# ORTHOGONALITY OF TYPES IN SEPARABLY CLOSED FIELDS

Z. Chatzidakis*, G. Cherlin*, S. Shelah***, G. Srour**, and C. Wood*

## §0. Introduction.

As the only examples currently known of stable but not superstable fields, separably closed fields hold a place of some interest. In the present paper we concentrate on aspects of the analysis of 1-types. In §1 we give preliminary and background information, and also construct families of mutually orthogonal types of rank 1. In §2 we give a class of more complicated examples of conjugate types which serves to prove that various dimensional order properties hold, hence many somewhat saturated models exist. In this section we also include a discussion of these various properties. In §3 we consider the relationship between U-rank and transcendence degree, and in §4 we give an example of a type of rank $\omega$.

Given that this paper has five authors, it should perhaps be expected that there is a nonstructure theorem concerning credits, but partial information is available. Bouscaren had raised the question of DOP for separably closed fields, in the context of the following problem: can one have a stable theory with stable pairs and with DOP (in contrast to her results for the superstable case [B]). Delon had observed that separably closed fields have stable pairs (indeed, that $SCF_n$ --see below--has a unique theory of distinct pairs, which is stable), and, upon Bouscaren's request, proved DOP in Fall 1985 [D]. Meanwhile, Chatzidakis, Cherlin and Wood were also considering separably closed fields, and Shelah spelled out to them what the various dimensional properties would require in this specific setting. In Fall 1985 (independently but after Delon), we produced the types $p_A$ in §2; the key inspiration for the examples which finally worked was Cherlin's, according to Chatzidakis and Wood. Srour was working on the analysis of 1-types for separably closed fields in a third part of the world; 1.2-1.4 and the discussion of accessible types in §3 come from his analysis. The five authors met in December 1985 in Chicago, and decided to amalgamate what we knew into the

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present paper. With regard to Bouscaren's original query, Poizat has recently observed that the theory of a single pairing function without loops has both stable pairs and DOP; this example makes very clear the crucial role of pairing.

## §1. Preliminaries.

In this section we summarize various facts about separably closed fields. For further details see [Sr] or [D]. For terminology from model theory, see [S].

In a field $F$ of characteristic $p \neq 0$, the set $F^p$ of pth powers of elements of $F$ forms a subfield. In describing 1-types over $F$, it is convenient to regard $F$ as a vector space over $F^p$, so we employ the following fairly standard terminology [Sr]. Throughout we fix $p$ a prime, and assume all fields are of characteristic $p$.

**Definitions.** Let $A \subset F$, $F$ a field. A **p-monomial of A** is any product of the form $a_1^{k_1} \cdots a_r^{k_r}$, where $a_1, \ldots, a_r$ are distinct elements of $A$ and where $0 \leq k_i < p$, for $i = 1, \ldots, r$. We say $A$ is **p-independent in F** if the set of all p-monomials of $A$ is linearly independent over $F^p$; moreover, a p-independent set $A$ is a **p-basis for F** if every element of $F$ is an $F^p$-linear combination of p-monomials of $A$. Similarly, for $m > 1$ we can define $p^m$-monomials and $p^m$-bases, looking at $F$ over $F^{p^m}$. We denote by $\mathbf{F^{p^\infty}}$ $(= \bigcap\{F^{p^m} | m < \omega\})$ the maximal perfect subfield of $F$.

**Remark.** If we write $[F : F^p] = p^n$, with $0 \leq n \leq \infty$, then any p-basis for $F$ has size $n$ (we do not distinguish infinite cardinalities), and p-bases always exist. In a language which includes + and $\cdot$, "$F$ has a p-basis of size $n$" is first order, and so $n$ is an elementary invariant of the theory of $F$.

**More definitions.** Given $F \subset K$ fields, we say that **K is separable over F** (or $\mathbf{F \subset_{sep} K}$) if p-independence in $F$ is preserved in $K$. We say $F$ is **separably closed** if $F$ has no proper separable algebraic extensions.

**Facts.**

(1) If $\alpha$ is separably algebraic over $F$, in the usual sense that the minimum polynomial $g$ of $\alpha$ over $F$ satisfies $g' \neq 0$, then $F \subset_{sep} F(\alpha)$, and a

p-basis for $F$ remains a p-basis for $F(\alpha)$.

(2) If $t$ is transcendental over $F$, then $F \subset_{\text{sep}} F(t)$, and if $A$ is a p-basis for $F$, then $A \cup \{t\}$ is a p-basis for $F(t)$.

(3) If $F \subset_{\text{sep}} K$ and $A \subset F$, $A \subset B \subset K$ are such that $A$ is a p-basis for $F$, $B$ a p-basis for $K$, then $B-A$ is algebraically independent over $F$.

(4) If $F \subset_{\text{sep}} K \subset_{\text{sep}} L$, then $F \subset_{\text{sep}} L$.

From (1) and (4) above, we see that every field $F$ can be extended uniquely to a **separable closure** $F^{\text{sep}}$, a field such that $F \subset_{\text{sep}} F^{\text{sep}}$, $F^{\text{sep}}$ is separably closed, and $F^{\text{sep}}$ is algebraic over $F$ --hence a p-basis for $F$ remains a p-basis in $F^{\text{sep}}$. From (2) we note that any non-algebraic extension which does not extend a p-basis must be more than a simple transcendental field extension; when we add $t$ we must also add coefficients for $t$ with respect to the original p-basis, and so on. It thus makes sense to incorporate into the language functions which pick out p-dependence coefficients.

