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ISRAEL JOURNAL OF MATHEMATICS 83 (1993), 369–374

NUMBER OF OPEN SETS FOR A TOPOLOGY WITH A COUNTABLE BASIS

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ABSTRACT

Let T be the family of open subsets of a topological space (not necessarily Hausdorff or even T_0). We prove that if T has a countable base and is not countable, then T has cardinality at least continuum.

Topological spaces are not assumed to be Hausdorff, or even T_0.

THEOREM 1: *Let T be the set of open subsets of a topological space, and suppose that T has a countable base B (more precisely, B is a countable subset of T which is closed under finite intersections, and the sets in T are the unions of subsets of B). Then the cardinality of T is either $2^{\aleph_0}$ or $\leq \aleph_0$.*

This answers a question of Kishor Kale. We thank Wilfrid Hodges for telling us the question and for writing up the proof from notes. In a subsequent work [2] we shall deal with the case $\lambda \leq |B| < 2^\lambda$, $|T| > |B|$, λ strong limit of cofinality $\aleph_0$ and prove that $|T| \geq 2^\lambda$.

* Partially supported by the Basic Research Fund, Israeli Academy of Sciences. Publication no. 454 done 6,8/1991.
Partially sponsored by the Edmund Landau Center for research in Mathematical Analysis, supported by the Minerva Foundation (Germany).
Received April 26, 1992 and in revised form July 29, 1992

The proof of Theorem 1 shows that if $|T| > \aleph_0$, then for some countable set Y of points, $\{U \cap Y : U \text{ open}\}$ has the cardinality of the continuum. In fact we can find one of a few "basic behaviours", most notably we can find points x_ν for $\nu \in {}^{\omega>}2$ such that for every $\eta \in {}^{\omega}2$ for some open set U_η we have $x_\nu \in U_\eta \Leftrightarrow \nu \triangleleft \eta$ (see [2]).

Our proof begins with some notation. A set Ω is given, together with a countable family B of subsets of Ω; $\Omega = \bigcup B$ and B is closed under finite intersections. We write T for the set of all unions of subsets of B. Thus T is a topology on Ω and B is a base for this topology.

We write X, Y etc. for subsets of Ω. We write $T(X)$ for the set $\{X \cap Y : Y \in T\}$, and likewise $B(X)$ with B in place of T. We say X is **small** if $|T(X)| \leq \omega$, and **large** otherwise.

LEMMA 2: *If $|\Omega| = \aleph_0$ and $|T| > \aleph_0$ then $|T| = 2^{\aleph_0}$.*

Proof: Identify Ω with the ordinal ω, and list the set B by a function ρ with domain ω, so that $B = \{\rho(m) : m < \omega\}$. Then a set X is in T if and only if

$$(\exists Y \subseteq \omega)(\forall n \in \omega)\ (n \in X \leftrightarrow \exists m(m \in Y \wedge n \in \rho(m))).$$

Thus T is an analytic set, and so its cardinality must be either $2^{\aleph_0}$ or $\leq \aleph_0$ (cf. Mansfield and Weitkamp [1] Theorem 6.3). $\square_2$

LEMMA 3: *Suppose Ω is linearly ordered by some ordering $\preceq$ in such a way that the sets in T are initial segments of Ω and any initial segment of the form $(-\infty, x)$ is open. If $|T| > \aleph_0$ then $|T| = 2^{\aleph_0}$.*

Proof: Suppose on the contrary that $x_0 < |T| < 2^{\aleph_0}$. As B is countable, the linear order has a countable dense subset D, but as $|T| < 2^{\aleph_0}$, the rationals are not embeddable in D, i.e. D is scattered. By Hausdorff's structure theorem for scattered linear orderings, D has at most countably many initial segments (cf. Mansfield and Weitkamp [1] Theorem 9.21), a contradiction. $\square_3$

Henceforth we assume that Ω is uncountable and large, and that $|T| < 2^{\aleph_0}$, and we aim for a contradiction. Replacing Ω by a suitable subset if necessary, we can also assume:

HYPOTHESIS: *The cardinality of Ω is $\aleph_1$.*

Finally we can assume without loss that if x, y are any two distinct elements of Ω then there is a set in B which contains one but not the other. (Define x

and y to be **equivalent** if they lie in exactly the same sets in B. Choose one representative of each equivalence class.)

