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## A non-reflexive Whitehead group <sup>☆</sup>

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### Abstract

We prove that it is consistent that there is a non-reflexive Whitehead group, in fact one whose dual group is free. We also prove that it is consistent that there is a group  $A$  such that  $\text{Ext}(A, \mathbb{Z})$  is torsion and  $\text{Hom}(A, \mathbb{Z}) = 0$ . As an application we show the consistency of the existence of new co-Moore spaces. © 2001 Elsevier Science B.V. All rights reserved.

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### 0. Introduction

This paper is motivated by a theorem and a question due to Martin Huber. He proved [8] in ZFC that if  $A$  is  $\aleph_1$ -coseparable (that is,  $\text{Ext}(A, \mathbb{Z}^{(\omega)}) = 0$ ), then  $A$  is reflexive (that is, the natural map of  $A$  to its double dual  $A^{**} = \text{Hom}(\text{Hom}(A, \mathbb{Z}), \mathbb{Z})$  is an isomorphism). He asked whether it is provable in ZFC that every Whitehead group  $A$  (i.e.,  $\text{Ext}(A, \mathbb{Z}) = 0$ ) is reflexive. This is true in any model where every Whitehead group is free. It is also true for Whitehead groups of cardinality  $\aleph_1$  in a model of  $\text{MA} + \neg\text{CH}$  (because they are  $\aleph_1$ -coseparable: cf. [4, Corollary XII.1.12]). Moreover, it is true in the original models of GCH where there are non-free Whitehead groups (cf. [10,11], [4, Theorem XII.1.9]). It was left as an open question in [4, p. 455] whether every Whitehead group is reflexive. Here we give a strong negative answer:

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**Theorem 0.1.** *It is consistent with ZFC that there is a strongly non-reflexive strongly  $\aleph_1$ -free Whitehead group  $A$  of cardinality  $\aleph_1$ .*

A group  $A$  is strongly non-reflexive if  $A$  is not isomorphic to  $A^{**}$ . In fact, the example  $A$  has the property that  $A^*$  is free of rank  $\aleph_2$  (i.e., isomorphic to  $\mathbb{Z}^{(\aleph_2)}$ ) so  $A^{**}$  is isomorphic to the product  $\mathbb{Z}^{\aleph_2}$ ; it is therefore not isomorphic to  $A$  since its cardinality is  $2^{\aleph_2} > \aleph_1$ . (See Theorem 1.5 and Corollary 1.6 of Section 1.)

If  $\text{Ext}(A, \mathbb{Z}) = 0$ , then  $A$  is separable [5, Theorem 99.1] and hence  $A^*$  is non-zero. However, using the same methods we can also prove:

**Theorem 0.2.** *It is consistent with ZFC that there is a non-free strongly  $\aleph_1$ -free group  $A$  of cardinality  $\aleph_1$  such that  $\text{Ext}(A, \mathbb{Z})$  is torsion and  $\text{Hom}(A, \mathbb{Z}) = 0$ .*

It is not a theorem of ZFC that there is a non-free torsion-free group  $A$  such that  $\text{Ext}(A, \mathbb{Z})$  is torsion. Indeed, in any model where every Whitehead group is free — a hypothesis which is consistent with CH or  $\neg$ CH (cf. [9]) — if  $A$  is not free, then  $\text{Ext}(A, \mathbb{Z})$  is not torsion [7,3], [4, Theorem XII.2.4].

Theorems 1.5 and 0.2 provide new examples of possible co-Moore spaces (see Section 6). In particular, we answer a question in [6, p. 46] by showing that it is consistent that for any  $n \geq 2$  there is a co-Moore space of type  $(F, n)$ , where  $F$  is a free group of rank  $\aleph_2$ .

