Groupwise density cannot be much bigger than the unbounded number

Saharon Shelah∗

Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem 91904, Israel
and Department of Mathematics, Hill Center – Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA

Received 1 June 2007, revised 17 September 2007, accepted 18 September 2007
Published online 1 July 2008

Key words Cardinal characteristics of the continuum, unbounded number, groupwise density.
MSC (2000) 03E17, 03E05, 03E20

We prove that \( g \) (the groupwise density number) is smaller or equal to \( b + \), the successor of the minimal cardinality of an unbounded subset of \( \omega \). This is true even for the version of \( g \) for groupwise dense ideals.

© 2008 WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim

1 Introduction

In the present note we are interested in two cardinal characteristics of the continuum, the unbounded number \( b \), and the groupwise density number \( g \). The former cardinal belongs to the oldest and most studied cardinal invariants of the continuum (see, e. g., van Douwen [9] and Bartoszyński and Judah [2]) and it is defined as follows.

Definition 1.1
(a) The partial order \( \leq J_{bd} \omega \) on \( \omega \) is defined by
\[
f \leq J_{bd} \omega g \text{ if and only if } (\exists N < \omega)(\forall n > N)(f(n) \leq g(n)).
\]
(b) The unbounded number \( b \) is defined by
\[
b = \min \{|F| : F \subseteq \omega \text{ has no } \leq J_{bd} \omega \text{-upper bound in } \omega \}.
\]

The groupwise density number \( g \), introduced by Blass and Laflamme in [4], is perhaps less popular but it has gained substantial importance in the realm of cardinal invariants. For instance, it has been studied in connection with the cofinality \( \text{cf}(\text{Sym}(\omega)) \) of the symmetric group on the set \( \omega \) of all integers, see Thomas [8] or Brendle and Losada [5]. The cardinal \( g \) is defined as follows.

Definition 1.2
(a) We say that a family \( A \subseteq [\omega]^{\aleph_0} \) is groupwise dense whenever
\[(a1) \ B \subseteq A, B \in [\omega]^{\aleph_0} \text{ implies } B \in A;
(a2) \text{ for every increasing sequence } \langle m_i : i < \omega \rangle \in \omega \text{ there is an infinite set } U \subseteq \omega \text{ such that}
\[
\bigcup \{[m_i, m_{i+1}) : i \in U\} \in A.
\]
(b) The groupwise density number \( g \) is the minimal cardinal \( \theta \) for which there is a sequence \( \langle A_\alpha : \alpha < \theta \rangle \) of groupwise dense subsets of \([\omega]^{\aleph_0}\) such that
\[
(\forall B \in [\omega]^{\aleph_0})(\exists \alpha < \theta)(\forall A \in A_\alpha)(B \not\subseteq^* A).
\]
(Recall that for infinite sets \( A \) and \( B \), \( A \subseteq^* B \) means \( A \setminus B \) is finite.)

∗ e-mail: shelah@math.huji.ac.il
The unbounded number $b$ and the groupwise density number $g$ can be in either order, see Blass [3] and more Mildenberger and Shelah [7, 6], the latter article gives a bound on $g$. However, as we show in Theorem 2.3, $g$ cannot be bigger than $b^+$.  

**Notation 1.3** Our notation is rather standard and compatible with that of classical textbooks on set theory (like Bartoszyński and Judah [2]). We will keep the following rules concerning the use of symbols.

1. $A, B, \mathcal{U}$ (with possible sub- and superscripts) denote subsets of $\omega$, infinite if not said otherwise.
2. $m, n, t, k, i, j$ are natural numbers; $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \xi$ are ordinals, $\theta$ is a cardinal.

### 2 The result

**Lemma 2.1** For some cardinal $\theta \leq b$ there is a sequence $\langle B_{\zeta,t} : \zeta < \theta, t \in I_\zeta \rangle$ such that the following hold:

(a) $B_{\zeta,t} \in [\omega]^{\aleph_0}$.

(b) If $\zeta < \theta$ and $s \neq t$ are from $I_\zeta$, then $B_{\zeta,s} \cap B_{\zeta,t}$ is finite (so $|I_\zeta| \leq 2^{\aleph_0}$).

