lambda, \underset{\sim}{R}_\alpha), \text{ and } \langle A_\beta : \beta < \mu \rangle \text{ belong to } N_{\alpha,j},$$

$\| N_{\alpha,j} \| < \lambda$, $N_{\alpha,j} \cap \lambda$ is an ordinal and $N_{\alpha,j}$ increasing continuous in j; $\langle N_{\alpha,i} : i \leqq j \rangle \in N_{\alpha,j+1}$. Note that this is done in V_a, so $\langle N_{\alpha,i} : i < j \rangle \in V_a$ and even $\langle \langle N_{\beta,i} : i < \lambda \rangle : \beta \leqq \alpha \rangle \in V_a$.

Define $\xi_\alpha(i) = \sup(N_{\alpha,i} \cap \lambda)$. Note that $\xi_\alpha(i)$ is always a limit ordinal and $\langle \xi_\alpha(i) : i < \lambda \rangle$ is increasing continuous. As $\underset{\sim}{A}_\alpha$ is a stationary subset of λ, w.l.o.g.

$$(**) \qquad \xi_\alpha(i) \in A_\alpha \quad \text{for every non-limit } i < \lambda.$$

We let $A'_\alpha = \{\xi_\alpha(i+1) : i < \lambda\}$ and note that $A'_\alpha \in V_a$.

Now we come to the main point: defining Q_α (in $V_a^{P_\alpha}$):

(A) A member of $\underset{\sim}{Q}_\alpha$ will consist of $< \kappa$ many atomic conditions (see (B)) with no two of them explicitly contradictory (see (C)).

(B) There are two kinds of atomic conditions:

  (I) $\underset{\sim}{f}_\alpha(i) = j$ where $i < j$, $j \in A'_\alpha$ and $j \in \{\xi_\alpha(\gamma+1) : \kappa i \leqq \gamma < \kappa i + \kappa\}$ (or if you want, the sequence $\langle \alpha, 0, i, j \rangle$, is a condition).

(II) $i \notin \operatorname{Rang} \underset{\sim}{f}_\alpha$.

(C) We shall have to say when two atomic conditions are explicitly contradictory; this occurs just in one of the following three cases:

(α) One-to-one: $\underset{\sim}{f}_\alpha(i_1) = j_1$ and $\underset{\sim}{f}_\alpha(i_2) = j_2$ when $i_1 = i_2, j_1 \neq j_2$ or $i_1 \neq i_2, j_1 = j_2$.

(β) Embedding: $\underset{\sim}{f}_\alpha(i_1) = j_1$ and $\underset{\sim}{f}_\alpha(i_2) = j_2$ when $V_a^{P_\alpha} \Vdash$ "$i_1 \underset{\sim}{R}_\alpha i_2 \equiv \neg\, j_1 \underset{\sim}{R}_0 j_2$".

(γ) Range : $\underset{\sim}{f}_\alpha(i) = j$ and $j \notin \operatorname{Rang} \underset{\sim}{f}_\alpha$.

The order is inclusion.

Explanations. The demand in (B)(I) is in order that $\underset{\sim}{Q}_\alpha$ satisfies the κ^+-c.c. Each $i < \lambda$ should have only κ many possible images. Why in (B)(I), $j \in A'_\alpha$? For reasons similar to those in the club method (see [1]).

1.5. FACT. If P_α satisfies the κ^+-c.c. then $\underset{\sim}{Q}_\alpha$ gives an embedding.

We want to prove (in $V_a^{P_\alpha}$) that $\Vdash_{Q_\alpha}$ "(λ, R_α) is embeddable into (λ, R_0)". We have a natural name for exemplifying this: f_α (defined by $f_\alpha(i) = j$ iff $[\underset{\sim}{f}_\alpha(i) = j]$ belongs to the generic subset of Q_α). It is (forced to be) a partial function from λ to λ by 1.4(B)(I), one-to-one by 1.4(C)(α) and an embedding by 1.4(C)(β). But we should still prove that for every $i < \lambda$, $\Vdash_{Q_\alpha}$ "$i \in \operatorname{Dom} \underset{\sim}{f}$". This is equivalent to proving that for every $q \in Q_\alpha$ for some j, $q \cup \{[\underset{\sim}{f}_\alpha(i) = j]\} \in Q_\alpha$ (assuming q itself has no such member). By 1.4(B)(I) we have κ many candidates for j:

$$B = \{\xi_\alpha(j+1) : \kappa i < j + 1 < \kappa i + \kappa\}.$$

The only difficult demand comes from 1.4(C)(β). As $B \in V_a$ (as $\langle N_{\alpha,i} : i < \lambda \rangle \in V_a$), and as forcing by P_α adds no new subset of λ of cardinality $< \kappa$, by the definition of Q_0, κ many $j \in B$ satisfy this, so we finish the proof of 1.5.

Now the rest of the proof is dedicated to proving that P_α satisfies the κ^+-c.c. assuming this holds for all $\beta < \alpha$. For this we shall derive more detailed information on $\bar{Q}^\alpha$ (using the fact that all $\underset{\sim}{Q}_\beta$, $\beta < \alpha$, were defined as above).

1.6. NICE DENSE SUBSETS OF P_α. For a function p with $\operatorname{dom}(p) \subseteq \alpha$ and such that for all $\alpha \in \operatorname{dom}(p)$, $p(\alpha) \in \underset{\sim}{Q}_\alpha$, define $\operatorname{Dom}(p) = \{\alpha \in \operatorname{dom} p : p(\alpha) \neq 0\}$. We use the variant of $(<\kappa)$-support iteration in which

$$\begin{aligned} P_\beta = \{\, p : {} & p \text{ a function with domain } \subseteq \beta, \operatorname{Dom}(p) \text{ is of power } < \kappa \\ & \text{and } p \restriction \gamma \Vdash_{P_\gamma} \text{“}\underset{\sim}{p}(\gamma) \text{ is a member of } \underset{\sim}{Q}_\gamma \text{ and is set} \\ & \text{of atomic condition of } \underset{\sim}{Q}_\gamma\text{”, for } \gamma \in \operatorname{Dom} p \}, \end{aligned}$$

Let

$$D^0_\beta = \{\, p \in P_\beta : \text{for each } \gamma \in \operatorname{Dom} p,\ p(\gamma) \text{ is an actual set of atomic conditions of power } < \kappa \}.$$

Note that not every function p with domain a subset of β, of power $<\kappa$, each $p(\gamma)$ a set of $<\kappa$ atomic conditions of the forms mentioned in 1.4(B), is in D^0_β: we need $p \restriction \gamma \Vdash_{P_\gamma}$ “$p(\gamma) \in \underset{\sim}{Q}_\gamma$” for each $\gamma \in \operatorname{dom} p$.

