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Mathematical Logic

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On nice equivalence relations on ${}^{\lambda}2$

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Abstract. Let E be an equivalence relation on the powerset of an uncountable set, which is reasonably definable. We assume that any two subsets with symmetric difference of size exactly 1 are not equivalent. We investigate whether for E there are many pairwise non equivalent sets.

Annotated content

§0 Introduction

§1 Dichotomical results on nice equivalence relations
[Assume E is a $\Pi^1_1[\lambda]$-equivalence relation on ${}^{\lambda}2$ such that η, ν are not E-equivalent whenever they differ in exactly one place. Assume further that this holds even after adding a λ-Cohen subset of λ. If $\lambda = \lambda^{<\lambda} \geq \beth_\omega$ (alternatively, E is more nicely defined or other requirement on λ) then E has a perfect set (so 2^λ elements) of pairwise non E-equivalent members of ${}^{\lambda}2$. There are related results.]

§2 Singular of uncountable cofinality
[Assume $\lambda = \lambda^{<\kappa} > \mathrm{cf}(\lambda) = \kappa > \aleph_0$. We find on ${}^{\kappa}\lambda$ quite nice equivalence relations for which the parallel of the results of §1 fail badly. If λ is strong limit, we can use ${}^{\lambda}2$.]

§3 Countable cofinality: positive results
[Assume that $\lambda > \mathrm{cf}(\lambda) = \aleph_0$ and λ is the limit of measurables, or just a related property (which consistently holds for $\aleph_\omega = \beth_\omega$) is satisfied. We prove the parallel of the result in §1 on ${}^{\omega}\lambda$.]

§4 The countable cofinality case: negative results

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[We show that if our universe is far enough from large cardinals (and close to **L**) then we can build counterexamples as in §2.]

§5 On $r_p(\mathrm{Ext}(G, \mathbb{Z}))$

[We return to the p-rank of the abelian group $\mathrm{Ext}(G, \mathbb{Z})$ where G is torsion free abelian group ($\aleph_1$-free, without loss of generality). We show that if κ is compact, λ strong limit (singular) cardinal $> \kappa$ and $r_p(\mathrm{Ext}(G, \mathbb{Z})) \geq \lambda$ then $r_p(\mathrm{Ext}(G, \mathbb{Z})) \geq 2^\lambda$. This is preserved by adding κ Cohens, κ super-compact. If GCH holds above κ we have a complete characterization of $\{\mathrm{Ext}(G, \mathbb{Z}) : G\}$.]

§0. Introduction

The main topic here is the possible generalizations of the following theorem from [Sh 273] on "simple" equivalence relation on ${}^\omega 2$ to higher cardinals.

Theorem 0.1. *1) Assume that*

(a) E is a Borel 2-place relation on ${}^\omega 2$
(b) E is an equivalence relation
(c) if $\eta, \nu \in {}^\omega 2$ and $(\exists! n)(\eta(n) \neq \nu(n))$, then η, ν are not E-equivalent.

Then there is a perfect subset of ${}^\omega 2$ of pairwise non E-equivalent members.

2) Instead of "E is Borel", "E is analytic (or even a Borel combination of analytic relations)" is enough.

3) If E is a Π^1_2 relation which is an equivalence relation satisfying clauses (b) + (c) also in $\mathbf{V}^{\mathrm{Cohen}}$, then the conclusion of (1) holds.

In [Sh 273], Theorem 0.1 was used to prove a result on the homotopy group: if X is a Hausdorff metric topological space which is compact, separable, arc-connected, and locally arc-connected, and the homotopy group is not finitely generated then it has the cardinality of the continuum; the proof of 0.1 used forcing in [Sh 273], see [PaSr98] without the forcing.

We may restrict E to be like the natural equivalence relation in presenting $r_p(\mathrm{Ext}(G, \mathbb{Z}))$ or just closer to group theory as in Grossberg Shelah [GrSh 302], [GrSh 302a], Mekler-Roslanowski-Shelah [MRSh 314], [Sh 664]. In §5 we say somewhat more. We here continue [Sh 664] but do not rely on it.

Turning to ${}^\lambda 2$ the problem split according to the character of λ and the "simplicity" of E. If E is Π^1_1 and $\lambda = \lambda^{<\lambda}$ and $\lambda \geq \beth_\omega$ (or just $(Dl)_\lambda$ holds), a generalization holds. If E is Σ^1_1 and $\lambda = \lambda^{<\lambda}$, the generalization in general fails; all this in §1. Now if λ is singular, strong limit for simplicity, it is natural to consider ${}^{\mathrm{cf}(\lambda)}\lambda$ instead of ${}^\lambda 2$. If λ has uncountable cofinality we get strong negative results in §2. If λ has countable cofinality, and is the limit of "somewhat large cardinals", e.g. measurable cardinals, (but $\lambda = \aleph_\omega$ may be O.K., i.e. consistently) the generalization holds (in §3), but if the universe is close to **L** (e.g. in **L** there is no weakly compact) then we get negative results (see §4). Note that theorems of the form "if E has many equivalence classes it has continuum many equivalence classes" do not generalize well, see [ShVs 719] even for λ weakly compact.

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Definition 0.2. *For a cardinal λ let $\mathcal{B}_\lambda$ be ${}^{\lambda}2$ (or ${}^{\lambda}\lambda$ or ${}^{cf(\lambda)}\lambda$); we write $\mathcal{B}$ for such set.*

1) For a logic $\mathcal{L}$ we say that E is a $\mathcal{L}$-nice, (say 2-place for simplicity) relation on $\mathcal{B}$ if there is a model M with universe λ and finite vocabulary τ, and unary function symbols $F_1, F_2 \notin \tau$ (denoting possibly partial unary functions), such that letting $\tau^+ = \tau \cup \{F_1, F_2\}$, for some sentence $\psi = \psi(F_1, F_2)$ in $\mathcal{L}(\tau^+)$ we have

$\circledcirc$ *for any $\eta_1, \eta_2 \in \mathcal{B}$ letting $M_{\eta_1,\eta_2} = (M, \eta_1, \eta_2)$ be the τ^+-model expanding M with $F_\ell^{M_{\eta_1,\eta_2}} = \eta_\ell$ for $\ell = 1, 2$ we have*
$\eta_1 E \eta_2 \Leftrightarrow (M, \eta_1, \eta_2) \models \psi$.
We may write $M \models \psi[\eta_1, \eta_2]$ and $\psi[\eta_1, \eta_2, M]$ or $\psi(x, y, M)$ or write $a \subseteq \lambda$ coding M instead of M.

2) E is a Π^1_1-relation on $\mathcal{B}$ means that above we allow ψ to be of the form $(\forall X \subseteq \lambda)\varphi$ where φ is first order or even in inductive logic (i.e. we have variables on sets and are allowed to form the first fix point for $\varphi(x, X)$ where $X_1 \subseteq X_2 \wedge \varphi(x, X_1) \to \varphi(x, X_2)$); if we allow just first order φ we say "strictly", if we allow $\varphi \in L$ we say L-strictly. Similarly Σ^1_1, Π^1_2, projective; writing nice means $\mathcal{L}$ is $\mathbb{L}$(induction) i.e. first order + definition by induction. We may write $E \in$ nice$(\mathcal{B}_\lambda)$, $\Sigma^1_1[\mathcal{B}]$ etc, and may replace $\mathcal{B}$ by λ if this holds for some $\mathcal{B}$. We write very nice for $\mathcal{L}$-nice when $\mathcal{L}$ is $\mathbb{L}$ first order logic.

We note the obvious (by now) relation (on $\mathbb{L}$ see below)

Fact 0.3. 1) If $\lambda = \lambda^{<\kappa}$ and R is a [strictly] $\mathbb{L}_{\lambda^+,\kappa}$-nice relation on $\mathcal{B}_\lambda$ then R is a [strictly] Σ^1_1-relation and also a [strictly] Π^1_1-relation (with parameter a relation of λ, of course).

2) If R is a $\mathbb{L}_{\lambda^+,\kappa}$(induction)-nice relation on $\mathcal{B}_\lambda$ and $\kappa > \aleph_0$, then R is $\mathbb{L}_{\lambda^+,\kappa}$-strictly a Π^1_1-relation on $\mathcal{B}_\lambda$

3) If $\mathrm{cf}(\lambda) > \aleph_0$ then if R is $\mathbb{L}$(induction)-nice relation on $\mathcal{B}_\lambda$ then R is strictly Σ^1_1-nice (hence being Σ^1_1 is equivalent to being strictly Σ^1_1).

Proof. (1), (2) Recall that in the definition of Π^1_1, φ was allowed to be a formula in $\mathbb{L}$(induction).

3) It is well known that a linear order $<^*$ on such λ is a well ordering iff for every $\alpha < \lambda$, $<^* \restriction \{\beta : \beta < \alpha\}$ is isomorphic to $(\gamma, <)$ for some $\gamma < \lambda$ (e.g. [Na85]). $\square_{0.3}$

Notation.

$(\forall^* i < \delta)$ means "for every large enough $i < \delta$".

J^{bd}_δ is the ideal of bounded subsets of δ.

$\mathcal{L}$ denotes a logic, $\mathcal{L}(\tau)$ denotes the language (i.e, a set of formulas, for the logic $\mathcal{L}$ in the vocabulary τ), $\mathbb{L}$ denotes first order logic, $\mathbb{L}_{\lambda,\kappa}$ denotes the extension of $\mathbb{L}$ by allowing $\bigwedge\limits_{\alpha<\alpha(*)} \varphi_\alpha$ (when $\alpha(*) < \lambda$) and $(\forall x_0, \ldots, x_i)_{i<\alpha(*)}\varphi$ for $\alpha(*) < \kappa$.

Definition 0.4. *Let $(D\ell)_\lambda$ means that λ is regular, uncountable and there is a sequence $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda\rangle$ such that $\mathcal{P}_\alpha$ is a family of $< \lambda$ subsets of α and for every $X \subseteq \lambda$ the set $\{\delta < \lambda : X \cap \delta \in \mathcal{P}_\delta\}$ is stationary; hence $\lambda = \lambda^{<\lambda}$. (By [Sh 460], $\lambda = \lambda^{<\lambda} \geq \beth_\omega \Rightarrow (D\ell)_\lambda$ and (by Kunen) $\lambda = \mu^+ \Rightarrow (D\ell)_\lambda \equiv \diamondsuit_\lambda$).*

Definition 0.5. *$\mathcal{Q} \subseteq {}^{\lambda}2$ is called perfect or λ-perfect* *if:*

(a) $\mathcal{Q} \neq \emptyset$
(b) if $\eta \in \mathcal{Q}$ *then* $\{\ell g(\eta \cap \nu) : \nu \in \mathcal{Q} \backslash \{\eta\}\} \subseteq \lambda$ *is an unbounded subset of* λ
(c) the set $\{\eta \restriction \zeta : \eta \in \mathcal{Q}$ *and* $\zeta \leq \lambda\}$ *is closed under the union of* $\triangleleft$*-increasing sequences.*

Equivalently, $\mathcal{Q} = \{\rho_\eta : \eta \in {}^{\lambda}2\}$ *such that*

$(a)'$ $\rho_\eta \in {}^{\lambda}2$
$(b)'$ $\eta_1 \neq \eta_2 \in {}^{\lambda}2 \Rightarrow \rho_{\eta_2} \neq \rho_{\eta_2}$
$(c)'$ *if* $\eta_0, \eta_1, \eta_2 \in {}^{\lambda}2$ *are distinct and* $(\eta_1 \cap \eta_2) \triangleleft (\eta_1 \cap \eta_0)$ *(so* $\eta_1 \cap \eta_2 \neq \eta_1 \cap \eta_0$*) then* $(\rho_{\eta_1} \cap \rho_{\eta_2}) \triangleleft (\rho_{\eta_1} \cap \rho_{\eta_0})$ *and* $\rho_{\eta_1}(\ell g(\rho_{\eta_1} \cap \rho_{\eta_2})) = \eta_1(\ell g(\eta_1 \cap \eta_2))$.

§1. Dichotomical results on nice equivalence relations on ${}^{\lambda}2$

We here continue [Sh 664, §2], the theorem and most proofs can be read without it. The claims below generalize [Sh 273].

Claim 1.1. *Assume*

$\boxtimes_1(a)$ $\lambda = \lambda^{<\lambda}$ *and* $\lambda \geq \beth_\omega$ *or just* $(Dl)_\lambda$ *(see 0.3)*
(b) *E is a nice 2-place relation on* ${}^{\lambda}2$
$(c)(\alpha)$ *E is an equivalence relation on* ${}^{\lambda}2$
(β) *if* $\eta, \nu \in {}^{\lambda}2$ *and* $(\exists! \alpha < \lambda)(\eta(\alpha) \neq \nu(\alpha))$ *then* $\neg(\eta E \nu)$.

Then E has 2^λ *equivalence classes, moreover a perfect set of pairwise non E-equivalent members of* ${}^{\lambda}2$.

Proof. Note that

⊛ If $\mathbb{P}$ is a λ-complete forcing (or just λ-strategically complete) then $\Vdash_{\mathbb{P}}$ "clauses (c), (α), (β) are still true".

So we can apply 1.2 below. $\square_{1.1}$

A relative is

Claim 1.2. *Assume*

$\boxtimes_2(a), (c)$ *as in* $\boxtimes_1$
(b) *E is a* $\Pi^1_1[\lambda]$ *2-place relation on* ${}^{\lambda}2$*, say defined by* $(\forall Z)\varphi(x, y, Z, \bar{a})$ *see Definition 0.2*
$(c)^+ = (c)^+_{Cohen}$ *if* $\mathbb{P} = ({}^{\lambda>}2, \triangleleft)$*, i.e.* λ*-Cohen,* *then* *in* $\mathbf{V}^{\mathbb{P}}$ *clauses (c) from 1.1 still hold.*

Then the conclusion of 1.1 holds.

Proof. Stage a. Let $(\eta_0, \eta_1) \in {}^{\lambda}2 \times {}^{\lambda}2$ be generic over $\mathbf{V}$ for the forcing $\mathbb{Q} = ({}^{\lambda>}2) \times ({}^{\lambda>}2)$. Now do we have $\mathbf{V}[\eta_0, \eta_1] \models$ "$\eta_0 E \eta_1''$? If so, then for some $(p_0, p_1) \in ({}^{\lambda>}2) \times ({}^{\lambda>}2)$ we have $(p_0, p_1) \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta}_0 E \underset{\sim}{\eta}_1''$, let $\alpha < \lambda$ be $> \ell g(p_0), \ell g(p_1)$ and by clause $(c)^+(\beta)$ in $\mathbf{V}[\eta_0, \eta_1]$ we can find $\eta_1' \in {}^{\lambda}2$ such that $\eta_1' \restriction \alpha = \eta_1 \restriction \alpha$, and for some $\beta \in (\alpha, \lambda)$, $\eta_1' \restriction [\beta, \lambda) = \eta_1 \restriction [\beta, \lambda)$, (here $\beta = \alpha + 1$ is O.K. but not so in some generalizations) and $\mathbf{V}[\eta_0, \eta_1] \models \neg(\eta_1' E \eta_1)$.

So $\mathbf{V}[\eta_0, \eta_1] \models$ "$\neg(\eta_0 E \eta'_1)$" (again as in $\mathbf{V}[\eta_0, \eta_1]$, E is an equivalence relation by clause $(c)^+$ and we are assuming for the time being that $\mathbf{V}[\eta_0, \eta_1] \models$ "$\eta_0 E \eta_1$"). But also (η_0, η'_1) is generic over $\mathbf{V}$ for $({}^{\lambda>}2) \times ({}^{\lambda>}2)$ with (p_0, p_1) in the generic set and $\mathbf{V}[\eta_0, \eta_1] = \mathbf{V}[\eta_0, \eta'_1]$ so we get a contradiction to $(p_0, p_1) \Vdash$ "$(\underset{\sim}{\eta}_0 E \underset{\sim}{\eta}_1)$". Hence

$\circledast_1 \Vdash_{({}^{\lambda>}2)\times({}^{\lambda>}2)}$ "$\neg(\underset{\sim}{\eta}_0 E \underset{\sim}{\eta}_1)$".

<u>*Stage B*</u>: Let χ be large enough and let $N \prec (\mathcal{H}(\chi), \in)$ be such that $\|N\| = \lambda$, $N^{<\lambda} \subseteq N$ and the definition of E belongs to N. Note that

$\circledast_2$ if $(\eta_0, \eta_1) \in ({}^{\lambda}2) \times ({}^{\lambda}2)$ (and is in $\mathbf{V}$) and $N[\eta_0, \eta_1] \models$ "$\neg(\eta_0 E \eta_1)$", <u>then</u> $\neg(\eta_0 E \eta_1)$.
[Why? As E is Π^1_1, in $N[\eta_0, \eta_1]$, there is a witness $\in {}^{\lambda}2$ for failure, and it also witnesses in $\mathbf{V}$ that $\neg(\eta_0 E \eta_1)$.]

Clearly to finish proving 1.1, it suffices to prove

Subclaim 1.3. *1) Assume $\lambda = \lambda^{<\lambda}$ and $(Dl)_\lambda$.*

If $\mathcal{H}(\lambda) \subseteq N$, $N^{<\lambda} \subseteq N$, $\|N\| = \lambda$ and $N \models ZFC^-$, <u>then</u> there is a perfect $\mathcal{Q} \subseteq {}^{\lambda}2$ such that for any $\eta_0 \neq \eta_1$ from $\mathcal{Q}$ the pair (η_0, η_1) is generic over N for $[({}^{\lambda>}2) \times ({}^{\lambda>}2)]^N$.

2) Assume that λ is regular and

(a) $\mathcal{T}$ is a tree with λ levels each of cardinality $< \lambda$ and 2^λ λ-branches (or just $\geq \mu$) and
(b) $\bar{N} = \langle N_\alpha : \alpha < \lambda\rangle$ is $\subseteq$-increasing, $\bar{N} \restriction (\alpha+1) \in N_{\alpha+1}, \mathcal{T} \in N_0$ and $\alpha \subseteq N_\alpha$, $\|N_\alpha\| < \lambda$ and $N = \bigcup\limits_{\alpha<\lambda} N_\alpha$ and $\mathcal{T}_{\leq\alpha} \in N_{\alpha+1}$ (if λ is regular it is enough that $\bar{N} \restriction (\alpha+1) \in N$, $\mathcal{T}_{\leq\alpha} \in N$)
(c) $<^*$ is a well ordering of N such that $<^* \restriction N_\alpha \in N_{\alpha+1}$.

<u>*Then*</u> *for some $X \subseteq {}^{\lambda}2$, $|X| = 2^\lambda$ (or just $|X| = \mu$) and $\eta_0 \neq \eta_1 \in X \Rightarrow$ the pair (η_0, η_1) is generic over N for $({}^{\lambda>}2) \times ({}^{\lambda>}2)$.*

3) Like part (2) but we weaken clause (a) to

$(a)'$ $\mathcal{T}$ is a tree with λ levels each of cardinality $\leq \lambda$ and $\bar{Y} = \langle Y_\alpha : \alpha \leq \lambda\rangle$, Y_α is a set of $< \lambda$ nodes of $\mathcal{T}$ of level α if $\alpha < \lambda$ and a set of λ-branches of $\mathcal{T}$ if $\alpha = \lambda$ and $|Y_\lambda| \geq \mu$ and $\eta \neq \nu \in Y_\lambda \Rightarrow (\exists \alpha < \lambda)(\eta \restriction \alpha, \nu \restriction \alpha \in Y_\alpha)$.

Remark 1.4. Such $\mathcal{T}$ is called a λ-Kurepa tree and much is known on its existence (and non existence). E.g. if λ is strong limit then such $\mathcal{T}$ exists.

Proof. 1) Let $\langle \mathcal{P}_\alpha : \alpha < \lambda\rangle$ be such that $\mathcal{P}_\alpha \subseteq \mathcal{P}(\alpha)$, $|\mathcal{P}_\alpha| < \lambda$, and for every $X \subseteq \lambda$ the set $\{\alpha : X \cap \alpha \in \mathcal{P}_\alpha\}$ is stationary. So by coding we can find $\mathcal{P}'_\alpha \subseteq \{(\eta_0, \eta_1) : \eta_0, \eta_1 \in {}^{\alpha}2\}$ of cardinality $< \lambda$ such that for every $\eta_0, \eta_1 \in {}^{\lambda}2$ the set $\{\alpha < \lambda : (\eta_0 \restriction \alpha, \eta_1 \restriction \alpha) \in \mathcal{P}'_\alpha\}$ is stationary. Lastly, let $\langle \mathcal{I}_\alpha : \alpha < \lambda\rangle$ list the dense open subsets of $({}^{\lambda>}2) \times ({}^{\lambda>}2)$ which belong to N. Now we define by induction on $\alpha < \lambda$, $\langle \rho_\eta : \eta \in {}^{\alpha}2\rangle$ such that:

(a) $\rho_\eta \in {}^{\lambda>}2$
(b) $\beta < \ell g(\eta) \Rightarrow \rho_{\eta\restriction\beta} \lhd \rho_\eta$
(c) $\rho_\eta\hat{}\langle\ell\rangle \lhd \rho_{\eta\hat{}\langle\ell\rangle}$
(d) if α is a limit ordinal and $(\eta_0, \eta_1) \in \mathcal{P}'_\alpha$, $\ell_0 < 2$, $\ell_1 < 2$ and $\eta_0\hat{}\langle\ell_0\rangle \neq \eta_1\hat{}\langle\ell_1\rangle$ <u>then</u> $(\rho_{\eta_0\hat{}\langle\ell_0\rangle}, \rho_{\eta_1\hat{}\langle\ell_1\rangle}) \in \bigcap_{\beta\le\alpha} \mathcal{I}_\beta$.

