Math. Log. Quart. **53**, No. 6, 636 – 641 (2007) / **DOI** 10.1002/malq.200610053

Two cardinal models for singular μ

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Received 9 July 2006, revised 20 July 2007, accepted 23 July 2007
Published online 1 October 2007

Key words Set theory, pcf theory, colorings, identities.
MSC (2000) 03E05

We deal here with colorings of the pair (μ^+, μ), when μ is a strong limit and singular cardinal. We show that there exists a coloring c with no refinement. It follows that the properties of colorings of (μ^+, μ) when μ is singular differ in an essential way from the case of regular μ (although the identities may be the same).

1 Introduction

Identities (or identifications) were first defined by Shelah in the late 60's. The purpose was dual. On the one hand, they may be used as a tool for solving problems in model theory. On the other hand, there is interest in them within the realm of set theory.

The basic connection between identities and questions of model theory (especially the compactness question of various pairs of cardinals) or mathematical logic (like the subject of generalized quantifiers) is formulated in [1]. It is used in a much more sophisticated context, in [3]. But here, we are interested in pure set theoretical considerations.

Shelah proved, in the first part of [4] (i. e., Sections 0 and 1), that the set of identities $\mathrm{ID}_2(\mu^+, \mu)$ has the property of 2-simplicity in the case of a regular cardinal μ such that $\mu = \mu^{<\mu}$. A natural example is the pair $(\aleph_1, \aleph_0)$.

Now one may ask whether the assumption on μ is necessary. We shall prove here that it can hardly be avoided. We will take a singular μ such that $2^{<\mu} = \mu$. Even under that assumption, we will see that there is $c : [\mu^+]^n \longrightarrow \mu$ which is not computable from any coloring $d : [\mu^+]^m \longrightarrow \mu$ when $m < n$.

Let us describe now the structure of this article. In Section 1, we give some definitions and basic facts about identities. In Section 2, we state the main theorem and establish some preliminary results used in its proof. In Section 3, we prove the main theorem, using methods of pcf theory. Our proof will be independent of the value of 2^μ.

Let us try to explain the idea. Assume $\kappa = \mathrm{cf}(\mu) < \mu$. Let $\langle \lambda_i : i < \kappa \rangle$ be an increasing sequence of regular cardinals with limit μ. Let $J = J_\kappa^{\mathrm{bd}}$ be the ideal of all the bounded subsets of κ. We use the assumption that

$$\mathrm{tcf}(\textstyle\prod_{i<\kappa} \lambda_i, J) = \mu^+$$

to prove our main theorem. The fact that there exists $\bar{\lambda} = \langle \lambda_i : i < \kappa \rangle$ such that $\mathrm{tcf}(\prod_{i<\kappa} \lambda_i, J) = \mu^+$ is a theorem of ZFC.

That brings us to a philosophical question about the meaning of analyzing the magnitude of 2^μ. It is clear that 2^μ can be manipulated by forcing. What do we do about this? In fact, several answers are possible. Pcf theory suggests that asking about the size of 2^μ is sometimes the wrong question.

Instead of looking at the value of 2^μ, about which there is a vast variety of consistency results, we should ask the right questions about the cardinality of products of cardinals, divided by an ideal. Section 3 here exemplifies the philosophical idea very well.

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2 Definitions

The basic notion that we need is identity:

Definition 2.1

(a) A *partial identity* $\boldsymbol{s}$ is a pair $(a, e) = (\mathrm{Dom}_{\boldsymbol{s}}, e_{\boldsymbol{s}})$. a is a finite set, and e is an equivalence relation on a subfamily of the subsets of a. We always require that e respects the cardinality of the subsets, i. e., $bec \Rightarrow |b| = |c|$.

(b) A *full identity* is an identity $\boldsymbol{s} = (a, e)$, where $\mathrm{Dom}(e) = \mathcal{P}(a)$. We might say just "identity", instead of full identity.