LEMMA 4: *If for each $n < \omega$, X_n is a small subset of Ω, then $\bigcup_{n<\omega} X_n$ is small.*

Proof: Each X_n has a countable subset Y_n such that if V, W are elements of T with $V \cap X_n \neq W \cap X_n$ then there is some element $y \in Y_n$ which is in exactly one of V, W. Now if V, W are elements of T which differ on $\bigcup_{n<\omega} X_n$, then they already differ on some X_n and hence they differ on $Y = \bigcup_{n<\omega} Y_n$. But Y is countable; so Lemma 2 implies that either Y is small or $|T(Y)| = 2^{\aleph_0}$. The latter is impossible since $|T| < 2^{\aleph_0}$, and so Y is small, hence $\bigcup_{n<\omega} X_n$ is small. $\square_4$

Our main argument lies in the next lemma, which needs some further notation. Let Z be a subset of Ω. The Z-**closure** of a subset X of Z is the set $\mathrm{cl}_Z(X)$ of all elements y of Z such that every set in B which contains y meets X. Given an element x of Z and a subset X of Z, we write $\mathrm{back}_Z(x, X)$ for the set $\{y \in Z : y \notin X \cup \mathrm{cl}_Z\{x\}\}$.

LEMMA 5: *Suppose Z is a large subset of Ω. Then there are an element x of Z and a set $X \in B$ such that $x \in X$ and $\mathrm{back}_Z(x, X)$ is large.*

Proof: Assume Z is a counterexample; we shall reach a contradiction. By a Z-**rich set** we mean a subset N of $Z \cup \wp(Z)$ such that

- N is countable.
- If $x \in N$ and $X \in B$ then $\mathrm{back}_Z(x, X) \in N$.
- If U is a subset of Z which is a member of N and is small, and V, W are elements of T such that $V \cap U \neq W \cap U$, then there is some element of $N \cap U$ which lies in exactly one of V and W.

Since Z has cardinality at most ω_1 (hence by Lemma 2 equal to ω_1), we can construct a strictly increasing continuous chain $\langle N_i : i < \omega_1 \rangle$ of Z-rich sets, such that $Z \subseteq \bigcup_{i<\omega_1} N_i$.

Let us say that an element x of Z is **pertinent** if there is some $i < \omega_1$ such that $x \in N_{i+1} \setminus N_i$, and x lies in some small subset of Z which is in N_i. If z is not pertinent, we say it is **impertinent**.

We claim that if V, W are any two distinct members of $T(Z)$ then some impertinent element is in exactly one of V and W. For this, consider the least $i < \omega_1$

such that some element z of $N_{i+1} \setminus N_i$ is in the symmetric difference of V and W. If z is pertinent, then by the last clause in the definition of Z-rich sets, some element of N_i already distinguishes V and W, contradicting the choice of i. This proves the claim.

Now let I be the set of all impertinent elements of Z. Since Z is large, the previous claim implies that I is large. Thinning the chain if necessary, we can arrange that for each $i < \omega_1$, $N_{i+1} \setminus N_i$ contains infinitely many elements of I.

We can partition I into countably many sets, so that for every $i < \omega_1$, each set meets $I \cap (N_{i+1} \setminus N_i)$ in exactly one element. By Lemma 4 above, since I is large, at least one of these partition sets must be large. Let J be a large partition set. We define a binary relation $\preceq$ on J by:

$$x \preceq y \Leftrightarrow \text{ for all } U \in B, \text{ if } y \in U \text{ then } x \in U.$$

We shall reach a contradiction with Lemma 3 by showing that $\preceq$ is a linear ordering and $T(J)$ is a set of initial segments of J under $\preceq$ which contains all the initial segments of the form $\{x : x \prec y\}$.

The relation $\preceq$ is clearly reflexive and transitive. We made it antisymmetric by assuming that no two distinct elements of Ω lie in exactly the same sets in B. We must show that if x and y are distinct elements of Z then either $x \preceq y$ or $y \preceq x$.

Let x, y be a counterexample, so that there are sets $X, Y \in B$ with $x \in X \setminus Y$ and $y \in Y \setminus X$. By symmetry and the choice of J we can assume that for some $i < \omega_1$, $x \in N_i$ and $y \in N_{i+1} \setminus N_i$. Since y is impertinent, no small set containing y is in N_i. In particular $\mathrm{back}_Z(x, X)$ contains y and hence is not both small and in N_i. But since N_i is Z-rich, it contains $\mathrm{back}_Z(x, X)$. Also we assumed that Z is a counterexample to the lemma; this implies that $\mathrm{back}_Z(x, X)$ is small. We have a contradiction.