(c) For every $B \in [\omega]^{\aleph_0}$ the set $\{ (\zeta, t) : \zeta < \theta \& t \in I_\zeta \& B \subseteq B_{\zeta,t} \}$ is of cardinality $2^{\aleph_0}$.

**Proof.** This lemma is a weak version of the celebrated base-tree theorem of Bohuslav Balcar and Petr Simon with $\theta = \omega$ which is known to be $\leq b$, see Balcar and Simon [1, Theorem 3.4, p. 350]. However, for the sake of completeness of our exposition, let us present a proof.

Let $(f_\zeta : \zeta < b)$ be a $\leq_{\text{pab}}$-increasing sequence of members of $\omega^\omega$ with no $\leq_{\text{pab}}$-upper bound in $\omega^\omega$. Moreover we demand that each $f_\zeta$ is increasing (clearly, this does not change $b$). By induction on $\zeta < b$ choose sets $T_\zeta$ and systems $\langle B_{\zeta,\eta} : \eta \in T_{\zeta+1} \rangle$ such that the following hold:

(i) $T_\zeta \subseteq \zeta(2^{\aleph_0})$, and if $\eta \in T_{\zeta+1}$, then $B_{\zeta,\eta} \in [\omega]^{\aleph_0}$.

(ii) If $\eta \in T_\zeta$ and $\varepsilon < \zeta$, then $\eta \cap \varepsilon \in T_\varepsilon$.

(iii) If $\zeta$ is a limit ordinal, then

$$T_\zeta = \{ \eta \in \zeta(2^{\aleph_0}) : (\forall \varepsilon < \zeta)(\eta \cap \varepsilon \in T_\varepsilon) \& (\exists A \in [\omega]^{\aleph_0})((\forall \varepsilon < \zeta)(A \subseteq B_{\varepsilon,\eta}(\varepsilon+1))) \}.$$  

(iv) If $\varepsilon < \zeta$ and $\eta \in T_{\zeta+1}$, then $B_{\zeta,\eta} \subseteq B_{\varepsilon,\eta}(\varepsilon+1)$.

(v) For $\eta \in T_{\zeta+1}$ and $m_1 < m_2$ from $B_{\zeta,\eta}$ we have $f_\zeta(m_1) < m_2$.

(vi) If $\eta \in T_\varepsilon$, then the set $\{ B_{\varepsilon,\nu} : \eta < \nu \in T_{\varepsilon+1} \}$ is an infinite maximal subfamily of

$$\{ A \in [\omega]^{\aleph_0} : (\forall \varepsilon < \nu)(A \subseteq B_{\varepsilon,\eta}(\varepsilon+1)) \}$$

consisting of pairwise almost disjoint sets.

It should be clear that the choice is possible. Note that for some limit $\zeta < b$ we may have $T_\zeta = \emptyset$ (and then also $T_\zeta = \emptyset$ for $\xi > \zeta$). Also, if we define $T_\emptyset$ as in (iii), then it will be empty (remember clause (v) and the choice of $\langle f_\zeta : \zeta < b \rangle$).

The lemma will readily follow from the following fact.

**Fact 2.2** For every $A \in [\omega]^{\aleph_0}$ there is $\xi < b$ such that $| \{ \eta \in T_{\xi+1} : B_{\xi,\eta} \cap A \text{ is infinite} \} | = 2^{\aleph_0}$.

To show Fact 2.2 let $A \in [\omega]^{\aleph_0}$ and define

$$S = \bigcup_{\zeta < b} \{ \eta \in T_\zeta : (\forall \varepsilon < \zeta)(A \cap B_{\varepsilon,\eta}(\varepsilon+1) \text{ is infinite}) \}.$$  

Clearly $S$ is closed under taking the initial segments and $\emptyset \in S$. By the “maximal” in clause (vi), we have that

(i) if $\eta \in S \cap T_\zeta$, where $\zeta < b$ is non-limit or $\text{cf}(\zeta) = \aleph_0$, then $(\exists \nu)(\eta \cap \nu \in T_{\zeta+1} \cap S)$.