1.7. Definition. $D^1_\beta = \{\, p : p$ is a function with domain $\subseteq \beta$ of power $<\kappa$ and for $\gamma \in \operatorname{Dom} p$, $p(\gamma)$ satisfies the demands for $p(\gamma) \in Q_\gamma$ in 1.4(A), (B), (C) except possibly “there are no two atomic conditions in $p(\gamma)$ which are explicitly contradictory by 1.4(C)(β)”$\}$.

For $p \in D^1_\beta$, $\gamma \notin \operatorname{Dom} p$, let $p(\gamma) = \varnothing$.

We define an order on D^1_β:

$$p \leqq r \text{ \textit{iff} for every } \gamma,\ p(\gamma) \subseteq r(\gamma).$$

1.8. Fact. (1) D^0_β is a dense subset of P_β and $D^0_\beta \subseteq D^1_\beta$.

(2) On $D^1_\beta \cap P_\beta$ the orders of P_β and of D^1_β coincide.

(3) For $p \in D^1_\beta$, $p \in D^0_\beta$ iff for every $\gamma \in \operatorname{Dom} p$ and $[\underset{\sim}{f}_\gamma(i_1) = j_1]$, $[\underset{\sim}{f}_\gamma(i_2) = j_2]$ in $p(\gamma)$

$$p \restriction \gamma \Vdash_{P_\gamma} \text{“} i_1 \underset{\sim}{R}_\gamma i_2 \text{ iff } j_1 \underset{\sim}{R}_0 j_2 \text{”}$$

(prove $p \restriction \gamma \in P_\gamma$ by induction).

(4) If $p \in D^1_\beta$, $w \subseteq \operatorname{Dom} p$ then $p \restriction w \in D^1_\beta$.

1.9. Fact.

(1) If ($\beta \leqq \alpha$ and) $p_\zeta \in P_\beta$ for $\zeta < \delta < \kappa$, and $p_\zeta \leqq p_\xi$ for $\zeta < \xi < \delta$ and p is defined by: $\operatorname{Dom}(p) = \bigcup_{\zeta<\delta} \operatorname{Dom}(p_\zeta)$ and for $\gamma \in \operatorname{Dom}(p)$, $p(\gamma) = \bigcup_{\zeta<\delta} p_\zeta(\gamma)$ (remember Q_γ is ordered by inclusion, and $p_\zeta(\gamma) = \varnothing$ for $\gamma \notin \operatorname{Dom}(p_\zeta)$) *then* $p \in P_\beta$ and $p_\zeta \leqq p$ for $\zeta < \delta$ (remember the beginning of 1.6). We say in such cases $p = \bigcup_{\zeta<\delta} p_\zeta$.

(2) If $p_\zeta \in D^1_\beta$ for $\zeta < \delta < \kappa$, $p_\zeta \leqq p_\xi$ for $\zeta \leqq \xi < \delta$ and let $p = \bigcup_{\zeta<\delta} p_\zeta$ be defined by $\operatorname{Dom}(p) = \bigcup_{\zeta<\delta} \operatorname{Dom}(p_\zeta)$, $p(\beta) = \bigcup_{\zeta<\delta} p_\zeta(\beta)$, *then*

$$p \in D^1_\beta \text{ and } p_\zeta \leqq p \text{ for } \zeta < \delta.$$

(3) In (2), if $p_\zeta \in D^0_\beta$ for $\zeta < \delta$, *then*

$$p \in D^0_\beta.$$

(4) If $p^1, p^2 \in D^0_\beta$, and for every $\gamma \in \operatorname{Dom} p^1 \cap \operatorname{Dom} p^2$, $p^1(\gamma) \subseteq p^2(\gamma)$ or

$p^2(\gamma) \subseteq p^1(\gamma)$, then $p^1 \cup p^2 \in D^0_\beta$ where $(p^1 \cup p^2)(\gamma) = p^1(\gamma) \cup p^2(\gamma)$ for $\gamma \in \mathrm{Dom}\, p^1 \cup \mathrm{Dom}\, p^2$ and $p^1 \leqq (p^1 \cup p^2)$, $p^2 \leqq (p^1 \cup p^2)$.

Now we continue with

1.10. Definition. For $\gamma \leqq \alpha$, $q \in \underset{\sim}{Q}_\gamma$, and ordinal $\delta < \lambda$ (usually but not always limit) we let:

$$\text{if } \gamma > 0\text{:}\quad \begin{aligned} q^{[\delta]} = \{[\underset{\sim}{f}_\gamma(i) = j] : {} & [\underset{\sim}{f}_\gamma(i) = j] \in q \\ & \text{and for some } \underset{\sim}{\varepsilon} < \lambda, \\ & j < \xi_\gamma(\underset{\sim}{\varepsilon}) < \delta\} \\ \cup \{[j \notin \mathrm{Rang}\, \underset{\sim}{f}_\gamma] : {} & [j \notin \mathrm{Rang}\, \underset{\sim}{f}_\gamma] \in q \\ & \text{and for some } \underset{\sim}{\varepsilon} < \lambda, j < \xi_\gamma(\underset{\sim}{\varepsilon}) < \delta\}; \\ q^{(\delta)} = \{[j \notin \mathrm{Rang}\, \underset{\sim}{f}_\gamma] : {} & [j \notin \mathrm{Rang}\, \underset{\sim}{f}_\gamma] \in q\} \cup q^{[\delta]} \end{aligned}$$

$$\text{if } \gamma = 0\text{:}\quad q^{[\delta]} = q^{(\delta)} = q \restriction \delta, \text{ i.e., if } \underset{\sim}{q} = (w, \gamma), \text{ then}$$
$$q^{[\delta]} = q^{(\delta)} = (w \cap \delta, r \restriction (w \cap \delta)).$$

1.11. Definition. (1) For $p \in P_\alpha$, and ordinal $\delta < \lambda$, let $p^{[\delta]}$ be a function with domain $\mathrm{Dom}\, p$ and $p^{[\delta]}(\gamma) = (p(\gamma))^{[\delta]}$.