__Definitions__. For $n > 0$, let $\{m_i | i = 0, \ldots, p^n-1\}$ be some fixed indexing of the set of p-monomials in variables $x_1, \ldots, x_n$. For each $i$ we introduce an $(n+1)$-place function $\boldsymbol{\lambda_i} (= \boldsymbol{\lambda_{i,n}})$ satisfying

$(*)_{i,n}$ $\quad \lambda_i(y, x_1, \ldots, x_n) = c_i$, if $x_1, \ldots x_n$ are p-independent and also

$$y = \Sigma\, c_i^p\, m_i,$$

$\lambda_i(y, x_1, \ldots, x_n) = 0$, otherwise.

Let **L** be the language of fields $(+, \cdot, 0, 1, {}^{-1}, -)$ together with all the $\lambda_{i,n}$'s. Then any field $F$ of characteristic $p$ can be expanded uniquely to a structure $F'$ for **L** such that the $(*)_{i,n}$'s hold. Also, if $F \subset K$, then for the resulting expansions $F'$ and $K'$ we have $F' \subset K'$ if and only if $F \subset_{\text{sep}} K$. Also, given $F \subset_{\text{sep}} K$ and $\alpha \in K$, the **L**-substructure of $K'$ generated by $F \cup \{\alpha\}$ is the least subfield of $K$ containing $F \cup \{\alpha\}$ over which $K$ is separable. We denote this **L**-substructure, the closure of $F \cup \{\alpha\}$ under the $\lambda$'s, by $\mathbf{F\langle\alpha\rangle}$, reserving $F(\alpha)$ for the usual field-theoretic extension. It should also be clear to the reader by now that this language is unnecessarily complicated if $[F : F^p] = p^n < \infty$, for then all the $\lambda_{i,m}$'s are $0$ for $m > n$, and $\{\lambda_{i,n} | 0 \le i < p^n\}$ would suffice.

Let **$SCN_n$** denote the theory of separably closed fields of characteristic $p$ and invariant $n$, $n \le \infty$. The basic model theoretic result about $SCF_n$, due to Ersov, can be stated in the present setting as follows.

**Theorem 1.** (Ersov [E]). $SCF_n$ is complete. As a theory in **L**, including $(*)_{i,n}$'s, $SCF_n$ has prime model extensions (separable closures) and admits elimination of quantifiers.

We remark that Dan Haran has recently worked out an elimination procedure for this theory.

We now turn our attention to 1-types over a model of $SCF_n$.

