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SEMIPROPER FORCING AXIOM IMPLIES MARTIN MAXIMUM BUT NOT PFA$^+$

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Abstract. We prove that MM (Martin maximum) is equivalent (in ZFC) to the older axiom SPFA (semiproper forcing axiom). We also prove that SPFA does not imply SPFA$^+$ or even PFA$^+$ (using the consistency of a large cardinal).

For an ordinal $\alpha \leq \omega_1$ and a property Pr of forcing notions, we make the following definition:

1. DEFINITION. $\mathrm{MA}_\alpha(\mathrm{Pr})$ is the assertion that ***if*** P is a proper forcing notion satisfying Pr, for $i < \omega_1$, $I_i \subseteq P$ is dense and for $\beta < \alpha$, $\underset{\sim}{S}_i$ a P_i-name $\Vdash_P$ "$\underset{\sim}{S}_i$ a stationary subset of ω_1", *then* for some directed $G \subseteq P$: (i) for $i < \omega_1$, $G \cap I_i \neq \emptyset$, and (ii) for $\beta < \alpha$, $\{\zeta < \omega_1 : (\exists p \in G)[p \Vdash_P \text{"}\zeta \in \underset{\sim}{S}_\beta\text{"}]\}$ is stationary.

By this notation, PFA (proper forcing axiom) is MA_0 (proper), and PFA$^+$ is MA_1 (proper). On PFA see Baumgartner [1] and Shelah [5].

When semiproperness was discovered and the semiproper iteration lemma was proved (see [4] or [5, Chapter X]) it was clear from CON(ZFC + supercompact) that $\mathrm{SPFA} =^{\mathrm{def}} \mathrm{MA}_0$(semiproper) is consistent (as well as $\mathrm{SPFA}^+ = \mathrm{MA}_1$(semiproper) or MA_{ω_1}(semiproper)).

Foreman, Magidor and Shelah [2] proved the consistency of MM = Martin maximum, which is MA (not destroying stationarity of subsets of ω_1). We can define MM$^+$ in a natural way.

It was proved there, in fact quite early, that MM$^+$ and SPFA$^+$ are equivalent; note that by [5] every semiproper forcing preserves stationary subsets of ω_1, hence $\mathrm{MM}^+ \Rightarrow \mathrm{SPFA}^+$; for the other direction it was proved that, assuming SPFA$^+$, every forcing notion P not destroying stationary subsets of ω_1 is semiproper, by applying SPFA to the following $\underset{\sim}{S}$ and forcing notion Q: $Q = \{f : f$ a function from some $\alpha < \omega$ into $H(\lambda)\}$, where $\lambda \geq (2^P)^+$, $P \in H(\lambda)$, and $\underset{\sim}{S} = \{\delta$: for some $f \in \underset{\sim}{G}_P$, Dom $f = \delta$, and Rang(f) is a counterexample to "P semiproper"$\}$. (So $\mathrm{MA}_1(\aleph_1$-complete) suffices for the equivalence of the two conditions on forcing notion.)

We prove here that just SPFA implies MM.

Magidor and Todorčević ask whether $\mathrm{SPFA} \nvdash \mathrm{SPFA}^+$. Magidor proved that $\mathrm{PFA} \nvdash \mathrm{PFA}^+$ (by forcing PFA, and then adding a stationary subset of

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$\{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_0\}$ which does not reflect). Independently, Beaudoin proved this.[1] We proved here that SPFA$\nvdash$ SPFA$^+$, and even SPFA$\nvdash$ PFA$^+$ [see Theorem 5; by Remark 6A2) supercompact suffices; by 6A3), for example, MA_1 (semiproper)$\nvdash$ MA_2 (semiproper); by 6A4) properness is not productive; and 6A5)), answering a question of Beaudoin, prove SPFA $\vdash MA_1$ ($\aleph_1$-complete)].

For completeness we deduce the consistency of SPFA and even MA_{ω_1} (semiproper) from the semiproper preservation lemma.

2. DEFINITION. Let κ be a supercompact cardinal. We call $f: \kappa \to H(\kappa)$ a *Laver function* if for every cardinal λ and $x \in H(\lambda)$, there is a normal fine ultrafilter D on $\underline{P}_{<\kappa}(H(\kappa))$ such that the set

$$A_D(x) =^{\mathrm{def}} \{a \in P_{<\kappa}(H(\lambda)) : x \in a, a \cap \kappa \in \kappa, \text{ and in the Mostowski collapse } MC_a \text{ of } a, x \text{ is mapped to } f(a \cap \kappa)\}$$

is in D.

By [3], if κ is a supercompact cardinal, we can assume that a Laver function for it exists.

3. LEMMA. *Suppose κ is supercompact and f^* a Laver function for it. Define by induction on $\zeta \le \kappa$ an RCS iteration (see [5, Chapter X, §1]) $\bar{Q}^\zeta = \langle P_i, \underset{\sim}{Q}_j : i \le \zeta, j < \zeta\rangle$, $\bar{Q}^\zeta \in H(\kappa)$ when $\zeta < \kappa$, as follows:*

If *$f^*(i)$ is a P_i-name, $\Vdash_{P_i}$ "$f^*(i)$ semiproper", i limit,* ***then*** *$\underset{\sim}{Q}_i = f^*(i)$.*

Otherwise *$Q_i =$ Levy$(\aleph_1, 2^{2^{\aleph_1}})$.*

Then $\Vdash_{P_\kappa}$ "SPFA".