(2) We can make those definitions even for $p \in D^1_\beta$.

1.12. Fact. (1) For any ordinals $\gamma > 0$, δ and $q \in \underset{\sim}{Q}_\gamma$, $q^{[\delta]} = \varnothing$ or for some $\underset{\sim}{\varepsilon}$, $q^{[\delta]} = q^{[\xi_\gamma(\underset{\sim}{\varepsilon})]}$, $\xi_\gamma(\underset{\sim}{\varepsilon}) \leqq \delta$ and $\underset{\sim}{\varepsilon}$ limit or $q^{[\delta]} = q^{[\xi_\gamma(\underset{\sim}{\varepsilon})+1]} = q^{[\xi_\gamma(\underset{\sim}{\varepsilon}+1)]}$, $\xi_\gamma(\underset{\sim}{\varepsilon}) < \delta$, $\underset{\sim}{\varepsilon}$ successor.

(2) If $p \in D^1_{\beta+1}$, $\underset{\sim}{\varepsilon} \leqq \lambda$ and
$(\forall a)[a \subseteq N_{\beta,\underset{\sim}{\varepsilon}} \wedge |a| < \kappa \Rightarrow a \in N_{\beta,\underset{\sim}{\varepsilon}}]$, then
$(p^{[\xi_\beta(\underset{\sim}{\varepsilon})]} \restriction |N_{\beta,\underset{\sim}{\varepsilon}}|) \in N_{\beta,\underset{\sim}{\varepsilon}}$.

(3) If $p \in D^1_\beta$, $\delta < \lambda$, then $p^{[\delta]} \in D^1_\beta$ and $p^{(\delta)} \in D^1_\beta$.

(4) $p^{[\delta]} \leqq p^{(\delta)} \leqq p$ (in D^1_β).

(5) If $p \in D^0_\beta$, $p \leqq r \in D^1_\beta$, $r = r^{(\delta)}$, $r^{[\delta]} \leqq p$, then $r \in D^0_\beta$.

(6) If δ is limit ordinal then for $p \in D^1_\beta$,
$p^{[\delta]} = \bigcup \{p^{[\alpha]} : \alpha < \delta\}$.

(7) If $\delta_1 \leqq \delta_2$, $p \in D^1_\beta$ *then* $p^{[\delta_1]} \leqq p^{[\delta_2]}$.

1.13. Definition. Let

$$\begin{aligned} D_\gamma = \{p \in D^0_\gamma : {} & \text{for every } \delta < \lambda, p^{[\delta]} \in D^0_\gamma; \\ & \text{moreover, if } 0 < \beta \in \mathrm{Dom}\, p \text{ and for } l = 1, 2, \\ & [f_\beta(i_l) = j_l] \in p(\beta), i_1, i_2 < \xi_\beta(\underset{\sim}{\varepsilon}) \textit{ then} \\ & p^{[\xi_\beta(\underset{\sim}{\varepsilon})]} \restriction \beta \Vdash_{P_\beta} \text{“} i_1 \underset{\sim}{R}_\beta i_2 \text{” or } p^{[\xi_\beta(\underset{\sim}{\varepsilon})]} \restriction \beta \Vdash_{P_\beta} \text{“} \neg\, i_1 \underset{\sim}{R}_\beta i_2 \text{”}\}. \end{aligned}$$

1.13A. Fact. The second condition ("Moreover ...") in the definition of D_γ implies the first.

Proof. Prove by induction on $\beta \leqq \gamma$ that for every δ, $(p \restriction \beta)^{[\delta]} \in D_\beta$.

1.14. The Crucial Claim. D_β is a dense subset of P_β (for $\beta \leqq \alpha$).

Proof. We prove this by induction on β.

Case i. $\beta = 0$.
Nothing to prove.

Case ii. $\beta = 1$.
Clearly (as Q_0 is so simple).

Case iii. β limit of cofinality $\geqq \kappa$.
Trivial, as $P_\beta = \bigcup_{\gamma < \beta} P_\gamma$.

Case iv. β limit of cofinality $< \kappa$.
Let $p \in P_\beta$ and we shall find $q \in D_\beta$, $p \leqq q$.

Let $\beta = \bigcup_{\zeta < \operatorname{cf}\beta} \beta(\zeta)$, $\beta(\zeta)$ increasing continuous and let $\beta(\operatorname{cf}\beta) \overset{\text{def}}{=} \beta$. We define by induction on $\zeta \leqq \operatorname{cf}\beta$ a condition $q_\zeta \in P_{\beta(\zeta)}$, $q_\zeta \in D_{\beta(\zeta)}$, q_ζ increasing and $p \restriction \beta(\zeta) \leqq q_\zeta$. For $\zeta = 0$ use the induction hypothesis on $\beta(0)$ (and $p \restriction \beta(0)$). For limit, $q_\zeta \overset{\text{def}}{=} \bigcup_{\xi < \zeta} q_\xi$ is in $D_{\beta(\zeta)}$ (as for each δ, $q_\zeta^{[\delta]} = \bigcup_{\xi < \zeta} q_\xi^{[\delta]}$ is in $D^0_{\beta(\zeta)}$ by Fact 1.9(3)), and clearly it is $\geqq p \restriction \beta(\zeta)$ (as $\langle \beta(\xi) : \xi \leqq \operatorname{cf}(\beta) \rangle$ is increasing continuous). For successor ζ, use the induction hypothesis and 1.9(4). So q_ζ is as required. This applies in particular to $\zeta = \operatorname{cf}\beta$.