One may wonder, why do we distinguish between full identities and partial identities? Well, in many cases we are interested in colorings of the type $c : [\lambda]^n \longrightarrow \mu$ when n is constant. Analyzing those colorings helps us to understand identities with e defined only on subsets of a with cardinality n. Those are partial identities, of course.

Definition 2.2 Let (a, e) be an identity (or a partial identity). Then we say that $\lambda \rightarrow (a, e)_\mu$ if for every function $f : [\lambda]^{<\aleph_0} \longrightarrow \mu$ there is a one-to-one mapping $h : a \longrightarrow \lambda$ such that $bec \Rightarrow f(h"b) = f(h"c)$.

Notice that the requirement of $\lambda \rightarrow (a, e)_\mu$ relates to every function f. So the next definition, which depends only on the pair (λ, μ), makes sense:

Definition 2.3 $\mathrm{ID}(\lambda, \mu) := \{(a, e) : (a, e) \text{ is an identity, and } \lambda \rightarrow (a, e)_\mu\}$.

But we might be interested also in the identities of a specific function f:

Definition 2.4 Let $f : [\lambda]^{<\aleph_0} \longrightarrow \mu$ be a function.

$$\begin{aligned}\mathrm{ID}(f) := \{(a, e) : \ & (a, e) \text{ is an identity, and there exists a one-to-one mapping } h : a \longrightarrow \lambda \\ & \text{such that } bec \Rightarrow f(h"b) = f(h"c)\}.\end{aligned}$$

Notice that $\mathrm{ID}(\lambda, \mu) = \bigcap\{\mathrm{ID}(f) : f \text{ is a function from } [\lambda]^{<\aleph_0} \text{ into } \mu\}$.

One of the basic tools for investigating identities is the notion of refinement. The idea is to compute the values of a coloring $c : [\lambda]^n \longrightarrow \mu$, with a coloring $d : [\lambda]^m \longrightarrow \mu$, when $m < n$.

Definition 2.5 Let $m < n < \omega$, (λ, μ) a pair of infinite cardinals. Let $c : [\lambda]^n \longrightarrow \mu$ and $d : [\lambda]^m \longrightarrow \mu$ be colorings. We say that d *refines* c if for any $\alpha_0, \dots, \alpha_{n-1} < \lambda$ with no repetitions, and any $\beta_0, \dots, \beta_{n-1} < \lambda$ with no repetitions, the condition $(*)$ is satisfied. This means:

$(*)$ Suppose for every $u \in [n]^m$, it is true that $d(\{\alpha_\ell : \ell \in u\}) = d(\{\beta_\ell : \ell \in u\})$. Then

$$c(\{\alpha_0, \dots, \alpha_{n-1}\}) = c(\{\beta_0, \dots, \beta_{n-1}\}).$$

3 The main theorem

Let μ be a singular cardinal, $\mu = 2^{<\mu}$, so μ is strong limit. We deal, in this section, with the pair (μ^+, μ). We state now the main theorem, proving it in the next section.

Main Theorem 3.1 *Assume:*

(a) *μ is a singular cardinal.*

(b) $2^{<\mu} = \mu$.

(c) $n \in [2, \omega)$.

Then there is a coloring $c : [\mu^+]^{n+1} \longrightarrow \mu$ such that no $d : [\mu^+]^n \longrightarrow \mu$ is a refinement for c.

Before beginning the proof, we recall the parallel situation for a regular μ. If $\mu = \mu^{<\mu}$, and $c : [\mu^+]^{<\aleph_0} \longrightarrow \mu$ is a coloring, then there is $d : [\mu^+]^2 \longrightarrow \mu$ which is a refinement of c. We do not need the assumption of order on the ordinals in the domain of c.

That theorem is the main claim in [4, Section 1]. It follows, quite immediately, that $\mathrm{ID}_2(\mu^+, \mu)$ is 2-simple (those notions are defined there). So here we show that colorings of (μ^+, μ), when μ is singular, behave much differently.