PROOF. By the semiproperness iteration lemma, P_κ is semiproper, and also it satisfies the κ-c.c. Let $\underset{\sim}{Q}$ be a P_κ-name of a semiproper forcing, and λ a regular cardinal such that $\underset{\sim}{Q} \in \tilde{H}(\lambda)$; without loss of generality $\Vdash_{P_\kappa}$ "$2^{|\underset{\sim}{Q}|} < \lambda$". Let I_i $(i < \omega_1)$ and $\underset{\sim}{S}_\beta$ $(\beta \le \omega_1)$ be given as in Definition 1 (i.e., they are P_κ-names of such objects). Apply Definition 2 to $x = \underset{\sim}{Q}$, and get D as there. Choose $a \in A_D(x)$ such that $\langle I_i : i < \omega_1\rangle$ and $\langle \underset{\sim}{S}_\beta : \beta \le \omega_1\rangle$ belong to a, $(a, \in)$ is isomorphic to some $(H(\chi), \in)$, $\mu \overset{\mathrm{df}}{=} a \cap \kappa \in \kappa$ and $\bar{Q}^\kappa \in a$. Easily $Q_\mu = f^*(\mu)$ in V^{P_μ}, and $\underset{\sim}{Q}_\mu$ is isomorphic to $a \cap \underset{\sim}{Q}$, so we can easily finish.

4. THEOREM. *Martin maximum $\equiv$ SPFA (i.e., proved in ZFC).*

PROOF. As every semiproper forcing preserves stationary subsets of ω_1, clearly MM $\Rightarrow$ SPFA. So we assume

$(*)_0$ SPFA.

It suffice to prove that every forcing notion P satisfying $(*)_1$ is semiproper, where

$(*)_1$ The forcing notion P preserves stationary subsets of ω_1.

We assume $(*)_1$. Without loss of generality the set of members (= conditions) of P is a cardinal $\lambda(0)$. Too generously, let $\lambda(l+1) = (2^{|H(\lambda(l))|})^+$. Let $<_l^* = <_{\lambda(l)}^*$

[1] It is folklore that in the usual forcing for PFA (or SPFA) any subsequent reasonable forcing preserves PFA (or SPFA). Magidor and Beaudoin refine this, showing that starting from a model of PFA, forcing a stationary subset of $\{\delta < \omega_2 : \mathrm{cf}(\delta) = \aleph_0\}$ by $P = \{h : h$ a function from some $\alpha < \omega_2$ to $\{0, 1\}$ such that for no $\delta < \omega_2$ is it true that $\mathrm{cf}(\delta) = \omega_1$ and $h^{-1}(\{1\}) \cap \delta$ is a stationary subset of $\delta\}$ (ordered by inclusion) produces a stationary subset of $\{\delta < \omega_2 : \mathrm{cf}(\delta) = \aleph_0\}$ of ω_2 which does not reflect, and this still preserves PFA but easily makes PFA$^+$ (and SPFA) fail.

We can also start with $V \vDash$ SPFA, and force a function ω (by initial segments of power $\aleph_1$) as produced in the proof of Theorem 5. The proof is much like the proof of Theorem 5. Another way is to force $h: \omega_2 \to \omega_1$ such that no $h^{-1}(\{\alpha\}) \cap \delta$ is stationary in δ, where $\alpha < \omega_1$, $\delta < \omega_2$, and cf $\delta \ne \aleph_1$.

be a well-ordering of $H(\lambda_l)$ extending $<^*_{\lambda(m)}$ for $m < l$. Let

$$K_P^{\text{neg}} =^{\text{def}} \{N: N \prec (H(\lambda(2)), \in, <^*_2), \|N\| = \aleph_0, P \in N \text{ (hence } \lambda_0, \lambda_1 \in N) \text{ and } \neg(\forall p \in P \cap N)(\exists q)[p \leq q \in P \text{ and } q \text{ semigeneric for } (N, P)]\}$$

and

$$K_P^{\text{pos}} =^{\text{def}} \{N: N \prec (H(\lambda(2)), \in, <^*_2), \|N\| = \aleph_0, P \in N, \text{ and } \neg(\exists N')[N \prec N' \in K_P^{\text{neg}} \text{ and } N \cap \omega_1 = N' \cap \omega_1]\}.$$

We now define a forcing notion Q

$$Q =^{\text{def}} \{\langle N_i: i \leq \alpha\rangle: \alpha < \omega_1, N_i \in K_P^{\text{neg}} \cup K_P^{\text{pos}}, N_i \in N_{i+1}, \text{ and } N_i \text{ increasing continuous}\}.$$

A. *Fact. If* $P \in M_0 \prec (H(\lambda(3)), \in, <^*_{\lambda(3)})$, $\|M_0\| = \aleph_0$, *then there is* M_1, $M_0 \prec M_1 \prec (H(\lambda(3)), \in, <^*_{\lambda(3)})$, $\|M_1\| = \aleph_0$, $M_0 \cap \omega_1 = M_1 \cap \omega_1$ *and* $M_1 \restriction H(\lambda(2)) \in K_P^{\text{neg}} \cup K_P^{\text{pos}}$.