Case v. $\beta = \gamma + 1, \gamma > 0$.
So suppose $p \in P_\beta$ and we shall find $p^1 \geqq p$, $p^1 \in D_\beta$, First, by Fact 1.8(1) there is $p_1 \geqq p$, $p_1 \in D^0_\beta$. Second, by the induction hypothesis there is $r \in D_\gamma$, $r \geqq p_1 \restriction \gamma$. As $p_1 \in D^0_\beta$ by 1.12(1) there is an increasing continuous sequence $\langle \delta(\theta) : \theta \leqq \theta(*) \rangle$ of ordinals $< \lambda$, $\delta(0) = 0$, $\theta(*) < \kappa$, $\delta(\theta + 1) \in \{ \xi_\gamma(\underset{\sim}{\varepsilon} + 1) : \underset{\sim}{\varepsilon} < \lambda \}$ such that, if $[f_\gamma(i) = j] \in p_1(\gamma)$, then $\operatorname{Min}\{\underset{\sim}{\varepsilon} : \xi_\gamma(\underset{\sim}{\varepsilon}) > i\}$ is $\delta(\theta + 1)$ for some θ (ε is necessarily non-limit).

We now define by induction on $\theta \leqq \theta(*)$ a condition r_θ such that:

(*) (i) $r_\theta \in D_\gamma$,
   (ii) $r_0 \geqq p_1 \restriction \gamma$,
   (iii) r_θ is increasing continuous,
   (iv) if $[f_\gamma(i_1) = j_1]$, $[f_\gamma(i_2) = j_2]$ belongs to $p(\gamma)$ and $i_1, i_2 < \delta(\theta)$ then $r_\theta^{[\delta(\theta)]}$ determines the truth value of $i_1 \underset{\sim}{R}_\gamma i_2$.

If we succeed we shall finish to prove the crucial claim: $p_1 \cup r_{\theta(*)}$ is as required: for each δ choose θ such that $\delta \geqq \delta(\theta)$, $p_1(\gamma)^{[\delta]} = p_1(\gamma)^{[\delta(\theta)]}$ so by (*) above

$$r^{[\delta(\theta)]}_{\theta(*)} \Vdash_{P_\gamma} \text{“}p_1(\gamma)^{[\delta(\theta)]} \in \underset{\sim}{Q}_\gamma\text{”}$$

but $r^{[\delta(\theta)]}_{\theta(*)} \leqq r^{[\delta]}_{\theta(*)}$ hence

$$r^{[\delta]}_{\theta(*)} \Vdash_{P_\gamma} \text{“}p_1(\gamma)^{[\delta]} = p_1(\gamma)^{[\delta(\theta)]} \in \underset{\sim}{Q}_\gamma\text{”}$$

but $r_{\theta(*)} \in D_\gamma$ hence $r^{[\delta]}_{\theta(*)} \in D^0_\gamma$ hence $r^{[\delta]}_{\theta(*)} \cup p_1^{[\delta]} \in D^0_{\gamma+1}$ as required.

So for proving the crucial claim we just have to carry the induction definition of r_θ as to satisfy (*). For $\theta = 0$, $r_\theta = r$ (remember $p_1(\gamma)^{[0]} = \emptyset$), for θ limit $r_\theta = \bigcup_{\sigma<\theta} r_\sigma$, and there are no problems (see 1.9). So let $\theta > 0$ be a successor ordinal.

Let $\underset{\sim}{\varepsilon}$ be such that $\xi_\gamma(\underset{\sim}{\varepsilon}) = \delta(\theta)$ (exists as θ is a successor ordinal). So clearly $p_1(\gamma)^{[\delta(\theta)]} \subseteq N_{\gamma,\underset{\sim}{\varepsilon}}$ (and if $(\forall \chi < \lambda)\chi^{<\chi} < \lambda$ then w.l.o.g. $p_1(\gamma)^{[\delta(\theta)]} \in N_{\gamma,\underset{\sim}{\varepsilon}}$; otherwise this is not necessarily true). But for every finite subset u of $p_1(\gamma)^{[\delta(\theta)]}$, $u \in N_{\gamma,\underset{\sim}{\varepsilon}}$ and let

$$I_u = \{r \in P_\gamma : r \in D_\gamma, \text{ and } \textit{either } r \Vdash_P \text{“}u \text{ satisfies 1.4(C)(β)”} \text{ or } r \Vdash_P \text{“}u \text{ fails 1.4(C)(β)”}\}$$

and let $\{u_{\theta,\sigma} : \sigma < \sigma_\theta\}$ list the finite subsets of $p_1(\gamma)^{[\delta(\theta)]}$. Clearly for each $u_{\theta,\sigma}$, $I_{u_{\theta,\sigma}}$ belongs to $N_{\gamma,\underset{\sim}{\varepsilon}}$ and it is a dense subset of P_γ. As P_γ satisfies the κ^+-c.c., necessarily $N_{\gamma,\underset{\sim}{\varepsilon}} \cap I_{u_{\theta,\sigma}}$ is a predense subset of P_γ. So we can define by induction on $\sigma \leqq \sigma_\theta$, $r_{\theta,\sigma} \in D_\gamma$, $r_{\theta,\sigma}$ increasing, and $q_{\theta,\sigma} \in I_{u_{\theta,\sigma}} \cap N_{\gamma,\underset{\sim}{\varepsilon}}$, such that $q_{\theta,\sigma} \leqq r_{\theta,\sigma}$.

Now $q_{\theta,\sigma} \Vdash$ “$u_{\theta,\sigma}$ fail 1.4(C)(β)” is impossible as $q_{\theta,\sigma}$ is compatible with r_θ ($r_{\theta,\sigma+1}$ exemplifies this) hence with $r_0 \geqq p_1 \restriction \gamma$, but $p_1 \restriction \gamma \Vdash_{P_\gamma}$ “$p_1(\gamma) \in \underset{\sim}{Q}_\gamma$” hence $p_1 \restriction \gamma \Vdash_{P_\gamma}$ “$u_{\theta,\sigma} \in \underset{\sim}{Q}_\gamma$”.