Thus it follows that $\preceq$ is a linear ordering of J, and the definition of $\preceq$ then implies that $T(J)$ is a set of initial segments of $\preceq$. As B separates points, every set $\{x : x \prec y\}$ is open. This contradicts Lemma 3 and so proves the present lemma. $\square_5$

Proof of Theorem 1: Now we can finish the proof of the theorem. We shall find elements x_n of Ω and sets $X_n \in T$ $(n < \omega)$ such that $x_m \in X_n$ if and only if $m = n$. By taking arbitrary unions of the sets X_n it clearly follows that $|T| = 2^\omega$.

We define x_n and X_n by induction on n. Writing Z_{-1} for Ω and Z_n for $\text{back}_{Z_{n-1}}(x_n, X_n)$, we require that $x_{n+1} \in Z_n$ and each set Z_n is large. Since Ω is large, Lemma 5 tells us that we can begin by choosing x_0 and X_0 so that $\text{back}_\Omega(x_0, X_0)$ is large.

After x_n and X_n have been chosen, we use Lemma 5 again to choose x_{n+1} in Z_n and Y_{n+1} in B so that $x_{n+1} \in Y_{n+1}$ and $\text{back}_{Z_n}(x_{n+1}, Y_{n+1})$ is large. For each $m \leq n$, x_{n+1} is in Z_m and hence it is not in $\text{cl}_{Z_{m-1}}\{x_m\}$, so that there is some set $U_m \in B$ which contains x_{n+1} but not x_m. Put $X_{n+1} = \bigcap_{m \leq n} U_m \cap Y_{n+1}$. (Note that this is the one place where we use the fact that B, and hence also T, is closed under finite intersections.) Since $X_{n+1} \subseteq Y_{n+1}$, $\text{back}_{Z_n}(x_{n+1}, X_{n+1})$ is large.

We must show that this works. First, $x_n \in X_n$ for each n by construction. Next, if $m \leq n$ then x_{n+1} is in Z_m and hence it is not in X_m. Finally if $m \leq n$ then $x_m \notin X_{n+1}$ by the definition of X_{n+1}. $\square_1$

The following theorem has a similar proof. We omit details, except to say that (i) "countable" is replaced by "of cardinality at most $|B|$", and ω_1 by $|B|^+$, and (ii) a more complicated analogue of Lemma 2 is needed.

THEOREM 6: *Let T be the set of open subsets of a topological space Ω (not neccessarily Hausdorff, nor even T_0), and suppose that T has a base B which is closed under finite intersections, and $|T| > |B| + \aleph_0$. Then*

(1) there are $x_n \in \Omega$ and $X_n \in B$ for $n < \omega$ such that for all $m, n < \omega$, $x_n \in X_m$ iff $m = n$, and

(2) $|T| \geq 2^{\aleph_0}$.

One naturally asks whether we can let B in Theorem 1 be any set such that T is the set of unions of sets in B, without the requirement that B is closed under finite intersections. The answer is no, for the following reason.

LEMMA 7: *Suppose there is a tree S with δ levels, μ nodes and exactly λ branches of length δ, where $\lambda \geq \mu$; suppose also that S is normal (i.e. at each limit level there are never two or more nodes with the same predecessors). Then there are a set Ω of cardinality λ and a family of μ subsets of Ω which has exactly λ unions.*

CONSTRUCTION: *Let Ω be a set of λ branches of length δ; for each $s \in S$ let U_s be $\{x \in \Omega : s \notin x\}$. Lastly let B be the family of sets $\{U_s : s \in S\}$, so that*

$|B| = \mu$. *Now the sets in* T *are: members of* B, Ω *itself and complements of singletons; so* $|T| = \lambda$. $\square_7$

Thus by starting with the full binary tree of height ω, we can build examples where B is countable and T is any cardinal between ω and 2^ω.

References

[1] Richard Mansfield and Galen Weitkamp, *Recursive Aspects of Descriptive Set Theory*, Oxford Univ. Press, 1985.

[2] S. Shelah, *Cardinalities of topologies with small base*, Publ. 454A, to appear.