PROOF. As $P \in M_0$, $\lambda(0) \in M_0$; hence $\lambda(1), \lambda(2) \in M_0$ and $Q \in M_0$. We can assume $M_0 \restriction H(\lambda(2)) \notin K_P^{\text{pos}}$, so by the definition of K_P^{pos} there is N', $M_0 \restriction H(\lambda_2) \prec N' \in K_P^{\text{neg}}$, $|N'| = \aleph_0$, $N' \cap \omega_1 = M_0 \restriction H(\lambda_0) \cap \omega_1$; hence $N' \cap \omega_1 = M_0 \cap \omega_1$. As $\lambda(2) \in M_0$ and $M_0 \restriction H(\lambda(2)) \prec N'$, the Skolem hull in $(H(\lambda(3)), \in, <^*_{\lambda(3)})$ of $M_0 \cup (N' \cap H(\lambda(1)))$ has the same countable ordinals as N'. Let M_1 be the Skolem hull of $M_0 \cup (N' \cap H(\lambda(1))$ (exists as $<^*_{\lambda(3)}$ is a well-ordering of $H(\lambda(3))$). So by the above

$$M_1 \cap \omega_1 = N' \cap \omega_1 = M_0 \cap \omega_1,$$
$$M_0 \prec M_1 \prec (H(\lambda(3)), \in, <^*_{\lambda(3)}),$$
$$\|M_1\| = \aleph_0 \quad (\text{as } \|M_0\|, \|N'\| = \aleph_0).$$

Also $M_1 \cap H(\lambda(1)) = N' \cap H(\lambda(1))$ (same reason). We can conclude that $M_1 \restriction H(\lambda(2)) \in K_P^{\text{neg}}$ (thus finishing the proof of Fact A), as:

B. *Subfact. If* $P \in N^a$, $N^b \prec (H(\lambda(2)), \in, <^*_{\lambda(2)})$ *are countable and* $N^a \cap H(\lambda(1)) = N^b \cap H(\lambda(1))$, *then* $N^a \in K_P^{\text{neg}} \Leftrightarrow N' \in K_P^{\text{neg}}$ (just check the definition of K_P^{neg}) [really, even $N^1 \cap \omega_1 \subseteq N^0 \subseteq N^1$, $N^1 \prec (H(\lambda(2)), \in, <^*_{\lambda(2)})$, $N^0 \in K_P^{\text{neg}}$, implies $N^1 \in K_P^{\text{neg}}$].

C. *Fact. Q is a semiproper forcing.*

Let $Q, P \in M \prec (H(\lambda(3)), \in, <^*_{\lambda(3)})$, M countable. Let $p \in Q \cap M$. It is enough to prove that there is a q, $p \leq q \in Q$, semigeneric for (M, Q). By Fact A there is M_1 with $M \prec M_1 \prec (H(\lambda(3)), \in, <^*_{\lambda(3)})$, $\|M_1\| = \aleph_0$, $M \cap \omega_1 = M_1 \cap \omega_1$ and $M_1 \restriction H(\lambda(2)) \in K_P^{\text{neg}} \cup K_P^{\text{pos}}$. In M_1 we can find an increasing sequence of $q_n = \langle N_i: i \leq \alpha_n\rangle \in Q \cap M_1, q_n \leq q_{n+1}, q_0 = p$, such that for every Q-name $\underset{\sim}{\gamma}$ of a countable ordinal for some $n = n(\gamma)$ and $\alpha(\gamma) \in M_1$, $q_n \Vdash_Q$ "$\underset{\sim}{\gamma} = \alpha(\gamma)$". Now $q =^{\text{def}} \langle N_i: i \leq \bigcup_{n<\omega} \alpha_n\rangle$ with $N_{\cup_n \alpha_n} = \bigcup_{i < \cup \alpha_n} N_i$ will be (M_1, Q)-semigeneric if $(\bigwedge_n \bigvee_m \alpha_n < \alpha_m$ and) $\bigcup_{i<\cup\alpha_n} N_i \in K_P^{\text{neg}} \cup K_P^{\text{pos}}$. But it is quite easy to manage that $\alpha_m < \bigcup_n \alpha_n$ and that $\bigcup_{i<\cup\alpha_n} N_i = M_1 \restriction H(\lambda(2))$, and it belongs to $K_P^{\text{neg}} \cup K_P^{\text{pos}}$ by the choice of M_1. Now $q \geq q_0 = p$; and, as q is (M_1, Q)-semigeneric and $M \prec M_1, M \cap \omega_1 = M_1 \cap \omega_1$, q is also (M_1, Q)-semigeneric, as required.

D. *Conclusion. There is a sequence* $\langle N_i^*: i \in \omega_1\rangle$ *such that* $(\forall\alpha)[\langle N_i^*: i \leq \alpha\rangle \in Q]$.

PROOF. By Fact C and SPFA (and as $I_{\alpha_0} = \{\langle N_i: i \leq \alpha\rangle: \alpha \geq \alpha_0\}$ is dense in Q for every α_0, proved by induction on α_0: for $\alpha_0 = 0$ or $\alpha_0 = \beta + 1$ by Fact A, for limit α_0 by the proof of Fact C, or simpler).

E. *Note.* As $N^*_i \in N^*_{i+1}$, clearly $i \subseteq N^*_i$.

F. DEFINITION. $S = \{i < \omega_1 : N^*_i \in K_P^{\text{neg}}\}$.

G. *Fact. S is not stationary.*

Suppose it is; then for every $i \in S$ for some $p_i \in N^*_i \cap P$ there is no (N^*_i, P)-semigeneric q, $p_i \leq q \in P$. By Fodor's lemma (as N^*_i is increasing continuous), for some $p \in \bigcup N^*_i \cap P$, $S_p = \{i \in S: p_i = p\}$ is stationary.