The models for both theorems result from a finite support iteration of c.c.c. posets and are models of  $\text{ZFC} + \neg\text{CH}$ . (Other methods will be needed to obtain consistency with CH.) We begin the iteration with a poset which yields “generic data” from which the group  $A$  is defined; we then iterate the natural posets which insure that  $\text{Ext}(A, \mathbb{Z}) = 0$  (resp.  $\text{Ext}(A, \mathbb{Z})$  is torsion). The hard work is in proving that  $\text{Hom}(A, \mathbb{Z})$  is as claimed. We define the forcing and the group more precisely in the next section and then prove their properties in the succeeding sections.

## 1. The basic construction

The group-theoretic construction is a generalization of that in [4, XII.3.4]. Let  $E$  be a stationary and co-stationary subset of  $\omega_1$  consisting of limit ordinals, and for each  $\delta \in E$ , let  $\eta_\delta$  be a ladder on  $\delta$ , that is, a strictly increasing function  $\eta_\delta: \omega \rightarrow \delta$  whose range approaches  $\delta$ . Let  $F$  be the free abelian group with basis  $\{x_v: v \in \omega_1\} \cup \{z_{\delta,n}: \delta \in E, n \in \omega\}$ . Let  $g$  be a function from  $E \times \omega$  to the integers  $\geq 1$ . Let  $u$  be a function from  $E \times \omega$  to the subgroup  $\langle x_v: v \in \omega_1 \rangle$  generated by  $\{x_v: v \in \omega_1\}$  such that  $u(\delta, n)$  belongs to  $\langle x_v: v < \eta_\delta(n) \rangle$ . Let  $K$  be the subgroup of  $F$  generated by  $\{w_{\delta,n}: \delta \in E, n \in \omega\}$  where

$$w_{\delta,n} = 2^{g(\delta,n)} z_{\delta,n+1} - z_{\delta,n} - x_{\eta_\delta(n)} - u(\delta, n). \quad (1.1)$$

Let  $A = F/K$ . Then clearly  $A$  is an abelian group of cardinality  $\aleph_1$ . Notice that because the right-hand side of (1.1) is 0 in  $A$ , we have for each  $\delta \in E$  and  $n \in \omega$  the following

relations in  $A$ :

$$2^{g(\delta,n)}z_{\delta,n+1} = z_{\delta,n} + x_{\eta_\delta(n)} + u(\delta, n) \quad (1.2)$$

and

$$2^{\sum_{j=0}^n g(\delta,j)}z_{\delta,n+1} = z_{\delta,0} + \sum_{k=0}^n 2^{\sum_{j=0}^{k-1} g(\delta,j)}(x_{\eta_\delta(k)} + u(\delta, k)). \quad (1.3)$$

Here, and occasionally in what follows, we abuse notation and write, for example,  $z_{\delta,n+1}$  instead of  $z_{\delta,n+1} + K$  for an element of  $A$ . For each  $\alpha < \omega_1$ , let  $A_\alpha$  be the subgroup of  $A$  generated by

$$\{x_v: v < \alpha\} \cup \{z_{\delta,n}: \delta \in E \cap \alpha, n \in \omega\}. \quad (1.4)$$

Then, by (1.3), for each  $\delta \in E$ ,  $z_{\delta,0} + A_\delta$  is non-zero and divisible in  $A_{\delta+1}/A_\delta$  by  $2^m$  for all  $m \in \omega$ . Thus  $A_{\delta+1}/A_\delta$  is not free and hence  $A$  is not free. (In fact  $\Gamma(A) \supseteq \tilde{E}$ .) Moreover,  $A$  is strongly  $\aleph_1$ -free; in fact, for every  $\alpha < \omega_1$ , using Pontryagin's criterion [4, IV.2.3] we can show that  $A/A_{\alpha+1}$  is  $\aleph_1$ -free for all  $\alpha \in \omega_1 \cup \{-1\}$ .