Let us go back to the claim. We shall start with a general lemma, which asserts the existence of a bounding function under some reasonable assumptions.

Lemma 3.2 *Let μ be a singular strong limit cardinal, and $n \in [2,\omega]$. Then we can find $\theta_n < \mu$ and a function $g_n : [\theta_n]^n \longrightarrow \mathrm{cf}(\mu)$ such that for every $f : [\theta_n]^{n-1} \longrightarrow \mathrm{cf}(\mu)$ there exists $u_f \in [\theta_n]^n$ such that*

$$v \in [u_f]^{n-1} \Rightarrow f(v) < g_n(u_f).$$

Proof. Let $\kappa = \mathrm{cf}(\mu)$, $\theta_2 = \kappa^+$, and $\theta_{n+1} = \beth_{n-1}(\kappa^+)$ for every $n \in [2,\omega)$.

We prove this result by induction on n. We separate the proof into two cases. In the first case $n = 2$, and then we build directly the desired g_2, using the fact that $\kappa^+ > \kappa$. In the second case we consider $n > 2$, and we use an induction hypothesis.

Case 1: $n = 2$.

So we need $g_2 : [\kappa^+]^2 \longrightarrow \kappa$, which dominates any $f : \kappa^+ \longrightarrow \kappa$. For every $\alpha < \kappa^+$, let $h_\alpha : \alpha \longrightarrow \kappa$ be a one-to-one mapping. Define for every $\alpha < \beta < \kappa^+$ $(= \theta_2)$ the following function:

$$g_2(\{\alpha,\beta\}) = h_\beta(\alpha).$$

Let us try to show that g_2 is as required. Assume that f is a function from κ^+ into κ. By the pigeon hole principle, we can choose $\gamma < \kappa$ such that $S := \{\alpha < \kappa^+ \,:\, f(\alpha) = \gamma\}$ is of cardinality κ^+. We choose also an ordinal $\beta_* \in S$ such that $|S \cap \beta_*| = \kappa$.

Notice that

$$|\{\{\alpha,\beta_*\} \,:\, g_2(\{\alpha,\beta_*\}) \le \gamma\}| = |\{\{\alpha,\beta_*\} \,:\, h_{\beta_*}(\alpha) \le \gamma\}| < \kappa,$$

since $\gamma < \kappa$, β_* is constant, and h_{β_*} is one-to-one. But $|S \cap \beta_*| = \kappa$, hence one may choose $\alpha_* \in S \cap \beta_*$ such that $g_2(\{\alpha_*,\beta_*\}) > \gamma$.

On the other hand, $f(\alpha_*) = f(\beta_*) = \gamma$ (since both α_* and β_* were taken from S). Define $u_f = \{\alpha_*,\beta_*\}$, and we are done.

Case 2: $n > 2$.

By the induction hypothesis, θ_ℓ and $g_\ell : [\theta_\ell]^\ell \longrightarrow \kappa$ satisfy the lemma for $\ell = n-1$.

Let $\langle f'_\alpha \,:\, \alpha \in [\theta_\ell,\theta_n)\rangle$ enumerate all the functions from $[\theta_n]^{n-1}$ into κ. Define $g_n : [\theta_n]^n \longrightarrow \kappa$ as follows. If $\alpha_0,\ldots,\alpha_{n-2} < \theta_\ell \le \alpha_{n-1} < \theta_n$, then let $g_n(\{\alpha_0,\ldots,\alpha_{n-1}\})$ be

$$\max\{f'_{\alpha_{n-1}}(\{\alpha_0,\ldots,\alpha_{n-2}\}) + 1, g_\ell(\{\alpha_0,\ldots,\alpha_{n-2}\})\}.$$

In any other case, let g_n be zero.