Now if $\eta \in S$ and $\text{lg}(\eta)$ is non-limit or $\text{cf}(\text{lg}(\eta)) = \aleph_0$, then there are $<\text{-incomparable } \nu_0, \nu_1 \in S$ extending $\eta$, i.e., $\eta < \nu_0$ and $\eta < \nu_1$. [Why? As otherwise $S_0 = \{ \nu \in S : \eta \leq \nu \}$ is linearly ordered by $<$, so let $\nu = \bigcup S_\mu$. It follows from (ii) that $\text{lg}(\eta) > \text{lg}(\nu)$ is a limit ordinal (of uncountable cofinality). Moreover, by (iv) + (vi),

$$\text{lg}(\eta) \leq \varepsilon < \text{lg}(\vartheta) \Rightarrow A \cap B_{\text{lg}(\eta),\vartheta}(\text{lg}(\eta)+1) = A \cap B_{\varepsilon,\vartheta}(\varepsilon+1)$$

Hence, by (iii) + (ii), $\vartheta \in T_{\text{lg}(\eta)}$, so necessarily $\text{lg}(\vartheta) < b$. Using (vi) again we may conclude that there is $\vartheta' \in S$ properly extending $\vartheta$, getting a contradiction.]
Consequently, we may find a system \( \langle \eta_\varrho : \varrho \in {}^\omega \omega \rangle \subseteq S \) such that for every \( \varrho \in {}^\omega \omega \)
1. \( k < \text{lg}(\varrho) \Rightarrow \eta_\varrho k < \eta_{\varrho+} \).
2. \( \eta_{\varrho^{-} }, \eta_{\varrho^{-} (1)} \) are \( \leftrightarrow \)-incomparable.
For \( \varrho \in {}^\omega \omega \) let
\[ \zeta(\varrho) = \sup \{ \text{lg}(\eta_\varrho) : \varrho \subseteq \nu \in {}^\omega \omega \} \].
Pick \( \varrho \) such that \( \zeta(\varrho) \) is the smallest possible (note that \( \text{cf}(\zeta(\varrho)) = \aleph_0 \)). Now it is possible to choose a perfect sub-
tree \( T^* \) of \( {}^\omega \omega \) such that
\[ \nu \in \lim(T^*) \Rightarrow \sup \{ \text{lg}(\eta_\nu n) : n < \omega \} = \zeta(\varrho). \]
We finish by noting that for every \( \nu \in \lim(T^*) \) we have that
\[ \bigcup \{ \eta_\nu n : n < \omega \} \in T^*_\zeta(\varrho) \cap S \]
and there is \( n^\ast \in T^*_\zeta(\varrho) + 1 \cap S \) extending \( \bigcup \{ \eta_\nu n : n < \omega \} \).

**Theorem 2.3** \( g \leq b^+ \).

**Proof.** Assume towards contradiction that \( g > b^+ \).
Let \( \langle f_\alpha : \alpha < b \rangle \subseteq {}^\omega \omega \) be a \( \leq \text{cf} b^+ \)-increasing sequence with no \( \leq \text{cf} b^+ \)-upper bound. We also demand that all functions \( f_\alpha \) are increasing and \( f_\alpha(n) > n \) for \( n < \omega \). Fix a list \( \langle \vec{m}_\xi : \xi < 2^{\aleph_0} \rangle \) of all sequences
\[ \vec{m} = \langle m_\ell : \ell < \omega \rangle \]
such that \( 0 = m_0 \) and \( m_\ell + 1 < m_{\ell+1} \).
For \( \alpha < b \) we define:
\[ n_{\alpha,0} = 0, \quad n_{\alpha,\ell+1} = f_\alpha(n_{\alpha,\ell}) \quad (\text{for } \ell < \omega), \quad \vec{n}_\alpha = \langle n_{\alpha,\ell} : \ell < \omega \rangle; \]
\[ \vec{n}_0 = \langle 0, n_{0,2}, n_{0,4}, \ldots \rangle = \langle n_{0,\ell}^0 : \ell < \omega \rangle, \quad \vec{n}_1 = \langle 0, n_{0,3}, n_{0,5}, n_{0,7}, \ldots \rangle = \langle n_{1,\ell}^0 : \ell < \omega \rangle. \]