We begin with a model  $V$  of ZFC where GCH holds, choose  $E \in V$ , and define the group  $A$  in a generic extension  $V^{Q_0}$  using generic ladders  $\eta_\delta$ , and generic  $u$  and  $g$ . Specifically:

**Definition 1.1.** Let  $Q_0$  be the set of all finite functions  $q$  such that  $\text{dom}(q)$  is a finite subset of  $E$  and for all  $\gamma \in \text{dom}(q)$ ,  $q(\gamma)$  is a triple  $(\eta_\gamma^q, u_\gamma^q, g_\gamma^q)$ , where for some  $r_\gamma^q \in \omega$ :

- $\eta_\gamma^q$  is a strictly increasing function:  $r_\gamma^q \rightarrow \gamma$ ;
- $u_\gamma^q: \{\gamma\} \times r_\gamma^q \rightarrow \langle x_v: v \in \omega_1 \rangle$  such that for all  $n < r_\gamma^q$ ,  $u_\gamma^q(\gamma, n) \in \langle x_v: v < \eta_\gamma(n) \rangle$ ; and
- $g_\gamma^q: \{\gamma\} \times r_\gamma^q \rightarrow \{n \in \omega: n \geq 1\}$ .

The partial ordering is defined by:  $q_1 \leq q_2$  if and only if  $q_1 \subseteq q_2$ ; note that we follow the convention that stronger conditions are larger. Clearly,  $Q_0$  is c.c.c. and hence  $E$  remains stationary and co-stationary in a generic extension. We now do an iterated forcing to make  $A$  a Whitehead group. We begin by defining the basic forcing that we will iterate.

**Definition 1.2.** Given a homomorphism  $\psi: K \rightarrow \mathbb{Z}$ , let  $Q_\psi$  be the poset of all finite functions  $q$  into  $\mathbb{Z}$  satisfying:

There are  $\delta_0 < \delta_1 < \dots < \delta_m$  in  $E$  and  $\{r_\ell: \ell \leq m\} \subseteq \omega$  such that  $\text{dom}(q) = \{z_{\delta_\ell, n}: \ell \leq m, n \leq r_\ell\} \cup \{x_v: v \in I_q\}$ , where  $I_q \subset \omega_1$  is finite and is such that for all  $\ell \leq m$

$$n < r_\ell \Rightarrow u(\delta_\ell, n) \in \langle x_v: v \in I_q \rangle \quad \text{and} \quad \eta_{\delta_\ell}(n) \in I_q \Leftrightarrow n < r_\ell \quad (1.5)$$

and for all  $\ell \leq m$  and  $n < r_\ell$ ,  $u(\delta_\ell, n) \in \langle x_v: v \in I_q \rangle$  and

$$\psi(w_{\delta_\ell, n}) = 2^{g(\delta_\ell, n)}q(z_{\delta_\ell, n+1}) - q(z_{\delta_\ell, n}) - q(x_{\eta_{\delta_\ell}(n)}) - q(u(\delta_\ell, n)). \quad (1.6)$$

(Compare with (1.1). The definition of  $q(u(\delta_\ell, n))$  is the obvious one, given that  $q$  should extend to a homomorphism.) Moreover, we require of  $q$  that for all  $\ell \neq j$

in  $\{0, \dots, m\}$ ,

$$\eta_{\delta_j}(k) \neq \eta_{\delta_\ell}(i) \quad \text{for all } k \geq r_j \text{ and } i \in \omega. \quad (1.7)$$

We will denote  $\{\delta_0, \dots, \delta_m\}$  by  $\text{cont}(q)$  and  $r_\ell$  by  $\text{num}(q, \delta_\ell)$ . The partial ordering on  $Q_\psi$  is inclusion.

**Proposition 1.3.** (i) For every  $\delta \in E$  and  $k \in \omega$ ,  $D_{\delta,k} = \{q \in Q_\psi: \delta \in \text{cont}(q) \text{ and } k \leq \text{num}(q, \delta)\}$  is dense in  $Q_\psi$

(ii)  $Q_\psi$  is c.c.c.

Before proving Proposition 1.3, we prove a lemma:

**Lemma 1.4.** Given  $\{\delta_0, \dots, \delta_m\} \in E$ , integers  $r'_\ell$  for  $\ell \leq m$  and a finite subset  $I'$  of  $\omega_1$ , there are integers  $r''_\ell \geq r'_\ell$  for all  $\ell \leq m$  and a finite subset  $I''$  of  $\omega_1$  containing  $I'$  such that for all  $\ell \leq m$ :

- (a)  $\eta_{\delta_\ell}(n) \in I'' \Leftrightarrow n < r''_\ell$ ; and
- (b) for all  $n < r''_\ell$ ,  $u(\delta_\ell, n) \in \langle x_v: v \in I'' \rangle$ .