If $p \in G \subseteq P$, G generic over V, in $V[G]$ we can find an increasing continuous sequence $\langle N_i : i < \omega_1\rangle$ of elementary submodels of $(H(\lambda(2)^V), \in, <^*_{\lambda(1)}, G)$ (G as a predicate), $N^*_i \subseteq N_i$. As P preserves stationary subsets of ω_1, and $\{i: N^*_i \cap \omega_1 = N_i \cap \omega_1 = i\}$ is a club of ω_1 (in $V[G]$), and $S_p \subseteq \omega_1$ is stationary (in V, hence in $V[G]$), it follows that there is $\delta \in S_p$ with $N^*_\delta \cap \omega_1 = N_\delta \cap \omega_1 = \delta$. As this holds in $V[G]$, $p \in G$, clearly there is $q \in G$, $q \geq p$, such that $q \Vdash$ "δ and $\langle \underset{\sim}{N}_i : i < \omega_1\rangle$ are as above". But q is necessarily (N^*_i, P)-semigeneric (as $N^*_\delta \subseteq N_\delta$ have the same countable ordinals).

H. *Fact. P is semiproper.*

As S is not stationary, for some club $C \subseteq \omega_1$, $(\forall \delta \in C)\ N^*_\delta \in K_P^{\text{pos}}$. Now if $M \prec (H(\lambda(3)), \in, <^*_{*\lambda(3)})$ is countable, and P, $\langle N^*_i : i < \omega_1\rangle$, $C \in M$, then $M \cap \bigcup_{i<\omega_1} N^*_i = N^*_\delta$ for some $\delta \in C$; hence $N^*_\delta \prec M \restriction H(\lambda(2)) \prec (H(\lambda(2)), \in, <^*_{\lambda(2)})$ is countable, and so $M \restriction H(\lambda(2)) \notin K_P^{\text{neg}}$, i.e., for every $p \in P \cap M\ (= P \cap (M \restriction H(\lambda(2))))$ there is an (M, P)-semigeneric q, $p \leq q \in P$; this is enough.

5. THEOREM. *Suppose κ is a supercompact limit of supercompacts. Then, in some generic extension, SPFA holds but PFA^+ fails.*

PROOF. Let f^* be a Laver function for κ. Our proof will unfold as follows.

We shall first define a semiproper iteration $\bar{Q}^\kappa$, leaving one point for the end. Now $\Vdash_{P_\kappa}$ SPFA is as in the proof of Lemma 3. We then define in V^{P_κ} a proper forcing notion R and an R-name $\underset{\sim}{S}$, $\Vdash_R$ "$\underset{\sim}{S} \subseteq \omega_1$ is stationary". We then show, by filling the point left above, that for no directed $G \subseteq R$ is $\underset{\sim}{S}[G]$ well defined (i.e., $(\forall i < \omega_1)(\exists P \in G)[p \Vdash_R$ "$i \in \underset{\sim}{S}$" or $p \Vdash_R$ "$i \in \underset{\sim}{S}$"] and stationary (i.e., $\{i < \omega_1 : (\exists p \in G)\ p \Vdash_R$ "$i \in \underset{\sim}{S}$"$\}$ is stationary).

We define by induction on $\zeta \leq \kappa$ an RCS iteration (see [5, Chapter X, §1]) $\bar{Q}^\zeta = \langle P_i, \underset{\sim}{Q}_j : i \leq \zeta, j < \zeta\rangle$, and if $\zeta \neq \kappa$, $\bar{Q}^\zeta \in H(\kappa)$, which is a semiproper iteration (i.e. for $i < j \leq \zeta$, i nonlimit, P_j/P_i is semiproper) and, if $\zeta = \delta$, δ limit, also a P_ζ-name W_δ as follows:

(a) Suppose ζ is nonlimit, let $\kappa_\zeta < \kappa$ be the first supercompact $> |P_\zeta|$, so κ_ζ is a supercompact cardinal even in V^{P_ζ}, and let Q_ζ be a semiproper forcing notion of power κ_ζ collapsing κ_ζ to $\aleph_2$ such that in V^P:

$\Vdash_{\overline{P_\zeta * Q_\zeta}}$ "any forcing notion not destroying stationary subsets of ω_1 is semiproper"

(it exists e.g. by Lemma 3 and Theorem 4; but really $Q_\zeta = \text{Levy}(\aleph_1, < \kappa_\zeta)$ (in V^{P_ζ}) is okay, as

$$\Vdash_{\overline{P_\zeta * Q_\zeta}} \text{"MA}_{\omega_1}(\aleph_1\text{-complete)"}$$

and $\text{MA}_1(\aleph_1$-complete) implies (by [2]) the required statement.

(b) Suppose ζ is limit. If $f^*(\zeta)$ is a P_ζ-name, $\Vdash_{P_\zeta}$ "$f^*(i)$ is semiproper", then let $\underset{\approx}{Q}^0_\zeta = f^*(\zeta)$. If $f^*(\zeta)$ is not like that, let $\underset{\approx}{Q}^0_\zeta =$ the trivial forcing.

Let $\underset{\approx}{Q}^1_\zeta$ be defined later, so that:

(∗) *If $\xi < \zeta$, $A \in V^{P_\xi}$, $A \subseteq \omega_1$, and A is stationary in V^{P_ζ} (equivalently in V^{P_ξ} when ξ is nonlimit), then A is stationary in $V^{P_\zeta * Q^0_\zeta * Q^1_\zeta}$.*

We let $\underset{\sim}{Q}_\zeta = \underset{\sim}{Q}^0_\zeta * \underset{\sim}{Q}^1_\zeta * \underset{\sim}{Q}^2_\zeta$; $\underset{\sim}{Q}^2_\zeta$ is the addition of $(\aleph_1 + 2^{\aleph_0})^{V^{P_\zeta}}$ Cohen reals with finite support.