There is no problem to carry the definition (using $|\mathcal{P}'_\alpha| < \lambda = \mathrm{cf}(\lambda)$) and $\{\bigcup_{\alpha<\lambda} \rho_{\eta\restriction\alpha} : \eta \in {}^\lambda 2\}$ is a perfect set as required.

2) Similar. We choose by induction on α, $\langle\rho_\eta : \eta \in \mathcal{T}_\alpha\rangle$ such that (a),(b),(c) above holds and

$(d)'$ if $\eta_0 \neq \eta_1$ are in $\mathcal{T}_{\alpha+1}$ then $(\rho_{\eta_0}, \rho_{\eta_1}) \in \cap\{\mathcal{I} : \mathcal{I}$ is a dense open subset of $\mathbb{P}$ and belongs to $N_\alpha\}$
(e) if $\langle\rho_\eta : \eta \in \mathcal{T}_\alpha\rangle$ is the $<^*_\chi$-sequence satisfying (a)–(d).

So $\langle\rho_\eta : \eta \in \mathcal{T}_\alpha\rangle$ can be defined from $\langle N_\beta : \beta < \alpha\rangle$.

The proof in part (2) is easier as we can assume that such a tree belongs to N.

3) Left to the reader.

So we have finished proving claim 1.2 hence claim 1.1. $\square_{1.3}, \square_{1.2}$

Claim 1.5. *1) In claims 1.1, 1.2 we can weaken clause (β) (in (c), $(c)^+$, call it $(c)^-$, $(c)^\pm$ respectively) to:*

$(\beta)^-$ if $\eta \in {}^\lambda 2$ and $\alpha < \lambda$ <u>then</u> for some $\beta \in (\alpha, \lambda)$ and $\rho \in {}^{[\alpha,\beta)}2$ the sequences $\eta, ((\eta \restriction \alpha)\hat{}\rho\hat{}\eta \restriction [\beta, \lambda))$ are not E-equivalent.

2) In claims 1.1, 1.2 and in 1.5(1), for any $\varepsilon^* \le \lambda$ we can replace E by $\langle E_\varepsilon : \varepsilon < \varepsilon^*\rangle$, each E_ε satisfying clauses (b) and (c), $(c)^+$, $(c)^-$, $(c)^\pm$ there respectively and in the conclusion:

$(*)$ there is a λ-perfect $\mathcal{Q}$ such that
(α) $\mathcal{Q} = \langle\rho_\eta : \eta \in {}^\lambda 2\rangle$ and
(β) if $\eta_1 \neq \eta_2$ are from ${}^\lambda 2$ then $\rho_{\eta_1} \neq \rho_{\eta_2}$ and $\varepsilon < \varepsilon^* \Rightarrow \neg(\rho_{\eta_1} E_\varepsilon \rho_{\eta_2})$
(γ) for $\eta \in {}^\lambda 2$ the set $\{\ell g(\rho_\eta \cap \rho_\nu) : \nu \in {}^\lambda 2\backslash\{\eta\}\}$ is a closed unbounded subset of λ.

3) In 1.2, 1.5(1),(2) we can weaken $(c)^+$ or $(c)^\pm$ to

$(*)$ for a stationary set of $N \in [\mathcal{H}(\lambda^+)]^\lambda$ there is (in $\mathbf{V}$) $\eta \in {}^\lambda 2$ which is Cohen over N such that $\Pi^1_1[\lambda]$ sentences are absolute from $N[\eta]$ to $\mathbf{V}$ (for $\Sigma^1_1[\lambda]$-sentences this is necessarily true) and clause (c) (or $(c)^-$) holds.

Proof. 1), 2) The same as the proof of 1.1.

3) The only place it makes a difference is in Stage A of the proof of Claim 1.1. We choose N, η as in $(*)$ of 1.5(3), and let $\eta_\ell = \langle\eta(2\alpha + \ell) : \alpha < \lambda\rangle$ in $N[\eta] = N[\eta_0, \eta_1]$ instead of working with $\mathbf{V}[\underset{\sim}{\eta}_0, \underset{\sim}{\eta}_1]$. $\square_{1.5}$

Now we would like not to restrict ourselves to $\Pi^1_1[\lambda]$-equivalence relations.

Claim 1.6. *1) Assume*
(a) $\lambda = \lambda^{<\lambda}$, $\mu \le 2^\lambda$
(b) E is a $\Pi^1_2[\lambda]$ 2-place relation on ${}^\lambda 2$, say definable by $(\forall Z_1)(\exists Z_2)$ $\varphi(x, y, Z_1, Z_2, a)$

$(c)(\alpha)$ *E is an equivalence relation on* ${}^{\lambda}2$

(β) *if* $\eta, \nu \in {}^{\lambda}2$ *and* $(\exists! \alpha < \lambda)(\eta(\alpha)) \neq \nu(\alpha))$ *then* $\neg(\eta E \nu)$

$(c)^+$ *if* $\eta \in {}^{\lambda}2$ *is generic over* $\mathbf{V}$ *for* $({}^{\lambda>}2, \triangleleft)$, *i.e. is a Cohen sequence over* $\mathbf{V}$ *<u>then</u> in* $\mathbf{V}[\eta]$, *clause (c) still holds*
(note that for $\rho_1, \rho_2 \in ({}^{\lambda}2)^{\mathbf{V}}$ *anyhow* $\mathbf{V} \models$ "$\rho_1 E \rho_2$" $\Leftrightarrow \mathbf{V}[\eta] \models$ "$\rho_1 E \rho_2$"*)*

(d) *for every* $A \subseteq \lambda$ *and* $\chi > 2^{\lambda}$ *there are* $N, \langle \rho_\varepsilon : \varepsilon < \mu \rangle$ *such that*

(i) $N \prec (\mathcal{H}(\chi), \in), N^{<\lambda} \subseteq N, \|N\| = \lambda, A \in N$

(ii) $\rho_\varepsilon \in {}^{\lambda}2$ *and* $[\varepsilon < \zeta \Rightarrow \rho_\varepsilon \neq \rho_\varepsilon]$

(iii) *for* $\varepsilon \neq \zeta$ *the pair* $(\rho_\varepsilon, \rho_\zeta)$ *is generic over* N *for the forcing notion* $({}^{\lambda>}2 \times {}^{\lambda>}2)$

(iv) $\Pi^1_1[\lambda]$ *formulas are preserved from* $N[\rho_\varepsilon, \rho_\zeta]$ *to* $\mathbf{V}$ *for* $\varepsilon < \zeta < \mu$.

<u>Then</u> E has $\geq \mu$ *equivalence classes.*

2) We can replace $\geq \mu$ *by "perfect" in the conclusion if in (d),* $\{\rho_\varepsilon : \varepsilon < \mu\} \subseteq {}^{\lambda}2$ *is perfect [see 0.5].*

3) We can replace ${}^{\lambda>}2$ *by a subtree* $\mathcal{T} \subseteq {}^{\lambda>}2$ *such that forcing with* $\mathcal{T}$ *adds no bounded subset to* λ.

Proof. By [Sh 664, 2.2t].

Definition 1.7. *Clause (d) of 1.6 is called "λ is $[\lambda, \mu)$-weakly Cohen-absolute: $[\lambda, \mu)$-w.c.a., in short" (as in [Sh 664, 2.1t]'s notation).*

Claim 1.8. *We can strengthen 1.6 just as 1.5 strenghthens 1.1.*

We may wonder when does clause (d) of 1.6 hold.

Claim 1.9. *1) Assume*

(i) $\lambda = \lambda^{<\lambda}$ *in* $\mathbf{V}$

(ii) $\mathbb{P}$ *is a forcing notion*

(iii) $\langle \underset{\sim}{\eta}_\varepsilon : \varepsilon < \mu \rangle$ *is a sequence of* $\mathbb{P}$*-names,*

(iv) $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\eta}_\zeta \neq \underset{\sim}{\eta}_\varepsilon \in {}^{\lambda}2$ *for* $\varepsilon < \zeta < \mu$"

(v) if $A \subseteq \lambda, p \in \mathbb{P}, \chi$ *large enough then there are* $N \prec (\mathcal{H}(\chi), \in), \|N\| = \lambda, N^{<\lambda} \subseteq N, \{A, p\} \in N$ *and* q *such that* $p \leq q \in \mathbb{P}, q$ *is* $(N, \mathbb{P})$*-generic,* $q \Vdash$ "$({}^{\lambda>}2)^{\mathbf{V}^{\mathbb{P}}} \subseteq N[\underset{\sim}{G}_{\mathbb{P}}]$" *and* $\mathbb{P}' \lessdot \mathbb{P}$ *such that* $q \Vdash_{\mathbb{P}}$ *"for some* $u \in [\mu]^{\mu}$*, for every* $\varepsilon \neq \zeta$ *from* u*, the pair* $(\underset{\sim}{\eta}_\varepsilon, \underset{\sim}{\eta}_\zeta)$ *is generic over* $N[\underset{\sim}{G}_{\mathbb{P}'}]$ *for* $({}^{\lambda>}2 \times {}^{\lambda>}2)^{\mathbf{V}^{\mathbb{P}}}$ *and the forcing* $\mathbb{P}/(\mathbb{P}' + \underset{\sim}{\eta}_\varepsilon + \underset{\sim}{\eta}_\zeta)$ *is* λ*-complete (or at least* λ*-strategically complete).*

<u>Then</u> λ *is* (λ, μ)*-w.c.a. (see 1.7) in the universe* $\mathbf{V}^{\mathbb{P}}$.

Proof. Straightforward.

§2. Singulars of uncountable cofinality

In this section we show that the natural generalization of 0.1 usually provably fails badly for ${}^{\mathrm{cf}(\lambda)}\lambda$, λ singular of uncountable cofinality.

Claim 2.1. *Assume*

(a) $\lambda > \kappa = cf(\lambda) > \aleph_0$

(b) $2^{\kappa} + \lambda^{<\kappa} = \lambda$.

Then there is E such that

(α) *E is an equivalence relation on ${}^\kappa\lambda$*
(β) *E is very nice* [1] *(see Definition 0.2)*
(γ) *if $\eta_1, \eta_2 \in {}^\kappa\lambda$ and $(\forall^* i < \kappa)(\eta_1(i) = \eta_2(i))$ then $\eta_1 E \eta_2 \Leftrightarrow \eta_1 = \eta_2$*
(δ) *E has exactly λ equivalence classes.*

Obervation 2.2. In 2.1, and in the rest of this section: (of course, we have to translate the results; we leave it as an exercise to the reader).

1) We can restrict ourselves to $\prod_{i<\kappa} \lambda_i$ where $i < \kappa \Rightarrow \lambda_i < \lambda = \sum_{j<\kappa} \lambda_j$, see the proof; similarly in 2.4.

2) We can consider ${}^\kappa\lambda$ as a subset of ${}^\lambda 2$, in fact a very nice one:

we identify $\eta \in {}^\kappa\lambda$ with $\nu_\eta \in {}^\lambda 2$ when $\nu_\eta(i) = 1 \Leftrightarrow i \in \{\mathrm{pr}(\zeta, \eta(\zeta)) : \zeta < \kappa\}$ for any choice of a pairing function pr, in fact, any one to one function pr from $\kappa \times \lambda$ onto λ is O.K.

3) If λ is strong limit we can identify $\prod_{i<\kappa} \lambda_i$ with ${}^\lambda 2$ as follows: without loss of generality $\lambda_i = 2^{\mu_i}$ with μ_i increasing, let $\langle g^i_\varepsilon : \varepsilon < {}^{\mu_i}2\rangle$ list the functions from $[\bigcup_{j<i} \mu_j, \mu_i)$ to $\{0,1\}$ and we identify $\eta \in \prod_{i<\kappa} \lambda_i$ with $\bigcup_{i<\kappa} g^i_{\eta(i)} \in {}^\lambda 2$.

4) We can translate our results to any $\prod_{i<\kappa} \lambda_i$ when $\lambda_i \leq \lambda = \sum_{i<\kappa} \lambda_i = \limsup\langle \lambda_i : i < \kappa\rangle$.

5) The union of $\leq \lambda$ closed subsets of $({}^\kappa\lambda) \times ({}^\kappa\lambda)$ is very nice.

Proof of 2.2. (1),(2),(3) left to the reader.

4) Define the function F from $\bigcup_{\zeta\leq\kappa} {}^\zeta\lambda$ to $\bigcup_{\zeta\leq\kappa} \prod_{i<\zeta} \lambda_i$ by defining $F(\eta)$ by induction on $\ell g(\eta)$ as follows:

(a) $F(<>) = <>$
(b) $F(\eta\hat{\ }\langle\alpha\rangle)$ is $F(\eta)\hat{\ }\rho_{\eta,\alpha}$ when: $\varepsilon_{\eta,\alpha} = \mathrm{Min}\{\varepsilon : \alpha < \lambda_{\ell g(F(\eta))+\varepsilon}\}$, $\rho_{\eta,\alpha} = 0_{\varepsilon_{\eta,\alpha}}\hat{\ }\langle 1+\alpha\rangle$
(c) for η of limit length, $F(\eta) = \bigcup_{\varepsilon<\ell g(\eta)} F(\eta \restriction \varepsilon)$.

Clearly $\ell g(\eta) \leq \ell g(F(\eta))$ and η, ν are $\triangleleft$-incomparable implies $F(\eta), F(\nu)$ are $\triangleleft$-incomparable, so F is one to one. Also F maps ${}^\kappa\lambda$ into $\prod_{i<\kappa} \lambda_i$ continuously so Range(F) is a closed set.

Also, when $\mathrm{cf}(\kappa) > \aleph_0$ for any $\eta, \nu \in {}^\kappa\lambda$ we have $(\forall^*\varepsilon)(\eta(\varepsilon) = \nu(\varepsilon)) \Leftrightarrow (\forall^*\varepsilon)((F(\eta))(\varepsilon) = F(\nu))(\varepsilon))$.

This is enough to translate 2.1 to $\prod_{i<\kappa} \lambda_i$ instead of ${}^\kappa\lambda$.

Alternatively, we can repeat the proof.

5) Why is it very nice? Assume $E = \cup\{E_i : i < i(*)\}$, $i(*) \leq \lambda$ and each E_i is a closed subset of $({}^\kappa\lambda) \times ({}^\kappa\lambda)$. Let $\{\nu_\alpha : \alpha < \lambda\}$ list ${}^{\kappa>}\lambda$ with no repetitions, and we define a model M:

its universe is λ
F_0 is unary, $F_0(\alpha) = \ell g(\nu_\alpha)$

[1] In fact we have a closed division of ${}^\kappa\lambda$ to ${}^\kappa 2$ sets such that E refines this division and on each part E is closed, see 2.2(5)

F_1 is binary, $F_1(\alpha, \varepsilon) = \beta$ iff $\nu_\beta = \nu_\alpha \restriction (\min\{\varepsilon, \ell g(\nu_\alpha))$
R is a three-place relation, $R^M(\alpha, \beta, i)$ iff for some $(\eta_0, \eta_1) \in E_i$ we have $\nu_\alpha \triangleleft \eta_0$, $\nu_\beta \triangleleft \eta_1$
P is unary predicate $P^M = i(*)$
$<$ is binary relation, the order on λ.

Now for $f, g : \kappa \to \lambda$ we have

$$f E g \text{ } \underline{\text{iff}} \text{ } (M, f, g) \models (\exists i)[P(i) \;\&\; (\forall \varepsilon < \kappa)(\exists \alpha, \beta)(F_0(\alpha) = \varepsilon \;\&\; F_0(\beta) = \varepsilon$$
$$\& \; R(\alpha, \beta, i) \;\&\; (\forall \zeta < \varepsilon)[f(\zeta) = F_1(\alpha, \zeta) \;\&\; g(\zeta) = F_1(\beta, \zeta)]).$$

Normally we do not elaborate such things. $\square_{2.2}$

Proof of 2.1. We choose $\bar{\lambda} = \langle \lambda_i : i < \kappa \rangle$, nondecreasing, i.e. $i < j \Rightarrow \lambda_i \le \lambda_j$ with limit λ, (e.g. $\lambda_i = \lambda$ which is the case stated in the claim) let $\mu_j = \prod_{i<j} \lambda_i$ so $\mu_j \le \lambda$ and let $\bar{f}^i = \langle f^i_\alpha : \alpha < \mu_i \rangle$ list $\prod_{j<i} \lambda_j$ or be just a set of representatives of $\prod_{j<i} \lambda_j / J^{\mathrm{bd}}_i$.

For every $\eta \in \prod_{i<\kappa} \lambda_i$ let

(a) for limit $i < \kappa$ let $\alpha_i(\eta) = \ \mathrm{Min}\{\alpha : \eta \restriction i = f^i_\alpha \bmod J^{\mathrm{bd}}_i\}$
(b) for $\varepsilon < \kappa$ let $B_\varepsilon(\eta) = \{i : i < \kappa$ is a limit ordinal, $\varepsilon < i$ and $f^i_{\alpha_i(\eta)}(\varepsilon) = \eta(\varepsilon)\}$ and lastly
(c) $A(\eta) = \{\varepsilon < \kappa : B_\varepsilon(\eta)$ is not stationary$\}$.

Now we define two binary relations E_0, E_1 on $\prod_{i<\kappa} \lambda_i$:

(d) $\eta_1 E_0 \eta_2$ iff for every $\varepsilon < \kappa$ we have $B_\varepsilon(\eta_1) = B_\varepsilon(\eta_2)$
(e) $\eta_1 E_1 \eta_2$ iff $\eta_1 E_0 \eta_2 \& \eta_1 \restriction A(\eta_1) = \eta_2 \restriction A(\eta_2)$.

Clearly

(α) E_0 is an equivalence relation on $\prod_{i<\kappa} \lambda_i$ with $\le 2^\kappa \le \lambda$ classes
(β) E_1 is an equivalence relation on ${}^\kappa\lambda$, refining E_0
(γ) E_0, E_1 are very nice; in details:
 (a) E_0 is a closed subset of $(\prod_{i<\kappa} \lambda_i) \times (\prod_{i<\kappa} \lambda_i)$ under the initial segment topology, that is, for $(\eta_0, \eta_1) \in (\prod_{i<\kappa} \lambda_i) \times (\prod_{i<\kappa} \lambda_i)$ the family $\{u^\varepsilon_{(\eta_0 \restriction \varepsilon, \eta_1 \restriction \varepsilon)} : \varepsilon < \kappa\}$ where $u^\varepsilon_{\bar{\rho}} = \{\{(\nu_0, \nu_1) \in (\prod_{i<\kappa} \lambda_i) \times (\prod_{i<\kappa} \lambda_i) : (\nu_0 \restriction \varepsilon, \nu_1 \restriction \varepsilon) = \bar{\rho}\}$ is a neighborhood basis of (η_0, η_1)
 [Why? as the truth value of $i \in B_\varepsilon(\eta)$ is determined by $\eta \restriction i$ for $\varepsilon < \kappa$, $i < \kappa$]
 (b) E_1 is the union of $\le 2^\kappa$ closed subsets of $({}^\kappa\lambda) \times ({}^\kappa\lambda)$ under the initial segment topology
 [Why?
  (a) as if $(\eta_0, \eta_1) \in \prod_{i<\kappa} \lambda_i \times \prod_{i<\kappa} \lambda_i \backslash E_0$, then for some $\varepsilon < \kappa$ and $i < \kappa$, we have $(i \in B_\varepsilon(\eta_0)) \equiv (i \notin B_\varepsilon(\eta_1))$ so $\varepsilon < i < \kappa$ and so $u = u^i_{(\eta_0 \restriction \varepsilon, \eta_1 \restriction \varepsilon)}$ is a nb of (η_0, η_1) and by the definition of $B_\varepsilon(-)$ we have $u \cap E_0 = \emptyset$ hence $u \cap E_1 = \emptyset$
  (b) for $\bar{B} = \langle B_\varepsilon : \varepsilon < \kappa \rangle$, $B_\varepsilon \subseteq \kappa$
  let $\Gamma_{\bar{B}} = \{\eta \in \prod_{i<\kappa} \lambda_i : B_\varepsilon(\eta) = B_\varepsilon$ for every $\varepsilon < \kappa\}$.