We will show that (g_n,θ_n) satisfies the claim. For this, assume f is a function from $[\theta_n]^{n-1}$ into κ. Clearly, $f \restriction [\theta_\ell]^{n-1}$ appears in the enumeration above. Let $\alpha_* \in [\theta_\ell,\theta_n)$ be an ordinal such that $f \restriction [\theta_\ell]^{n-1} \equiv f'_{\alpha_*}$. Define $f^- : [\theta_\ell]^{\ell-1} \longrightarrow \kappa$ as follows:

$$(\forall v \in [\theta_\ell]^{\ell-1})(f^-(v) = f(v \cup \{\alpha_*\})).$$

By the induction hypothesis, there is $u_{f^-} = \{\alpha_0,\ldots,\alpha_{\ell-1}\}$ as required for ℓ and g_ℓ, i. e., if $v_m = u_{f^-} \setminus \{\alpha_m\}$ for every $m \le \ell-1$, then $f^-(v_m) < g_\ell(u_{f^-})$. At last, we can define $u_f := u_{f^-} \cup \{\alpha_*\}$. We claim that u_f is as required:

$(*)_1$ $m \le \ell-1 \Rightarrow f(v_m \cup \{\alpha_*\}) = f^-(v_m) < g_\ell(u_{f^-}) \le g_n(u_{f^-} \cup \{\alpha_*\}) = g_n(u_f)$.

$(*)_2$ $f(u_{f^-}) = f'_{\alpha_*}(u_{f^-}) < g_n(u_{f^-} \cup \{\alpha_*\}) = g_n(u_f)$.

So, again, we are done. □

Moving back to the the main theorem, we try to create a coloring c with no refinement. It is, somehow, more convenient to work with functions that encode the information that the refinement captures, instead of dealing with the refinement itself. That is the idea behind the next lemma.

Lemma 3.3 *Let μ be an infinite cardinal, $c : [\mu^+]^{n+1} \longrightarrow \mu$ a coloring, and $d : [\mu^+]^n \longrightarrow \mu$ a refinement of c. One can find $F : [\mu^+]^{n+1} \longrightarrow \mu$ such that if $\alpha_0,\ldots,\alpha_n < \mu$ with no repetitions, and for $0 \le \ell \le n$ we write*

$$d(\{\alpha_0,\ldots,\alpha_n\} \setminus \{\alpha_\ell\}) = \gamma_\ell < \mu,$$

then $F(\gamma_0,\ldots,\gamma_n) = c(\{\alpha_0,\ldots,\alpha_n\})$.

Proof. Let E be the equivalence relation that is determined by c, i. e.,

$$\{\alpha_0, \dots, \alpha_n\} E \{\beta_0, \dots, \beta_n\} \quad \text{iff} \quad c(\{\alpha_0, \dots, \alpha_n\}) = c(\{\beta_0, \dots, \beta_n\}).$$

For any equivalence class of E, choose a representative. If $\{\alpha_0, \dots, \alpha_n\} \in [\mu^+]^{n+1}$, define

$$\gamma_\ell^\alpha = d(\{\alpha_0^*, \dots, \alpha_n^*\} \setminus \{\alpha_\ell^*\})$$

when $\{\alpha_0^*, \dots, \alpha_n^*\}$ is the representative of the equivalence class $\{\alpha_0, \dots, \alpha_n\}/E$.

Define $F(\gamma_0^\alpha, \dots, \gamma_n^\alpha) = c(\{\alpha_0, \dots, \alpha_n\})$ whenever $\{\alpha_0, \dots, \alpha_n\} \in [\mu^+]^{n+1}$. For every other $(n+1)$-tuple in $[\mu^+]^{n+1}$, define F to be zero. One can verify easily that F is well-defined and satisfies the requirements above, because of the assumption that d refines c.

[Let us explain more thoroughly why F is a well-defined function from $[\mu^+]^{n+1}$ into μ. Assume $\langle \gamma_0, \dots, \gamma_n \rangle$ belongs to $[\mu^+]^{n+1}$. Choose a representative for every equivalence class of E. We split the definition into two cases.