(c) For ζ limit we also have to define $W_{\zeta+1}$.

(i) W_ζ is a function whose domain is $A_\zeta = \{a: a \subseteq \zeta + 1, \zeta \in a \in V^{P_\zeta}$, and a is a countable set of limit ordinals$\}$.

(ii) For $a \in A_\zeta$, $W_\zeta(a) = \langle H_{\zeta,i}(a): i < \mathrm{otp}(a)\rangle$, where $H_{\zeta,i}(a)$ is a function from $\mathrm{otp}(a)$ to $\{0, 1\}$ (where $\mathrm{otp}(a)$ is the order type of a).

(iii) For every $\xi \in a \in A_\zeta$, $a \cap (\xi + 1) \in A_\xi$, and for $i \leq \mathrm{otp}(a \cap (\xi + 1))$, $H_{\xi,i}(a \cap (\xi + 1))$ is $H_{\zeta,i}(a)$ restricted to $\mathrm{otp}(a \cap (\xi + 1))$.

(iv) If $a \in A_\zeta$, we use the Cohen reals from $\underset{\sim}{Q}^2_\zeta$ to choose the values of $H_{\zeta,i}(a)(\alpha)$ for $i = \mathrm{otp}(a \cap \zeta)$ or $\alpha = \mathrm{otp}(a \cap \zeta)$.

Clearly P_κ is semiproper, satisfies the κ-c.c., and $|P_\kappa| = \kappa$. As in the proof of Lemma 3, SPFA holds in V^{P_κ}. Now we want to show that PFA$^+$ fails. The "components" of R and of the proof are not new. In $V_0 = V^{P_\kappa}$ let $T = \bigcup\{A_\delta: \delta < \kappa$ (limit)$\}$, and let $<_T$ be the order: being initial segment. The forcing we shall get by composition.

Let R_0 be Levy$(\aleph_1, \aleph_2)$ (in V_0). In $V_1 = V_0^{R_0}$, let R_1 be the Cohen forcing; in $V_2 =^{\mathrm{def}} V_1^{R_1}$ let R_2 be Levy$(\aleph_1, 2^{\aleph_2})$. Let $V_3 = V_2^{R_2}$. Forcing with $R_1 * \underset{\sim}{R}_2$ does not add κ-branches to T, so let us specialize it, i.e., let $\{B_i: i < i^*\} \in V_1$ be its κ-branches in $(V^{P_\kappa})^{R_0}$, so $|i^*| \equiv \aleph_1$ in V_3. Let $B'_i \subseteq B_i$ for $i < i^*$ be end segments, pairwise disjoint, and let

$$R_3 = \{f: f \text{ a finite function from } T \text{ to } \omega_1 \text{ such that } [x < y \in T \wedge f(x) = f(y) \rightarrow (\exists i)(x, y \in B_i)]\}.$$

Let $V_4 = V_3^{R_3}$. In V, for limit $\delta < \omega_1$, let η_δ be an ω-sequence converging to δ. Without loss of generality $\eta_{\delta_1}(n) = \eta_{\delta_2}(m)$ implies $n = m$ and $\eta_{\delta_1} \restriction n = \eta_{\delta_2} \restriction n$. Let $\mathbf{Z} = \{\eta: \eta$ a finite sequence of ordinals $<\omega_1$, such that $C_\eta =^{\mathrm{def}} \{\delta < \omega_1: \eta \lessdot \eta_\delta\}$ is stationary$\}$. So $(\forall \eta \in \mathbf{Z})(\exists^{\aleph_1} \nu \in \mathbf{Z})[\eta \lessdot \nu]$. Also $C^* = \{\delta < \omega_1$: for arbitrarily large $n < \omega$, for $\aleph_1$ ordinals $i < \omega_1$, $(\eta_\delta \restriction n)^\wedge\langle i\rangle \in \mathbf{Z}\}$ contains a club of ω_1, so by renaming (as we do not insist on η_δ being increasing) without loss of generality for every limit $\delta < \omega_1$ and $n < \omega$, for $\aleph_1$ ordinals i $(\eta_\delta \restriction n)^\wedge\langle i\rangle \in \mathbf{Z}$. Now for every stationary $A \subseteq \omega_1$ from V_0, $\{\eta \in \mathbf{Z}: A \cap C_\eta$ stationary$\}$ has power $\aleph_1$. Hence in V_1 we can find C^*_η stationary, for $\eta \in \mathbf{Z}$ such that for every $A \in V_0$, $A \subseteq \omega_1$ stationary, for some η, $|C^*_\eta - A| \leq \aleph_0$, and $C^*_{\eta \wedge \langle i\rangle} \cap C^*_{\eta \wedge \langle j\rangle} = \varnothing$ for $i \neq j$, $C^*_{\eta \wedge \langle i\rangle} \subseteq C^*_\eta$ and for $n \leq \lg(\eta)/2$, $C^*_\eta \subseteq C_{\langle \eta(0), \eta(2), \eta(4), \ldots, \eta(2n-2)\rangle}$. So let (in V_4)

$$R_4 = \{(u, w): w \text{ a finite set of limit ordinals } < \omega_1, u \text{ a finite subset of } \mathbf{Z}, \text{ and } w \cap C^*_\eta = \varnothing \text{ for } \eta \in u\}.$$

Easily R_4 satisfies the $\aleph_1$-c.c.; in fact for every $\aleph_1$ conditions there are $\aleph_1$ pairwise compatible (and more). Let $\underset{\sim}{S}^0 = \bigcup\{w: (\exists u)[(u, w) \in \underset{\sim}{G}_{R_4}]\}$. It easily can be shown to be stationary (in $V_4^{R_4}$).