Now $\langle \Gamma_{\bar{B}} : \bar{B} \in {}^{\kappa}\mathcal{P}(\kappa)\rangle$ list the E_0-equivalence classes (and $\emptyset$) and each $E_1 \restriction \Gamma_{\bar{B}}$ is closed.]

(δ) if $\eta_1, \eta_2 \in {}^{\kappa}\lambda$ and $\eta_1 E_0 \eta_2$ <u>then</u> $A(\eta_1) = A(\eta_2)$
[Why? Check the definitions]

(ε) for $\eta \in {}^{\kappa}\lambda$, $A(\eta)$ is a bounded subset of κ
[why? otherwise let $C = \{\delta < \kappa : \delta = \sup(A(\eta) \cap \delta)\}$, it is a club of κ, and for each $i \in C$ there is $j_i < i$ such that $\eta \restriction [j_i, i) = f^i_{\alpha_i(\eta)} \restriction [j_i, i)$, clearly j_i exists by the definition of $\alpha_i(\eta)$. By Fodor lemma, for some $j(*) < \kappa$ the set $S_{j(*)} = \{i \in C : j_i = j(*)\}$ is stationary, now choose $\varepsilon \in A(\eta) \backslash j(*)$, so clearly $B_\varepsilon(\eta)$ includes $S_{j(*)} \backslash \varepsilon$ hence is a stationary subset of κ hence by the definition of $A(\eta)$ clearly ε does not belong to $A(\eta)$, contradiction.]
So clearly

(ζ) E_1 has $\leq (\prod_{i<\kappa} \lambda_i / E_0) + \Sigma\{\prod_{j<i} \lambda_j : i < \kappa\} \leq \lambda$ equivalence classes.
Now

(η) if $\eta_1, \eta_2 \in {}^{\kappa}\lambda$ and $\eta_1 = \eta_2 \mod J^{\mathrm{bd}}_\kappa$ <u>then</u> for every limit $i < \kappa$ large enough we have $\alpha_i(\eta_1) = \alpha_i(\eta_2)$
[why? let $i^* = \sup\{j + 1 : \eta_1(j) \neq \eta_2(j)\}$ so by the assumption, if i is a limit ordinal and $i \in (i^*, \kappa)$ then $\eta_1 \restriction i = \eta_1 \restriction i \mod J^{\mathrm{bd}}_i$ hence $\alpha_i(\eta_1) = \alpha_i(\eta_2)$ by the definition of $\alpha_i(-)$, which is the desired conclusion of (η).]

(θ) if $\eta_1, \eta_2 \in {}^{\kappa}\lambda$ and $\eta_1 = \eta_2 \mod J^{\mathrm{bd}}_\kappa$ <u>then</u> $\eta_1 E_1 \eta_2 \Leftrightarrow \eta_1 = \eta_2$
[why? if $\eta_1 = \eta_2$ clearly $\eta_1 E_1 \eta_2$; so assume $\eta_1 E_1 \eta_2$ and we shall show that $\eta_1 = \eta_2$, i.e. $\varepsilon < \kappa \Rightarrow \eta_1(\varepsilon) = \eta_2(\varepsilon)$. By the definition of E_1 we have $\eta_1 E_0 \eta_2$ hence by clause (δ) we have $A(\eta_1) = A(\eta_2)$, call it A. If $\varepsilon \in A$, by the definition of E_1 we have $\eta_1 \restriction A = \eta_2 \restriction A$ hence $\eta_1(\varepsilon) = \eta_2(\varepsilon)$. So assume $\varepsilon \in \kappa \backslash A$, first we can find $j^* < \kappa$ such that for every limit $i \in (j^*, \kappa)$ we have $\alpha_i(\eta_1) = \alpha_i(\eta_2)$, it exists by clause ($\eta$). Second, the sets $B_\varepsilon(\eta_1), B_\varepsilon(\eta_2)$ are stationary (as $\varepsilon \notin A(\eta_\ell)$) and equal (as $\eta_1 E_0 \eta_2$); so we can find $i \in B_\varepsilon(\eta_1) \cap B_\varepsilon(\eta_2)$ which satisfy $i > j^*$. Now $\eta_1(\varepsilon) = f^i_{\alpha_i(\eta_1)}(\varepsilon)$ by the definition of $B_\varepsilon(\eta_1)$ as $i \in B_\varepsilon(\eta_1)$ and $\alpha_i(\eta_1) = \alpha_i(\eta_2)$ as $i > j^*$ and $f^i_{\alpha_i(\eta_2)}(\varepsilon) = \eta_2(\varepsilon)$ by the definition of $B_\varepsilon(\eta_2)$ as $i \in B_\varepsilon(\eta_2)$; together $\eta_1(\varepsilon) = \eta_2(\varepsilon)$. So we have completed the proof that $\varepsilon < \kappa \Rightarrow \eta_1(\varepsilon) = \eta_2(\varepsilon)$ thus proving $\eta_1 = \eta_2$ as required.]

(ι) E_1 has $\geq \lambda_i$ equivalence classes for any $i < \kappa$
[why? let $\eta^* \in \prod_{j<\kappa} \lambda_j$ and for $\alpha < \lambda_i$ let $\eta^*_\alpha \in {}^{\kappa}\lambda$ be defined by $\eta^*_\alpha(\varepsilon)$ is α if $\varepsilon = i$ and is $\eta^*(\varepsilon)$ otherwise. By clause (θ) we have $\alpha < \beta < \lambda_i \Rightarrow \neg \eta^*_\alpha E_1 \eta^*_\beta$, hence $|\prod_{j<\kappa} \lambda_j / E_1| \geq \lambda_i$.]

(κ) E_1 has exactly λ equivalence classes
[why? by clause (ι), E_1 has $\geq \sup\{\lambda_i : i < \kappa\} = \lambda$ equivalence classes and by clause (ζ), E has $\leq \lambda$ equivalence classes.]

We could have defined E_0 as

($*$) $\eta_1 E_0 \eta_2$ iff for every $\varepsilon < \kappa$ we have $B_\varepsilon(\eta_1) = B_\varepsilon(\eta_2) \mod \mathcal{D}_\kappa$ where $\mathcal{D}_\kappa$ is the club filter on κ.

This causes no change except that E_0 is not a closed subset of $({}^{\kappa}\lambda) \times ({}^{\kappa}\lambda)$, but a union of 2^κ ones. $\square_{2.1}$

Claim 2.3. *Assume*

(a) $\lambda > \kappa = cf(\lambda) > \aleph_0$
(b) $2^\kappa + \lambda^{<\kappa} = \lambda$
(c) $\lambda \le \theta \le \lambda^\kappa$.

Then there is E such that

(α) *E is an equivalence relation on* ${}^\kappa\lambda$
(β) *E is very nice* [2]
(γ) *if* $\eta_1, \eta_2 \in {}^\kappa\lambda$ *and* $\eta_1 = \eta_2$ *mod* $J^{\rm bd}_\kappa$ *then* $\eta_1 E \eta_2 \Leftrightarrow \eta_1 = \eta_2$
(δ) *E has* θ *equivalence classes.*

Proof. Let $\bar\lambda$ be as in the proof of 2.1 except that we add $i < \kappa \Rightarrow \prod_{j<i} \lambda_j \le \lambda_i$, (this holds if e.g. if $i < \kappa \Rightarrow \lambda_i = \lambda$). We can find a tree $\mathcal{T} \subseteq {}^{\kappa>}\lambda$ with λ nodes and exactly θ κ-branches ([Sh 262]); we can easily manage that $\eta \ne \nu \in \lim_\kappa(\mathcal{T}) \Rightarrow (\exists^\kappa i < \kappa)(\eta(i)) \ne \nu(i))$. We proceed as in the proof of 2.1, but in the definition of E_1 we add

$$\eta_1 \in \lim_\kappa(T) \equiv \eta_2 \in \lim_\kappa(\mathcal{T}) \ \& \ (\eta_1 \in \lim_\kappa(\mathcal{T}) \rightarrow \eta_1 = \eta_2).$$

$\square_{2.3}$

Claim 2.4. *In Claim 2.1 we can replace clauses* $(\beta), (\gamma)$ *by*

$(\beta)_1$ *E is very nice (even the union of* $\le \lambda$ *closed sets minus the union of* $\le \lambda$ *closed sets)*
$(\gamma)_1$ *for every* $\eta^* \in {}^\kappa\lambda$, *the set* $\{\eta \in {}^\kappa\lambda : \eta = \eta^* \mathrm{mod} J^{\rm bd}_\kappa\}$ *is a set of representatives for the family of E-equivalence classes.*

Proof. Let $\bar\lambda$ be as there but $\varepsilon < \kappa \Rightarrow \kappa^+ \le \lambda_i$. Let K_i be a group, with universe λ_i and unit 0_{K_i}. Let $<^*$ be a well ordering of ${}^\kappa(\mathcal{P}(\kappa))$. For every $\eta \in \prod_{i<\kappa} \lambda_i$ let

$$\Xi_\eta = \{\langle B_\varepsilon(\nu) : \varepsilon < \kappa\rangle : \nu \in \prod_{i<\kappa} \lambda_i \text{ and } \nu = \eta \text{ mod } J^{\rm bd}_\kappa\}.$$

So Ξ_η is a nonempty subset of ${}^\kappa(\mathcal{P}(\kappa))$ and let $\bar B^*_\eta = \langle B^*_{\eta,\varepsilon} : \varepsilon < \kappa\rangle$ be its $<^*$-first member. Note that

$\boxdot$ for $\eta_1, \eta_2 \in \prod_{i<\kappa} \lambda_i$ if $\eta_1 = \eta_2$ mod $J^{\rm bd}_\kappa$ then $\bar B^*_{\eta_1} = \bar B^*_{\eta_2}$ and $\Xi_{\eta_1} = \Xi_{\eta_2}$

let $\Theta_\eta = \{\nu \in \prod_{i<\kappa} \lambda_i : B_\varepsilon(\nu) = B^*_{\eta,\varepsilon}$ for every $\varepsilon < \kappa$ and $\nu = \eta$ mod $J^{\rm bd}_\kappa\}$.
Now note

$(*)_0$ $\Theta_\eta \ne \emptyset$.
[Why? By the definition of Ξ_η, $\bar B^*_\eta$ and Θ_η.]
$(*)_1$ if $\nu \in \Theta_\eta$ then for every limit $i < \kappa$ large enough we have $\alpha_i(\nu) = \alpha_i(\eta)$.
[Why? As $\nu = \eta$ mod $J^{\rm bd}_\kappa$.]
$(*)_2$ if $\nu_1, \nu_2 \in \Theta_\eta$ and $\varepsilon < \kappa$, then for every limit i large enough we have: $\alpha_i(\nu_1) = \alpha_i(\nu_2)$ hence $f^i_{\alpha_i(\nu_1)}(\varepsilon) = f^i_{\alpha_i(\nu_2)}(\varepsilon)$.

[2] in fact, again union of $\le 2^\kappa$ closed sets of pairs

Now for $\eta \in \prod_{i<\kappa} \lambda_i$ we define $\rho_\eta \in \prod_{i<\kappa} \lambda_i$ by

$$\rho_\eta(\varepsilon) \text{ is : } \begin{cases} f^i_{\alpha_i(\eta)}(\varepsilon) \text{ for every } i \in B^*_{\eta,\varepsilon} \text{ large enough} \\ \qquad \underline{\text{if}}\ B^*_{\eta,\varepsilon} \text{ is stationary} \\ 0_{K_i} \qquad \underline{\text{if}}\ B^*_{\eta,\varepsilon} \text{ is not stationary.} \end{cases}$$

It is easy to see that

$(*)_3$ if $\eta \in \prod_{i<\kappa} \lambda_i$ then $\rho_\eta(\varepsilon) = \eta(\varepsilon)$ for every $\varepsilon < \kappa$ large enough.
[Why? We can find $\nu \in \prod_{i<\kappa} \lambda_i$ such that $\nu = \eta \bmod J^{\mathrm{bd}}_\kappa$ and $\langle B_\varepsilon(\nu) : \varepsilon < \kappa \rangle = \bar{B}^*_\eta$. Now apply (ε) inside the proof of 2.1.] hence

$(*)_4$ $\rho_\eta = \eta \bmod J^{\mathrm{bd}}_\kappa$

$(*)_5$ if $\eta_1, \eta_2 \in \prod_{i<\kappa} \lambda_i$ and $\eta_1 = \eta_2 \bmod J^{\mathrm{bd}}_\kappa$ <u>then</u> $\rho_{\eta_1} = \rho_{\eta_2}$.

Lastly, we define the equivalence relation E:
for $\eta_1, \eta_2 \in \prod_{i<\kappa} \lambda_i$ we define:

$\boxdot$ $\eta_1 E \eta_2$ <u>iff</u> (for every $i < \kappa$ we have $K_i \models$ "$\eta_1(i)(\rho_{\eta_1}(i))^{-1} = \eta_2(i)(\rho_{\eta_2}(i))^{-1}$").

Now clearly

$(*)_6$ if $\eta_1, \eta_2 \in \prod_{i<\kappa} \lambda_i$ and $\eta_1 = \eta_2 \bmod J^{\mathrm{bd}}_\kappa$ then $\eta_1 E \eta_2 \Leftrightarrow \eta_1 = \eta_2$.
[Why? By $(*)_5$ we have $\rho_{\eta_1} = \rho_{\eta_2}$, call it ρ; we are done by $\boxdot$ and the properties of groups (i.e. $x_1 y^{-1} = x_2 y^{-1} \Leftrightarrow x_1 = x_2$.]

$(*)_7$ if $\eta \in \prod_{i<\kappa} \lambda_i$ then $\{\eta' : \eta' \in \prod_{i<\kappa} \lambda_i$ and $\eta' = \eta \bmod J^{\mathrm{bd}}_\kappa\}$ is a set of representatives of the E-equivalence classes.
[Why? Let $\eta, \nu \in \prod_{i<\kappa} \lambda_i$ and we shall define $\eta' \in \prod_{i<\kappa} \lambda_i$ such that $\eta' \in \nu/E$ and $\eta' = \eta \bmod J^{\mathrm{bd}}_\kappa$. For $i < \kappa$ we choose $\eta'(i) \in K_i$, i.e. $< \lambda_i$ such that $K_i \models$ "$\eta'(i)(\rho_\eta(i))^{-1} = \nu(i)(\rho_\nu(i))^{-1}$.
[Why this is solvable? As K_i is a group and $\rho_\eta(i), \nu(i), \rho_\nu(i)$ are well defined members of K_i.] Also we know that $\nu = \rho_\nu \bmod J^{\mathrm{bd}}_\kappa$ by $(*)_4$ hence for some $i_1 < \kappa$ we have $i \in [i_1, \kappa) \Rightarrow \nu(i) = \rho_\nu(i)$; this implies that $i \in [i_1, \kappa) \Rightarrow \eta'(i) = \rho_\eta(i)$, so $\eta' = \rho_\eta \bmod J^{\mathrm{bd}}_\kappa$; however $\rho_\eta = \eta \bmod J^{\mathrm{bd}}_\kappa$ hence $\eta' = \eta \bmod J^{\mathrm{bd}}_\kappa$, as required. Hence $\rho_{\eta'} = \rho_\eta$ so by the definition of η' we have $K_i \models$ "$\eta'(i)(\rho_{\eta'}(i))^{-1} = \nu(i)(\rho_\nu(i))^{-1}$" which means that $\eta' E \nu$, so we have proved $(*)_7$.]

Lastly, how complicated is E? Define a two-place relation E^* on $\prod_{i<\kappa} \lambda_i$:

$\boxdot_2$ $\eta_1 E^* \eta_2$ <u>iff</u>
  (a) $\bar{B}^*_{\eta_1} = \bar{B}^*_{\eta_2}$.

Clearly

$(*)_8$ E^* is an equivalence relation on ${}^\kappa\lambda$ and is the union of $\le \lambda$ closed minus the union of $\le \lambda$ closed subsets of $(\prod_{i<\kappa} \lambda_i) \times (\prod_{i<\kappa} \lambda_i)$ with $\le 2^\kappa$ equivalence classes

$(*)_9$ on each E^*-equivalence class the function $\eta \mapsto \rho_\eta$ is continuous (even under the Tichonov topology, even more)

$(*)_{10}$ if Y_1, Y_2 are E^*-equivalence classes, then $E \cap (Y_1 \times Y_2)$ is closed (even under the Tichonov topology).

Now check. $\square_{2.4}$

We may like to weaken the cardinal arithmetic assumptions.

Remark 2.5. Assume that $\kappa = \theta^+$ and instead the ideal J^{bd}_κ we use the ideal $[\kappa]^{<\theta}$. Then we can define $\alpha_j(\eta)$ for $\eta \in \prod_{i<\kappa} \lambda_i$ and $j < \kappa$ if $\mathrm{cf}(j) = \mathrm{cf}(\theta)$. Let $\alpha_j(\eta)$ be $\mathrm{Min}\{\alpha : f^j_\alpha = \eta \restriction j \bmod J_j\}$ where $J_j = \{A \subseteq j$: for some $i < j$ we have $|A \backslash i| < \theta\}$ so J_j replaces J^{bd}_j in the earlier proof.

So $\eta = \nu \bmod [\kappa]^{<\theta}$ implies that $\alpha_j(\eta) = \alpha_j(\nu)$ for all suitable j. There are no marked changes.

Now

$(*)$ if $\eta_1 E^* \eta_2$ then $B_\varepsilon(\eta_1) = B_\varepsilon(\eta_2)$, $\Xi_{\eta_1} = \Xi_{\eta_2}$ and $\bar{B}^*_{\eta_1} = \bar{B}^*_{\eta_2}$

$\boxdot_0$ E^* can serve as well and it is an equivalence relation with $\leq 2^\kappa$ equivalence classes, each closed even under the Tichonov topology.

We can use $\lambda > \kappa \geq \theta$, $J = [\kappa]^{<\theta}$ but in general the number of ideals necessary is κ^θ. Most interesting is the case $\theta = \aleph_0$ dealt with in the next claim.

Claim 2.6. *1) Assume*

(a) $\lambda > \kappa = cf(\lambda) > \aleph_0$
(b) $\kappa^{\aleph_0} < \lambda = \lambda^{\aleph_0}$.

Then the results 2.1, 2.4 and 2.3 holds if we replace the ideal J^{bd}_κ by the ideal $[\kappa]^{<\aleph_0}$.

2) This applies also to 2.3 if

(c) $\lambda \leq \theta \leq \lambda^\kappa$ and there is a tree $\mathcal{T}$ with λ nodes and κ-branches.

3) The natural topology for (1) + (2) is the $\aleph_1$-box product.

Proof. Without loss of generality $\lambda_i > \kappa^{\aleph_0}$, $\langle \lambda_i : i < \kappa \rangle$ as in the proof of 2.1. Let $\langle D_i : i < \kappa^{\aleph_0} \rangle$ list the subsets of κ of order type ω and let $\bar{f}^i = \langle f^i_\alpha : \alpha < \prod_{j \in D_i} \lambda_j \rangle$ list $\prod_{j \in D_i} \lambda_j$ (or just a set of representatives modulo $J^{\mathrm{bd}}_{D_i}$). For $\eta \in \prod_{\varepsilon<\kappa} \lambda_\varepsilon$ let

$(a)'$ $\alpha_i(\eta) = \mathrm{Min}\{\alpha : \eta \restriction D_i = f^i_\alpha \bmod J^{\mathrm{bd}}_{D_i}\}$ for $i < \kappa^{\aleph_0}$
$(b)'$ for $\varepsilon < \kappa$ let $B_\varepsilon(\eta) = \{i < \kappa^{\aleph_0} : \varepsilon \in D_i$ and $\eta(\varepsilon) = f_{\alpha_i(\eta)}(\varepsilon)\}$
$(c)'$ $A(\eta) = \{\varepsilon < \kappa : B'_\varepsilon(\eta)$ is finite$\}$
$(d)'$ $B_\varepsilon(\eta) = \{i \in B'_\varepsilon(\eta) : i \cap B'_\varepsilon(\eta)$ is finite$\}$.

With those choices the proofs are similar. $\square_{2.6}$

Claim 2.7. *1) If $2^{\aleph_0} < \lambda = \lambda^{<\kappa}$, $\aleph_0 < \kappa = \mathrm{cf}(\lambda) < \lambda$, then we can find E as in 2.1(α), (β), (δ) (but not necessarily (γ)) and*

$(\gamma)^*$ if $\eta \in {}^\kappa\lambda$ and $i < \kappa$ then $X_{\eta,i} = \{\nu \in {}^\kappa\lambda : (\forall j)(j < \kappa \;\&\; j \neq i \rightarrow \nu(j) = \eta(j)\}$ is a set of representatives for E.