So $q_{\theta,\sigma} \Vdash_{P_\gamma}$ “$u_{\theta,\sigma}$ satisfies 1.4(C)(β)” hence $r_{\theta,\sigma} \Vdash_{P_\gamma}$ “$u_{\theta,\sigma}$ satisfies 1.4(C)(β)” hence $r_{\theta,\sigma_\theta} \Vdash_{P_\gamma}$ “every finite $u \subseteq p_1(\gamma)^{[\delta(\theta)]}$ satisfies 1.4(C)(β)” so $r_{\theta,\sigma_\theta} \Vdash_{P_\gamma}$ “$p_1(\gamma)^{[\delta(\theta)]} \in \underset{\sim}{Q}_\gamma$”. But for every $\sigma < \sigma_\theta$, $q_{\theta,\sigma} \leqq (r_{\theta,\sigma})^{[\delta(\theta)]} \leqq (r_{\theta,\sigma_\theta})^{[\delta(\theta)]}$, hence $(r_{\theta,\sigma_\theta})^{[\delta(\theta)]} \Vdash_{P_\gamma}$ “$p_1(\gamma)^{[\delta(\theta)]} \in \underset{\sim}{Q}_\gamma$”. So let $r_{\theta+1} \overset{\text{def}}{=} r_{\theta,\sigma_\theta}$, and it is as required, so we have proved 1.14.

1.15. Fact. Suppose $\delta = \xi_\beta(\underset{\sim}{\varepsilon})$ and $p \in P_\alpha$, $\beta < \alpha$, $p = p^{[\delta]}$.

(1) If $\underset{\sim}{\iota}$ is a P_β-name of an ordinal, $\underset{\sim}{\iota} \in N_{\beta,\underset{\sim}{\varepsilon}}$ *then* for some q, $p \leqq q \in P_\alpha$, $q = q^{[\delta]}$ and q force a value for $\underset{\sim}{\iota}$, and $p \restriction [\beta,\alpha) = q \restriction [\beta,\alpha)$.

(2) We can do this simultaneously to $<\kappa$ such names.

(3) If $u \subseteq \delta$, $|u| < \kappa$, *then* there is q, $p \leqq q \in P_\alpha$, $q = q^{[\delta]}$ and q forces a value to $\underset{\sim}{R}_\beta \restriction u$.

Proof. (1) As is the definition of $q_{\theta,\sigma+1}$ in the proof of 1.14.

(2) By 1.15(1) and 1.9.

(3) By 1.15(2).

1.16. **Main Lemma.** *P_α satisfies the κ^+-c.c.*

Proof. Let $p_\zeta \in P_\alpha$ for $\zeta < \kappa^+$, and for $\zeta \neq \xi$ the conditions p_ζ, p_ξ are not compatible, and we shall eventually derive a contradiction. Clearly we can replace $\langle p_\zeta : \zeta < \kappa^+ \rangle$ by $\langle p'_\zeta : \zeta < \kappa^+ \rangle$ if $p'_\zeta \geqq p_\zeta$, and by $\langle p_\zeta : \zeta \in A \rangle$ if $A \subseteq \kappa^+$, $|A| = \kappa^+$. We shall use this freely.

W.l.o.g. for every ζ:

(a) $p_\zeta \in D_\alpha$.

(b) $0 \in \mathrm{Dom}\, p_\zeta$.

(c) If $\beta \neq \gamma \in \mathrm{Dom}\, p_\zeta$, $j \in A'_\beta \cap A'_\gamma$, *then* j belongs to the universe of $p_\zeta(0)$.

(d) If $[j \notin \mathrm{Rang}\, \underset{\sim}{f_\beta}] \in p_\zeta(\beta)$ or $[\underset{\sim}{f_\beta}(i) = j] \in p_\zeta(\beta)$ for some β and i, *then* j belongs to the universe of $p_\zeta(0)$.

(e) If $[\underset{\sim}{f_\beta}(i) = j] \in p_\zeta(\beta)$ and $j_1 \in A'_\beta$, $\kappa i < j_1 < j$ then j_1 belongs to the universe of $p_\zeta(0)$.

(f) If j belongs to the universe of $p_\zeta(0)$ and $\beta \in \mathrm{Dom}(p_\zeta)$ *then* $[j \notin \mathrm{Rang}\, \underset{\sim}{f_\beta}] \in p_\zeta(\beta)$ or $(\exists i)([\underset{\sim}{f_\beta}(i) = j] \in p_\zeta(\beta))$.

We can easily find $\zeta < \xi < \kappa^+$ such that:

$$\text{if } \beta \in \mathrm{Dom}(p_\zeta) \cap \mathrm{Dom}(p_\xi) \text{ then } p_\zeta(\beta) \cup p_\xi(\beta) \text{ belongs to } D^1_\alpha.$$

Let $w = \{\delta(\theta) : \theta < \theta(*)\}$ where $\theta(*) < \kappa$ be such that:

(1) $\delta(\theta)$ is increasing continuous.

(2) $\delta(0) = 0$, $\delta(\theta) < \lambda$.

(3) $\mathrm{Dom}(p_\zeta(0)) \cup \mathrm{dom}(p_\xi(0)) \subseteq \{\delta(\theta) : \theta < \theta(*)\}$.

(4) If $\delta(\theta)$ is limit then $\delta(\theta + 1) = \delta(\theta) + 1$.

Note. $(\forall \theta)[\theta \text{ limit } \theta \leqq \theta(*) \rightarrow \mathrm{cf}(\delta(\theta)) = \mathrm{cf}\, \theta]$.

Now we shall define r_θ by induction on $\theta \leqq \theta(*)$ such that:

(A) $r_\theta \in D^0_\alpha$, r_θ increasing continuous (in θ).