Let $V_5 = V_4^{R_4}$.

Let $\langle S_\gamma: \gamma < \omega_1\rangle \in V$ be a partition of ω_1 into stationary sets. In V_1 let $\langle \zeta_\alpha: \alpha < \omega_1\rangle$ be an increasing continuous sequence of ordinals converging to κ, and for $\alpha < \omega_1$, $j < i^*$, let $H^{(j)}_\alpha = \bigcup\{H_{\zeta_\beta,\alpha}(a)$: for some $\beta < \omega_1$, $a \in B_j \cap \mathrm{Dom}\, W_{\zeta_\beta}$, $\alpha \leq \mathrm{otp}(a)\}$. In V^5 we now define R_5: it is the product with finite support of $R^5_{\alpha,i}$ $(\alpha < \omega_1, i < i^*)$, where the aim of $R^5_{\alpha,i}$ is making ω_1 the union of $\aleph_0$ sets, on each of which $H^{[j]}_{\alpha}$ is constantly 0 (or 1) if

$$(\exists\gamma < \omega_1)[\alpha \in S_\gamma \wedge \gamma \in S^0] \quad (\text{or } (\exists\gamma < \omega_1)[\alpha \in S_\gamma \wedge \gamma \notin S^0]).$$

Now $R^5_{\alpha,i}$ is just the set of finite functions h from ω_1 to ω so that on each $h^{-1}(\{n\})$ the coloring $H^{[j]}_{\alpha}$ is constantly 0 (or 1) (as required above).

Lastly, let $R = R_0 * \underset{\sim}{R}_1 * \underset{\sim}{R}_2 * \underset{\sim}{R}_3 * \underset{\sim}{R}_4 * \underset{\sim}{R}_5$. We define $\underset{\sim}{S}$ such that $\underset{\sim}{S}^0 \subseteq \underset{\sim}{S} \subseteq \underset{\sim}{S}^0 \bigcup \{\gamma + 1 : \gamma < \omega_1\}$ and, if $G \subseteq R$ is directed and $\underset{\sim}{S}[G]$ well defined, then all relevant information is decided (i.e., what is needed below):[2]

Fact. The forcing R is proper (in V_0).

As properness is preserved by composition, we just have to check R_i in V_i. The only nontrivial one is R_5. For this it suffices to show that the product of any finitely many $R^5_{\alpha,i}$ satisfies the $\aleph_1$-c.c. Let $m < \omega$, and let the (α_l, i_l) for $l < m$ be distinct. It is easy to show that in V_1

(*) *If for each $l < m$, u^l_γ and w^l_γ $(\gamma < \omega_1)$ are pairwise disjoint subsets of ω_1, then for some $\gamma(1), \gamma(2) < \omega_1$, for each even $l < m$*

$$[x \in u^l_{\gamma(1)}, y \in w^l_{\gamma(2)} \Rightarrow H^{[i_l]}_{\alpha_l}(x, y) = 0]$$

and for each odd $l < m$

$$[x \in u^l_{\gamma(1)}, y \in w^l_{\gamma(2)} \Rightarrow H^{[i_l]}_{\alpha_l}(x, y) = 1].^{2.5}$$

The subsequent forcing by R_1, R_2, R_3, R_4 preserves the satisfaction of (*), and it implies that any finite product of $R^5_{\alpha,i}$ satisfies the $\aleph_1$-c.c.[3]

Clearly R is proper in V_0, and $\Vdash_R$ "$\underset{\sim}{S} \subseteq \omega_1$ is stationary". Suppose $G \subseteq R$ is directed and $\underset{\sim}{S}[G]$ well defined. Then we can have that for some $\delta < \kappa$, G and $\underset{\sim}{S}[G]$ over V^{P_δ} are similar enough to G_R and $\underset{\sim}{S}[G_R]$ over V^{P_κ}. So $V_0 \vDash \text{cf}(\delta) = \aleph_1$. But we have some freedom left in choosing $\underset{\sim}{Q}^1_\delta$. We define it by a semiproper iteration first to collapse δ to $\aleph_1$ if necessary, then (if cf $\delta = \aleph_1$) fix the set of branches of $T \cap V^{P_\delta}$, and lastly for each 2-place symmetric function $f: \omega_1 \to \{0, 1\}$, if there is a semiproper forcing (in $H(\kappa)$) producing $A_{f,n} \subseteq \omega_1$, such that $\omega_1 = \bigcup_n A_{f,n}$ and $f \restriction A_{f,n}$ is constant, then such $A_{f,n}$ already exists. As stationarity of subsets of ω_1 is preserved, though we may have various candidates for the directed $G \subseteq R$, there is at most one candidate S^δ for $\{\delta : \delta \in S, \delta < \omega_1 \text{ limit}\}$ (as for the coding we use stationary $S_\gamma \subseteq \omega_1$). We shoot a club through the complement of S^δ. Now all the previous forcings were proper or semiproper, and the last one does not destroy stationary subsets of ω_1 from $\bigcup_{\beta<\delta}(\mathscr{P}(\omega_1) \cap V^{P_\beta})$(see the definitions of C^*_n and R_4), which is okay, because for every successor ordinal $\xi < \zeta$, $P_\zeta * Q_\zeta / P_\xi$ does not destroy stationarity of subsets of ω_1 from V^{P_ξ}, hence is semiproper.