**Definitions.** Let $\alpha \in K$, where $F \subset_{sep} K$, with $F, K \models SCF_n$. We say that **$\alpha$ has an F-tree** if the extension $F\langle\alpha\rangle$ does not extend a p-basis of $F$ (which is always the case when $n < \infty$). In this case the type **$typ(\alpha, F)$** of $\alpha$ over $F$ is completely determined by the tree of coefficients of $\alpha$ with respect to any chosen p-basis for $F$. To illustrate, let $p = 2$ and $n = 1$, and take $u \in F-F^p$ as p-basis. Then the tree of $\alpha$ over $F$ looks like:

```
              α = α0
             / \     \
          α00         α01
          / \         / \
      α000  α001  α010    α011
      / \    / \   / \     / \
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where $\alpha_0 = {\alpha_{00}}^2 + {\alpha_{01}}^2 u$, and, for any string $\eta$ of 0's and 1's,

$$\alpha_\eta = \alpha^2_{\eta\frown 0} + \alpha^2_{\eta\frown 1} u.$$

Notice this gives polynomial relations between the levels of the tree, such as

$$\alpha_0 = (\alpha_{000})^4 + (\alpha_{010})^4 u + (\alpha_{001})^4 u^2 + (\alpha_{011})^4 u^3.$$

These can also be viewed as expressions for $\alpha$ in terms of the $2^m$-basis $u$ of $F$ over $F^{2^m}$, with coefficients the $\alpha_\eta$'s where $\text{length}(\eta) = m$.

We also define **$F^{(m)}(\alpha)$**, the m-th level subfield of $F\langle\alpha\rangle$, to be the field $F(\{\alpha_\eta \mid \text{length}(\eta) = m\})$, so $F\langle\alpha\rangle = \bigcup\{F^{(m)}(\alpha) \mid m \in \omega\}$.

It is easy to check whether a type **$p(x)$** given by such a tree is consistent, since one need only check that the polynomial relation on the $x_\eta$'s at any level is consistent.

If $p$ is a type with an F-tree and $\varphi$ is an automorphism of $F$ fixing a p-basis of $F$, then we get a type $q = \varphi^*p$ conjugate to $p$ by replacing $x_\eta$ by $\varphi(x_\eta)$, for each $\eta$ a finite string of $0, \ldots, (p^n-1)$'s.

In the next section we will construct types of transcendence degree 1. By transcendence degree of $p$ over $F$ we mean here the usual algebraic transcendence degree of $F\langle\alpha\rangle$ over $F$, where $\alpha$ realizes $p$. These will necessarily have U-rank 1, and their convenience for us stems from the fact that "forking" means "becomes algebraic over". Thus the following observation, due--at least--to Delon and to Srour, is of use.

**Lemma 2.** Let $\alpha$ realize $p$ over $F$, where $p$ has an F-tree and $F \models SCF_n$. If $\beta$ is separably algebraic over $F\langle\alpha\rangle$, then $\beta$ has an F-tree, all of whose entries are separably algebraic over $F\langle\alpha\rangle$. Moreover, if $\beta$ has degree $d$ over $F^{(r)}(\alpha)$, then $\beta_\eta$ has degree $\leq d$ over $F^{(r+m)}(\alpha)$, where $m = \text{length}(\eta)$.

Proof: Our notation implies that we are working with respect to some fixed p-independent $u_1, \ldots, u_n$ $F$, and $\beta_{\eta\hat{i}}$ is the p-th root of the i-th coefficient of $\beta_\eta$ with respect to $u_1, \ldots, u_n$, $i = 0, \ldots, p^n-1$. Since $F\langle\alpha\rangle = \cup\{F^{(k)}(\alpha) | k \in \omega\}$ it follows from $\beta$ separably algebraic over $F\langle\alpha\rangle$ that $\beta$ is separably algebraic over $F^{(r)}(\alpha)$, for some $r$. Since $\beta$ is also purely inseparable over $F^{(r)}(\alpha)(\beta^p)$, it follows that $\beta \in F^{(r)}(\alpha)(\beta^p)$. If $\beta$ has degree $d$ over $F^{(r)}(\alpha)$, then so does $\beta^p$. Applying each $\lambda_i$ once gives us that $\beta_{0\hat{i}} \in \lambda_i(F^{(r)}(\alpha)(\beta^p)) \subset F^{(r+1)}(\alpha)(\beta) = F^{(r+1)}(\alpha)(\beta^p)$. Continuing for $m$ steps, we get $\beta_\eta \in F^{(r+m)}(\alpha)(\beta^p)$ as above, of degree over $F^{(r+m)}(\alpha)$ bounded by $d$. □

**Corollary 3.** If $\beta$ is separably algebraic over $F\langle\alpha\rangle$, $F$ and $\alpha$ are as in the lemma, then the transcendence degree of $F\langle\beta\rangle$ over $F$ is at most that of $F\langle\alpha\rangle$ over $F$.

Proof: Immediate. □

**Corollary 4.** For all $n > 0$, there are $2^{\aleph_0}$ pairwise orthogonal 1-types over any model $F$ of $SCR_n$.

Proof: Let $u \in F - F^p$ and let $\sigma : \mathbb{N} \to \mathbb{N}$. We describe an F-tree with all coefficients of the form $\lambda_i(\_, u)$ --i.e., a type $p^\sigma$ with all entries in its tree p-dependent on u:

$$x_0 = x_1{}^p + (x_1{}^{\sigma(0)})^p u$$

$$x_1 = x_2{}^p + (x_2{}^{\sigma(1)})^p u$$

$$\dots$$

$$x_n = x_{n+1}{}^p + (x_{n+1}{}^{\sigma(n)})^p u.$$

Then $x_{n+1}$ is transcendental over F and algebraic of degree exactly $p\,\sigma(n)$ over $F(x_n)$ --see [J]. Also, $\mathbf{p}^\sigma$ has rank 1, since the corresponding extension has transcendence degree 1 over F. Moreover, none of this changes if we extend F to a larger model.