(B) $r_\theta = r_\theta^{[\delta(\theta)]}$.

(C) $p_\zeta^{[\delta(\theta)]} \leqq r_\theta$, $p_\xi^{[\delta(\theta)]} \leqq r_\theta$.

(D) $p^{[\delta(\theta)]} \subseteq r_\theta$ where

$p = p_a \cup p_b$ where

$p_a(\beta) = \{[j \notin \mathrm{Rang}\, f_\beta] : \beta \in \mathrm{Dom}\ p_\zeta \cup \mathrm{Dom}\ p_\xi$ and
$\neg\, (\exists\, i)([f_\beta(i) = j] \in p_\zeta(\beta) \cup p_\xi(\beta))$
and $j \in \mathrm{dom}(p_\zeta(0)) \cup \mathrm{dom}(p_\xi(0))\}$,

$p_b(\beta) = \{[j \notin \mathrm{Rang}\, f_\beta] : \beta \in \mathrm{Dom}\ p_\zeta \cup \mathrm{Dom}\ p_\xi$ and
$\neg\, (\exists\, i)([f_\beta(i) = j] \in p_\zeta(\beta) \cup p_\xi(\beta))$
and for some $\gamma \in (\mathrm{Dom}\ p_\zeta \cup \mathrm{Dom}\ p_\xi)$, $\gamma \neq \beta$ and $j \in A'_\beta \cap A'_\gamma\}$.

Case I. $\theta = 0$.
Trivial.

Case II. $\theta = 1$.
Use 1.9(4) for $p_\zeta^{[\delta(1)]}$, $p_\xi^{[\delta(1)]}$.

Case III. θ limit.
So $\delta(\theta) = \bigcup_{\sigma < \theta} \delta(\sigma)$, and $\bigcup_{\sigma < \theta} r_\sigma$ is as required.

Case IV. $\theta = \sigma + 1$, $\delta(\sigma)$ non-limit > 0.
Trivial.

Case V. $\theta = \sigma + 1$, $\sigma > 0$, not Case IV.
Let $u_\zeta = \{\beta < \alpha : \beta \in \mathrm{Dom}(p_\zeta)$, and $(\exists\, i)([\underset{\sim}{f}_\beta(i) = \delta(\sigma)] \in p_\zeta(\beta))\}$ so u_ζ has cardinality $< \kappa$ and, for $\beta \in u_\zeta$, let $i = i_{\zeta,\beta}$ be such that $[\underset{\sim}{f}_\beta(i_{\zeta,\beta}) = \delta(\sigma)] \in p_\zeta(\beta)$; similarly for ξ.

1.16A. FACT. There is q_σ such that
(1) $r_\sigma \leqq q_\sigma \in D_\alpha$,
(2) $q_\sigma^{[\delta(\sigma)]} = q_\sigma$,
(3) for $\beta \in u_\zeta$, $q_\sigma \restriction \beta$ forces a value for
$\underset{\sim}{R}_\beta \restriction (\{i : (\exists j)([\underset{\sim}{f}_\beta(i) = j] \in q_\sigma)\} \cup \{i_{\zeta,\beta}\})$,
(4) for $\beta \in u_\xi$, $q_\sigma \restriction \beta$ forces a value for
$\underset{\sim}{R}_\beta \restriction (\{i : (\exists j)([\underset{\sim}{f}_\beta(i) = j] \in q_\sigma)\} \cup \{i_{\xi,\beta}\})$.

PROOF. By 1.15 and closure under union.

We now want to define r_θ. Let r_θ be defined as follows:
for $\beta < \alpha$, $\beta \neq 0$,

$$r_\theta(\beta) = q_\sigma(\beta) \cup p_\zeta^{[\delta(\theta)]}(\beta) \cup p_\xi^{[\delta(\theta)]}(\beta) \cup p^{[\delta(\theta)]}(\beta);$$

for $\beta = 0$,
$r_\theta(0)$ has universe $\mathrm{dom}(q_\sigma(0)) \cup \{\delta(\sigma)\}$, extends $q_\sigma(0)$, and $p_\zeta^{[\delta(\theta)]}(0)$, $p_\xi^{[\delta(\theta)]}(0)$ and:

(*) suppose $\beta \in \mathrm{Dom}(p_\zeta)$, $[\underset{\sim}{f}_\beta(i) = j] \in q_\sigma$ and $[\underset{\sim}{f}_\beta(i_{\zeta,\beta}) = \delta(\sigma)] \in p_\zeta(\beta)$ *then* $q_\sigma \restriction \beta \Vdash i\underset{\sim}{R}_\beta i_{\zeta,\beta}$ iff $q_0 \restriction \beta \not\Vdash \neg\, i\underset{\sim}{R}_\beta i_{\zeta,\beta}$ iff $r_\theta(0) \vDash jR_0\delta(\sigma))$

and

(**) similarly for ξ.

Note that $q_\sigma \restriction \beta \Vdash i\underset{\sim}{R}_\beta i_{\zeta,\beta}$ iff $q_\sigma \restriction \beta \not\Vdash \neg\, i\underset{\sim}{R}_\beta i_{\zeta,\beta}$ by 16A. We should verify that r_θ is as required.

Point (i). Why is $r_\theta(0)$ well defined?

A priori we may have two conflicting demands on the truth value of $j_1 R_0 j_2$. We have five sources of such demands: q_σ $p_\zeta^{[\delta(\theta)]}(0)$, $p_\xi^{[\delta(\theta)]}(0)$, (*) and (**).

The first and second do not contradict as $q_\sigma = q_\sigma^{[\delta(\sigma)]}$ whereas $(p_\zeta^{[\delta(\theta)]})^{[\delta(\sigma)]} = p_\zeta^{[\delta(\sigma)]} \leqq r_\sigma \leqq q_\sigma$. Similarly the first and third do not contradict.

The second and third do not contradict by the choice of $\zeta < \xi$, i.e., as $p_\zeta \cup p_\xi \in D_\alpha^1$ so $p_\zeta(0) \cup p_\xi(0) \in Q_0$.