Observe that if \( \vec{m} \in {}^\omega \omega \) is increasing, then for every large enough \( \alpha < b \) we have:
(a) \( \exists \exists j < \omega \)(\( m_{\ell+1} > f_\alpha(m_\ell) \)), and hence
(b) for at least one \( \ell \in \{ 0, 1 \} \) we have
\[ (\exists \exists j < \omega)(\exists j < \omega)(m_\ell, m_{\ell+1} ; n_{\alpha,j}^\ell, n_{\alpha,j+1}^\ell). \]

Now for \( \xi < 2^{\aleph_0} \) we put:
\[ \gamma(\xi) = \min \{ \alpha < b : (\exists \exists j < \omega)(f_\alpha(m_\xi, j) > m_{\xi,j+1}) \}; \]
\[ \ell(\xi) = \min \{ \ell : (\exists \exists \ell j < \omega)(\exists j < \omega)(m_\xi, m_{\xi,j}) \subseteq [n_{\gamma(\xi), j}^\ell, n_{\gamma(\xi), j+1}^\ell]) \}; \]
\[ \mathcal{U}_\xi^\ell = \{ \ell : (\exists \exists \ell j < \omega)(m_{\xi,j}) \subseteq [n_{\gamma(\xi), j}^\ell, n_{\gamma(\xi), j+1}^\ell]) \}. \]

Note that \( \gamma(\xi) \) is well defined by (a), and so also \( \ell(\xi) \) is well defined (by (b)). Plainly, \( \mathcal{U}_\xi^\ell \) is an infinite subset of \( \omega \).
Now for each \( \xi < 2^{\aleph_0} \), we may choose \( \mathcal{U}_\xi^2 \) so that \( \mathcal{U}_\xi^2 \subseteq \mathcal{U}_\xi^1 \) is infinite and for any \( i_1 < i_2 \) from \( \mathcal{U}_\xi^2 \) we have
\[ (\exists j < \omega)(m_{\xi,i_1+1} < n_{\gamma(\xi), j}^\ell & n_{\gamma(\xi), j+1}^\ell < m_{\xi,i_2}). \]

Let a function \( g_\xi : \mathcal{U}_\xi^2 \longrightarrow \omega \) be such that
\[ (\star) \; i_1 \in \mathcal{U}_\xi^2 \& g_\xi(i_1) = j \Rightarrow [m_{\xi,i_1}, m_{\xi,i_1+1}] \subseteq [n_{\gamma(\xi), j}^\ell, n_{\gamma(\xi), j+1}^\ell]) \).

Clearly, \( g_\xi \) is well defined and one-to-one. (This is very important, since it makes sure that the set \( g_\xi[\mathcal{U}_\xi^2] \) is infinite.)
Fix a sequence $\bar{B} = \{B_{\zeta,t} : \zeta < \theta, t \in I_{\zeta}\}$ given by Lemma 2.1 (so $\theta \leq b$ and $\bar{B}$ satisfies the demands in Lemma 2.1(a) – (c)). By Lemma 2.1(c), for every $\xi < 2^{\aleph_0}$, the set
\[ \{ (\zeta, t) : \zeta < \theta \text{ and } t \in I_{\zeta} \text{ and } B_{\zeta,t} \cap g_\xi[U^2_\xi] \text{ is infinite} \} \]
has cardinality continuum.

Now for each $\beta < b^+$ and $\xi < 2^{\aleph_0}$ we choose a pair $(\zeta_{\beta,\xi}, t_{\beta,\xi})$ such that
\[ (\ast)_2 \zeta_{\beta,\xi} < \theta \text{ and } t_{\beta,\xi} \in I_{\zeta_{\beta,\xi}} \]
\[ (\ast)_3 B_{\zeta_{\beta,\xi}, t_{\beta,\xi}} \cap g_\xi[U^2_\xi] \text{ is infinite, and} \]
\[ (\ast)_4 t_{\beta,\xi} \notin \{ t_{\alpha,\xi} : \epsilon < \xi \text{ or } \epsilon = \xi \& \alpha < \beta \}. \]

To carry out the choice we proceed by induction first on $\xi < 2^{\aleph_0}$, then on $\beta < b^+$. As there are $2^{\aleph_0}$ pairs $(\zeta, t)$ satisfying clauses $(\ast)_2 + (\ast)_3$, whereas clause $(\ast)_4$ excludes $(b^+ + |\xi|) \times \theta < 2^{\aleph_0}$ pairs (recalling that towards contradiction we are assuming $b^+ < g \leq 2^{\aleph_0}$), there is such a pair at each stage $(\beta, \xi) \in b^+ \times 2^{\aleph_0}$.