2) If $2^{\kappa^{\aleph_0}} \leq \lambda = \lambda^{\aleph_0}, \aleph_0 < \kappa = \mathrm{cf}(\lambda) < \lambda, 1 \leq \theta \leq \lambda$ *and* $(\forall \mu < \lambda)[(\mu + \theta)^{<\kappa} \leq \lambda)$, *then we can find* E *as in 2.1*(α), (β) *and*

$(\gamma)^*$ *if* $\eta \in {}^{\kappa}\lambda$ *and* $i < \kappa$ *then* $X_{\eta,i}$ *contains a set of representatives*
$(\delta)^*$ E *has* θ *equivalence classes.*

Proof. 1) First the proof in short.

We choose $\lambda_i = \lambda$ for $i < \kappa$. We let K be a group with universe λ and let $\langle D_j : j < \kappa^{\aleph_0} \rangle$ be as in the proof of 2.6 and define E by: $\eta E \nu$ iff $K \models \prod_{i \in A(\eta)} (\eta(i)(\rho_\eta(i))^{-1}) = \prod_{i \in A(\nu)} (\nu(i)(\rho_\nu(i))^{-1})$. We give a more detailed proof below.

2) First, the proof in short. We choose λ_i but $\theta \leq \lambda_i$; without loss of generality each λ_i is a subgroup of K but we use equality in cosets of $xK_1 = yK_1$, K_1 a subgroup of K such that $[K : K_1] = \theta$ and $a, c \in K \Rightarrow \{abcK_1 : b \in \{\varepsilon : \varepsilon < \lambda_0\}\} = \{bK_1; b < \lambda\}$.

Now in detail (for (2) so including a proof of (1)).

We repeat the proof of 2.4.2.6, so for $\eta \in {}^{\kappa}\lambda$ we let $\Xi_\eta = \{\langle B_\varepsilon(\nu) : \varepsilon < \kappa \rangle : \nu \in {}^{\kappa}\lambda$ and $\nu = \eta \bmod [\kappa]^{<\aleph_0}\}$ where $B_\varepsilon(\nu) = \{j < \kappa^{\aleph_0} : f_{\alpha_{j(\nu)}}(\varepsilon) = \nu(\varepsilon)\}$ and let $\bar{B}^*_\nu$ be the $<^*$-first member of Ξ_ν and let $\Theta_\eta = \{\nu \in {}^{\kappa}\lambda : B_\varepsilon(\nu) = B^*_{\eta,\varepsilon}$ for every $\varepsilon < \kappa$ and $\nu = \eta \bmod [\kappa]^{<\aleph_0}\}$ and for $\eta \in {}^{\kappa}\lambda$ let $\rho_\eta \in {}^{\kappa}\lambda$ be defined by

(a) $\rho_\eta(\varepsilon) = f_{\alpha_{j(\eta)}}(\varepsilon)$ if $(\exists \nu)(\nu \in \Theta_\eta \ \& \ \nu(\varepsilon) = f_{\alpha_{j(\eta)}}(\varepsilon)) \ \& \ j \in B^*_{\eta,\varepsilon}$
(b) $\rho_\eta(\varepsilon) = 0$ if there are no j, ν as in (a).

Easily $\rho_\eta \in {}^{\kappa}\lambda$ is well defined and $\rho_\eta = \eta \bmod [\kappa]^{<\aleph_0}$.

Lastly, let $a_\eta = \{\varepsilon < \kappa : \eta(\varepsilon) \neq \rho_\eta(\varepsilon)\}$ and we define the two-place relation E on $\prod_{i<\kappa} \lambda_i$ by $\eta_1 E \eta_2$ <u>iff</u> $a_{\eta_1} = a_{\eta_2}$ & $(\prod_{i \in a_{\eta_1}} \eta_1(\varepsilon)\rho_\eta(i)^{-1} K_1 = (\prod_{\varepsilon \in a_{\eta_2}} \eta_2(\varepsilon)$ $\eta_2(\varepsilon)^{-1})K_1$. Is this well defined? The product $\prod_{\varepsilon \in a_{\eta_\ell}} \eta_\ell(\varepsilon)$ is a finite product in the group K, so in general we have to choose an order of $\langle \eta_\ell(\varepsilon) : \varepsilon \in a_{\eta_\ell} \rangle$, i.e., of a_{η_i}. We use the most natural choice: the order on κ (if K is abelian clearer). Obviously E is an equivalence relation on $\prod_{\varepsilon<\kappa} \lambda_\varepsilon$ and it has $|\{xK_1 : \lambda \in K\}| = [K : K_1]$ equivalence classes. Now suppose that $\eta \in {}^{\kappa}\lambda$ and $\varepsilon < \kappa$ and we shall prove that $X_{\eta,\varepsilon}$ is the set of representatives for E, recall $X_{\eta,\varepsilon}$ is defined in $(\gamma)^*$ of 2.7(1). Let $a^- = a_\eta \cap \varepsilon$, $a^+ = a_\eta \backslash (\varepsilon + 1)$, let $g^- = \prod_{i \in a^-} (\eta(i)(\rho_\eta(i))^{-1})$ and $g^+ = \prod_{i \in a^+} (\eta(i)(\rho_\eta(i))^{-1})$, so:

$(*)$ $g^-, g^+ \in K$ again well defined as a^-, a^+ are finite
$(**)$ if $\nu \in X_{\eta,\varepsilon}$ then $a_\nu \subseteq a_\eta \cup \{\varepsilon\}$ and $\prod_{i \in a_\nu} (\nu(i)(\rho_\nu(i))^{-1}) = g^- \nu(\varepsilon) g^+ \in K$, the product in K, of course.

Now for part (1), g_ν the sequence $\langle \nu(\varepsilon) : \nu \in X_{\eta,\varepsilon} \rangle$ lists K without repetition (as the universe of K is λ) hence (by basic group theory), $\langle g^{-1} \nu(\varepsilon) g^+ : \nu \in X_{\eta,\varepsilon} \rangle$ lists K without repetitions hence $\langle \prod_{i \in a_\nu} (\nu(i)(\rho_\nu(i))^{-1}) : \nu \in X_{\alpha,\varepsilon} \rangle$ lists K without repetitions, so if we use the trivial K_1, $X_{\eta,i}$ is a set of representatives of E, as required.

For (2) the sequence $\langle g_\nu K_2 : \nu \in X_{\eta,\varepsilon} \rangle$ lists $\{xK_1 : x \in K\}$ possibly with repetition. $\square_{3.5}$

Concluding Remark 2.8. 1) Instead of $\langle J^{\rm bd}_{D_i} : i < \kappa^{\aleph_0}\rangle$ we can use $\langle (D_i, J_i) : i < i^{\bar\lambda}\rangle$, $D_i \subseteq \kappa$, J_i an ideal on D_i such that $|\prod_{\varepsilon\in D_i} \lambda_\varepsilon/J_i| \le \lambda$, $I = \{D \subseteq \kappa$: for every $i < i^*$ we have $D \cap D_i \in J_i\}$ is included in $J^{\rm bd}_\kappa$. The author has not pursued this.

2) Assume K is a group of cardinality λ, K_1 a subgroup and $[K : K_1] = \theta \le \lambda$. Then we can find $B \subseteq K$, $|B| = \theta$ such that if K' is a subgroup of K including B such that

$\circledast_{K,K_1,K'}$ if $a, c \in K'$ then $\{acK_1 : b \in K'\} = \{bK_1 : b \in K\}$.
[Why? Let $\{b_i : i < \theta\}$ be such that $\{b_iK_1 : i < \theta\} = \{bK_1 : b \in K\}$ and let $B = \{b_i : i < \theta\}$. If $B \subseteq K' \subseteq K$ and $ac \in K'$ and $i < \theta$ there is $b' \in K'$ such that $ab'c = b_i$ so $ab'cK_1 = b_iK_1$.]

§3. Countable cofinality: positive results

We first phrase sufficient conditions which relate to large cardinals. Then we prove that they suffice. The proof of 3.1 is presented later in this section.

Lemma 3.1. *Assume*

- (a) *λ is strong limit of cofinality $\aleph_0$*
- (b) *λ is a limit of measurables, or just*
- $(b)^-$ *for every $\theta < \lambda$ for some μ, χ satisfying $\theta \le \mu \le \chi < \lambda$, there is a (χ, μ, θ)-witness (see Definition 3.2 below)*
- (c) *E is a nice equivalence relation on ${}^{\omega}\lambda$ (or has enough absoluteness, as proved in 3.12), i.e., fact 3.13, so being $\Delta^1_1(\lambda)$ is enough*
- (d) *if $\eta, \nu \in {}^{\omega}\lambda$ and $(\exists! n)(\eta(n) \ne \nu(n))$ then $\neg(\eta E \nu)$.*

Then E has 2^λ equivalence classes, moreover if $\lambda_n < \lambda_{n+1} < \lambda = \Sigma_{n<\omega}\lambda_n$ then there is a subtree of ${}^{\omega>}\lambda$ isomorphic to $\bigcup_m \prod_{n<m} \lambda_n$, whose ω-branches are pairwise non E-equivalent (even somewhat more, see 3.17).

Remark. For the simplest example of "witness" defined below see 3.4(2) so a witness is a weak form of λ being measurable.

Definition 3.2. *1) We say $(\mathbb{Q}, s_1, s_2)$ is a (λ, μ, θ)-witness <u>if</u> ($\lambda \ge \mu \ge \theta$ and):*

- (a) *$\mathbb{Q}$ is a θ-complete forcing notion*
- (b) *s_1 is a function from $\mathbb{Q}$ to $\mathcal{P}(\lambda)\backslash\{\emptyset\}$*
- (c) *s_2 is a function from $\mathbb{Q}$ to $\{A : A \subseteq \{(\alpha, \beta) : \alpha < \beta < \lambda\}\}$*
- (d) *if $\mathbb{Q} \models$ "$p \le q$" then $s_\ell(q) \subseteq s_\ell(p)$ for $\ell = 1, 2$*
- (e) *$(\alpha, \beta) \in s_2(p) \Rightarrow \{\alpha, \beta\} \subseteq s_1(p)$ for $p \in \mathbb{Q}$*
- (f) *for every $p \in \mathbb{Q}$ there is q such that $p \le q \in \mathbb{Q}$ and $(\forall\beta)(\exists\alpha, \gamma)[\beta \in s_1(q) \to (\alpha, \beta) \in s_2(p)$ & $(\beta, \gamma) \in s_2(p)]$*
- (g) *if $p \in \mathbb{Q}$ and $A \subseteq \lambda \times \lambda$, <u>then</u> for some q we have $p \le q \in \mathbb{Q}$ and $(s_2(q) \subseteq A) \vee (s_2(q) \cap A = \emptyset)$*
- (h) *if $p \in \mathbb{Q}$ then for some $Y \in [\lambda]^\mu$ for every $\alpha < \beta$ from Y we have $(\alpha, \beta) \in s_2(p)$ (hence $Y \subseteq s_1(p)$).*

2) We say $(\mathbb{Q}, s_1, s_2)$ is a $(\lambda, \mu, \theta, \varrho)$-witness if ϱ is a cardinal $\leq \lambda$ and we can strengthen clause (g) to [3]

$(g)^+_\varrho$ *if $f : {}^2\lambda \to \varrho$ and $p \in \mathbb{Q}$ then for some q we have $p \leq q \in \mathbb{Q}$ and $f \restriction s_2(q)$ is constant.*

3) We call $(\mathbb{Q}, s_1, s_2)$ uniform (λ, μ, θ)-witness if $\lambda = \cup\{s_1(p) : p \in \mathbb{Q}\}$ and for every $p \in \mathbb{Q}$ and $\alpha < \lambda$ for some q we have $p \leq q \in \mathbb{Q}$ and $s_1(q) \cap \alpha = \emptyset$.

Similarly "a uniform $(\lambda, \mu, \theta, \varrho)$-witness".

4) We replace ϱ by $< \varrho$ if we demand only $(g)^+_{<\varrho}$ which means that $Rang(f)$ is a subset of ϱ of cardinality $< \varrho$. We write "$< \mu$" instead of "μ" if in clause (h) of Definition 3.2(1) we demand just that for each $\alpha < \mu$ there is $Y \subseteq \lambda$ of order type α and as there (so μ can be an ordinal).

Definition 3.3. *1) We say that $(\mathbb{Q}, \bar{s})$ is a $(\lambda, \mu, \theta, \varrho; n)$-witness if $\lambda \geq \mu \geq \theta, \lambda \geq \varrho$ and $\bar{s} = \langle s_m : m = 1, \ldots, n\rangle$ and*

- (a) *$\mathbb{Q}$ is a θ-complete forcing*
- (b) *s_m is a function from $\mathbb{Q}$ to $\mathcal{P}(\{\bar{\alpha} : \bar{\alpha} = \langle \alpha_\ell : \ell < m\rangle \in {}^m\lambda$ and $\alpha_\ell < \alpha_{\ell+1} < \lambda$ for $\ell < m - 1\})$*
- (c) *if $\mathbb{Q} \models$ "$p \leq q$" and $m \in \{1, \ldots, n\}$ then $s_m(q) \subseteq s_m(p)$*
- (d) *if $\langle \alpha_\ell : \ell < m + 1\rangle \in s_{m+1}(p)$ and $k < m + 1$ then $\langle \alpha_\ell : \ell < k\rangle\char`^\langle \alpha_\ell : \ell = k + 1, \ldots, m\rangle \in s_m(p)$*
- (e) *for every $m \in \{1, \ldots, n - 1\}, k < m$ and $p \in \mathbb{Q}$ there is q satisfying $p \leq q \in \mathbb{Q}$ and $(\forall \bar{\alpha} \in s_m(q))(\exists \bar{\beta} \in s_{m+1}(p))[\bar{\alpha} = (\bar{\beta} \restriction k)\char`^(\bar{\beta} \restriction [k+1, m))]$*
- $(f)^+$ *if $m \in \{1, \ldots, n\}$ and $f : {}^m\lambda \to \varrho$ and $p \in \mathbb{Q}$ then for some q we have $p \leq q \in \mathbb{Q}$ and $f \restriction s_m(q)$ is constant*
- (g) *if $p \in \mathbb{Q}$ then for some $Y \in [\lambda]^\mu$ every increasing $\bar{\alpha} \in {}^nY$ belongs to $s_n(p)$.*

2) $(\mathbb{Q}, \bar{s})$ is a $(\lambda, \mu, \theta, \varrho; \omega)$-witness is defined similarly (i.e., $\bar{s} = \langle s_m : m \in [1, \omega)\rangle$).

3) If $\varrho = 2$ we may omit it, as in Definition 3.2. Also "uniform" and "$< \varrho$" means as in Definition 3.2.

We first give some basic facts on witnesses, including cases of existence.

Claim 3.4. *1) If $(\mathbb{Q}, \bar{s})$ is a $(\lambda, \mu, \theta; n)$-witness and $\varrho < \theta, n < \omega$, then $(\mathbb{Q}, \bar{s})$ is a $(\lambda, \mu, \theta, 2^\varrho; n)$-witness.*

2) If $\mathcal{D}$ is a normal ultrafilter on λ so λ is a measurable cardinal and we choose, $\mathbb{Q} = (\mathcal{D}, \supseteq), s_1(A) = A, s_2(A) = \{(\alpha, \beta) : \alpha < \beta$ are from $A\}$, then $(\mathbb{Q}, s_1, s_2)$ is a uniform $(\lambda, \lambda, \lambda, < \lambda)$-witness.

3) If in (2), $s_m(A) = \{\bar{\alpha} : \bar{\alpha} = \langle \alpha_\ell : \ell < m\rangle$ is increasing, $\alpha_\ell \in A\}, \bar{s} = \langle s_{m+1} : 1 + m \leq n\rangle$ and $n \leq \omega$ then $(\mathbb{Q}, \bar{s})$ is a $(\lambda, \lambda, \lambda, < \lambda; n)$-witness.

4) If there is a $(\lambda, \mu, \theta, \varrho; n)$-witness and $2^{<\theta} \leq \lambda$, then there is such $(\mathbb{Q}, \bar{s})$ with $|\mathbb{Q}| \leq 2^\lambda$.

5) Definition 3.2(1) is the case $n = 2$ of Definition 3.3(1) that is, $(\mathbb{Q}, s_1, s_2)$ is a $(\lambda, \mu, \theta, \varrho)$-witness iff $(\mathbb{Q}, (s_1, s_2))$ is a $(\lambda, \mu, \theta, \varrho; 2)$-witness.

[3] note that $(g)^+_\varrho$ is equal to (g) if $\varrho = 2$

6) If $(\mathbb{Q}, \bar{s})$ is a (λ, μ, θ)-witness and $p \in \mathbb{Q}$, then we can find q such that $p \leq q \in \mathbb{Q}$ and for every $\beta \in s_1(q)$ there are $\alpha_1 < \alpha_2 < \beta$ such that $(\alpha_1, \beta), (\alpha_2, \beta) \in s_2(p)$ (this strengthens clause (f) of 3.2).

Proof. Easy.

1) Checking Definition 3.3 the least easy clause is $(f)^+$, so assume $m \in \{1, \dots, n\}$ and $p \in \mathbb{Q}$ and f is a function from ${}^m\lambda$ to 2^{ϱ} and we should find q satisfying $p \leq q \in \mathbb{Q}$ and $f \restriction s_m(q)$ is constant. Let h be a one to one function from 2^{ϱ} into ${}^{\varrho}2$ and define $f_\varepsilon : {}^m\lambda \to \{0, 1\}$ for $\varepsilon < \varrho$ by $f_\varepsilon(\bar{s}) = (h(f(\bar{s})))(\varepsilon)$. Now we choose $p_\varepsilon \in \mathbb{Q}$, increasing (by $\leq_{\mathbb{Q}}$) by induction on $\varepsilon \leq \varrho$ such that $p_0 = p$, $f_\varepsilon \restriction s_m(p_{\varepsilon+1})$ is constant, say is ℓ_ε. For $\varepsilon = 0$ this is trivial, for ε successor use "$(\mathbb{Q}, \bar{s})$ is $(\lambda, \mu, \theta; n)$-witness, i.e. clause $(f)^+$ in Definition 3.3". For ε a limit ordinal we use "$\mathbb{Q}$ is θ-complete, i.e., clause (a) in Definition 3.3 for $(\mathbb{Q}, \bar{s})$ is a $(\lambda, \mu, \theta; n)$-witness, recall $\varrho < \theta$.

Lastly, let $q = p_\varrho$ so we are done.

2), 3) Note that $\mathbb{Q}$ is λ-complete as $\mathcal{D}$ is λ-complete as $\mathcal{D}$ is a λ-complete ultrafilter (being normal) and clause $(f)^+$ holds because if $f_n : [\lambda]^n \to \mu$ and $\mu < \lambda$ then for some $A_n \in \mathcal{D}$ we have $f \restriction [A]^n$ is constant (see, e.g., [J]) and as $\mathcal{D}$ is closed under intersection of $< \lambda$ (hence of $\aleph_0$) we are done (if $p \in \mathbb{Q}$, let $q = p \cap \bigcap\limits_{n<\omega} A_n$).

4) Let $(\mathbb{Q}, \bar{s})$ be a $(\lambda, \mu, \theta, \varrho; n)$-witness and let χ be large enough. Choose an elementary submodel N of $(\mathcal{H}(\chi), \in)$ to which $(\mathbb{Q}, \bar{s})$ satisfying $\|N\| = 2^\lambda$, $[N]^\lambda \subseteq N$ so $2^\lambda \subseteq N$.

Lastly, choose $\mathbb{Q}' = \mathbb{Q} \restriction N$ and $s'_m = s_m \restriction \mathbb{Q}'$. Now check that $(\mathbb{Q}', \langle s'_{m+1} : m < n\rangle)$ is a $(\lambda, \mu, \theta, \varrho; n)$-witness recalling $\mu, \theta, \varrho \leq \lambda$.

5) Read the definitions.

6) For $\ell \in \{0, 1, 2\}$ let $A_\ell = \{\alpha \in s_1(p)$: the number $|\{\alpha' < \alpha : (\alpha', \alpha) \in s_2(p)\}|$ is equal to ℓ or $\ell = 2$ and the number is $\geq \ell\}$.

So $\langle A_0, A_1, A_2\rangle$ is a partition of $s_1(p)$.