In the first case, there is no representative of the form $\{\alpha_0^*, \dots, \alpha_n^*\}$ such that $d(\{\alpha_0^*, \dots, \alpha_n^*\} \setminus \{\alpha_\ell^*\}) = \gamma_\ell$ for every $0 \le \ell \le n$. In that case, we have defined $F(\gamma_0, \dots, \gamma_n) = 0$. Notice that a different choice of the representatives would not change this fact, so F is well-defined in that case.

In the other case, there is a representative $\{\alpha_0^*, \dots, \alpha_n^*\}$ with $d(\{\alpha_0^*, \dots, \alpha_n^*\} \setminus \{\alpha_\ell^*\}) = \gamma_\ell$ for any $0 \le \ell \le n$.

We show that the definition of F does not depend on the way that we choose the representatives. Suppose that we choose $\{\beta_0^*, \dots, \beta_n^*\}$ as a representative of the same equivalence class, and $d(\{\beta_0^*, \dots, \beta_n^*\} \setminus \{\beta_\ell^*\}) = \gamma_\ell$ for every $0 \le \ell \le n$. It means that for every $u \in [n+1]^n$ we have $d(\{\alpha_\ell^* : \ell \in u\}) = d(\{\beta_\ell^* : \ell \in u\})$. By Definition 2.5, we must infer that $c(\{\alpha_0^*, \dots, \alpha_n^*\}) = c(\{\beta_0^*, \dots, \beta_n^*\})$. This fact enables us to define

$$F(\gamma_0, \dots, \gamma_n) = c(\{\alpha_0^*, \dots, \alpha_n^*\})$$

without any problem of ambiguity. So F is well-defined also in that case.] □

Remark 3.4 The $(n+1)$-tuples in the domain of F might be with repetitions. So we write $F(\gamma_0, \dots, \gamma_n)$ and not $F(\{\gamma_0, \dots, \gamma_n\})$. We observe also that F is symmetric, i. e., its value does not depend on the order of the ordinals in the $(n+1)$-tuple.

4 The pcf advantage

Theorem 4.1 *Assume:*

(a) *μ is a singular cardinal.*

(b) $2^{<\mu} = \mu$.

(c) $m \in [2, \omega)$.

Then there exists $c : [\mu^+]^{m+1} \longrightarrow \mu$ such that no $d : [\mu^+]^m \longrightarrow \mu$ refines it.

Proof. Denote $\kappa = \mathrm{cf}(\mu) < \mu$, and $\theta = \theta_m = \beth_{m-2}(\kappa^+)$. Let $J = J_\kappa^{\mathrm{bd}}$, the ideal of bounded subsets of κ. By [2, Chapter II, Main Claim 2.3], one can choose an increasing sequence of regular cardinals $\langle \lambda_i : i < \kappa \rangle$, $\theta < \lambda_0$, and $\mu = \bigcup_{i<\kappa} \lambda_i$ such that $\mu^+ = \mathrm{tcf}(\prod_{i<\kappa} \lambda_i, J)$.

Let $\langle g_\alpha^* : \alpha < \mu^+ \rangle$ exemplify it. We may assume that the sequence of the g_α^*'s is strictly increasing. We are going to define a coloring with no refinement, using the g_α^*'s. But we need some other functions.

$(*)_0$ Let $\bar{f}^\theta = \langle f_\alpha^\theta : \alpha < \mu^+ \rangle$ be a sequence of functions such that:

(i) $f_\alpha^\theta : [\theta]^m \longrightarrow \kappa$, for any $\alpha < \mu^+$.

(ii) For every $f : [\theta]^m \longrightarrow \kappa$, we have:

$$\mu^+ = \sup\{\alpha : f_\alpha^\theta = f\}.$$

(The meaning of (ii) is that every f_α^θ appears μ^+ times in the sequence. It enables us to pick a specific function from a high enough level in the sequence.)