**Proof.** The proof is by induction on  $m \geq 0$ . If  $m = 0$  we can take

$$r''_0 = \max\{r'_0, \max\{k + 1: \eta_{\delta_0}(k) \in I'\}\}$$

and take  $I''$  to be a minimal extension of  $I' \cup \{\eta_{\delta_0}(n): n < r''_0\}$  satisfying (b); then (a) holds because  $u(\delta, n) \in \langle x_v: v < \eta_\delta(n) \rangle$ . If  $m > 0$ , without loss of generality, we can assume that  $\delta_0 < \delta_1 < \dots < \delta_m$ . Let

$$r''_m = \max\{r'_m, \max\{k + 1: \eta_{\delta_m}(k) \in I'\}, \min\{k: \eta_{\delta_m}(k) > \delta_{m-1}\}\}.$$

As in the case  $m = 0$ , there exists  $\tilde{I}$  containing  $I'$  such that (a) and (b) hold for  $\tilde{I}$  for  $\ell = m$ . Then apply the inductive hypothesis to  $\{\delta_0, \dots, \delta_{m-1}\}$ ,  $\tilde{I}$ , and the  $r'_\ell$  ( $\ell < m$ ) to obtain  $r''_\ell$  for  $\ell < m$  and a minimal  $I''$ .  $\square$

For  $q \in Q_\psi$  and  $\alpha \in \omega_1$ , let  $q \upharpoonright \alpha$  denote the restriction of  $q$  to

$$\text{dom}(q) \cap (\{z_{\delta,n}: \delta < \alpha, n \in \omega\} \cup \{x_v: v < \alpha\}).$$

Say that  $\tau$  occurs in  $q$  if  $\tau \in \text{cont}(q)$  or  $x_\tau \in \text{dom}(q)$ .

**Proof of Proposition 1.3.** (i) Given  $\delta \in E$ ,  $k \in \omega$  and  $p \in Q_\psi$ , we need  $q \geq p$  such that  $q \in D_{\delta,k}$ . Let  $\text{cont}(p) = \{\delta_0, \dots, \delta_m\}$ . We consider two cases. The first is that  $\delta \in \text{cont}(p)$ , that is,  $\delta = \delta_j$  for some  $j \leq m$ . We can assume that  $k > \text{num}(p, \delta_j)$ . Apply Lemma 1.4 with  $I' = I_p$ ,  $r'_j = k$ , and  $r'_\ell = \text{num}(p, \delta_\ell)$  for  $\ell \neq j$  to obtain  $I''$  and  $r''_\ell$ . Then we can define  $q$  to be the extension of  $p$  with  $\text{cont}(q) = \text{cont}(p)$  and  $\text{num}(q, \delta_\ell) = r''_\ell$  and  $I_q = I''$ . Since (1.5) and (1.7) hold, we can inductively define  $q(x_{\eta_{\delta_\ell}(i)})$  and  $q(z_{\delta_\ell, i+1})$  for  $r'_\ell \leq i < r''_\ell$  (setting  $q(x_v) = 0$  for  $v \in I_q \setminus \bigcup \{\text{rge}(\eta_{\delta_\ell}): \ell \leq m\}$  if not already defined) so that (1.6) holds. Note that (1.7) continues to hold.