Define a function f from A_1 to λ: for $\alpha \in A_1$, $f(\alpha)$ is the unique $\alpha' < \alpha \in s_2(p)$. It is known (and easy) that we can find a partition $\langle B_1, B_2, B_3\rangle$ of A_1 such that $\ell \in \{1, 2, 3\}$ & $\alpha \in B_\ell \Rightarrow f(\alpha) \notin B_\ell$. Let $B_0 = A_0$, $B_4 = A_2$, so $\langle B_0, \dots, B_4\rangle$ is a partition of $\bigcup_{\ell=0}^{2} A_\ell$ that is of $s_1(p)$. By clause (g) of Definition 3.2 (applied three times, see 3.4(1)) we can find $\ell(*) < 5$ and $q \in \mathbb{Q}$ such that $p \leq q \in Q$ and $s_1(q) \subseteq B_{\ell(*)}$. s $s_2(q) \neq \emptyset$ necessarily $\ell(*) = 4$ and so we are done. $\square_{3.4}$

Something of the "largeness" remains if we collapse a large cardinal, see, e.g., [JMMP]. We shall need

Claim 3.5. *1) Assume*

(a) $2 \leq n < \omega$ *and* $\lambda = \beth_{n-1}(\theta)^+$
(b) *θ is a compact cardinal or just a λ-compact cardinal*
(c) $\mu = \mu^{<\mu} < \theta$
(d) $\mathbb{P} = \text{Levy}(\mu, < \theta)$.

Then in $\mathbf{V}^{\mathbb{P}}$ (and of course in $\mathbf{V}$), there is a $(\lambda, \mu, \theta; n)$-witness $(\mathbb{Q}, \bar{s})$ which is even a $(\lambda, \mu, \theta, < \mu; n)$-witness.

2) If there are λ_n for $n < \omega$, $\lambda_n < \lambda_{n+1}$ and λ_n is $2^{(2^{\lambda_n})^+}$-compact and $\lambda = \Sigma\{\lambda_n : n < \omega\}$, then for some set forcing $\mathbb{P}$, in $\mathbf{V}^{\mathbb{P}}$ the cardinal $\lambda = \beth_\omega = \aleph_\omega$ is dichotomically good (see Definition 3.8 below).

Proof. By [Sh 124].

Remark 3.6. 1) In fact we can weaken the consistency strength considerably. Assume that (G.C.H. holds for simplicity) and:

(a) $\langle \mu_n : n < \omega \rangle$ is strictly increasing sequence of cardinals
(b) $\mu_n \le \lambda_n < \mu_{n+1}$
(c) D_{n+1} is a μ_{n+1}-complete ultra filter on $I_{n+1} = \{a \subseteq \lambda_{n+1} : |a| = \lambda_{n-1}, \min(a) < \mu_{n+1}\}$
(d) let $\iota_{n+1} : I_{n+1} \to \lambda_{n+1}$ is $\iota_{n+1}(a) = \min(\mu_{n+1} \cap a)$ and if $A \in D_{n+1}$, $f : A \to \mu_{n+1}$ is regressive, i.e., $f(a) < \iota_{n+1}(a)$ then f is constant on some $B \in D_{n+1}$, $B \subseteq A$
(e) if $g : [\lambda_{n+1}]^n \to \mu_n$ then $\{a \in I_{n+1} : g \restriction \{[a \backslash \{\min(a)\}]^n$ is constant$\} \in D_{n+1}$
(f) $\mathbb{Q}_0 = \text{Levy}(\aleph_0, \mu_0)$, $\mathbb{Q}_{n+1} = \text{Levy}(\lambda_n^{++}, < \mu_{n+1})$, $\mathbb{Q} = \prod_{n<\omega} \mathbb{Q}_n$.

Then $\mathbf{V}^{\mathbb{Q}}$ is as required in 3.5.

2) If μ_n is $\mu_n^{+(n+2)}$-hyper-measurable, $\lambda = \mu_n^{+n+1}$ then there is $\mathbf{j}_n : \mathbf{V} \to M_n$, μ_n is the critical cardinal of $\mathbf{j}_n$, $M_n^{\mu_n} \subseteq M_n$, $j_n(\mu_n) > \mu_n^{+(n+2)}$. So in $\mathbf{V}$ we can find $b \in [j_n(\mu_n) \setminus \mu_n]^{\lambda_{n-1}}$ such that $f : [\mu_n]^n \to \lambda_{n-1} \Rightarrow \mathbf{j}_n(f) \restriction [b]^n$ is constant. Let $a = \{\mu_n\} \cup b$ so $a \in M_n$ and $D_n = \{A \subseteq [\mu_n]^{\lambda_{n-1}} : a \in \mathbf{j}_n(A)\}$.

Those D_n are as required for $\lambda_n = \mu_n$.

Toward proving Lemma 3.1 assume (from 3.10 till the end of this section) that

Hypothesis 3.7. $\mathfrak{m} = \langle \lambda_n, \mu_n, \theta_n, \mathbb{P}_n, s_{n,1}, s_{n,2} \rangle_{n<\omega} = \langle \lambda_n^{\mathfrak{m}}, \mu_n^{\mathfrak{m}}, \theta_n^{\mathfrak{m}}, \mathbb{P}_n^{\mathfrak{m}}, s_{n,1}^{\mathfrak{m}}, s_{n,2}^{\mathfrak{m}} \rangle_{n<\omega}$ satisfies $\lambda = \Sigma\{\lambda_n : n < \omega\}$ and $\aleph_1 + \{2^{\lambda_\ell} : \ell < n\} < \theta_n \le \lambda_n$ and $(\mathbb{P}_n, s_{n,1}, s_{n,2})$ is a $(\lambda_n, < \mu_n^+, \theta_n)$-witness and it follows that $\mu_n < \mu_{n+1}$ and $\lambda = \sum_n \{\mu_n : n < \omega\}$.

Definition 3.8. *We call λ dichotomically good if there is $\mathfrak{m}$, i.e., there are $\lambda_n, \mu_n, \theta_n, \mathbb{P}_n, s_{n,1}, s_{n,2}$ as in 3.7.*

The hypothesis 3.7 is justified because

Obervation 3.9. If λ satisfies (a) + (b) or at least (a) + (b)$^-$ of Lemma 3.1 then λ is dichotomically good. Also consistently G.C.H. and $\aleph_\omega$ is dichotomically good and w log E is on $\prod_{n<w} \lambda_n$

Proof. By 3.4(2) we know $(b) \Rightarrow (b)^-$ in 3.1, now read the definitions. Second those by 3.5

Definition 3.10. *1) We define the forcing notion $\mathbb{Q}_1$ (really $\mathbb{Q}_1 = \mathbb{Q}[\mathfrak{m}]$) as follows:*

(a) $\mathbb{Q}_1 = \{p : p = (\eta, \bar{A}) = (\eta^p, \bar{A}^p)$ *such that letting* $\mathbf{n}^p = \mathbf{n}(p) = \ell g(\eta)$ *we have* $\mathbf{n}^p < \omega$, $\eta^p \in \prod_{\ell<\mathbf{n}[p]} \lambda_\ell$ *and* $\bar{A}^p = \langle A_\ell^p : \ell \in [\mathbf{n}(p), \omega) \rangle$ *and* $A_\ell^p \in \mathbb{P}_\ell\}$

(*b*) $p \le_{\mathbb{Q}_1} q$ *iff* $\eta^p \trianglelefteq \eta^q$ (*so* $\mathbf{n}(p) \le \mathbf{n}(q)$ *and* $[\ell \in [\mathbf{n}(q), \omega) \Rightarrow \mathbb{P}_\ell \models$ "$A^p_\ell \le A^q_\ell$"] *and* $[\mathbf{n}(p) \le \ell < \mathbf{n}(q) \Rightarrow \eta^q(\ell) \in s_1(A^p_\ell)]$

(*c*) *We define the* $\mathbb{Q}_1$*-name* $\underset{\sim}{\eta}$ *by:* $\underset{\sim}{\eta}[G] = \cup\{\eta^p : p \in \underset{\sim}{G}_{\mathbb{Q}_1}\}$

(*d*) *We define*

(α) $p \le^{\mathbb{Q}_1}_{pr} q$ *iff* $p \le_{\mathbb{Q}_1} q$ & $\mathbf{n}(p) = \mathbf{n}(q)$

(β) $p \le^{\mathbb{Q}_1}_{apr} q$ *iff* $p \le_{\mathbb{Q}_1} q$ & $\bigwedge_{\ell \ge \mathbf{n}(q)} (A^q_\ell = A^p_\ell)$

(γ) $p \le^{\mathbb{Q}_1}_{pr,n} q$ *iff* $p \le^{\mathbb{Q}_1}_{pr} q$ *and* $\bar{A}^p \restriction [\mathbf{n}(p), n) = \bar{A}^q \restriction [\mathbf{n}(p), n)$ *if* $n \ge \mathbf{n}(p)$.

2) We define the forcing notion $\mathbb{Q}_2$ *(really* $\mathbb{Q}_2[\mathfrak{m}]$*) by:*

(*a*) $\mathbb{Q}_2 = \big\{p : p = (\eta_0, \eta_1, \bar{A}) = (\eta^p_0, \eta^p_1, \bar{A}^p)$ *where for some* $\mathbf{n}(p) < \omega$ *we have:* $\eta^p_0, \eta^p_1 \in \prod_{\ell < \mathbf{n}(p)} \lambda_\ell$ *and* $\bar{A}^p = \langle A^p_\ell : \ell \in [\mathbf{n}(p), \omega)\rangle$ *and* $A^p_\ell \in \mathbb{P}_\ell\big\}$

(*b*) $p \le_{\mathbb{Q}_2} q$ *iff*

(*i*) $\mathbf{n}(p) \le \mathbf{n}(q)$

(*ii*) $\eta^p_\ell \trianglelefteq \eta^q_\ell$ *for* $\ell = 0, 1$

(*iii*) $A^q_\ell \subseteq A^p_\ell$ *for* $\ell \in [n(q), \omega)$

(*iv*) *the pair* $(\eta^q_0(\ell), \eta^q_1(\ell))$ *is from* $s_2(A^p_\ell)$ *for* $\ell \in [\mathbf{n}(p), \mathbf{n}(q))$

(*c*) *we define the* $\mathbb{Q}_2$*-name* $\underset{\sim}{\eta}_\ell$ *(for* $\ell = 0, 1$*) by* $\underset{\sim}{\eta}_\ell[G] = \cup\{\eta^p_\ell : p \in \underset{\sim}{G}_{\mathbb{Q}_2}\}$

(*d*) *we define*

(α) $p \le^{\mathbb{Q}_2}_{pr} q$ *iff* $p \le_{\mathbb{Q}_1} q$ & $\mathbf{n}(p) = \mathbf{n}(q)$ *and*

(β) $p \le^{\mathbb{Q}_2}_{apr} q$ *iff* $p \le_{\mathbb{Q}_2} q$ & $\bigwedge_{\ell \ge n(q)} A^q_\ell = A^p_\ell$ *and*

(γ) $p \le^{\mathbb{Q}_2}_{pr,n} q$ *iff* $p \le^{\mathbb{Q}_2}_{pr} q$ *and* $\bar{A}^p \restriction [\mathbf{n}(p), n) = \bar{A}^q \restriction [\mathbf{n}(p), n)$ *if* $n > \mathbf{n}(p)$.

3) If for a fixed $k < \omega$*, we have* $(\mathbb{P}_n, \bar{s}^n)$ *is a* $(\lambda_n, \mu_n, \theta_n; k)$*-witness for* $n < \omega$ *then we can define* $\mathbb{Q}_k$ *naturally.*

4) If $(\mathbb{P}_n, \bar{s}^n)$ *is a* $(\lambda_n, \mu_n, \theta_n; n)$*-witness for* $n < \omega$ *then we can define* $\mathbb{Q} = \{(\eta, \bar{A}) : n < \omega, \eta(\ell) \in {}^{\ell}(\lambda_\ell)$ *and* $\bar{A} = \langle A_\ell : \ell \in [n, \omega), A_\ell \in \mathbb{P}_\ell\}\}$ *with the natural order.*

Remark. 1) We shall not pursue here parts (3) and (4) of Definition 3.10 because we deal with equivalence relations which are binary. We can prove parallel theorems for relations with higher arity using 3.10(3),(4).

2) In the definition of the set of elements p of $\mathbb{Q}_2$, why don't we ask $(\forall \ell < \mathbf{n}^p)(\eta^p_0(\ell) < \eta^p_1(\ell))$? To be able to construct the perfect set, but, of course, $p \Vdash_{\mathbb{Q}_2}$ "$\underset{\sim}{\eta}_0(\ell) < \underset{\sim}{\eta}_1(\ell)$ for $\ell \in [\mathbf{n}(p), \omega)$".

3) Those forcing notions are in the (large) family of relatives of Prikry forcing.

Fact 3.11. Let $\ell \in \{1, 2\}$.

0) For $p, q \in \mathbb{Q}_\ell$ we have:

(*i*) $p \le^{\mathbb{Q}_\ell}_{pr} q \Rightarrow p \le_{\mathbb{Q}_\ell} q$

(*ii*) $p \le^{\mathbb{Q}_\ell}_{apr} q \Rightarrow p \le q$

(*iii*) $p \le^{\mathbb{Q}_\ell}_{pr,n+1} q \Rightarrow p \le^{\mathbb{Q}_\ell}_{pr,n} q \Rightarrow p \le^{\mathbb{Q}_\ell}_{pr} q$.

1) If $p \leq_{\mathbb{Q}_\ell} r$ <u>then</u> for some q we have $p \leq^{\mathbb{Q}_\ell}_{pr,\mathbf{n}(q)} q \leq^{\mathbb{Q}_\ell}_{apr} r$.

2) If $\bar p = \langle p_i : i < \alpha\rangle$ is $\leq^{\mathbb{Q}_\ell}_{pr}$-increasing and $\alpha < \theta_{\mathbf{n}(p_0)} (= \theta^{\mathfrak{m}}_{\mathbf{n}(p_0)})$, <u>then</u> $\bar p$ has a $\leq^{\mathbb{Q}_\ell}_{pr}$-upper bound; similarly for $\leq^{\mathbb{Q}_\ell}_{pr,n}$ and $\alpha < \theta_n$.

3) If $\underset{\sim}{\tau}$ is a $\mathbb{Q}_\ell$-name of an ordinal and $p \in \mathbb{Q}_\ell$, <u>then</u> for some q and n we have:

(a) $p \leq_{pr} q$
(b) if $q \leq_{apr} r$ and $\mathbf{n}(r) \geq n$, <u>then</u> r forces a value to $\underset{\sim}{\tau}$.

4) In (3), if $\Vdash_{\mathbb{Q}_\ell}$ "$\underset{\sim}{\tau} < \omega$ or just $< \alpha^* < \theta_{n(p)}$" <u>then</u> without loss of generality $n = \mathbf{n}(p)$.

Proof. Easy.

Claim 3.12. *Recall that E is a nice definition of a two-place relation on $\prod_{n<\omega}\lambda_n$. Then forcing by $\mathbb{Q}_2$ preserves "E is an equivalence relation on $\prod_{n<\omega}\lambda_n$ satisfying clause (d) of 3.1" or more exactly the definition E defines in $\mathbf{V}^{\mathbb{Q}_2}$ an equivalence relation on $\prod_{n<\omega}\lambda_n$ satisfying clause (d) of 3.1 (and, of course, $E^{(\mathbf{V}^{\mathbb{Q}_2})} \restriction (\prod_{n<\omega}\lambda_n)^{\mathbf{V}} = E^{\mathbf{V}}$).*

Proof. Assume toward contradiction that $p^* \Vdash_{\mathbb{Q}_2}$ "$\underset{\sim}{\nu}_0, \underset{\sim}{\nu}_1, \underset{\sim}{\nu}_2 \in \prod_{\ell<\omega}\lambda_\ell$ form a counterexample, that is: $\underset{\sim}{\nu}_0 E \underset{\sim}{\nu}_1 \wedge \underset{\sim}{\nu}_1 E \underset{\sim}{\nu}_2 \wedge \neg \underset{\sim}{\nu}_0 E \underset{\sim}{\nu}_2$ or $\neg \underset{\sim}{\nu}_0 E \underset{\sim}{\nu}_0$ or $\underset{\sim}{\nu}_0 E \underset{\sim}{\nu}_1 \wedge \neg \underset{\sim}{\nu}_1 E \underset{\sim}{\nu}_0$ or $\underset{\sim}{\nu}_0 E \underset{\sim}{\nu}_1 \wedge (\exists! n)(\underset{\sim}{\nu}_0(n) \neq \underset{\sim}{\nu}_1(n))$".

Choose χ large enough and $\bar N = \langle N_n : n < \omega\rangle$, N such that:

$\circledast^{\chi}_{\bar N}$ (i) $N_n \prec_{\mathbb{L}_{\lambda_n^+,\lambda_n^+}} (\mathcal{H}(\chi), \in)$ and $\|N_n\| = 2^{\lambda_n}$ and $\{p^*, E, \underset{\sim}{\nu}_0, \underset{\sim}{\nu}_1, \underset{\sim}{\nu}_2, N_0, \ldots, N_{n-1}\}$ belong to N_n

(ii) $N_n \in N_{n+1}$ hence $N_n \prec_{\mathbb{L}_{\lambda_n^+,\lambda_n^+}} N_{n+1}$ and $N = \bigcup\limits_{n<\omega} N_n$ so $N \prec (\mathcal{H}(\chi), \in)$.

Now we choose p_n by induction on $n < \omega$ such that:

$(*)(i)$ $p_0 = p^*$,

(ii) $p_n \in N_n \cap \mathbb{Q}_2$ and $\mathbf{n}(p_n) = \max\{n, \mathbf{n}(p^*)\}$

(iii) $p_n \leq p_{n+1}$

(iv) if $\underset{\sim}{\tau} \in N_n$ is a $\mathbb{Q}_2$-name of an ordinal <u>then</u> for some $k_n(\underset{\sim}{\tau}) > n+1$ we have: if $p_{n+1} \leq q$ and $\mathbf{n}(q) \geq k_n(\underset{\sim}{\tau})$ then q forces a value to $\underset{\sim}{\tau}$.

This is possible by 3.11(2),(3). Now let $G = \{q : q \in N \cap \mathbb{Q}_2$ and $q \leq p_n$ or just $p_n \Vdash$ "$q \in \underset{\sim}{G}$" for some $n\}$; it is a subset of $\mathbb{Q}_2^N$ generic over N. (Why? If $N \models$ "$\mathcal{I} \subseteq \mathbb{Q}_2$ is dense" then $\mathcal{I} \subseteq \mathbb{Q}_2$ is dense and there is $\mathcal{I}' \subseteq \mathcal{I}$, a maximal antichain of $\mathbb{Q}_2$ which belongs to N hence to some N_n; there is $g \in N_n$, a one to one function from $\mathcal{I}'$ onto $|\mathcal{I}'|$, so it defines a $\mathbb{Q}_2$-name $\underset{\sim}{\tau}$ by $\underset{\sim}{\tau}[G] = \gamma \Leftrightarrow (\forall q)(q \in \mathcal{I}' \cap G \rightarrow f(q) = \gamma) \Leftrightarrow (\exists q)(q \in \mathcal{I}' \cap G \,\&\, f(q) = \gamma)$, so $k_n(\underset{\sim}{\tau}) < \omega$ is well defined (see clause (iv) above) and so $p_{k_n(\underset{\sim}{\tau})}$ forces a value to $\underset{\sim}{\tau}$ hence forces $q \in \underset{\sim}{G}$ for some $q \in \mathcal{I}' \subseteq \mathcal{I}$, hence $q \in G$ so $G \cap \mathcal{I} \neq \emptyset$ as required). Now by straightforward absoluteness argument, $\underset{\sim}{\nu}_0[G], \underset{\sim}{\nu}_1[G], \underset{\sim}{\nu}_2[G] \in \prod_{\ell<\omega}\lambda_\ell$ give contradiction to an assumption.

In details let $\nu_\ell = \underset{\sim}{\nu}_\ell[G]$. Let M be the Mostowski collapse of N, so there is an isomorphism g from N onto M. Clearly $\lambda_n \subseteq N_n$ hence $\lambda \subseteq N$ hence $\lambda + 1 \subseteq N$

so $g(x) = x$ if $x \in \lambda + 1$ or $x \subseteq \lambda + 1$ or $x \in \mathcal{H}(\lambda)$. Clearly $G^* = g''(G)$ is a generic subset of $\mathbb{Q}_2^* = (g(\mathbb{Q}_2))^M$ and $M^* = M[G]$ is a generic extension of M (for $g(\mathbb{Q}_2)^M$) and so

$\boxdot_1$ M^* is a transitive model of enough set theory (i.e. of ZC if χ is strong limit) which includes $\mathcal{H}(\lambda) \cup \{\lambda, \mathcal{H}(\lambda)\} \cup \{\langle \lambda_n : n < \omega \rangle\}$.