6A. Remarks. 1) Magidor and, independently, Beaudoin proved the consistency of PFA + $\neg$PFA$^+$.

2) Can we reduce the assumption to "κ is supercompact"? Yes, as, say in (b) we let $\underset{\sim}{Q}^0_\zeta = f^*(\zeta)$ only if: $f^*(\zeta)$ is a P_ζ-name, $\Vdash_{P_\zeta}$ "$f^*(\zeta)$ is semiproper" and for some

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[2] Including a well ordering of ω_1 of order type $\underset{\sim}{\zeta}_\alpha$ for $\alpha < \omega_1$.

[2.5] Because for $\zeta < \kappa$, $\{\langle l, u^l_\gamma, w^l_\gamma\rangle : \gamma < \omega, l < m\}$ belongs to V^{P_ζ} and to $H(\zeta)$, $V^{P_\zeta} \vDash$ "$\zeta = \aleph_2$" and remembers the way we use Cohen reals to define H.

[3] The least trivial is why R_3 preserves it. This is because for any $p_i \in R_3 (i < \omega_1)$ there are disjoint uncountable $S_1, S_2 \subseteq \omega_1$ such that if $i < j$ for $i \in S_1$ and $j \in S_2$, then p_i and p_j are compatible. This suffices; it also holds for R_1 and R_4.

$\lambda_\zeta < \kappa$, $f^*(\zeta) \in H(\lambda_\zeta)$, and ζ is $\beth_8(\lambda_\zeta)$-supercompact. This does not change the proof of Lemma 3. Now we let Q_κ = shooting a club (of order type κ) through $\{i < \kappa : \models \text{cf } i = \aleph_0$ or $V \models$ "i is strongly inaccessible in V and $\beth_8(\lambda_\zeta)$-supercompact"$\}$ (by initial segments). Now it is folklore that, for such Q_κ, $V^{P_\kappa * Q_\kappa} \models$ SPFA, and clearly $V^{P_\kappa * Q_\kappa} \models \neg \text{PFA}^+$. But there is a small cheating above: in (b) (more exactly in the definition of Q^1_δ which appears in the end of the proof) as the result of an iteration we ask "is there a semiproper forcing in $H(\kappa)$ such that...", and this defeats our desire that $Q^1_\delta \in H(\beth_8(\lambda_\zeta))$. We want to be able to "decipher" the possible "codings" fast, i.e., by a forcing notion of small cardinality.

We let $\gamma_{\alpha,j}$ be 0 if $(\exists \gamma < \omega_1)[\alpha \in S_\gamma \wedge \gamma \in S^0]$ and 1 otherwise, and let

$$R^5_{\alpha,j} = \{(w,h): w \text{ is a finite subset of } \omega_1 \text{ and } h \text{ is a finite function from the family of nonempty subsets of } w \text{ to } \omega \text{ such that: } \textbf{\textit{if }} u_1, u_2 \in \mathrm{Dom}(h),\ h(u_1) = h(u_2) \textbf{\textit{ then }} |u_1| = |u_2| \text{ and } [\zeta = u_1 - u_2 \wedge \xi \in u_2 - u_1 \wedge \zeta < \xi \Rightarrow H^{[j]}_\alpha(\zeta,\xi) = \gamma_{\alpha,j}]\}.$$

Now for every δ, we define Q^1_δ by the following (finite) iteration: collapse δ to $\aleph_1$, add a Cohen real, collapse $2^{\aleph_1}$, specialize the relevant tree so we know the $H^{[j]\alpha} \restriction \delta$ $(j < \omega_1, \alpha < \delta)$, R_4, R_5, and now force MA (proper forcing of cardinality $\aleph_1$).

The resulting forcing is not too large, and it essentially determines[4] the $\gamma_{\alpha,j}$ (i.e., we can find $\gamma^0_{\alpha,j}$ so that if we have an appropriate G, the values of the $\gamma_{\alpha,j}$ will be $\gamma^0_{\alpha,j}$). So we have only one candidate for $\underset{\sim}{S}[G]$, namely S_δ, and if it is not disjoint to any stationary subset of ω_1 from V^{P_δ}, we end the iteration by shooting a club through $\omega_1 - S_\delta$ (δ has enough supercompactness so that $(P_\delta/P_j) * Q_\delta$ is semiproper for every nonlimit $j < \delta$).

3) We can similarly prove that if $\alpha(0), \alpha(1) \le \omega_1$ and $|\alpha(0)| < |\alpha(1)|$, then $\text{MA}_{\alpha(0)}$ (semiproper) $\nvdash \text{MA}_{\alpha(1)}$ (proper).

4) Observe that properness is not productive, i.e. (provably in ZFC) there are two proper forcings whose product is not proper.