$(*)_1$ Let $h : [\theta]^m \longrightarrow \kappa$ be a dominating function, as given in Lemma 3.2, i. e., for every $g : [\theta]^{m-1} \longrightarrow \kappa$, there exists $v_g \in [\theta]^m$ such that

$$(\forall \gamma \in v_g)[g(v_g \setminus \{\gamma\}) < h(v_g)].$$

Now denote $n = m + 1$, and define $c : [\mu^+]^n \longrightarrow \mu$ as follows:

(i) If $v \in [\theta]^m$ and $\alpha \in [\theta, \mu^+)$, then

$$c(v \cup \{\alpha\}) := g^*_\alpha(\max\{h(v), f^\theta_\alpha(v)\} + 1) + 1.$$

(ii) For any $u \in [\mu^+]^n$ that does not fall in (i), define $c(u) = 0$.

Assume towards a contradiction that $d : [\mu^+]^m \longrightarrow \mu$ refines c. By Lemma 3.3, there is $F : [\mu^+]^n \longrightarrow \mu$ which computes c from the values of d. We will reach the desired contradiction using F. We need some more functions:

$(*)_2$ For every $j < \kappa$ and any $\alpha < \mu^+$, we define $f^*_{\alpha,j} : [\theta]^{m-1} \longrightarrow \kappa$ as follows:

$$f^*_{\alpha,j}(v) = \min\{i < \kappa : i > j \text{ and } \lambda_i > d(v \cup \{\alpha\})\}.$$

$(*)_3$ Let $f^{**} : [\theta]^m \longrightarrow \kappa$ be defined by:

$$f^{**}(v) = \min\{i < \kappa : d(v) < \lambda_i\}.$$

We add also some functions of a different form:

$(*)_4$ Define $g' \in \prod_{i<\kappa} \lambda_i$ by

$$g'(k) = \sup\{\lambda_k \cap \operatorname{Rang}(d \restriction [\theta]^m)\} \cup \bigcup_{j<k} \lambda_j.$$

$(*)_5$ Define $g'' \in \prod_{i<\kappa} \lambda_i$ by

$$g''(k) = g'(k) \cup \sup\{\lambda_k \cap \operatorname{Rang}(F \restriction [g'(k)]^n)\}.$$

Everything is ready now. Since $g'' \in \prod_{i<\kappa} \lambda_i$, we can pick an ordinal $\alpha_0 < \mu^+$ such that $g'' <_J g^*_{\alpha_0}$. By $(*)_0$, we can choose an ordinal $\alpha_1 \in [\theta, \mu^+)$, $\alpha_0 < \alpha_1 < \mu^+$, such that $f^\theta_{\alpha_1} \equiv f^{**}$. Clearly, $g'' <_J g^*_{\alpha_1}$, so by the nature of the ideal J, there exists $j(*) < \kappa$ such that

$$g'' \restriction [j(*), \kappa) < g^*_{\alpha_1} \restriction [j(*), \kappa).$$

Choose $v_* \in [\theta]^m$ such that for every $\gamma \in v_*$ it is true that $f^*_{\alpha_1,j(*)}(v_* \setminus \{\gamma\}) < h(v_*)$ (v_* exists, by $(*)_1$).

From the definition of $f^*_{\alpha_1,j(*)}$, it follows that

$\odot_0$ $$\gamma \in v_* \Rightarrow d((v_* \setminus \{\gamma\}) \cup \{\alpha_1\}) < \lambda_{f^*_{\alpha_1,j(*)}(v_* \setminus \{\gamma\})} < \lambda_{h(v_*)}.$$

Let $i(*) = \max\{h(v_*), f^\theta_{\alpha_1}(v_*)\}$. By the definition of the f^*'s,

$$\gamma \in v_* \Rightarrow j(*) < f^*_{\alpha_1,j(*)}(v_* \setminus \{\gamma\}),$$

and since $f^*_{\alpha_1,j(*)}(v_* \setminus \{\gamma\}) < h(v_*)$, we know that $j(*) < h(v_*)$. So $j(*) < i(*)$. We need this for bounding the values of the coloring d, because $\odot_0$ implies now that

$$\gamma \in v_* \Rightarrow d((v_* \setminus \{\gamma\}) \cup \{\alpha_1\}) < \lambda_{h(v_*)} \le \lambda_{i(*)}.$$