Also easily in M^*, $\underset{\sim}{\nu}_\ell[G^*] = \nu_\ell$, so as $g(p^*) \in G^*$, clearly (E stands for the formula defining it, its parameter a subset of λ so it is mapped by g to itself):

$$\begin{aligned} M^* \models \text{“}&\nu_0, \nu_1, \nu_2 \in \Pi\{\lambda_n : n < \omega\} \text{ and}\\ &\nu_0 E \nu_1 \ \&\ \nu_1 E \nu_2 \ \&\ \neg \nu_0 E \nu_1 \text{ or}\\ &\neg \nu_0 E \nu_0 \text{ or } \nu_0 E \nu_1 \ \&\ \neg \nu_1 E \nu_0 \text{ or}\\ &\nu_0 E \nu_1 \ \&\ (\exists! n)(\nu_0(n) \neq \nu_1(n))\text{”}. \end{aligned}$$

So it is enough to prove (see Lemma 3.1, clause (c)). $\square_{3.12}$

Fact 3.13. Assume M^* satisfies $\boxdot_1$ above, E is a nice two-place relation on $\Pi\{\lambda_n : n < \omega\}$ so a definition with parameter a subset of λ (equivalently: a model on λ) as in Definition 0.2(1).

Then

$\boxdot_2$ if M^* satisfies “$\eta_1 E \eta_2$ & $\neg \eta_3 E \eta_4$ and $\eta_0, \eta_1, \eta_2, \eta_3 \in \Pi\{\lambda_n : n < \omega\}$” then so does $\mathbf{V}$.

Proof. Immediate.

In fact

Obervation 3.14. Assume

$(a)(i)$ λ^* is strong limit of cofinality $\aleph_0$,
(ii) $\lambda^* = \sum_{n<\omega} \lambda^*_n$
(iii) $\lambda^*_n < \lambda^*_{n+1}$ for $n < \omega$, for simplicity $2^{\lambda^*_n} < \lambda^*_{n+1}$
$(b)(i)$ $\mathbb{Q}$ is a forcing notion
(ii) $\leq_{pr}$ is included in $\leq_{\mathbb{Q}}$
(iii) $\mathbf{n} : \mathbb{Q} \to \omega$ is a function satisfying for each n the set $\mathcal{I}_n = \{p \in \mathbb{Q} : \mathbf{n}(p) \geq n\}$ is a dense subset of $\mathbb{Q}$
(iv) for $p \in \mathbb{Q}$, $\{q \in \mathbb{Q} : p \leq_{pr} q\}$ is $\lambda^*_{\mathbf{n}(p)}$-complete
(v) $\mathbb{Q}$ has pure decidability for $\mathbb{Q}$-names of truth values
(vi) if $p \in \mathbb{Q}$ and $\underset{\sim}{\tau}$ is a $\mathbb{Q}$-name of an ordinal, then there are $m < \omega$ and q satisfying: $p \leq_{pr} q$ and $(q \leq r \ \&\ m \leq \mathbf{n}(r)) \Rightarrow$ (r forces a value to $\underset{\sim}{\tau}$)
(c) $N, \langle N_n : n < \omega \rangle$ as in the proof of 3.12 for $\langle \lambda^*_n : n < \omega \rangle$, $\{\mathbb{Q}, \leq, \leq_{pr}\} \in N_0$.

Then there is $G \subseteq \mathbb{Q}^N$ generic over N hence $\mathcal{H}(\lambda)^{N[G]} = \mathcal{H}(\lambda) = \mathcal{H}(\lambda)^N$.

Proof. Should be clear.

Claim 3.15. *Assume that F is a permutation of $(\prod_{\ell<n(*)} \lambda_\ell) \times (\prod_{\ell<n(*)} \lambda_\ell)$ and let $\mathbb{Q}_2^{\geq n(*)} = \{p \in \mathbb{Q}_2 : \mathbf{n}(p) \geq n(*)\}$. We let $\hat{F}$ be the following function from*

$\mathbb{Q}_2^{\geq n(*)}$ *to* $\mathbb{Q}_2^{\geq n(*)}$

$$\hat{F}(p) = q \text{ iff } \begin{array}{l} \mathbf{n}(q) = \mathbf{n}(p) \\ (\eta_0^q \restriction n(*), \eta_1^q \restriction n(*)) = F((\eta_0^p \restriction n(*), \eta_1^p \restriction n(*))) \\ \eta_0^q \restriction [n(*), \mathbf{n}(p)) = \eta_0^p \restriction [n(*), \mathbf{n}(p)) \\ \eta_1^q \restriction [n(*), \mathbf{n}(p)) = \eta_1^p \restriction [n(*), \mathbf{n}(p)) \\ \bar{A}^q = \bar{A}^p. \end{array}$$

Then the following holds:

1) For $p \in \mathbb{Q}_2^{\geq n(*)}$, $\hat{F}(p)$ is well defined $\in \mathbb{Q}_2^{\geq n(*)}$.

2) $\hat{F}$ is a permutation of $\mathbb{Q}_2^{\geq n(*)}$ preserving $\leq, \leq_{pr}, \leq_{pr,n}, \leq_{apr}$ and their negations, and $F \mapsto \hat{F}$ is a group homomorphism (hence embedding).

3) If $G \subseteq \mathbb{Q}_2$ is generic over $\mathbf{V}$ then

(*a*) *$\hat{F}(G) =: \{r \in \mathbb{Q}_2$: for some $q \in G \cap \mathbb{Q}_2^{\geq n(*)}$ we have $r \leq \hat{F}(q)\}$ is a subset of $\mathbb{Q}_2$ generic over $\mathbf{V}$*

(*b*) *$G = \{p \in \mathbb{Q}_2$: there is $q \in \mathbb{Q}_2^{\geq n(*)}$ such that $p \leq_{\mathbb{Q}_2} q$ and $\hat{F}(q) \in \hat{F}(G)\}$*

(*c*) *and $\mathbf{V}[\hat{F}(G)] = \mathbf{V}[G]$ and even $N[\hat{F}(G)] = N[G]$ if, e.g., $N \prec (\mathcal{H}(\chi), \in), \mathbb{Q}_2 \in N, F \in N, \lambda \subseteq N$.*

Proof. Easy.

Claim 3.16. $\Vdash_{\mathbb{Q}_2}$ "$\neg \underset{\sim}{\eta}_0 E \underset{\sim}{\eta}_1$".

Proof. If not, let $p \in \mathbb{Q}_2$ be such that $p \Vdash_{\mathbb{Q}_2}$ "$\underset{\sim}{\eta}_0 E \underset{\sim}{\eta}_1$". Now by clause (f) of Definition 3.2(1), we can find p_1 such that:

(*i*) $\mathbb{Q}_2 \models p \leq_{pr} p_1$

(*ii*) if $\mathbf{n}(p) \leq n < \omega$ and $\beta \in s_1(A_n^{p_1})$ then for some α, γ we have $(\alpha, \beta), (\beta, \gamma) \in s_2(A_n^p)$.

Let $G_1 \subseteq \mathbb{Q}_2$ be generic over $\mathbf{V}$ such that $p_1 \in G_1$ and let $\eta_\ell = \underset{\sim}{\eta}_\ell[G_1]$ for $\ell = 1, 2$ so $\mathbf{V}[G_1] \models \eta_0 E \eta_1$. By 3.12 in $\mathbf{V}[G_1]$, $\underset{\sim}{E}$ is still an equivalence relation satisfying clause (d) of 3.1 and trivially $n \in [\mathbf{n}(p), \omega) \Rightarrow \eta_1(n) \in s_1(A_n^{p_1})$. Let $n^* =: \mathbf{n}(p)$, we can find $\alpha^* < \lambda_{n^*}$ such that $\alpha^* < \eta_1(n^*), \alpha^* \neq \eta_0(n^*)$ and $(\alpha^*, \eta_1(n^*)) \in s_2(A_{n^*}^p)$. Let us define $\eta_0' \in \prod_{n<\omega} \lambda_n$ by $\eta_0'(n)$ is α^* if $n = n^*$ and $\eta_0(n)$ otherwise; as $\alpha^* < \eta_1(n^*) < \eta_0(n^*)$ necessarily $\eta_0 \neq \eta_0'$.

Now the pairs $(\eta_0 \restriction (n(*) + 1), \eta_1 \restriction (n(*) + 1))$ and $(\eta_0' \restriction (n(*)) + 1), \eta_1 \restriction (n(*) + 1))$ are from $(\prod_{n \leq n(*)} \lambda_n) \times (\prod_{n \leq n(*)} \lambda_n)$, so there is a permutation F of this set interchanging those two pairs and is the identity otherwise. Let $\hat{F}$ be the automorphism of $\mathbb{Q}_2^{\geq (n^*+1)}$ from Claim 3.15. Let $G_2 = \hat{F}(G_1)$. Now by 3.15:

$(*)_1$ G_2 is a generic subset of $\mathbb{Q}_2$ over $\mathbf{V}$

$(*)_2$ $\mathbf{V}[G_2] = \mathbf{V}[G_1]$

$(*)_3$ $\underset{\sim}{\eta}_0[G_2] = \eta_0', \underset{\sim}{\eta}_1[G_2] = \eta_1$.

By 3.12 (and the choice of η_0') we have

$(*)_4$ $\mathbf{V}[G_1] \models \neg \eta_0 E \eta'_0$.

As $p \le p_1 \in G_1$, by the choice of p clearly

$(*)_5$ $\mathbf{V}[G_1] \models$ "$\eta_0 E \eta_1$".

By the choice of p_1 and $(\alpha, \eta_1(n^*))$ clearly $p \le (\eta'_0 \restriction [-n(*)+1)), \eta'_1 \restriction (n(*)+1), \bar{A} \restriction [(n(*)+1, \omega) \in G_2$ so (using $(*)_1$)

$(*)_6$ $\mathbf{V}[G_2] \models$ "$(\underset{\sim}{\eta}_0[G_2]) E (\underset{\sim}{\eta}_1[G_2])$"

hence by $(*)_2 + (*)_3$ we have

$(*)_7$ $\mathbf{V}[G_2] \models$ "$\eta'_0 E \eta_1$".

Now $(*)_4 + (*)_5 + (*)_7$ contradict 3.12. $\square_{3.16}$

Claim 3.17. *1) Fix $\chi > \lambda$ large enough and choose $N_n \prec_{\mathbb{L}_{\lambda_n,\lambda_n}} (\mathcal{H}(\chi), \in)$ such that $\|N_n\| = 2^{\lambda_n}$, $\{E, \mathfrak{m}\} \cup \{N_\ell : \ell < n\}$ belongs to N_n (hence $\mathbb{Q}_2 \in N_n$), and let $N = \bigcup_{n<\omega} N_n$; (certainly can be done). <u>Then</u> we can find $\langle \rho_\nu : \nu \in \prod_{\ell<n} \mu_n$ and $n < \omega \rangle$ and*[4]

(α) $\rho_\nu \in \prod_{\ell<\ell g(\nu)} \lambda_\ell$
(β) $\nu_1 \lhd \nu_2 \Rightarrow \rho_{\nu_1} \lhd \rho_{\nu_2}$
(γ) if $\nu_1, \nu_2 \in \prod_{\ell<n} \lambda_\ell$ and $m \le k < n$, $\nu_1 \restriction m = \nu_2 \restriction m$ and $\nu_1(m) < \nu_2(m)$ then $\eta_{\nu_1}(k) < \eta_{\nu_2}(m)$
(δ) if $\nu \in \prod_{\ell<\omega} \mu_\ell$ then $\rho_\nu =: \bigcup_{n<\omega} \rho_{\nu \restriction n}$ is generic for $(N, \mathbb{Q}_1)$
(ε) if $\nu_0, \nu_1 \in \prod_{\ell<\omega} \mu_\ell$ and $\nu_0 <_{lex} \nu_1$ <u>then</u> $(\rho_{\nu_0}, \rho_{\nu_1})$ is generic for $(N, \mathbb{Q}_2)$ hence
(ζ) if $\nu_0 \ne \nu_1 \in \prod_{\ell<\omega} \mu_\ell$ <u>then</u> $\neg(\rho_{\nu_0} E \rho_{\nu_1})$.

2) Also, for some $p \in \mathbb{Q}_2$, $\mathbf{n}(p) = 0$ and non-principal ultrafilter D on ω we have

$(*)$ if $\eta, \nu \in \prod_{n<\omega} s_1(A^p_n)$ and $\eta/D \ne \nu/D$ <u>then</u> $\neg(\eta E \nu)$.

3) Moreover, there is a filter J on ω to which all co-finite subsets of ω belong and for $\eta, \nu \in \prod_{n<\omega} s_1(A^p_n)$ we have $\eta E \nu \Leftrightarrow \eta = \nu \bmod J$.

Proof. Let $M_0 \prec_{\mathbb{L}_{\aleph_1,\aleph_1}} N_0$ be such that $\|M_0\| = 2^{\aleph_0}$ and $\{E, \mathfrak{m}\} \in M_0$.

As above we choose p_n by induction on n such that:

$\boxtimes_1(i)$ $p_n \in \mathbb{Q}_2$
(ii) $p_n \in N_n$
(iii) $\mathbf{n}(p_0) = 0$
(iv) $p_n \le_{pr} p_{n+1}$ (hence $p_0 \le_{\mathrm{pr}} p_n$ so $\ell < \omega \Rightarrow \mathbf{n}(p_\ell) = 0$)
(v) for every $\mathbb{Q}_2$-name of an ordinal $\underset{\sim}{\tau} \in N_n$, for some $k_n(\underset{\sim}{\tau}) \in [n, \omega)$ we have: if $\mathbb{Q}_2 \models$ "$p_{n+1} \le q$" and $\mathbf{n}(q) \ge k_n(\underset{\sim}{\tau})$ then q forces a value to $\underset{\sim}{\tau}$

[4] why not $\nu \in \Pi\{\lambda_\ell : \ell < n\}$? First we like $\langle \rho_\nu(n) : \nu \in \Pi\{\lambda_\ell : \ell \le n\}\rangle$ to be increasing with ν (the ν's are linearly ordered by lexicographic order) so the order type is the ordinal product $\lambda_n \times \lambda_{n-1} \times \ldots \times \lambda$ has cardinality λ but order type $> \lambda$. Second and more seriously we intend to use clause (h) of Definition 3.2 which gives us Y of cardinality μ; note if we use 3.4(1) we get $\lambda_n = \mu_n$ but not if we use 3.5(1)

(*vi*) if $\underset{\sim}{\tau} \in M_0$ is a $\mathbb{Q}_2$-name of a natural number then p_0 forces a value to it.

Moreover,

(*vii*) if $n < \omega$ and $\eta_0, \eta_1 \in \prod_{\ell<n} \lambda_\ell$ then $p^1_{n+1} = (\eta_0, \eta_1, \langle A^{p_{n+1}}_\ell : \ell \in [n, \omega]\rangle \in \mathbb{Q}_2$ satisfies, too, clause (v).

We can find $p_\omega \in \mathbb{Q}_2$ such that $n < \omega \Rightarrow p_n \le_{pr} p_\omega$ and we can find p^* such that $p_\omega \le p^*$ and $(\forall n)(\forall \beta)(\exists \alpha, \gamma)[\beta \in s_1(A^{p^*}_n) \to (\alpha, \beta), (\beta, \gamma) \in s_2(A^{p_\omega}_n)]$ and we shall show that p^* is as required in parts (2) and (3), for p. Now clearly

$\boxtimes_2$ if $\eta_0, \eta_1 \in \prod_{n<\omega} \lambda_n$ and $(\forall \ell < 2)(\forall n < \omega)(\eta_\ell(n) \in s_1(A^{p^*}_n))$ and for every $n < \omega$ large enough $(\eta_0(n), \eta_1(n)) \in s_2(A^{p^*}_n)$ then
(*a*) for some subset G of $\mathbb{Q}^N_2$ generic over N to which p_0 belongs we have $\underset{\sim}{\eta}_0[G] = \eta_0, \underset{\sim}{\eta}_1[G] = \eta_1$
(*b*) $\neg \eta_0 E \eta_1$.
[Why? Let $k^* < \omega$ be such that $k^* \le k < \omega \Rightarrow \eta_0 \restriction (k^*) \ne \eta_1 \restriction (k^*)$, it exists by the definition of order on $\mathbb{Q}_2$. For every $k \ge k^*$ we define a condition $q = q^k_{\eta_0,\eta_1} \in \mathbb{Q}_2$ by: $\mathbf{n}(q) = k, \eta^q_0 = \eta_0 \restriction k, \eta^q_1 = \eta_1 \restriction k$ and $A^q_n = A^{p^*}_n$ for $n \in [k, \omega)$ and let $G_{\eta_0,\eta_1} =: \{r : r \in \mathbb{Q}_2, r \in N$ and $r \le_{\mathbb{Q}_2} q^k_{\eta_0,\eta_1}$ for some $k < \omega\}$. By $\boxtimes_1$ and the proof of 3.12 easily G_{η_0,η_1} is a subset of $\mathbb{Q}_2 \cap N$ generic over N, so clause (a) holds. By 3.16 clearly $N[G_{\eta_0,\eta_0}] \models \neg(\eta_0 E \eta_1)$. By using absoluteness (as in 3.12(1)), also clause (b) holds.]

This suffices for part (1), in detail: by clause (h) of Definition 3.2(1) recalling the μ^+_n is Hypothesis 3.7, we can find $Y_n \subseteq \lambda_n$ of order type $\mu_n \times \mu_{n-1} \times \cdots \times \mu_0$ from N_n such that for any $\alpha < \beta$ from Y_n the pair (α, β) belong to $s_2(A^{p^*}_n)$. Now we can choose by induction on n, $\langle \rho_\nu : \nu \in \prod_{\ell<n} \mu_\ell\rangle$ as required in (α), (β), (γ) of 3.17(1), they are as required.

We are left with proving part (2). For $B \subseteq \omega$ let $\underset{\sim}{\eta}_B$ be the following $\mathbb{Q}_2$-name:

$$\underset{\sim}{\eta}_B(n) \text{ is } \underset{\sim}{\eta}_1(n) \text{ if } n \in B \text{ and is } \underset{\sim}{\eta}_0(n) \text{ if } n \in \omega \backslash B.$$

Clearly $\underset{\sim}{\eta}_B$ is a $\mathbb{Q}_2$-name of a member of ${}^\omega\lambda$ and $\underset{\sim}{\eta}_B \in M_0$ (recall that $\|M_0\| = 2^{\aleph_0}$) hence for $B_1, B_2 \subseteq \omega$ the following $\mathbb{Q}_2$-name of a truth value, the truth value of $(\underset{\sim}{\eta}_{B_1} E \underset{\sim}{\eta}_{B_2})$, is decided by p_0, say it is $\mathbf{t}(B_1, B_2)$.

Define a two place relation E' on $\mathcal{P}(\omega)$: $B_1 E' B_2$ iff $\mathbf{t}(B_1, B_2) =$ truth.

Let $J = \{B \subseteq \omega : \mathbf{t}(\emptyset, B) = \text{truth}\}$, that is, $J = \{B : \emptyset E' B\}$.

Clearly

$(*)_0$ E' is an equivalence relation on $\mathcal{P}(\omega)$.
[Why? By E being (forced to be) an equivalence relation.]
$(*)_1$ $\omega \notin J$, moreover $[n, \omega) \notin J$.
[Why? By $\boxtimes_2$.]
$(*)_2$ if $B_1, B_2 \in J$ then $B_1 E' B_2$.
[Why? As E' is an equivalence relation.]