Lastly, for $\beta < b^+$ and $\xi < 2^{\aleph_0}$ we let
\[ (\ast)_5 U_{\beta,\xi} = g_\xi^{-1}[B_{\zeta_{\beta,\xi}, t_{\beta,\xi}}] \cap U^2_\xi \]
(it is an infinite subset of $\omega$) and we put
\[ (\ast)_6 A_{\beta,\xi} = \bigcup \{ m_{\xi,i}, m_{\xi,i+1} : i \in U_{\beta,\xi} \}, \text{ and } A_\beta = \{ A \in [\omega]^{\aleph_0} : \text{ for some } \xi < 2^{\aleph_0} \text{ we have } A \subseteq A_{\beta,\xi}^{\ast} \}. \]

By the choice of $(\tilde{m}_\xi : \xi < 2^{\aleph_0})$, $A_{\beta,\xi}^\ast$, and $A_\beta$ one easily verifies that for each $\beta < b^+$, $A_\beta$ is a groupwise dense subset of $[\omega]^{\aleph_0}$. Since we are assuming towards contradiction that $g > b^+$, there is an infinite $B \subseteq \omega$ such that
\[ (\forall \beta < b^+) (\exists A \in A_\beta)(B \subseteq^* A). \]

Hence for every $\beta < b^+$ we may choose $\xi(\beta) < 2^{\aleph_0}$ such that $B \subseteq^* A_{\beta,\xi(\beta)}^\ast$. Plainly,
\[ \gamma(\xi(\beta)) < b \text{ and } \zeta_{\beta,\xi(\beta)} < \theta \leq b \text{ and } \ell(\xi(\beta)) \in \{0, 1\}, \]
and therefore for some triple $(\gamma^*, \zeta^*, \ell^*)$ the set
\[ W := \{ \beta < b^+ : (\gamma(\xi(\beta)), \zeta_{\beta,\xi(\beta)}, \ell(\xi(\beta))) = (\gamma^*, \zeta^*, \ell^*) \} \]
is unbounded in $b^+$. Note that if $\beta \in W$, then
\[ (1) \quad B \subseteq^* A_{\beta,\xi(\beta)}^{\ast} \]
\[ = \bigcup \{ m_{\xi, i}, m_{\xi, i+1} : i \in U_{\beta,\xi(\beta)} \} \subset \bigcup \{ n_{g_{\xi}(i), j} : g_{\xi}(i) \text{ for some } i \in U_{\beta,\xi(\beta)} \} \subset \bigcup \{ n_{\gamma(\xi(\beta)), j} : j \in B_{\zeta_{\beta,\xi(\beta)}} \}. \]

[Why? By the choice of $(\beta, \xi(\beta))$, by $(\ast)_6$, and by $(\ast)_1$ as Dom$(g_{\xi(\beta)}) \subseteq U_{\beta,\xi(\beta)} \subseteq U_{\beta,\xi(\beta)}^{\ast}$; by $(\ast)_5$.]

Also, for $\beta \in W$ we have $\ell(\xi(\beta)) = \ell^*$, $\gamma(\xi(\beta)) = \gamma^*$, and $\zeta(\beta, \xi(\beta)) = \zeta^*$, so it follows from (1) that
\[ B \subseteq^* \bigcup \{ n_{\gamma^*, j}^{\ast} : j \in B_{\zeta^*, t_{\beta,\xi(\beta)}} \} \]
for every $\beta \in W$.