Next, the first and fourth do not contradict as $q_\sigma^{[\delta(\sigma)]} = q_\sigma$, so for every $j : jR_0\delta(\sigma) \notin q_\sigma(0)$ and $\neg\, jR_0\delta(\sigma) \notin q_\sigma(0)$. Similarly, the first and fifth do not contradict.

What about a contradiction between the second and fourth, i.e., $p_\zeta^{[\delta(\theta)]}(0)$ and an instance of (*)? Let the instance of (*) be $\beta \in \mathrm{Dom}(p_\zeta)$, $[\underset{\sim}{f}_\beta(i) = j] \in q_\sigma$, $[\underset{\sim}{f}_\beta(i_{\zeta,\beta}) = \delta(\sigma)] \in p_\zeta(\beta)$ and the contradiction is about $jR_0\delta(\sigma)$. So $\beta \in \mathrm{Dom}(p_\zeta)$ (by the last sentence), and $j \in \mathrm{dom}(p_\zeta(0))$ (as $p_\zeta(0)$ forces a truth value to $jR_0\delta(\sigma)$), hence by (f), $[j \notin \mathrm{Rang}\, \underset{\sim}{f}_\beta] \in p_\zeta(\beta)$ or $(\exists i)([\underset{\sim}{f}_\beta(i) = j] \in p_\zeta(\beta))$. In the first case

$$[j \notin \mathrm{Rang}\, \underset{\sim}{f}_\beta] \in p_\zeta(\beta)^{[\delta(\sigma)]} \leqq q_\sigma$$

(as $j < \delta(\sigma)$) contradiction, hence the second case occurs. So for some $i_1 [\underset{\sim}{f}_\beta(i_1) = j] \in p_\zeta(\beta)$, but then $\kappa i_1 < j < \delta(\sigma)$ clearly $[\underset{\sim}{f}_\beta(i_1) = j] \in q_\sigma$, hence $i_1 = i$. So $[\underset{\sim}{f}_\beta(i_{\zeta,\beta}) = \delta(\sigma)]$, $[\underset{\sim}{f}_\beta(i_1) = j]$ belongs to $p_\zeta(\beta)$ hence, as $p_\zeta \in D_\alpha$, $p^{[\delta(\sigma)]} \restriction \beta$ force a truth value for $i\underset{\sim}{R}_\beta i_{\zeta,\beta}$ equal to the one $p_\zeta(0)$ determine for $j\underset{\sim}{R}\delta(\sigma)$ and we get an easy contradiction as $p_\zeta^{[\delta(\sigma)]} \leqq q_\sigma$.

Similarly there is no contradiction between the third and fifth.

What about a contradiction between the second and fifth, i.e., between $p_\zeta^{[\delta(\theta)]}(0)$ and an instance of (**) which is $\beta \in \mathrm{Dom}(p_\xi)$, $[\underset{\sim}{f}_\beta(i) = j] \in q_\sigma(\beta)$, $[\underset{\sim}{f}_\beta(i_{\xi,\beta}) = \delta(\sigma)] \in p_\xi(\beta)$ and the contradiction is about $jR_\sigma\delta(\sigma)$? But in such a case by (d), $\delta(\sigma) \in \mathrm{dom}\, p_\xi(0)$. As $p_a(\beta)^{[\delta(\theta)]} \leqq q_\sigma(\beta)$ and $j \in \mathrm{Dom}(p_\zeta(0))$ and $[\underset{\sim}{f}_\beta(i) = j] \in q_\sigma(\beta)$, necessarily $j \in \mathrm{dom}\, p_\xi(0)$. So both p_ζ and p_ξ force truth values for $jR_0\delta(\sigma)$, but $p_\zeta \cup p_\xi \in D^1_\alpha$, hence it is the same and we get a contradiction between the third and fifth, and we finish by the previous case.

Similarly there is no contradiction between the third and fourth.

Next we deal with two instances of the fourth, i.e.,

(∗) for $l = 1, 2$, $\beta_l \in \mathrm{Dom}(p_\zeta)$,
$[f_{\beta_l}(i_l) = j_l] \in q_\sigma$ and $[f_{\beta_l}(i_{\zeta,\beta_l}) = \delta(\sigma)] \in p_\zeta(\beta_l)$.

Since there is a contradiction between $j_1R_0\delta(\sigma)$, $j_2R_0\delta(\sigma)$ it may be assumed that $j_1 = j_2$ and $\beta_1 \neq \beta_2$. But

$$\begin{gathered}\beta_1, \beta_2 \in \mathrm{Dom}(p_\zeta), \text{ hence by (c)}\\ A'_{\beta_1} \cap A'_{\beta_2} \subseteq \mathrm{dom}(p_\zeta(0)), \text{ but by the above}\\ \{j_1, \delta(\sigma)\} \subseteq A'_{\beta_1} \cap A'_{\beta_2}\end{gathered}$$

so $j_1R_0\delta(\sigma) \in p_\zeta(0)$ or $\neg\, j_1R_0\delta(\sigma) \in p_\zeta(0)$, so we get a contradiction between the second and one of the instances of (∗) with which we have already dealt.

Similarly there is no contradiction between two instances of the fifth.