In order for $\mathbf{p}^\sigma$ and $\mathbf{p}^\tau$ to fail to be orthogonal, then, the functions $\sigma$ and $\tau$ must be related. For example, suppose $\alpha$ realizes $\mathbf{p}^\sigma$ and $\beta$ realizes $\mathbf{p}^\tau$, where $\beta$ is algebraic over $F\langle\alpha\rangle$, say, $\beta$ has degree d over $F^{(k)}(\alpha) = F(\alpha_k)$. Then $\alpha_k$ is algebraic over $F\langle\beta\rangle$, say of degree d'. By the lemma, we have $\beta_1$ algebraic over $F(\alpha_{k+1})$ of degree $\leq d$ and $a_{k+1}$ is algebraic over $F(\beta_1)$ of degree $\leq d'$. By considering

$$F(\beta) \subset F(\beta, \alpha_k) \subset F(\beta, \alpha_{k+1}) \subset F(\beta_1, \alpha_{k+1})$$

and $F(\beta) \subset F(\beta_1) \subset F(\beta_1, \alpha_{k+1})$, we get that

$$p\,\tau(0) \leq [F(\beta_1, \alpha_{k+1}) : F(\beta)] \leq d'dp\sigma(k).$$

Similarly, $\tau(m) \leq d'd\,\sigma(k+m)$ and $\sigma(k+m) \leq d'd\,\tau(m)$.

Thus we see that $\sigma(k+m)$ and $\tau(m)$ are bounded by d'd times each other. It is easy to find $2^{\aleph_0}$ choices of $\sigma$ such that no pair is related in this way, we get $2^{\aleph_0}$-many orthogonal $\mathbf{p}^\sigma$'s. □

Of course the types in the above corollary are <u>not</u> conjugate, having distinctly different "shapes", and so for DOP there remains work to be done.

## §2. Orthogonal conjugate types.

In the present section we describe a large family of mutually orthogonal conjugate types of transcendence degree 1. These types will serve as witnesses to Shelah's "DOP" (dimensional order property), as well as to related properties: "DIDIP" (dimensional discontinuity property) and "$\omega$-DOP" (an infinitary strengthening of both) [S, Chapter 5] and [S2, §2]. A similar family of types was studied independently by Delon, using a different criterion for orthogonality.

Our methods require more extensive computation than Delon's, but do yield some additional information. It is not clear how to get the full $\omega$-DOP with less computation.

We assume that $M$ is a monster model of $SCF_n$, where $n$ and $p$ are fixed throughout. Fix $u \in M-M^p$. Our goal is to define a type $\mathbf{p}$ over $M^{p^\infty}(u)$ in such a way that its conjugates over $u$ are mutually orthogonal. More precisely, we associate to each countable sequence $A$ in $M^{p^\infty}$ a type $\mathbf{p}_A \in S_1(Au)$ of transcendence degree 1, in such a way that:

I. If $\varphi \in Aut(M/u)$, then $\mathbf{p}_{\varphi A} = \varphi * \mathbf{p}_A$.

II. If $A$ and $B$ are distinct sequences in $M^{p^\infty}$, then $\mathbf{p}_A \perp \mathbf{p}_B$.

The intention is that an element realizing $\mathbf{p}_A$ will be $p^m$-dependent on $u$ for all $m$, in a way that depends on $A$.

Before describing the construction in detail, we explore the model theoretic significance of such a family of types. Consider first the type $\mathbf{p}_A$ associated with a sequence $A = (a_1, a_2)$ of length 2. Let $A_i = \{a_i, u\}$, $i = 1, 2$. If $a_1, a_2$ are independent over $u$, then we have the following situation:

1. $\mathbf{p}_A \in S_1(A_1 \cup A_2)$.
2. $A_1$ is independent from $A_2$ over $A_1 \cap A_2$.
3. $\mathbf{p}_A \perp A_1$, $\mathbf{p}_A \perp A_2$.

Indeed, by a general model theoretic criterion, to see that $\mathbf{p}_A \perp A_i$ it suffices to take an independent conjugate $B$ of $A$ over $A_i$ and to check that $\mathbf{p}_A \perp \mathbf{p}_B$, which holds by II.

Now, conditions 1.-3. constitute a rather strong form of the DOP. According to the definition given in [S2, §2] we should take $A_1$, $A_2$, and $A_1 \cap A_2$ to be quite saturated models $(a, \kappa(T))$ --saturated, in fact) and we need only have $\mathbf{p}_A$ defined over the $(a, \kappa(T))$ --prime model over $A_1 \cup A_2$. One can easily move upward from the situation as we have described it by putting $A_1 \cap A_2$, $A_1$, $A_2$ inside $(a, \kappa(T))$ --saturated models $K_0$, $K_1$, $K_2$ taken as freely as possible, and then $\mathbf{p}_A$ can be replaced by a nonforking extension over the $(a, \kappa(T))$ --prime model $K$ over $K_1 \cup K_2$.

For the **$\omega$-DOP**, which has not previously been defined, we require a type $\mathbf{p}_K$ over the $(a, \tau(T))$--prime model over the union of a infinite family of independent models $K_i$ (all containing a fixed model $K_0$), satisfying $p_K \perp \cup\{K_i | i \neq j\}$ for each $j$. To do this with our examples we take $A = \{a_i | i \in \omega\}$ an infinite sequence of elements of $M^{p^\infty}$ independent over $u$ and let $A_i = \{a_i, u\}$, so that we have:

1. $p \in S_1(\cup\{A_i | i \in \omega\})$,
2. $A_i$ is independent from $\cup\{A_i | i \neq j\}$ over $u$, and
3. $p_A \perp \cup\{A_i | i \neq j\}$ for all $j \in \omega$.

Again, this situation can be blown up to a similar situation over models.