Consequently, if $\beta \neq \alpha$ are from $W$, then the sets
\[ \{ n_{\gamma^*, j}^{\ast} : j \in B_{\zeta^*, t_{\beta,\xi(\beta)}} \} \text{ and } \{ n_{\gamma^*, j}^{\ast} : j \in B_{\zeta^*, t_{\alpha,\xi(\alpha)}} \} \]
are not almost disjoint. Hence, as $\langle n_{\gamma^*, j}^{\ast} : j < \omega \rangle$ is increasing, necessarily the sets $B_{\zeta^*, t_{\beta,\xi(\beta)}}$ and $B_{\zeta^*, t_{\alpha,\xi(\alpha)}}$ are not almost disjoint. So applying Lemma 2.1(b) we conclude that $t_{\beta,\xi(\beta)} = t_{\alpha,\xi(\alpha)}$. But this contradicts $\beta \neq \alpha$ by $(\ast)_4$, and we are done.

www.mlq-journal.org  © 2008 WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim
Definition 2.4 We define a cardinal characteristic $g_\beta$ as the minimal cardinal $\theta$ for which there exists a sequence $(I_\alpha : \alpha < \theta)$ of groupwise dense ideals of $\mathcal{P}(\omega)$ (i.e., $I_\alpha \subseteq [\omega]^{\aleph_0}$ is groupwise dense and $I_\alpha \cup [\omega]^{<\aleph_0}$ is an ideal of subsets of $\omega$) such that

$$(\forall B \in [\omega]^{\aleph_0})(\exists \alpha < \theta)(\forall A \in A_\alpha)(B \not\subseteq^* A).$$

Observation 2.5 $2^{\aleph_0} \geq g_\beta \geq g$.

Theorem 2.6 $g_\beta \leq b^+$.

Proof. We repeat the proof of Theorem 2.3. However, for $\beta < b^+$ the family $\mathcal{A}_\beta \subseteq [\omega]^{\aleph_0}$ does not have to be an ideal. So let $I_\beta$ be an ideal on $\mathcal{P}(\omega)$ generated by $\mathcal{A}_\beta$ so also $I_\beta$ is the ideal generated by

$${\{ A^+_{\beta,\xi,\ell,k} : \xi < 2^{\aleph_0} \}} \cup [\omega]^{<\aleph_0}.$$ 

Lastly, let $I_\beta = I_\beta \setminus [\omega]^{<\aleph_0}$.

Assume towards contradiction that $B \in [\omega]^{\aleph_0}$ is such that

$$(\forall \alpha < b^+)(\exists A \in I_\alpha)(B \not\subseteq^* A).$$

So for each $\beta < b^+$ we can find $k_\beta < \omega$ and $\xi(\beta,0) < \xi(\beta,1) < \cdots < \xi(\beta,k_\beta) < 2^{\aleph_0}$ such that

$$B \subseteq^* \bigcup {A^+_{\beta,\xi,\ell,k} : k \leq k_\beta}.$$ 

Let $D$ be a non-principal ultrafilter on $\omega$ to which $B$ belongs. Then for every $\beta < b^+$ there exists $k(\beta) \leq k_\beta$ such that $A^+_{\beta,\xi,\ell,k(\beta)} \in D$. As in the proof there for some $(\gamma^*, \zeta^*, \ell^*, k^*, k(+))$ the following set is unbounded in $b^+$:

$$W := \{ \beta < b^+ : k(\beta) = k(+), k_\beta = k^*, \gamma(\beta,k(\beta)) = \gamma^*, \zeta(\beta,k(\beta)) = \zeta^*,$$

and $\ell(\beta,k(+)) = \ell^*}. $$

As there it follows that if $\beta \in W$, then

$$\bigcup {n^*_{\gamma^*,\ell^*,\ell^*+1} : j \in B_{\zeta^*,t(\beta,k(+))}}$$

belongs to $D$. But for $\beta \neq \alpha \in W$ those sets are not almost disjoint, whereas $(\zeta^*, t(\beta,k(+)) \neq (\zeta^*, t(\alpha,k(+))))$ are distinct, giving us a contradiction. 

Acknowledgements We would like to thank Shimon Garti and the anonymous referee for corrections. The author acknowledges support from the United States-Israel Binational Science Foundation (Grant no. 2002323). This is publication 887 of the author.

References