Let $\alpha_n^0 < \alpha_n^1 < \alpha_n^2 < \alpha_n^3 < \alpha_n^4 < \alpha_n^5$ be from Y_n for $n < \omega$ and for $h \in {}^\omega\{0,1,2,3,4\}$ let $\nu_h \in \prod_{n<\omega} \lambda_n$ be $\nu_h(n) = \alpha_n^{h(n)}$. If $g_1, g_2 \in {}^\omega\{1,2,3,4\}$ and $B \subseteq \omega$ we let $h_{g_1,g_2,B} \in {}^\omega\{1,2,3,4\}$ be $h_{g_1,g_2,B}(n) = \begin{cases} g_1(n) & \underline{\text{if}}\ n \notin B \\ g_2(n) & \underline{\text{if}}\ n \in B. \end{cases}$

Easily

$(*)_3$ if $g_1, g_2 \in {}^\omega\{1,2,3,4\}$ and $(\forall n < \omega)((n \in B_1 \backslash B_2) \vee (n \in B_2 \backslash B_1) \Rightarrow g_1(n) < g_2(n))$ and $B_1, B_2 \subseteq \omega$ $\underline{\text{then}}$ $B_1 E' B_2$ $\underline{\text{iff}}$ $\nu_{h_{g_1,g_2,B_1}} E \nu_{h_{g_1,g_2,B_2}}$
[Why? That is, let $h_\ell = h_{g_1,g_2,B_\ell}$ for $\ell = 1, 2$ and note that $n \in (B_1 \cap B_1) \cup (\omega \backslash B_1 \backslash B_2) \Rightarrow h_1(n) = h_2(n)$.
We define $\eta_0^*, \eta_1^* \in \prod_n y_n$ as follows:
(a) $\underline{\text{if}}$ $n \in (B_1 \backslash B_2) \cup (B_2 \backslash B_1)$ then $\eta_0^*(n) = \alpha^n_{g_1(n)}, \eta_1^*(n) = \alpha^n_{g_2(n)}$
(b) if $n \in B_1 \cap B_2$ then $\eta_0^*(n) = \alpha^n_{g_2(n)-1}, \eta_1^*(n) = \alpha^n_{g_2(n)}$
(c) if $n \in \omega \backslash B_1 \backslash B_2$ then $\eta_0^*(n) = \alpha^n_{g_1(n)}, \eta_1^*(n) = \alpha^n_{g_1(n)+1}$.
Now choose G as in clause (a) of $\boxtimes_2$ for (η_0^*, η_1^*) and note that
(d) $\nu_{h_1} = \underset{\sim}{\eta}_{B_1}[G], \nu_{h_2} = \underset{\sim}{\eta}_{B_2}[G]$.
[Why? Because as

$$n \in B_1 \backslash B_2 \Rightarrow (\underset{\sim}{\eta}_{B_1}(n), \underset{\sim}{\eta}_{B_2})[G] = (g_2(n), g_1(n)) = (\nu_{h_1}(n), \nu_{h_2}(n)) \text{ and}$$

$$n \in B_2 \backslash B_1 \Rightarrow (\underset{\sim}{\eta}_{B_1}(n), \underset{\sim}{\eta}_{B_2}(n))[G] = (g_1(n), g_2(n)) = (\nu_{h_1}(n), \nu_{h_2}(n))$$

and also for the other n's.]
Now clearly, $\nu_{h_1} E \nu_{h_2}$ $\underline{\text{iff}}$ $N[G] \models$ "$\underset{\sim}{\eta}_{B_1}[G] E \underset{\sim}{\eta}_{B_2}[G]$" which is equivalent to $B_1 E' B_2$, so we are done.]

$(*)_4$ if $B_1, B_2 \subseteq \omega$, $B = B_1 \cap B_2$ then $B_1 E' B_2 \Leftrightarrow B_1 E' B \ \&\ B_2 E' B$.
[Why? The implication $\Leftarrow$ holds as E is an equivalence relation so let us proof $\Rightarrow$. By the symmetry it is enough to show that $B_1 E' B$. We choose $h_\ell \in {}^\omega\{1,2,3,4\}$ for $\ell = 1,2,3$ by: if $n \in (B_1 \backslash B_2)$ then $(h_1(n), h_2(n), h_3(n)) = (2,3,1)$, if $n \in B_2 \backslash B_1$ then $(h_1(n), h_2(n), h_3(n)) = (1,2,3)$, if $n \in \omega \backslash (B_1 \cup B_2)$ or $n \in B_1 \cap B_2 = B$ then $(h_1(n), h_2(n), h_3(n)) = (1,1,1)$.
Now we choose functions $g_1^a, g_2^a, g_1^b, g_2^b, g_1^c, g_2^c \in {}^\omega\{1,2,3,4\}$ as follows: for $n < \omega$, the six-tuple $(g_1^a(n), g_2^a(n), g_2^b(n), g_2^b(n), g_1^c(n), g_2^c(n)$ is:
(i) $(1,2;1,3;2,3)$ $\underline{\text{if}}$ $n \in B_1 \backslash B_2$
(ii) $(1,3;2,3;1,2)$ $\underline{\text{if}}$ $n \in B_2 \backslash B_1$
(iii) $(1,1;1,1;1,1)$ $\underline{\text{if}}$ $n \in (B_1 \cap B_2) \cup (\omega \backslash (B_1 \cup B_2))$
So $\nu_{h_1} E \nu_{h_3}$ as we are asuming $B_1 E' B_2$, using $(*)_3$ for (g_1^a, g_2^a) the "only if" part. Also $\nu_{h_2} E \nu_{h_3}$ similarly using (g_1^b, g_2^b).
Together it follows that $\nu_{h_1} E \nu_{h_2}$ as E is an equivalence relation. Using $(*)_3$ again for (g_1^c, g_2^c) this time, by the "if" part it follows that $B_1 E' B$ as required.]

Similarly

$(*)_5$ if $B_1 \subseteq B_2 \subseteq \omega$ then $B_1 E' B_2 \Leftrightarrow (B_2 \backslash B_1) \in J$.
[Why? This follows by $(*)_3$.]

$(*)_6$ if $B_1 \subseteq B_2 \subseteq B_3 \subseteq \omega$ and $B_1 E' B_3$ then $B_1 E' B_2 \ \&\ B_2 E' B_3$
[Why? We define $h_1, h_2, h_3 \in {}^\omega\{1,2,3,4\}$ by:

(*a*) if $n \in B_3 \backslash B_2$ then $(h_1(n), h_2(n), h_3(n)) = (2, 2, 4)$

(*b*) if $n \in B_2 \backslash B_1$ then $(h_1(n), h_2(n), h_3(n)) = (2, 3, 4)$

(*c*) if $n \in \omega \backslash (B_3 \backslash B_1)$ then $(h_1(n), h_2(n), h_3(n)) = (1, 1, 1)$. Now $\nu_{h_1} E \nu_{h_3}$ as we are assuming $B_1 E' B_3$ using $(*)_3$ with (g_1^a, g_2^a) the "only if" part. Similarly $\nu_{h_2} E \nu_{h_3}$ using $(*)_3$ with (g_1^b, g_2^b).

As E is an equivalence relation we deduce $\nu_{h_1} E \nu_{h_2}$ hence $B_1 E' B_2$ by $(*)_3$ using (g_1^c, g_2^c) the "if" part.
By E' being an equivalence relation we can deduce $B_2 E' B_3$ so we are done.]

$(*)_7$ J is an ideal
[Why? If $B_1 \subseteq B_2$ & $B_2 \in J$ we have $\emptyset \subseteq B_1 \subseteq B_2$ & $\emptyset E' B_2$ so by $(*)_6$ we have $\emptyset E' B_1$ as required. If $B_1, B_2 \in J$ are disjoint members of J, then $\emptyset E' B_1$ & $\emptyset E' B_2$ by the definition J, so by E' being an equivalence relation $B_1 E' B_2$. Now $B_1 \in J$ and so by $(*)_5$ applied with $B_1, B_1 \cup B_2$ here standing for B_1, B_2 there we get $B_1 E'(B_1 \cup B_2)$ so by transitivity of E' we have $\emptyset E'(B_1 \cup B_2)$ which means $B_1 \cup B_2 \in J$.]

$(*)_8$ $\{0, \ldots, n\} \in J$
[Why? By $\boxtimes_2$.]

$(*)_9$ $B_1 E' B_2$ <u>iff</u> $B_1 \Delta B_2 \in J$
[Why? Let $B = B_1 \cap B_2$; if $B_1 \Delta B_2 \in J$ then we have $B_1 \backslash B, B_2 \backslash B \in J$ so by $(*)_5$ we have $B_1 E' B$ & $B E' B_2$ hence $B_1 E' B_2$. If $B_1 E' B_2$ then $B_1 E' B$ & $B_2 E B$ by $(*)_4$, hence $B_1 \backslash B, B_2 \backslash B \in J$ by $(*)_5$ so by $(*)_7$ $B_1 \Delta B_2 \in J$.]

So by $(*)_7 + (*)_2$ there is an ultrafilter D on ω disjoint to J, and by $(*)_8$ it is non-principal, and by $(*)_9$ it has the desired property so we have proved also part (2). Part (3) has been proved by $(*)_7 + (*)_9$. $\square_{3.17}$ $\square_{3.1}$

§4. The countable cofinality case: negative results

In the previous section we have gotten positive results, however, the assumptions are such that they may fail in ZFC (for every λ). Can we eliminate those assumptions? We below show that we cannot eliminate them: for reasonable λ the conclusion fails strongly (as in §2), if λ fails the free subset property (a well known property, see, e.g., [J]). So e.g. if $\neg 0^{\#}$, the results of §3 fail.

Claim 4.1. *Assume*

(*a*) $\lambda > \mathrm{cf}(\lambda) = \aleph_0$

(*b*) $(\forall \alpha < \lambda)[|\alpha|^{\aleph_0} < \lambda]$

(*c*) *there is an algebra $\mathfrak{B}$ with universe λ, with $< \lambda$ functions and with no infinite free subset, see Definition below.*

<u>Then</u> there is E such that

(α) *E is an equivalence relation on ${}^{\omega}\lambda$*

(β) *E is very nice (see Definition 0.2)*

(γ) if $\eta, \nu \in {}^{\omega}\lambda$ and $\eta =^* \nu$ (i.e. $(\exists^{<\aleph_0} n)(\eta(n) \neq \nu(n))$ <u>then</u> $\eta E \nu \Leftrightarrow \eta = \nu$
(δ) E has λ equivalence classes.

Definition 4.2. *A subset Y of an algebra $\mathfrak{B}$ is free <u>if</u>: $a \in Y \Rightarrow a \notin c\ell_{\mathfrak{B}}(Y \backslash \{a\})$ where $c\ell_{\mathfrak{B}}(Y')$ means the subalgebra of $\mathfrak{B}$ generated by Y'.*

Remark 4.3. 1) We can replace ${}^{\omega}\lambda$ by the set of increasing ω-sequences or by $\prod_{n<\omega} \lambda_n$ when $\lambda_n < \lambda_{n+1} < \lambda = \sum_{m<\omega} \lambda_m$ or by $\{A \subseteq \lambda : (\forall n)(\exists! \alpha)(\alpha \in A \ \& \ \sum_{\ell<n} \lambda_\ell \leq \alpha < \lambda_n)\}$.

2) We can omit clause (b) if we weaken clause (γ). We can imitate 2.4 and 2.7, see 4.4 below.

Proof. Without loss of generality $\mathfrak{B}$ has $\aleph_0$ function and the individual constants $\{\alpha : \alpha < \lambda_0\}$ and there are no other individual constants. Let $\Sigma_n^* = \{\sigma(x_0, \ldots, x_{n-1}) : \sigma(x_0, \ldots, x_{n-1})$ a $\tau_{\mathfrak{B}}$-term$\}$ and $<_n^*$ a well ordering of Σ_n^* where $\lambda_0 < \lambda$, of course.

We define a two place E_0 on ${}^{\omega}\lambda$ by

$\eta E_0 \nu$ <u>iff</u>: if $n < \omega$ and
$k, k_1, \ldots, k_n < \omega$ then
(a) there is $\sigma(x_0, \ldots, x_{n-1}) \in \Sigma_n^*$ such that
$\eta(k) = \sigma(\eta(k_1), \ldots, \eta(k_n))$ iff there is
$\sigma(x_0, \ldots, x_{n-1}) \in \Sigma_n^*$ such that
$\nu(k) = (\sigma(\nu(k_1), \ldots, \nu(k_n))$
(b) if in (a) they hold then the $<_n^*$-first term $\sigma(x_0, \ldots, x_{n-1}) \in \Sigma_n^*$
such that $\eta(k) = \sigma(\eta(k_1), \ldots, \eta(k_n)$
is the $<_n^*$-first term $\sigma(x_0, \ldots, x_{n-1}) \in \Sigma_n^*$ such that
$\nu(k) = \sigma(\nu(k_1), \ldots, \nu(k_n))$.

So E_0 is an equivalence relation with $\leq \lambda_0^{\aleph_0} < \lambda$ equivalence classes. For $\eta \in {}^{\omega}\lambda$ let $A(\eta) = \{k :$ for some $k^* < \omega$ there are no $n < \omega, k_1, \ldots, k_n \in [k^*, \omega)$ and $\mathfrak{B}$-term $\sigma(x_1, \ldots, x_n)$ such that $\eta(k) = \sigma(\eta(k_1), \ldots, \eta(k_n))\}$.

Lastly, we define E_1 by

$$\eta E_1 \nu \text{ iff } \eta E_0 \nu \ \& \ \eta \restriction A(\eta) = \nu \restriction A(\nu).$$

The rest is as in §2. $\square_{4.1}$

Claim 4.4. *1) In 4.1 we can demand*
(δ) for each $\eta \in {}^{\omega}\lambda$, $\eta / J_{\omega}^{\rm bd}$ is a set of representatives of E.
2) We can weaken in 4.1 assumption (b) to
$(b)'$ $(\aleph_0 + |\tau(\mathfrak{B})|)^{\aleph_0} < \lambda$.
3) If in 4.1 we change clause (γ) in the conclusion to $(\gamma)^*$ below, we can omit clause (b) of the assumption
$(\gamma)^*$ for every $\eta \in {}^{\omega}\lambda$ the set $\langle \eta_{\alpha,n} : \alpha < \lambda \rangle$ is a set of representatives of E with no repetition where $\eta_{\alpha,n} \in {}^{\omega}\lambda$ is: $\eta_{\alpha,n}(\ell) = \alpha$ if $\ell = n$ and $\eta_{\alpha,n}(\ell) = \eta(\ell)$ otherwise.

Proof. 1) We imitate 2.4 only letting $\Xi_\eta = \{\{\langle k, k_1, \dots, k_n, \sigma\rangle : \nu(k) = \sigma(k_1, \dots, k_n)\} : \nu \in {}^\omega\lambda, \nu/J^{\rm bd}_\omega = \eta/J^{\rm bd}_\omega\}$.

2) The same proof.

3) For $\eta \in {}^\omega\lambda$ let $\mathbf{n}(\eta) < \omega$ be the minimal $n \in [\mathbf{n}(\eta), \omega) \Rightarrow c\ell_{\mathfrak{B}}\{\eta(\ell) : \ell \in [n, \omega)\} = c\ell_{\mathfrak{B}}\{\eta(\ell) : \ell \in [\mathbf{n}(\eta), \omega)\}$. Let K be an additive group with universe λ, K_1 a subgroup, $|K_1| = \lambda$, $[K : K_1] = \lambda$ and $\eta E \nu$ iff $\prod_{n<\mathbf{n}(\eta)} \eta(n) = \prod_{n<n(\nu)} \nu(\eta) \bmod K_1$.

$\square_{4.4}$

Remark. We can imitate in §2 the proof of 4.1: use a function $F : {}^\omega\lambda \to \lambda$ such that there is no infinite independent set for the algebra (λ, F); see [EH 71]

Question 4.5. 1) What about having $\sigma \in (\lambda, 2^\lambda)$ equivalence classes?

2) Assume, e.g., λ is strong limit singular and $2^\lambda > \lambda^+$, does λ have the free subset property? (See in [Sh 513]).

§5. On $r_p(\mathrm{Ext}(G, \mathbb{Z})$

Definition 5.1. *For an abelian group G and prime p let $r_p(G)$ be the rank of G/pG as a vector space over $\mathbb{Z}/p\mathbb{Z}$. Let $r_0(G)$ be the rank of $G/Tor(G)$.*

There has been much interest in $\mathrm{Ext}(G, \mathbb{Z})$ for G torsion free abelian group see [EM], and later [MRSh 314]. This group is divisible so the ranks $r_p(G)$ above and $r_0(\mathrm{Ext}(G, \mathbb{Z}))$ determine it up to isomorphism.

Instead using a definition of the abelian group $\mathrm{Ext}(G, \mathbb{Z})$, we quote (see [Fu]) a result which gives a characterization of the cardinal $r_p(\mathrm{Ext}(G, \mathbb{Z}))$ directly from G.

Claim 5.2. *For a torsion free abelian group G and prime p, $r_p(\mathrm{Ext}(G, \mathbb{Z}))$ is the rank of $Hom(G, \mathbb{Z}/pZ)/(Hom(G, \mathbb{Z})/p\mathbb{Z})$ where*

(*a*) *$Hom(G, \mathbb{Z}/p\mathbb{Z})$ is the abelian group of homomorphisms from G to $\mathbb{Z}/p\mathbb{Z}$,*
(*b*) *$Hom(G, \mathbb{Z})/p\mathbb{Z}$ is the abelian group of homomorphism h from G to $\mathbb{Z}/p\mathbb{Z}$ such that for some homomorphism g from G to $\mathbb{Z}$ we have $x \in G \Rightarrow g(x)/p\mathbb{Z} = h(x)$.*

More generally (see [Sh 664, §3] except separating g^*), the point is that asking what can $r_p(\mathrm{Ext}(G, \mathbb{Z}))$ be when G is an abelian group of cardinality λ, we can translate the situation to a λ-system:

Definition 5.3. *1) We say $\mathcal{Y} = (\bar{A}, \bar{K}, \bar{G}, \bar{D})$ is a λ-system <u>if</u>*

(*A*) *$\bar{A} = \langle A_i : i \le \lambda\rangle$ is an increasing sequence of sets, $A = A_\lambda = \cup\{A_i : i < \lambda\}$*
(*B*) *$\bar{K} = \langle K_t : t \in A\rangle$ is a sequence of finite groups*
(*C*) *$\bar{G} = \langle G_i : i \le \lambda\rangle$ is a sequence of groups, $G_i \subseteq \prod_{t\in A_i} K_t$, each G_i is closed (under the Tichonov topology) and $i < j \le \lambda \Rightarrow G_i = \{g \restriction A_i : g \in G_j\}$ and $G_\lambda = \{g \in \prod_{t\in A_\lambda} K_t : (\forall i < \lambda)(g \restriction A_i \in G_i)\}$, that is, G is the inverse limit of $\langle G_i : i < \lambda\rangle$ under the restriction maps*

(D) $\bar{D} = \langle D_\delta : \delta \leq \lambda$ *(a limit ordinal)*$\rangle$, D_δ *an ultrafilter on* δ *such that* $\alpha < \delta \Rightarrow [\alpha, \delta) \in D_\delta$*; the* D_δ*'s are used to choose limits canonically.*

Of course, formally we should write $A_i^{\mathcal{Y}}, K_t^{\mathcal{Y}}, G_\delta^{\mathcal{Y}}, D_\delta^{\mathcal{Y}}, g_i^{\mathcal{Y}}$*, etc., but if clear from the context we shall not write this.*

2) Let $\mathcal{Y}^-$ *be the same omitting* D_λ *and we call it a lean* λ*-system.*

3) We say $\bar{g}^*$ *is a* $\mathcal{Y}$*-candidate if*

$$(E)\bar{g}^* = \langle g_i^* : i < \lambda\rangle,\ g_i^* \in G_\lambda \text{ and } g_i^* \restriction A_i = e_{G_i} = \langle e_{K_t} : t \in A_i\rangle.$$

We can deduce the result of Sageev Shelah [SgSh 148] (if $|G| = \lambda$ is weakly compact ($> \aleph_0$) and p is prime, <u>then</u> $r_p(\mathrm{Ext}(G, \mathbb{Z})) \geq \lambda \Rightarrow r_p(\mathrm{Ext}(G, \mathbb{Z})) = 2^\lambda$). (We later get more.) For this note

Claim 5.4. *1) Assume*

(a) $\mathcal{Y}$ *is a* λ*-system*

(b) $\bar{H} = \langle H_i : i < \lambda\rangle$ *is a sequence of groups,* $\bar{\pi} = \langle \pi_{i,j} : i < j < \lambda\rangle$, $\pi_{i,j} \in \mathrm{Hom}(H_j, H_i)$*, commuting*

(c) $\bar{h} = \langle h_i : i < \lambda\rangle$, $h_i \in \mathrm{Hom}(H_i, G_i^{\mathcal{Y}})$*, and* $i < j < \lambda\ \&\ x \in H_j \Rightarrow (h_j(x)) \restriction A_i = h_i(\pi_{i,j}(x))$

(d) $H_\lambda, \pi_{i,\lambda}$ $(i < \lambda)$ *form the inverse limit of* $\langle H_i, \pi_{i,j} : i < j < \lambda\rangle$*, and* $h = h_\lambda$ *the inverse limit of* $\langle h_i : i < \lambda\rangle$

(e) E_h *is the following 2-place relation on* G_λ : $f_1 E_h f_2 \Leftrightarrow f_1 f_2^{-1} \in \mathrm{Rang}(h)$ *similarly* E_{h_α} *for* $\alpha < \lambda$*.*

<u>*Then*</u>

(α) $h \in \mathrm{Hom}(H_\lambda, G_\lambda)$

(β) *if* $(\forall i < \lambda)(|A_i| \leq \lambda\ \&\ |H_i| \leq \lambda\ \&\ |G_i| \leq \lambda)$*,* <u>*then*</u> E_h *is a* $\Sigma_1^1[\lambda]$*-equivalence relation on* G_λ

(γ) *if* $(\forall i < \lambda)(|A_i| < \lambda\ \&\ |H_i| < \lambda)$ *and* λ *is weakly compact uncountable cardinal,* <u>*then*</u>

  (a) *the 2-place relation* $E = E_h$ *on* G_λ *(from clause (e)) is a very nice equivalence relation*

  (b) *if* $f_1, f_2 \in G_\lambda$ *and* $f_1 f_2^{-1} \notin \mathrm{Rang}(h)$ *then for every* $\alpha < \lambda$ *large enough* $(f_1 \restriction A_i)(f_2 \restriction A_i)^{-1} \notin \mathrm{Rang}(h_i)$ *that is* $\neg(f_1 E_\lambda f_2) \Rightarrow (\forall^* \alpha < \lambda)\neg(f_1 \restriction A_i)$

(δ) *under* (γ)*'s assumptions, if* $[G : \mathrm{Rang}(h)] \geq \lambda$ *then* $[G : \mathrm{Rang}(h)] = 2^\lambda$.