The DOP may be viewed as a somewhat disguised procedure for encoding an arbitrary binary relation into a suitable model of a given theory. Namely, one builds a model $M^*$ containing the model $M_0$ and a large family of independent submodels $M_i$ containing $M_0$, with each pair $M_i$, $M_j$ isomorphic to $M_1$, $M_2$ for $i \neq j$. Letting $p_{ij}$ be the type corresponding to $p_K$, we may think of the $p_{ij}$-dimension of $M^*$ as an attribute of the pair $M_i$, $M_j$. The orthogonality hypothesis amounts to the statement that these attributes are independent. Thus an arbitrary symmetric graph can be encoded into $M^*$ in a sense which is sufficiently precise to yield the existence of many nonisomorphic models of specified cardinality. In a similar sense the $\omega$-DOP corresponds to an encoding of an arbitrary infinitary symmetric relation into a suitable model $M^*$.

One special case of the $\omega$-DOP would be relevant to the classification of sufficiently saturated models of a stable theory. This is the DIDIP [S2, §2]: there is an increasing sequence $M_n$ of $(a, \kappa(T))$-models, and a type $p_M$ over the $(a, \kappa(T))$-prime model $M$ over their union, with $p_M \perp M_n$ for all $n$. This property is independent of the DOP, and can be used to produce many quite saturated models of suitable cardinality. A superstable theory cannot exhibit this pathology, which is of practical significance only in the absence of the DOP, as an alternate source of many models.

In any case the $\omega$-DOP provides a universal source of pathology in the structural analysis of separably closed fields, strictly stronger that the DOP and DIDIP.

We now turn to the description of the specific types $p_A$ we have in mind.

For the construction and algebraic properties of $p_A$ below, we no longer need assume that $M$ is highly saturated. Let $\overline{G} = (G_i)_{i\geq 0}$ be a sequence of polynomials $G_i(x) \in Z(A)[x]$, and associate with $\overline{G}$ the type $p_{A,G}(X) \in S_1(Au)$ which expresses the condition:

(*) There is a sequence $x = x_0, x_1, x_2, \ldots$ with $x_0$ transcendental over $A \cup \{u\}$, and $x_i = x_{i+1}{}^p + G_i(x_{i+1})^p u$ for all $i$.

In other words, $p_{A,\overline{G}}$ says that $x$ is $p^m$-dependent on $u$ for all $m$, and describes the associated tree of coefficents explicitly, as in §1. Given the sequence $\overline{G}$, we may produce a sequence $\overline{H} = (H_i)$ with $H_i(x) \in Z(A)[u,x]$ for which (*) implies $x = H_i(x_i)$, namely:

$$H_0(x) = x,\ H_{i+1}(x) = H_i(x_{i+1}{}^p + G_i(x_{i+1})^p u).$$

The equation $x = H_i(x_i)$ amounts to a description of the expansion of $x$ with respect to the $p^i$-basis $\{u\}$. Notice that $p_A$ always has transcendence degree 1 over $A \cup \{u\}$.

In practice the polynomials $G_i$ will be of one of the forms:

(G1) $G_i = 0$,

(G2) $G_i = a_j$ for some $j = j(i)$,

(G3) $G_i = x^m$ for some $m = m(i)$.

Once the sequence $\overline{G}$ if fully specified, the types $p = p_{A,\overline{G}}$ are determined. We shall impose various constraints on the sequence $\overline{G}$, particularly with respect to the value of $m = m(i)$ (when defined), in order to ensure the correctness of our orthogonality assertion: $p_A \perp p_B$ for $A$ and $B$ distinct sequences from $M^{p^\infty}$.

Our first constraint, concerning the distribution of polynomials of the form (G2) in the sequence $\overline{G}$, is quite mild. Notice first that for $x$ a realization of $p_{A,\overline{G}}$, the type of $x_i$ (a solution of $H_i(y) = x$) is of the form $p_{A,\ell_i\overline{G}}$, where $\ell_i\overline{G} = (G_{i+j})_{j\geq 0}$, a left shift of $\overline{G}$. If $p_A = p_{A,\overline{G}}$, let us write $\Lambda_i p_A = p_{A,\ell_i\overline{G}}$. Then our first constraint is:

(C1) For distinct sequences $A$, $B$ in $M^{p^\infty}$, and any $i$, $j$, $\Lambda_i\, p_A \neq \Lambda_j p_B$.

(If we are willing to consider only pairs $A$, $B$ which are distinct as sets, we can use the simpler constraint, "for all $j$, $\{i \mid j = j(i)\}$ is infinite".)

The constraint (C1) is certainly necessary for the desired orthogonality conditions. Our intention is to combine it with the following condition, which will require the most work to achieve:

(C*) If $\bar{c} = c_1, c_2, \ldots$ is a sequence of $M^{p^\infty}(u)$-algebraically independent realizations of $p_{A_i}$, $i = 1, 2, \ldots,$ and if $d$ is a realization of $p_A$ which is algebraic over $M^{p^\infty}(u, \bar{c})$, then $d \in \bar{c}$.

**Lemma 1.** If the sequence $\bar{G}$ satisfies the constraints (C1) and (C*), then for $A$, $B$ distinct sequences in $M^{p^\infty}$, $p_A \perp p_B$.

Proof: By a general model theoretic principle, it suffices to check that for sequences $\bar{c}$, $\bar{d}$ of independent realizations of $p_A$, $p_B$ over $M^{p^\infty}(u)$, $\bar{c}$ is independent from $\bar{d}$ over $M^{p^\infty}(u)$. We may take $\bar{d} = \tilde{d}, e$ to be of finite length $\ell$ (so that $\tilde{d}$ is of length $\ell-1$) and suppose that $\bar{c}$ is independent from $\tilde{d}$ over $M^{p^\infty}(u)$. It then suffices to check that the type realized by $e$ over $M^{p^\infty}(u, \bar{c}, \tilde{d})$ is the nonforking extension of $p_B$ to this set.