The second case is when  $\delta \notin \text{cont}(p)$ . Let  $\delta_{m+1} = \delta$ . Choose  $r'_\ell$  for  $\ell \leq m+1$  so that  $r'_\ell \geq \text{num}(p, \delta_\ell)$  for  $\ell \leq m$  and such that (1.7) holds, that is,  $\eta_{\delta_j}(n) \neq \eta_{\delta_\ell}(i)$  for all  $n \geq r'_j$  and  $i \in \omega$  for all  $j \neq \ell \in \{0, \dots, m+1\}$ . Apply Lemma 1.4 to  $\{\delta_0, \dots, \delta_{m+1}\}$ ,  $I_p$ , and the  $r'_\ell$  to obtain  $r''_\ell$  for  $\ell \leq m+1$  and  $I''$ . Let  $I_q = I''$  and  $\text{num}(q, \delta_\ell) = r''_\ell$ . For  $\ell \leq m$  define  $q(x_{\eta_{\delta_\ell}(i)})$ ,  $q(z_{\delta_\ell, i+1})$  and  $u(\delta_\ell, i)$  for  $\text{num}(p, \delta_\ell) \leq i < r''_\ell$  by induction on  $i$  as in the first case. Define  $q(z_{\delta, r''_{m+1}+1}) = 0$  and define  $q(z_{\delta, n})$  for  $n \leq r''_{m+1}$  by “downward induction”, i.e.

$$q(z_{\delta, n}) = 2^{g(\delta, n)} q(z_{\delta, n+1}) - q(x_{\eta_\delta(n)}) - q(u(\delta, n)) - \psi(w_{\delta, n}).$$

(Setting  $q(x_\tau) = 0$  where not already defined, we can assume  $q(x_{\eta_\delta(n)})$  and  $q(u(\delta, n))$  are defined.)

(ii) Consider an uncountable subset  $\{q_v: v \in \omega_1\}$  of  $\mathcal{Q}_\psi$ . By the  $\Delta$ -system lemma we can assume that  $\{\text{cont}(q_v): v \in \omega_1\}$  forms a  $\Delta$ -system, i.e., there is a finite subset  $\Delta$  of  $E$  such that for all  $v \neq \mu$ ,  $\text{cont}(q_v) \cap \text{cont}(q_\mu) = \Delta$ . By renumbering an uncountable subset, we can assume that for all  $v$ , if  $\delta \in \text{cont}(q_v) \setminus \Delta$ , then  $\delta > v$ . Furthermore, by passing to a subset and using (i) we can assume that if  $\delta \in \text{cont}(q_v) \setminus \Delta$  and  $\eta_\delta(n) < v$ , then  $n < \text{num}(q_v, \delta)$ . By Fodor's Lemma we can assume that there exists  $\gamma \geq \max \Delta$  such that for all  $v$  and  $n$ , if  $\delta \in \text{cont}(q_v)$  and  $\eta_\delta(n) < v$ , then  $\eta_\delta(n) < \gamma$  and moreover such that if  $\tau \in I_{q_v}$  and  $\tau < v$ , then  $\tau < \gamma$ . We can also assume that for all  $\mu, v$ ,  $q_\mu \upharpoonright \mu = q_v \upharpoonright v$ . If we pick  $\mu < v$  such that  $\gamma < \mu$  and whenever  $\tau$  occurs in  $q_\mu$ , then  $\tau < v$ , then we will have that  $q_\mu \cup q_v \in \mathcal{Q}_\psi$ . Notice that (1.7) will be satisfied: if  $\delta \in \text{cont}(q_\mu) \setminus \Delta$  and  $\rho \in \text{cont}(q_v) \setminus \Delta$  and  $k \geq \text{num}(q_\mu, \delta)$  and  $m \geq \text{num}(q_v, \rho)$ , then  $\mu \leq \eta_\delta(k) < v \leq \eta_\rho(m)$ ; moreover, if  $i \in \omega$  and  $\eta_\rho(i) < v$ , then  $\eta_\rho(i) < \gamma < \mu \leq \eta_\delta(k)$ . Similarly it follows that (1.5) holds.  $\square$