[Let T be the tree $({}^{\omega_1 >}\omega_2, \lessdot)$; now one forcing, P, shoots a branch with supremum ω_2, e.g., $P = T$ (it is $\aleph_1$-complete). The second forcing, Q, guarantees that in any extension of V^Q, as long as $\aleph_1$ is not collapsed, T will have no ω_1-branch with supremum ω_2. Use $Q_1 * \underset{\sim}{Q}_2 * \underset{\sim}{Q}_3$, where Q_1 is Cohen forcing, $Q_2 = \text{Levy}(\aleph_1, \aleph_2)$ in V^{Q_1} (so it is well known that in $V^{Q_1 * Q_2}$, $\text{cf}(\omega_2^V) = \omega_1$, and T has no branch with supremum ω_2), and Q_3 is the appropriate specialization of T. We could have used the tree ${}^{\omega_1 >}2$, but then we should speak of "a branch of T which is not in V".]

5) Beaudoin asks whether SPFA $\nvdash \text{MA}_1$ ($\aleph_1$-complete). This is a natural question. Note that the proof of Theorem 5 shows that SPFA $\nvdash \text{MA}_1$ (finite iteration of $\aleph_1$-complete and c.c.c. forcing notions).

But $\aleph_1$-complete forcing would be a somewhat better counterexample. We have *Fact.* SPFA $\vdash \text{MA}_1$ ($\aleph_1$-complete).

[4] By the celebrated proof (of Todorčević [6]),

(*) If f is a two-place function from ω_1 to $\{0,1\}$, then, for some proper forcing Q of cardinality $\aleph_1$, $\Vdash_Q$ "there is an uncountable $A \subseteq \omega_1$ such that for $\alpha < \beta$ from A, $f(\alpha,\beta) = 0$ *or* there are $n < \omega$ and pairwise disjoint n-triples $\langle \alpha^\zeta_0, \ldots, \alpha^\zeta_n \rangle$ of ordinals $< \omega_1$ such that for every $\zeta < \xi < \omega_1$, for some $l \le n$, $f(\alpha^\zeta_0, \alpha^\xi_l) = 1$".

PROOF. Suppose $V \models$ SPFA, P is $\aleph_1$-complete forcing, $\underset{\sim}{S}$ a P-name, and $\Vdash_P$ "$\underset{\sim}{S} \subseteq \omega_1$ is stationary". For $i < \omega_2$ let $(P_i, \underset{\sim}{S}_i)$ be isomorphic to (P, S), and let P^* be the product of P_i $(i < \omega_1)$ with countable support; so $P_i \lessdot P^*$, P^* is $\aleph_1$-complete, and $\underset{\sim}{S}_i$ is a P^*-name.

Let $I = \{A \in V: A \subseteq \omega_1$, A stationary and $\Vdash_P$ "$\underset{\sim}{S} \cap A$ is not stationary$\}$. Let $\{A_i: i < i^*\} \subseteq I$ be a maximal antichain (i.e., the intersection of any two elements is not stationary).

So, by [2] and Theorem 4, $|i^*| = \omega_1$, and so there is an $A \subseteq \omega_1$ such that

(i) $\Vdash_P$ "$\underset{\sim}{S} \cap A$ is not stationary", and

(ii) for every $B \subseteq \omega_1 - A$ stationary, for some $p \in P$, $p \Vdash_{P^*}$ "$\underset{\sim}{S} \cap B$ is stationary".

Now $\omega_1 - A$ is stationary (as $\Vdash_P$ "$\underset{\sim}{S}$ is stationary"). Also, clearly,

(iii) for each $i < \omega_1$, and stationary $B \subseteq \omega_1 - A$ for some $p \in P_i \lessdot P^*$, $p \Vdash_{P^*}$ "$\underset{\sim}{S}_i \cap B$ is stationary".

As P^* is the product of the P_i with countable support, we have

(iv) for every stationary $B \subseteq \omega_1 - A$, $\Vdash_{P^*}$ "for some i, $\underset{\sim}{S}_i \cap B$ is stationary".

Let $\underset{\sim}{S}^*$ be the P^*-name: $\nabla_i \underset{\sim}{S}_i = \{\alpha < \omega_1 : (\exists i < \alpha)\ \alpha \in \underset{\sim}{S}_i\}$. So $\Vdash_{P^*}$ "for every stationary $B \subseteq \omega_1 - A$, $B \cap \underset{\sim}{S}^*$ is stationary".

In V^{P^*} let Q^* be shooting a club through $A \cup S^*$ (i.e., $Q^* = \{h: h$ an increasing continuous function from some nonlimit $\alpha < \omega_1$ into $A \cup S\}$). Now Q^* does not destroy any stationary subset of ω_1 from V (though it destroys some from V^{P^*}). So $P^* * \underset{\sim}{Q}^*$ does not destroy any stationary subsets of ω_1 from V; hence by Theorem 4 it is semiproper. Now if $G \subseteq P^* * \underset{\sim}{Q}^*$ is generic enough, for each $i < \omega_1$, $G \subseteq P_i$ is generic enough, $\underset{\sim}{S}_i[G]$ well-defined, and $\nabla_i \underset{\sim}{S}_i[G]$ includes $\omega_1 - A$ on a club. So for some i, $\underset{\sim}{S}_i[G]$ is stationary, and we finish.[5]

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[5] A similar proof works if $P = P^a * \underset{\sim}{P}^b$, where P^a satisfies the $\aleph_1$-c.c. and $\underset{\sim}{P}^b$ is $\aleph_1$-complete in V^{P^a}, if we use $P^* = \{f, f$ a function from ω_1 to $\underset{\sim}{P}$, $f(i) = (p_i, q_i)$, $|\{i: p_i \neq \varnothing\}| < \aleph_1$, $|\{i: q_i \neq \varnothing\}| < \aleph_1\}$. In short, we need that some product of copies of P is semiproper.