Now $u$ is a $p^\infty$-basis for $M^{p^\infty}(u, \bar{c}, \tilde{d})$ (in $M$), and for technical reasons it is convenient to close this field under the coordinate functions with respect to $\{u\}$, getting the field $L = M^{p^\infty}(u, c_{ij}, d_{ij})_{j\in\omega}$, where $H_j(c_{ij}) = c_i$, $H_j(d_{ij}) = d_j$. Then $L$ is algebraic over $M^{p^\infty}(u, \bar{c}, \tilde{d})$, and the nonforking extension of $p_B$ to $L$ is obtained by supplementing $p_B$ by the clause:

$x$ is transcendental over $L$.

Thus $e$ is independent from $\bar{c}$, $\tilde{d}$ over $M^{p^\infty}(u)$, unless $e$ is algebraic over $L$. In the former case, $\bar{c}$, $\bar{d}$ will be independent over $u$ by the calculus of independence, and in the latter case $e$ will be algebraic over $M^{p^\infty}(u, \bar{c}, \tilde{d})$, and (C*) applies: $e \in \bar{c} \cup \tilde{d}$. As $p_A \neq p_B$, $e \in \tilde{d}$, contradicting the supposed independence of the sequence $\bar{d}$. □

We have translated the original model theoretic problem into the purely algebraic one described by (C*). The next step will be to transform the constraint (C*) on $p_A$, $p_B$ into a simpler constraint on all the translates $\Lambda_i p_A$, $\Lambda_i p_B$. For this we need an additional constraint on the polynomials $H_i$. Make a slight change in notation: $H_i^A = H_i(x, \overline{a}, u)$, where $H_i \in \mathbb{Z}[x, \overline{y}, z]$. We require:

(C2) $\lim\inf_{i\to\infty}(\deg_z H_i)/p^i = 0$.

Whenever $G_i = 0$, we have $\deg_z H_i = \deg_z H_{i+1}$, so this constraint is easily met.

**Lemma 2.** Let $\overline{G}$ satisfy the constraint (C2), and let $\overline{c}$, $d$ be as described in (C*). Then for some $i$, the elements $\overline{c}'$, $d'$ defined by $H_i(c_j') = c_j$, $H_i(d') = d$ are algebraically dependent over $M^{p^\infty}$.

Proof: We begin with an algebraic relation $f(\overline{c}, d, u) = 0$ with $f \in M^{p^\infty}[\overline{x}, t, z]$. Let $\delta$ be the total degree of $f$, and choose $i$ so that $\delta \cdot \deg_z H_i < p^i$. We rewrite $f = 0$ in terms of $\overline{c}', d', u$ using $H_i$, getting $f^*(\overline{c}', d', u) = 0$ for a suitable polynomial $f^*$ whose degree in $u$ is less than $p^i$, while the $c_j'$, $d'$ occur to powers which are multiples of $p^i$. That is,

$$f^*(\overline{c}', d', u) = \Sigma_{j<p^i} f_j^*(\overline{c}', d')^{p^i} u^j, \text{ with } f_j^* \in M^{p^\infty}[\overline{x}, t].$$

Thus $f_j^* = 0$ for all $j < p^i$, and at least one of these relations is nontrivial.

Remark. There is a remarkable uniformity here--the single choice of $i$ --which depends heavily on the constraint (C2). Realizations $c$ of $\mathbf{p}_A$ or $c_j$ of $\Lambda_j \mathbf{p}_A$ are related by $c = H_j(c_j)$, a relation that depends essentially on $u$. When the relations are shifted suitably far down the tree, however, we get the relation $c_i = (c_{j+i})^{p^j}$, in which $u$ has vanished.

It remains to prove the following variant of (C*):

**Theorem 3.** The sequence $\overline{G}$ can be chosen so that for any $i$, the types $\mathbf{q}_A = \Lambda_i \mathbf{p}_A$ satisfy:

(C**) If $c_1, c_2, \ldots$ is a sequence of $M^{p^\infty}(u)$-algebraically independent realizations of $\mathbf{q}_{A_i}$, respectively, and if $d$ is a realization of $\mathbf{q}_A$ which is algebraic over $M^{p^\infty}(\overline{c})$, then $d \in \overline{c}$.

Proof: We will impose additional constraints of the following form on $\mathbf{p}_A$ (which may be easily rephrased as constraints on $\overline{G}$), for suitable values of $i, j, k, m$:

(C3. i,j,k,m) $$x_i = x_k^{p^{k-i}} + (x_k^m)^{p^{k-i}} u^{p^{j-i}}.$$

Specifically, for each fixed $m$ we want a sequence of values $i, j, k$ for which the constraint (C3. i,j,k,m) is valid, and so that $k-j$ and $j-i$ both tend to infinity. There is of course no difficulty in arranging this.

We now verify (C**). Consider the elements $\bar{c}$, d, which we take to be realizations of $p_{A_i}$, $p_B$. If $f(\bar{x}, z) \in M^{p^\infty}[\bar{x}, z]$ is an irreducible polynomial with $f(\bar{c}, d) = 0$, we claim first that if $d \notin \bar{c}$, then there is a value of $m$ for which:

(*) $f$ does not divide $\Sigma x_i m(\partial f/\partial x_i) + z^m(\partial f/\partial z)$.

Indeed, if (*) fails for all $m$, we get a relation of the form:

$$f \cdot \xi = \mathcal{U} \cdot \text{grad } f$$

where $\xi$ is a vector of polynomials and $\mathcal{U}$ is a van der Monde matrix. This then yields:

$$f \text{ divides } (\det \mathcal{U})(\partial f/\partial w), \ (w = x_i \text{ or } z).$$

If $f$ divides $\det \mathcal{U}$, then $f$ is necessarily of the form $z - x_i$, and $d \in \bar{c}$, as claimed. In the remaining case, $f$ divides $\partial f/\partial w$ for all $w = x_i$, $y_j$, or $z$ and then grad $f = 0$. In other words, $f = f^*(\bar{x}^p, z^p)$, $f^* \in M^{p^\infty}[\bar{x}, z]$, which is certainly not irreducible, a contradiction.

So we may fix a value of $m$ for which (*) holds, and take a triple $(i, j, k)$ for which (C3. i,j,k,m) holds, chosen such that