Lastly, what about a contradiction between (∗) and (∗∗)? So we assume

$$\beta_1 \in \mathrm{Dom}(p_\zeta),\ [\underset{\sim}{f}_{\beta_1}(i_1) = j_1] \in q_\sigma,$$

$$[\underset{\sim}{f}_{\beta_1}(i_{\zeta,\beta_1}) = \delta(\sigma)] \in p_\zeta(\beta_1);$$

$$\beta_2 \in \mathrm{Dom}(p_\xi),\ [\underset{\sim}{f}_{\beta_2}(i_2) = j_2] \in q_\sigma,$$

$$[\underset{\sim}{f}_{\beta_2}(i_{\xi,\beta_2}) = \delta(\sigma)] \in p_\xi(\beta_2).$$

As we can assume that there is a contradiction necessarily $\beta_1 \neq \beta_2$, $j_1 = j_2 \in A'_{\beta_1} \cap A'_{\beta_2}$, $j_1 < \delta(\sigma)$, and $[j_1 \notin \mathrm{Rang}\, \underset{\sim}{f}_{\beta_1}] \notin p_\zeta(\beta_1)$. Now $\beta_1 \in \mathrm{Dom}(p_\zeta)$ and $p_b^{[\delta(\sigma)]} \leqq p^{[\delta(\sigma)]} \leqq r_\sigma \leqq q_\sigma$, so by the definition of p_b, and the last sentence, necessarily $j_1 \in \mathrm{dom}(p_\zeta(0))$. Similarly, $j_1 \in \mathrm{dom}(p_\xi(0))$. Also $\delta(\sigma) \in \mathrm{dom}(p_\zeta(0))$ (as $[\underset{\sim}{f}_{\beta_1}(i_{\zeta,\beta}) = \delta(\sigma)] \in p_\zeta(\beta_1)$ and $\delta(\sigma) \in \mathrm{dom}(p_\xi(0))$. So p_ζ, p_ξ determine the truth value of $j_1R_0\delta(\sigma)$, and in the same way (as $p_\zeta \cup p_\xi \in D^1_\alpha$) we hence reduce the contradiction to a previous case.

Point (ii). Why does $r_\theta(\beta)$ (where $\beta > 0$) satisfy 1.4(C)(d) (one-to-oneness)?

There are three sources of atomic condition $[f_\beta(i) = j]$ for $r_\theta(\beta)$: q_σ, $p_\zeta^{[\delta(\theta)]}$, $p_\xi^{[\delta(\theta)]}$. The second and third cannot contradict as $p_\zeta \cup p_\xi \in D^1_\alpha$.

Suppose that the first and second contradict. As $p_\zeta^{[\delta(\sigma)]} \leqq q_\sigma$, the only possibility is that $[f_\beta(i_1) = \delta(\sigma)] \in p_\zeta$ contradict some member $[f_\beta(i_2) = j_2]$ of q_σ. As necessarily $j_2 < \delta(\sigma)$ we conclude $i_2 = i_1$.

As $[f_\beta(i_1) = \delta(\sigma)] \in p_\zeta(\beta)$, clearly for some $\underset{\sim}{\varepsilon}_0 : \delta(\sigma) = \xi_\beta(\underset{\sim}{\varepsilon}_0)$, $\kappa i_1 < \underset{\sim}{\varepsilon}_0 < \kappa i_1 + \kappa$, and $\kappa i_1 < \underset{\sim}{\varepsilon} < \underset{\sim}{\varepsilon}_0 \Rightarrow [\xi_\beta(\underset{\sim}{\varepsilon}) \notin \mathrm{Rang}\, f_\beta] \in p_\zeta(\beta)$ hence $\kappa i_1 < \underset{\sim}{\varepsilon} < \underset{\sim}{\varepsilon}_0 \Rightarrow [\xi_\beta(\underset{\sim}{\varepsilon}) \notin \mathrm{Rang}\, f_\beta] \in r_\sigma(\beta) \subseteq q_\sigma(\beta)$, contradicting $j_2 < \delta(\sigma) = \xi_\beta(\underset{\sim}{\varepsilon}_0)$, $[f_\beta(i_1) = j_2] \in q_\sigma(\beta)$.

Similarly the first and third do not contradict.

Point (iii). Why does $r_\theta(\beta)$ (where $\beta > 0$) satisfy 1.4(C)(β) (embedding)?

In the choice of q_σ (in the fact above) and (*), (**) of the definition of $r_\theta(0)$ take care of this.

Point (iv). Why does $r_\theta(\beta)$ (where $\beta > 0$) satisfy 1.4(C)(γ) (Rang)?

Left to the reader.

Point (v). Why do (A), (B), (C), (D) above hold?

See definition of r_0, r_θ.

DISCUSSION.

Question. Can we get a similar result for two cardinals λ_1, λ_2 simultaneously (when $\kappa < \lambda_1 < \lambda_2 < 2^\kappa$)?

Question. Can you classify countable first-order theories by, e.g., $T_1 \sim T_2$ *iff* for any universe of set theory (e.g., which you get by set forcing and cardinal λ; e.g., such that $(\exists \kappa)(\aleph_0 < \kappa = \kappa^{<\kappa} < \lambda < 2^\kappa)$) T_1 has a universal model of power λ *iff* T_2 has a universal model of power λ?

Question. For which classes, e.g., is there no universal model of power λ if $(\exists \kappa)(\aleph_0 < \kappa = \kappa^{<\kappa} < \lambda < 2^\kappa)$, or even if just $(\exists \mu < \lambda)(2^\mu > \lambda)$? (See [9], p. 86 on this.)

REFERENCES

1. U. Abraham, M. Rubin and S. Shelah, *On the consistency of some partition theorems for continuous coloring and the structure of* $\aleph_1$*-dense real order type*, Ann. Pure Appl. Logic **29** (1985), 123–206.
2. J. Baumgartner, *Almost disjoint sets, the dense set problem and the partition calculus*, Ann. Math. Logic **9** (1976), 401–439.
3. W. B. Easton, *Powers of regular cardinals*, Ann. Math. Logic **1** (1970), 139–178.
4. R. Grossberg and S. Shelah, *On universal locally finite groups*, Isr. J. Math. **44** (1983), 289–302.
5. A. H. Mekler, *Universal structures in power* $\aleph_1$, J. Symb. Logic, to appear.
6. S. Shelah, *Classification Theory and the Number of Non-isomorphic Models*, North-Holland, Amsterdam, 1978, 542 + xvi pp.
7. S. Shelah, *Independence results*, J. Symb. Logic **45** (1980), 563–573.
8. S. Shelah, *Some results in set theory*, Abstracts Am. Math. Soc. (1982), 522.
9. S. Shelah, *Universal graphs without instances of CH*, Ann. Pure Appl. Logic **26** (1984), 75–87.