2) If for $\varepsilon < \varepsilon(*) \leq \lambda$ *we have* $\langle H_i^\varepsilon : i < \lambda\rangle, \langle \pi_{i,j}^\varepsilon : i < j < \lambda\rangle, \langle h_i^\varepsilon : i \leq \lambda\rangle$ *are as in (a)–(e) above and* $\otimes$ *below (which follows for* λ *weakly compact) and* $i < \lambda \Rightarrow |H_i| + |A_i| < \lambda$*, and for every* $\alpha < \lambda$ *there are* $f_i^\alpha \in G_\lambda$ *(for* $i < \alpha$*) such that* $\neg(f_i^\alpha E_{h_\lambda^\varepsilon} f_j^\alpha)$ *for* $i < j < \alpha\ \&\ \varepsilon < \varepsilon(*)$*,* <u>*then*</u> *there are* $f_i \in G$ *for* $i < 2^\lambda$ *such that* $i < j < 2^\lambda\ \&\ \varepsilon < \varepsilon^* \Rightarrow \neg(f_i E_{h_\lambda^\varepsilon} f_j)$

$\otimes$ λ *is strong limit and for any* $f, g \in G_\lambda$ *and* $\varepsilon < \varepsilon(*)$ *such that* $fg^{-1} \notin \mathrm{Rang}(h_\alpha^\varepsilon)$ *for some* $\gamma < \lambda$ *we have* $(fg^{-1}) \restriction A_\gamma \notin \mathrm{Rang}(h_\gamma^\varepsilon)$.

Proof. Straightforward. 1) Clause (α): Easy.

Clause (β): By (b) of clause (γ) proved below it is enough (in Definition 0.2) to code E_i for every $i < \lambda$ and as λ is strong limit this is easy.

Clause (γ): The point is that if $f \in G_\lambda \setminus \mathrm{Rang}(h_\lambda)$ then for some $i < \lambda$ we have $\pi_{i,\lambda}(f) \in G_i \setminus \mathrm{Rang}(h_i)$ by the tree property of λ, (one of the equivalent forms of being "weakly compact").

Clause (δ): By part (2).

2) We shall show the proof such that it works for any strong limit except one point where we use weak compactness. For each $i < \lambda$, as λ is strong limit, let μ_α be $(\prod_{t \in A_\alpha} |K_t|)^+$ if λ regular, $\prod_{t \in A_\alpha} |K_t| + \mathrm{cf}(\lambda)$ if λ singular. By the assumption we can find $\langle f_i^\alpha : i < (2^{\mu_\alpha})^+ \rangle$ such that $f_i^\alpha \in G$ and $\varepsilon < \alpha \ \& \ i < j < \mu_\alpha \Rightarrow \neg(f_i^\alpha E_{h_\lambda^\varepsilon} f_j^\alpha)$. By the choice of α without loss of generality $i < (2^{\mu_\alpha})^+ \Rightarrow f_i^\alpha \restriction A_\alpha = f_0^\alpha \restriction A_\alpha$. By the weak compactness (i.e., see clause (γ) of part (1)) for any $i < j < \mu_\alpha$ there is $\gamma_\alpha(i,j) < \lambda$ such that $\varepsilon < \alpha \Rightarrow (f_i^\alpha (f_j^\alpha)^{-1}) \restriction A_{\gamma_\alpha(i,j)} \notin \mathrm{Rang}(h^\varepsilon_{\gamma_\alpha(i,j)})$.

If $\lambda = \mathrm{cf}(\lambda)$ let $\gamma_\alpha^* = \sup\{\gamma_\alpha(i,j) : i < j < (\mu_\alpha)^+\}$. Note if λ is regular then trivially $\gamma^*_\alpha < \lambda$ and if $\lambda > \mathrm{cf}(\lambda)$ by Erdös-Rado theorem without loss of generality $\gamma^*_\alpha = \sup\{\gamma_\alpha(i,j) : i < j < \mu_\alpha^+\} < \lambda$. So for some club E of λ we have $\alpha \in E \Rightarrow \gamma^*_\alpha < \mathrm{Min}(E \setminus (\alpha+1))$. Now for any $\rho \in \prod_{\alpha \in E} \mu^+_\alpha$ we define $\bar g_\rho = \langle g_{\rho,\alpha} : \alpha < \lambda \rangle$ as follows:

$g_{\rho,\alpha} \in G_\lambda$ is $f^\alpha_{\rho(\alpha)}$ if $\alpha \in E$ and is e_{G_λ} if $\alpha \notin E$ and let $f_\rho = f_{\bar g_\rho}$ be defined as in [Sh 664, §3]. Easily (see there)

⊛ $f_\rho \in G_\lambda$ and if $\rho_1 \restriction \alpha = \rho_2 \restriction \alpha, \alpha \in E, \rho_1(\alpha) \neq \rho_2(\alpha)$ and $\beta = \mathrm{Min}(E \setminus (\alpha + 1))$ then $\varepsilon < \varepsilon(*) \Rightarrow (f_{\rho_1} f_{\rho_2}^{-1}) \restriction A_\beta \notin \mathrm{Rang}(h^\varepsilon_\lambda)$.

Easily we can find $B_\alpha \in [\mu^+_\alpha]^{\mu^+_\alpha}$ for $\alpha \in E$ such that:

$(*)$ $\rho_1, \rho_2 \in \prod_{\beta \in E \cap \alpha} \mu^+_\beta$ and $\zeta_1 \neq \zeta_2 \in B_\alpha$ and $\varepsilon < \varepsilon(*)$ and $\beta = \mathrm{Min}(E \setminus (\alpha + 1))$ then
$(f_{\langle g_{\rho_1,\gamma}:\gamma<\alpha\rangle\hat{}\langle f^\alpha_{\alpha,\zeta_1}\rangle})(f_{\langle g_{\rho_2,\gamma}:\gamma<\alpha\rangle\hat{}\langle f^\alpha_{\zeta_1}\rangle})^{-1} \restriction A_\beta \notin \mathrm{Rang}(h_\varepsilon)$.

So restricting ourselves to $\langle f_\rho : \rho \in \prod_{\alpha \in E} B_\alpha \rangle$ we are done, that is, if $\varepsilon < \varepsilon(*)$ and $\rho_1 \neq \rho_2 \in \prod_{\alpha \in E} B_\alpha$ then we can find α such that $\rho_1 \restriction \alpha = \rho_2 \restriction \alpha, \rho_1(\alpha) = \rho_2(\alpha)$, so letting $\beta = \mathrm{Min}(E \setminus (\alpha+1)), \rho'_\ell = \rho_\ell \restriction (\alpha+1)$ for $\ell = 1,2$ we have $f_{\rho_\ell} \restriction A_\beta = f_{\rho'_\ell} \restriction A_\beta$ for $\ell = 1,2$ so $(f_{\rho'_1} \circ f^{-1}_{\rho'_2}) \restriction A_\beta \notin \mathrm{Rang}(h^\varepsilon_\beta)$ hence $(f_{\rho_1} f^{-1}_{\rho_2}) \restriction A_\beta \notin \mathrm{Rang}(h^\varepsilon_\beta)$ hence $f_{\rho_1} f^{-1}_{\rho_2} \notin \mathrm{Rang}(h^\varepsilon_\lambda)$ as required. $\square_{5.4}$

Remark 5.5. We can phrase 5.4(2) forgetting h^ε_λ, etc., using only $E_\varepsilon (\varepsilon < \lambda)$ and $E^\varepsilon_i = \{(f,g) \in G_\lambda \times G_\lambda : (fg^{-1}) \restriction A_i \in \mathrm{Rang}(h^\varepsilon_i)\}$.

Claim 5.6. *Assume*

$(A)(a)$ *λ is a strong limit cardinal and θ is a compact cardinal $< \lambda$*

(b) *K_i is a group for $i < \lambda$*

(c) *I is a directed partial order, $t \in I \Rightarrow A(t) \subseteq \lambda$ and $\bigcup_{t \in I} A(t) = \lambda$*

(d) *for $t \in I$, G_t is a subgroup of $\Pi\{K_i : i \in A(t)\}$*

(e) *for $s \le t$ from I we have $A(s) \subseteq A(t)$ and $f \in G_t \Rightarrow f \restriction A(s) \in G_s$*

(f) *G_∞ is the inverse limit of the G_t's, i.e, $\{f \in \prod_{i<\lambda} K_i : f \restriction A_t \in G_t$ for every $t \in I\}$*

$(B)(a)$ $\varepsilon(*) \le \lambda$

(b) *for* $\varepsilon < \varepsilon(*)$, $\langle H^\varepsilon_u, \pi^\varepsilon_{u,w} : u \le w$ *from* $I\rangle$ *is an inversely directed system of groups*

(c) $h^\varepsilon_u \in \mathrm{Hom}(H^\varepsilon_u, G_u)$ *for* $u \in I$, $\varepsilon < \varepsilon(*)$ *and* $x \in H^\varepsilon_\omega$ & $u \le w \Rightarrow h^\varepsilon_w(x) \restriction A(u)$, $h^\varepsilon_u(\pi^\varepsilon_{u,w}(x))$

(d) $H^\varepsilon_\infty, h^\varepsilon, h^\varepsilon_{\infty,u}$ *are the limit of the inverse system*

(e) E_ε *is the equivalence relation on* $G_\infty : f E_\varepsilon g \Leftrightarrow fg^{-1} \in \mathrm{Rang}(h^\varepsilon_\infty)$

(C) *for every* $\mu < \lambda$ *we can find* $\langle f^\mu_\alpha : \alpha < \mu\rangle$ *from* G_∞ *such that* $\varepsilon < \mu \cap \varepsilon(*)$ & $\alpha < \beta \Rightarrow \neg(f^\mu_\alpha E_\varepsilon f^\mu_\beta)$

(D) θ *is* $> \sup\limits_{i<\lambda} |K_i| + \sup\limits_{t\in I} |A(t)|$ *and also* $\sup_{t\in I, \varepsilon<\varepsilon(*)} |H^\varepsilon_t|$.

<u>Then</u> there are $f_\alpha \in G$ *for* $\alpha < 2^\lambda$ *such that* $\varepsilon < \varepsilon(*)$ & $\alpha < \beta < 2^\lambda \Rightarrow \neg(f_\alpha E_\varepsilon f_\beta)$.

Proof. Let $\kappa = \mathrm{cf}(\lambda)$, $\langle \lambda_i : i < \kappa\rangle$ be increasing with limit λ. We can choose by induction on $i < \lambda$, I_i, A_i such that

(α) $A_i \subseteq \lambda$, $|A_i| \le \theta + |i|$ and $j < i \Rightarrow A_j \subseteq A_i$, $\lambda_i \subseteq A_i$

(β) $I_i \subseteq I$ is directed, $|I_i| \le \theta + |i|$ and $j < i \Rightarrow I_j \subseteq I_i$ and $t \in I_i \Rightarrow A(t) \subseteq A_i$

(γ) if we restrict ourselves to A_i, I_i, there is a sequence $\langle f^i_\alpha : \alpha < \lambda_i\rangle$, such that $f^i_\alpha \in G^{I_i}_\infty = \mathrm{Lim}_{I_i}\langle G_u, f_{u,w} : u \le w$ from $I_i\rangle$ and $\varepsilon < \varepsilon(*)$ & $\alpha < \lambda_i \Rightarrow \neg(f^i_\alpha E^{I_i}_\varepsilon f^i_\beta)$ and $\gamma \in \bigcup\limits_{j<i} A_j \cap B_i$, $f^i_\alpha(\gamma) = e_{K_\gamma}$.

This is straightforward (see the proof of 5.9, first case). We can extend f^i_α to ${}^*f^i_\alpha \in G_\infty$ such that $i \in \lambda\backslash B_i \Rightarrow {}^*f^i_\alpha(i) = e_{K_i}$. Now we can apply the proof of 5.4. $\square_{5.6}$

Claim 5.7. *1) Assume*

(a) $\lambda > \mathrm{cf}(\lambda) = \kappa$, *and* κ *is a measurable cardinal, say* D *a normal ultrafilter on* κ

(b) *G is a torsion free abelian group*

(c) $|G| = \lambda$

(d) *p is a prime number.*

If $r_p(Ext(G, \mathbb{Z})) \ge \lambda$ *and* $\lambda = \lambda^{<\kappa}$ <u>*then*</u> $r_p(Ext(G, \mathbb{Z})) \ge \lambda^\kappa$.

2) Assume

(a) *of part (1)*

(b) $\langle G_i : i \le \kappa\rangle$ *is an increasing continuous sequence of torsion free abelian group*

(c) $\mu_i = r_p(Ext(G_i, \mathbb{Z}))$ *for* $i \le \kappa$.

<u>*Then*</u>

(α) *if* $f \in \mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ *but* $f \notin \mathrm{Hom}(G, \mathbb{Z})/p\mathbb{Z}$ *then for some* $i < \kappa$, $f \restriction G_i \in \mathrm{Hom}(G_i, \mathbb{Z}/p\mathbb{Z})$, $f \notin \mathrm{Hom}(G, \mathbb{Z})/p\mathbb{Z}$

(β) $\mu_\kappa \le \prod_{i<\kappa} \mu_i$.

Proof. 1) Let $\lambda = \sum\{\lambda_i : i < \kappa\}$, $i < j \Rightarrow \lambda_i < \lambda_j$. Let $\langle G_i : i < \kappa\rangle$ be an increasing sequence of pure subgroups of G with union G satisfying $i < \kappa \Rightarrow |G_i| = \lambda_i$. Now

(∗) if $g \in \operatorname{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ and $i < \kappa \Rightarrow g \restriction G_i \in \operatorname{Hom}(G_i, \mathbb{Z})/p\mathbb{Z}$ <u>then</u> $g \in \operatorname{Hom}(G, \mathbb{Z})/p\mathbb{Z}$.
[Why? Let $g \restriction G_i = h_i/p\mathbb{Z}$ where $h_i \in \operatorname{Hom}(G, \mathbb{Z})$ and let h a function from G to $\mathbb{Z}$ be defined as $h(x) = n \Leftrightarrow \{i < \kappa : h_i(x) = n\} \in D$. Clearly $h \in \operatorname{Hom}(G, \mathbb{Z})$ and $g = h/p\mathbb{Z}$, as required.]

The result follows by 5.4(2).

2) Similar. $\square_{5.7}$

A complimentary claim is

Claim 5.8. *Assume that $\langle G_i : i \leq \kappa\rangle$ is a purely increasing sequence of torsion free abelian groups, $\kappa = \operatorname{cf}(\kappa)$ for notational simplicity. $r'_p(G_i) = r_p(\operatorname{Ext}(G, \mathbb{Z}))$*

1) If $\langle r'_p(G_i) : i < \kappa\rangle$ is not eventually constant then for some closed unbounded set $C \subseteq \kappa$ we have

(a) $\langle r'_p(G_i) : i \in C\rangle$ is strictly increasing
(b) there are $\langle f^i_\alpha : i \in C, \alpha < r'_p(G_i)\rangle$ such that
  (α) $f'_\alpha \in \operatorname{Hom}(G_\kappa, \mathbb{Z}/p\mathbb{Z})$
  (β) $f^i_\alpha \restriction G_i$ is constantly zero (of the abelian group $\mathbb{Z}/p\mathbb{Z}$)
  (γ) if $i \in C$, $j = \operatorname{Min}(C\backslash(i+1))$ and $\alpha < \beta < r'_p(G_i)$ then $(f^i_\alpha - f^i_\beta) \restriction G_j \notin (\operatorname{Hom}(G, \mathbb{Z})/p\mathbb{Z})$; moreover, $\langle f^i_\alpha \restriction G_i) + (\operatorname{Hom}(G_j\mathbb{Z})/p\mathbb{Z} : \alpha < r'_p(G_i)\rangle$ is independent.

2) If $C \subseteq \kappa = \sup(C)$ and the sequence $\langle f^i_\alpha : i \in C, \alpha < \mu_i\rangle$ is as above <u>then</u> $r'_p(G_\kappa) \geq \prod_i \mu_i$.

Proof. Straight.

Conclusion 5.9. If

(a) λ is a strong limit cardinal and such that $(\alpha) \vee (\beta)$ where
  (α) λ is above some compact cardinal
  (β) $\operatorname{cf}(\lambda)$ is a measurable cardinal
(b) G is a torsion free abelian group and p is a prime.

<u>Then</u> $r_p(\operatorname{Ext}(G, \mathbb{Z})) \geq \lambda \Rightarrow r_p(\operatorname{Ext}(G, \mathbb{Z})) = 2^\lambda$.

Proof.
First Case. Let $\theta < \lambda$ be a compact cardinal.

For any $\mu < \lambda$ we can find a sequence $\langle f_i : i < \mu\rangle$ of members of Hom $(G, \mathbb{Z}/p\mathbb{Z})$ such that $i < j \Rightarrow f_j - f_i \in \{h/p\mathbb{Z} : h \in \operatorname{Hom}(G, \mathbb{Z})\}$. As θ is compact for $i < j < \mu$ we can find a pure subgroup $G_{i,j}$ of G of cardinality $< \theta$ such that $f_j \restriction G_{i,j} = f_i \restriction G_{i,j} \notin \{h/p\mathbb{Z} : h \in \operatorname{Hom}(G_{i,j}, \mathbb{Z})\}$.

Let G_μ be a pure subgroup of G of cardinality $\leq \mu + \theta$ which includes $\cup\{G_{i,j} : i < j < \mu\}$. So $r_p(\operatorname{Ext}(G_\mu, \mathbb{Z})) \geq \mu$. By 5.4(2) we are done.

Second Case. Should be clear by the two previous claims. $\square_{5.9}$

Remark 5.10. 1) So for λ strong limit singular the problem of the existence of G such that $|G| = \lambda$, $r_p(\mathrm{Ext}(G, \mathbb{Z})) = \lambda$ is not similar to the problem of the existence of M such that $\|M\| = \lambda$, $\mathrm{nu}(M) = \lambda$ where $\mathrm{nu}(M) = \{N/\cong : N$ is a model of cardinality $\|M\|$, $\mathbb{L}_{\infty,\lambda}$-equivalent to $M\}$.

That is, we know (in ZFC) that for λ strong limit singular of uncountable cofinality, for some model M of cardinality λ we have $\mathrm{nu}(M) = \lambda$ (see Shelah and Vaisanen [ShVs 644] and history there). Now 5.9 is a strong negation of the parallel of this result for $r_p(\mathrm{Ext}(G, \mathbb{Z}))$.

2) There has been much effort to characterize the class $\{\mathrm{Ext}(G, \mathbb{Z}) : G$ a torsion free abelian group$\}$ of abelian groups under the assumption $\mathbf{V} = \mathbf{L}$ (see [MRSh 314] and references there). We note another possible characterization (in a different model of ZFC).

Claim 5.11. *Assume κ is supercompact, $(\forall\mu)(\mu \geq \kappa \rightarrow 2^{\mu} < 2^{\mu^+})$ and $\mathbb{Q}$ is the forcing of adding κ Cohen reals. Then in $\mathbf{V}^{\mathbb{Q}}$ we have*

$\boxtimes_1$ *if G is a torsion free abelian group, p a prime and $r_p(\mathrm{Ext}(G, \mathbb{Z})) > 0$ then for some (pure) subgroup G' of G of cardinality $< 2^{\aleph_0}$ we have $r_p(Ext(G', \mathbb{Z})) > 0$*

$\boxtimes_2$ *if G is a torsion free abelian group, then $r_p(\mathrm{Ext}(G, \mathbb{Z}))$, if not finite, has the form 2^{μ}*

$\boxtimes_3$ *in (2) $r_p(\mathrm{Ext}(G, \mathbb{Z})) = 2^{fr-rk_{[p]}(G)}$, see below.*

Definition 5.12. *For a prime p.*

1) Let $\mathbf{K}_p = \{G : G$ is a torsion free abelian group such that even if we add $|G|^+$ Cohen reals still $r_p(Ext(G, \mathbb{Z})) = 0\}$.

2) For a torsion free abelian group G let

$$fr\text{-}rk_{[p]}(G) = Min\{rk(G') : G' \text{ is a pure subgroup of } G \text{ and } G/G' \in \mathbf{K}_p\}.$$

Proof. Essentially by [MkSh 418].

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