(1) $p^j \deg f < p^k$ and (2) $p^i \deg f < p^j$.

Consider the form of the relation $f(\bar{c}, d) = 0$ after the substitution $c_j = H_k(c_j')$, $d = H_k(d')$, say $f^*(\bar{c}', d') = 0$. As $\deg_u H_i < p^i$, we have $\deg_u H_k < p^j$ and hence $\deg_u f^* < p^k$, by (1).

So $f^*(\bar{c}', d') = \Sigma_{\ell < p^k} f_\ell(\bar{c}^*, d^*)u^\ell$, where $\bar{c}^* = \bar{c}'^{p^k}$, $d^* = d'^{p^k}$, and thus $f_\ell(\bar{c}^*, d^*) = 0$ for $\ell < p^k$. As $H_k(x) = x^{p^k}$ + terms of higher degree in $u$, $f_0 = f$.

Now consider $g = f_{p_j}$. Then $g(\bar{c}^*, d^*)$ is the coefficient of $u^{p^j}$ in $f(H_k(\bar{c}'), H_k(d'))$, and $H_k = H_i(x^{p^{k-i}} + x^{mp^{k-i}} u^{p^{j-i}})$, where $H_i(x)$ may be written more explicitly as a polynomial of the form $H_i^*(x^{p^i}, u)$, and $\deg_u H_i^* < p^i$. As $H_i^*(x, u) = x^{p^i}$ + terms of higher degree in $u$, $H_i(x^{p^{k-i}} + x^{mp^{k-i}} u^{p^{j-i}})$ is of the form:

$$x^{p^k} + x^{mp^k} u^{p^j} + \text{terms of u-degree less than } p^i \text{ or greater than } p^j.$$

By conditions (1) and (2), the coefficient of $u^{p^j}$ in $f(H_k(\bar{c}'), H_k(d'))$ is the same as the coefficient of $u^{p^j}$ in $f(\bar{c}^* + \bar{c}^{*m} u^{p^j}, d^* + d^{*m} u^{p^j})$, and by a Taylor series expansion we find that

$$g(\bar{c}^*, d^*) = \Sigma c_r^{*m}(\partial f/\partial x_r) + d^{*m}(\partial f/\partial z),$$

where all partial derivatives are evaluated at $\bar{c}^*$, $d^*$.

Now $f(\bar{c}^*, z)$ is the minimal polynomial for $d^*$ over $M^{p^\infty}(\bar{c}^*)$, and hence divides $g(\bar{c}^*, z)$. As $f$ is irreducible and $\bar{c}^*$ are algebraically independent, $f$ divides $g$. But $m$ was chosen to make this false. This contradiction finishes our theorem. □

By taking $\bar{G}$ to satisfy (C1), (C2), together with enough instances of (C3. i,j,k,m), we have the following:

**Corollary 4.** For any $0 < n \leq \infty$, $SCF_n$ has ω-DOP, hence DOP and DIDIP.

Proof: Immediate from the discussion at the beginning of the section. □

**Remark.** As a corollary of the proof of Theorem 3, we can reach conclusions similar to those of the theorem about types which behave like the $p_A$'s for arbitrarily long stretches; i.e., the proof applies locally. We shall make use of this phenomenon in the sections which follow.

## §3. Rank and transcendence degree.

In considering stability in an algebraic context, one question which arises is the relationship of model-theoretic rank to an algebraically "natural" rank (e.g., transcendence degree, in the case of separably closed fields). The aim of this section is to consider situations when U-rank and transcendence degree do not correspond; we begin by considering a condition under which the two ranks are the same.

**Definitions.** 1. Let $F \underset{sep}{\subset} K$ be separably closed, and let $\{K_\nu | \nu < \kappa\}$ be subfields of $K$ such that

(i) $K_\nu = F(B_\nu)^{sep}$ for some $B_\nu$ independent over $F$,

(ii) $K_\nu \subset K_{\nu'}$ for $\nu < \nu'$, and

(iii) $K = \bigcup \{K_\nu | \nu < \kappa\}$.

Then we say that $\{K_\nu | \nu < \kappa\}$ is a **tower** for $K$ over $F$.

2. Let **p** be a type over F, say **p** = typ($\alpha$, F). Then we say **p** is **accessible** from F if there exists a tower for $K = F\langle\alpha\rangle^{sep}$ over F.

**Examples.**

(i) If **p** is a type of transcendence degree 1 over F, then **p** is accessible.

(ii) If **p** = **p**(x) is the generic type--the one which satisfies no incidental relations over F (see [D])--then p is accessible, with $B_0 = \{x\}$ and $B_\nu = B_{\nu+1} = \{\lambda_i(b) | b \in B_\nu, i = 0, \dots, p^n-1\}$ in case n is finite, and $B_\nu = \{x\}$ for all $\nu$ when $n = \infty$.

**Theorem 1.** If $\alpha$ realizes an accessible type **p** of U-rank r over F, F separably closed, then the transcendence degree of $F\langle\alpha\rangle^{sep}$ over F is r, and **p** is a product of r accessible types of transcendence degree 1 over F.

Proof: Let $K = F\langle\alpha\rangle^{sep} = \bigcup_\nu F(B_\nu)^{sep}$ as above. Since $F\langle\alpha\rangle^{sep}$ cannot contain more than r independent elements, from some $\nu'$ on, $card(B_{\nu'}) = card(B_\nu) = m \leq r$, and $card(B_{\nu'}) = tr\ deg_F\ K$. Now $tr\ deg_F\ K \geq r$, since whenever **p** forks, the transcendence degree must drop. Thus $card(B_{\nu'}) = r$, and so every element of $B_{\nu'}$ is of transcendence degree 1 and U-rank 1, giving **p** as the desired product.□

We next construct examples of types where U-rank and transcendence degree differ, answering a question of Delon [D, p.45].

**Theorem 2.** Let F be separably closed, with $F \neq F^p$. Then there exists $\alpha$ such that $F\langle\alpha\rangle$ has transcendence degree 2 over F and typ($\alpha$, F) has U-rank 1.

Proof: Let $u \in F^p - F$, and take A and B distinct sequences from $F^{p^\infty}$, as in §2. We describe the tree of coefficients for a type $\mathbf{q} = \mathbf{q}_{A,B}$ over F:
Start with $x = y_1 + z_1 u$, where $y_1$ and $z_1$ are algebraically independent over F. Next begin to build the trees of types on $y_1$ and $z_1$ according to $\mathbf{p}_A$ and $\mathbf{p}_B$, respectively, as in §2. Cofinally often, however, introduce the following twist:

$$(*_n)\quad y_n = y_{n+1}{}^p + z_n{}^p u,\ z_n = z_{n+1}{}^p + y_n{}^p u,$$

then begin again, this time building $\mathbf{p}_A$ and $\mathbf{p}_B$ on $y_{n+1}$ and $z_{n+1}$. Notice that it is just the twist in $(*_n)$ which makes our type inaccessible. The only requirement on n's such that $(*_n)$ happens, is that they be sparse, so that we have arbitrarily long stretches in which we are constructing $p_A$ and $p_B$. To see that $(*_n)$ is consistent, we consider three steps:

1) add $y_{n+1}$, $z_{n+1}$ algebraically independent over F,
2) solve for w in $w - (y_{n+1} + w^p u)^p u - z_{n+1} = 0$ (which is OK since this is separable in w), and let $z_n$ be a solution for w, and
3) let $y_n = y_{n+1} + z_n^p u$.

our type q has transcendence degree 2 over F. In order for q to fork over K, where $F \underset{sep}{\subset} K$, the transcendence degree must drop. This makes $y_n$ and $z_n$ algebraically dependent over K for some, hence all, n. But since we have built arbitrarily long stretches of the construction of $p_A$ and $p_B$ into the description of q, and since $A \neq B$ we conclude that one, hence both, of $y_n$ and $z_n$ are algebraic over K (as remarked at the end of §2). Thus the only way for q to fork over K is via a realization of q in K, and q has U-rank 1. □

**Corollary 3.** There exist separably closed fields $F \underset{sep}{\subset} K$ such that K has transcendence degree 2 over F, and such that there is no L with $F \underset{sep}{\subset} L \underset{sep}{\subset} K$ with L of transcendence degree 1 over F.

Proof: Take $\alpha$ realizing q as in Theorem 2, and $K = F\langle\alpha\rangle^{sep}$. □

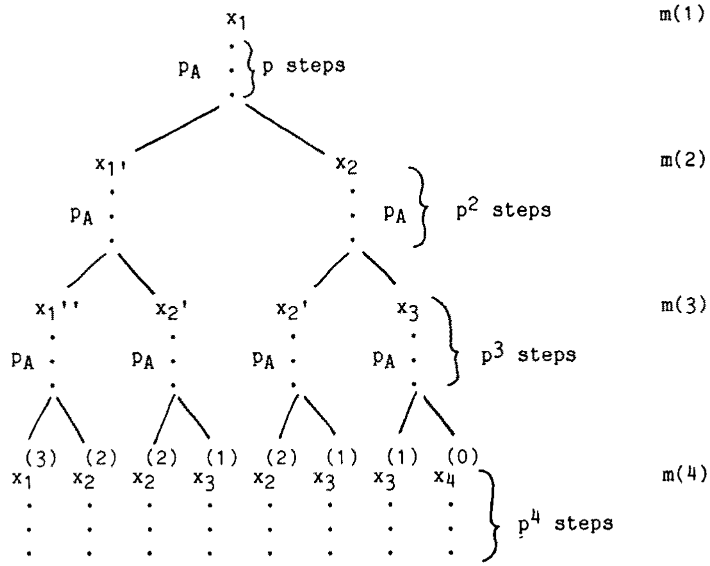
We remark that one can similarly concoct types of rank 1 and transcendence degree m, for any $m > 1$.

## §4. Types of rank $\omega$.

Here the information about the types in §2 is used a second time to control rank, again answering a question of Delon [D].

**Theorem 1.** Let F be separably closed, $F \neq F^p$. Then there exists a type over F of U-rank $\omega$.

Proof: Let $u \in F - F^p$ and let $p_A$ be as in §2, where A is any sequence from $F^{p^\infty}$. We construct the required type q according to the following picture:




To be precise, for each $i \geq 1$ and $j \geq 0$, we follow the construction of $\mathbf{p}_A$ below $x_i^{(j)}$ for $p^{i+j}$ steps, from $y_1 = x_i^{(j)}$ to $y_{p^{i+j}}$. Next introduce $x_i^{(j+1)}$ and $x_{i+1}^{(j)}$ via $y_{p^{i+j}} = (x_i^{(j+1)})^p + (x_{i+1}^{(j)})^p u$, and now continue with $p_A$ for $p^{i+j+1}$ steps below $x_i^{(j+1)}$ and $x_{i+j}^{(j)}$.

Consistency of this tree is easy to see, since for each $k$ we can take $k$ arbitrary elements $z_1, \ldots, z_k$, and build the coefficient tree back up from level $m(k) = p^{k-1} + p^{k-2} + \ldots + p + k - 1$ by letting $x_i$ $= z_i$, $i = 1, \ldots, k$.

Next we verify that $\mathbf{q}$ has rank $\leq \omega$. Let $\alpha$ be a realization of a forking extension of $\mathbf{q}$. Then at some level of the tree of coefficients for $\alpha$ (which we choose to be of the form $m(k)$ for some $k$), we must find an algebraic relation among the corresponding $\alpha_i^{(j)}$ 's, $i + j = k$. By Lemma 1.2, the algebraic dependence persists down the tree , so we may assume that $k$ is large enough that, using the $\mathbf{p}_A$'s as in Section 2, we conclude some $\alpha_i^{(j)}$ is actually algebraic. Once $\alpha_i^{(j)}$ is known to be algebraic, the entire tree has finite transcendence degree, since all the $a_{i'}^{(j')}$ for $i' > i$ and $j' > j$ become algebraic once $\alpha_i^{(j)}$ is. It then follows that for all subsequent $m(k')$ we get that $\alpha_1^{(k'-1)}, \ldots, \alpha_{i-1}^{(k'-i+1)}, \alpha_{k'-j+1}^{(j-1)}, \ldots, \alpha_{k'}^{(0)}$ generate this level, hence the transcendence degree is at worst $i - 1 + j$, for all $k' \geq k$. Thus the resulting type of $\alpha$ has rank at most $i - 1 + j$, and so **q** has rank $\leq \omega$.

To see that any finite rank is achievable, note that making $x_i^{(j)}$ algebraic gives rise to no restriction on the choices for $x_{i'}^{(j')}$ for $i' < i$ and $j' < j$, so that the type can continue to fork $i + j - 1$ times. Thus **q** has rank exactly $\omega$, as desired. □

**Remark.** Similar constructions can be made to produce types of other countable ranks.

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Current addresses of authors:

Z. Chatzidakis, Dept. of Math, Princeton U., Princeton, N.J. 08544
G. Cherlin, Dept. of Math, Rutgers U., Hill Center, Busch Campus, New Brunswick, N.J. 08903
S. Shelah, Hebrew U., Math Inst., 91904 Jerusalem
G. Srour, Dept. of Math, Simon Fraser U., Burnaby, B.C. V5A1S6
C. Wood, Dept. of Math, Wesleyan U., Middletown, CT 06457