Now  $P = \langle P_i, \dot{Q}_i: 0 \leq i < \omega_2 \rangle$  is defined to be a finite support iteration of length  $\omega_2$  so that for every  $i \geq 1$   $\Vdash_{P_i} \dot{Q}_i = \mathcal{Q}_{\dot{\psi}_i}$  where  $\Vdash_{P_i} \dot{\psi}_i$  is a homomorphism:  $K \rightarrow \mathbb{Z}$  and the enumeration of names  $\{\dot{\psi}_i: 1 \leq i < \omega_2\}$  is chosen so that if  $G$  is  $P$ -generic and  $\psi \in V[G]$  is a homomorphism:  $K \rightarrow \mathbb{Z}$ , then for some  $i \geq 1$ ,  $\dot{\psi}_i$  is a name for  $\psi$  in  $V^{P_i}$ . Then  $P$  is c.c.c. and in  $V[G]$  every homomorphism from  $K$  to  $\mathbb{Z}$  extends to one from  $F$  to  $\mathbb{Z}$ . This means that  $\text{Ext}(A, \mathbb{Z}) = 0$ , that is,  $A$  is a Whitehead group (see, for example, [4, p. 8]). We claim moreover that:

**Theorem 1.5.** *In  $V[G]$   $A^*(=\text{Hom}(A, \mathbb{Z}))$  is free of cardinality  $\aleph_2$ .*

As a consequence we can conclude:

**Corollary 1.6.** *In  $V[G]$   $A$  is strongly non-reflexive.*

**Proof.** Since  $A^*$  is isomorphic to  $\mathbb{Z}^{(\aleph_2)}$ ,  $A^{**}$  is isomorphic to  $\mathbb{Z}^{\aleph_2}$  and hence not isomorphic to  $A$  because its cardinality is different. We remark also that  $A^{**}$  is not slender, but  $A$  is slender since it is a Whitehead group — see [4, Proposition XII.1.3, p. 345]).  $\square$

The next three sections are devoted to a proof of Theorem 1.5. The fact that  $A^*$  has cardinality  $2^{\aleph_1}$  is a consequence of a result of Chase [1, Theorem 5.6]; by standard arguments it can be seen that in  $V[G]$   $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ . Let  $G_v = \{p \upharpoonright v : p \in G\}$ , so that  $G_v$  is  $P_v$ -generic. To prove that  $\text{Hom}(A, \mathbb{Z})^{V[G]}$  is free, it suffices to prove that

- (I)  $\text{Hom}(A, \mathbb{Z})^{V[G_1]} = 0$ ;
- (II) for every limit  $\beta \leq \omega_2$ ,  $\text{Hom}(A, \mathbb{Z})^{V[G_\beta]} = \bigcup_{i < \beta} \text{Hom}(A, \mathbb{Z})^{V[G_i]}$ ; and
- (III) for all  $i < \omega_2$ ,  $\text{Hom}(A, \mathbb{Z})^{V[G_{i+1}]} / \text{Hom}(A, \mathbb{Z})^{V[G_i]}$  is free, and in fact is either 0 or  $\mathbb{Z}$ .

We shall prove (I) immediately, and then prove the other two parts in the next three sections.

**Proof of (I).** Notice first that, by (1.3), if  $h \in \text{Hom}(A, \mathbb{Z})$  and  $h(x_\mu) = 0$  for all  $\mu \in \omega_1$ , then  $h$  is identically zero. So suppose, to obtain a contradiction, that there exists a  $Q_0$ -name  $\dot{h}$  and  $p \in G_1$  such that

$$p \Vdash \dot{h} \in \text{Hom}(A, \mathbb{Z}) \wedge \dot{h}(x_\mu) = m$$

for some  $\mu \in \omega_1$  and some non-zero integer  $m$ . Choose  $d$  such that  $2^d$  does not divide  $m$ . For each  $\delta \in E$  there exists  $p_\delta \geq p$  and  $c_\delta \in \mathbb{Z}$  such that