This fact tells us what happens if we drop one ordinal from v_*, adding α_1 instead. We also know what happens if we omit α_1 and keep v_*:

$$d(v_* \cup \{\alpha_1\} \setminus \{\alpha_1\}) = d(v_*) < \lambda_{f^{**}(v_*)} = \lambda_{f^\theta_{\alpha_1}(v_*)} \le \lambda_{i(*)}.$$

This follows from the definition of f^{**} in $(*)_3$, and the choice of α_1, which implies that $f^\theta_{\alpha_1} \equiv f^{**}$.

We can finish the proof now as follows. Define

$$W := \{F(\zeta_0, \ldots, \zeta_{n-1}) : \zeta_0 < \cdots < \zeta_{n-1} < \lambda_{i(*)}\},$$
$$W^+ := \{F(\zeta_0, \ldots, \zeta_{n-1}) : \zeta_0 < \cdots < \zeta_{n-1} < g'(i(*) + 1)\},$$

and get $W \subseteq W^+$ and also $W^+ \cap \lambda_{i(*)+1} \subseteq g''(i(*) + 1)$ (by $(*)_4$ and $(*)_5$).

By virtue of F''s definition, we have $c(v_* \cup \{\alpha_1\}) \in W^+$. On the other hand, by the choice of c in (i) of $(*)_1$,

$$c(v_* \cup \{\alpha_1\}) = g^*_{\alpha_1}(\max\{h(v_*), f^\theta_{\alpha_1}(v_*)\} + 1) + 1 = g^*_{\alpha_1}(i(*) + 1) + 1 < \lambda_{i(*)+1}.$$

So $c(v_* \cup \{\alpha_1\}) \in W^+ \cap \lambda_{i(*)+1} \subseteq g''(i(*) + 1)$.

But $j(*) < i(*) + 1$, so $g''(i(*) + 1) < g^*_{\alpha_1}(i(*) + 1) + 1$, a contradiction. □

Remark 4.2

(a) Combine Theorem 4.1 with the main claim of [4, Section 1], and one has (almost) a full picture for the pair (μ^+, μ).

(b) One may wonder about the assumption $2^{<\mu} = \mu$. As a matter of fact, the proof of Theorem 4.1 depends only on the fact that $\theta = \beth_{m-2}(\kappa^+) < \mu$. Of course, we want this for every $m < \omega$, but this is still a weaker assumption.

(c) We can also ask what happens for other pairs of cardinals. We will try, in a subsequent paper, to shed light on the pair (μ^{+n}, μ).

Acknowledgements The first author would like to thank Badri Kriheli, for the encouragement. We offer our deepest thanks to the referee, for many helpful comments and improvements.

The research of the second author is supported by the United States-Israel Binational Science Foundation. This paper is publication 891 of the second author.

References

[1] S. Shelah, Two cardinal compactness. Israel J. Math. **9**, 193 – 198 (1971). MR MR0302437 (46 #1581).

[2] S. Shelah, Cardinal Arithmetic. Oxford Logic Guides 29 (The Clarendon Press, Oxford University Press, Oxford Science Publications, 1994). MR MR1318912 (96e:03001).

[3] S. Shelah, The pair $(\aleph_n, \aleph_0)$ may fail $\aleph_0$-compactness. In: Logic Colloquium ’01, pp. 402 – 433. Lecture Notes in Logic 20 (ASL, 2005). MR MR2143906 (2006i:03069).

[4] S. Shelah, Two cardinals models with gap one revisited. Math. Logic Quart. **51**, 437 – 447 (2005). MR MR2163755 (2006j:03053).