$$p_\delta \Vdash \dot{h}(z_{\delta,0}) = c_\delta.$$

By Fodor's Lemma and a  $\mathcal{A}$ -system argument, there exist  $\delta_1 \neq \delta_2 > \mu$  such that  $c_{\delta_1} = c_{\delta_2}$ , and if (for convenience of notation) we let  $p^i = p_{\delta_i}$ ,  $r_{\delta_1}^{p^1} = r_{\delta_2}^{p^2} = r$ ,  $\eta_{\delta_1}(n) = \eta_{\delta_2}(n)$ ,  $u(\delta_1, n) = u(\delta_2, n)$  for all  $n < r$  and  $p^1$  and  $p^2$  are compatible. Then there is a condition  $q \in Q_0$  such that  $p^i \leq q$  for  $i = 1, 2$  and

$$q \Vdash \eta_{\delta_1}(r) = \eta_{\delta_2}(r) \wedge g(\delta_1, r) = d = g(\delta_2, r) \wedge u(\delta_1, r) = x_\mu \wedge u(\delta_2, r) = 0.$$

Now consider a generic extension  $V[G'_1]$ , where  $q \in G'_1$ . By subtracting (1.3) for  $n = r$  and  $\delta = \delta_2$  from (1.3) for  $n = r$  and  $\delta = \delta_1$  and applying  $h$  we obtain that (in  $V[G'_1]$ )  $2^d$  divides  $h(u(\delta_1, r)) - h(u(\delta_2, r)) = h(x_\mu) - h(0) = m$ . But this is a contradiction of the choice of  $d$ .  $\square$

## 2. Preliminaries

Before beginning the proof proper of (II) and (III), we prove a crucial proposition that we will need. For a fixed  $m \in \omega$  and  $S \subseteq E$ , let  $Z_m[S]$  denote the pure closure in  $A$  of the subgroup generated by  $\{z_{\delta,m} + K : \delta \in S\}$ . For  $t \in \omega$ , let  $Z_{m,t}[S]$  denote  $Z_m[S] + 2^t A$ .

**Proposition 2.1.** *In  $V[G]$ , for all  $m, t \in \omega$  and all stationary  $S \subseteq E$ ,  $A/Z_{m,t}[S]$  is a finite group.*

**Proof.** The proof is by contradiction. Suppose that  $q^* \in G$  such that for some  $m, t \in \omega$  and some  $\dot{S}$

$$q^* \Vdash_P \dot{S} \text{ is a stationary subset of } E \text{ and } A/Z_{m,t}[\dot{S}] \text{ is infinite.}$$

Let  $S'$  be the set of all  $\delta \in E$  such that  $q^*$  does not force “ $\delta \notin \dot{S}$ ”; then  $S' \in V$  is a stationary subset of  $E$ . For each  $\delta \in S'$  choose  $p_\delta \geq q^*$  such that  $p_\delta \Vdash \delta \in \dot{S}$ .

We can assume that each  $p_\delta$  satisfies:

( $\dagger$ )  $0 \in \text{dom}(p_\delta)$ ;  $\delta \in \text{dom}(p_\delta(0))$ ; for each  $j \in \text{dom}(p_\delta)$ ,  $p_\delta(j)$  is a function in  $V$  and not just a name;  $r_\gamma^{p_\delta(0)}$  ( $=r_\delta$ ) is independent of  $\gamma \in \text{dom}(p_\delta(0))$ ; if  $j \in \text{dom}(p_\delta) \setminus \{0\}$ ,  $\gamma \in \text{cont}(p_\delta(j))$  implies  $\gamma \in \text{dom}(p_\delta(0))$  and  $\text{num}(p_\delta(j), \gamma)$  ( $=r'_{\delta,j}$ ) is  $\leq r_\delta$  and independent of  $\gamma$ . Moreover, if  $\gamma \in \text{dom}(p_\delta(0))$  and  $\gamma > \delta$ , then  $\eta_\gamma(r'_{\delta,j}) > \delta$ .

When we say that “ $v$  occurs in  $p$ ” we mean that  $p(v)$  is non-empty, or  $v$  occurs in  $p(j)$  for some  $j > 0$  or  $v \in \text{dom}(p(0)) \cup \{\eta_\gamma(n) : \gamma \in \text{dom}(p(0)), n < r_\gamma^{p(0)}\}$ , or  $u(\gamma, n) \notin \langle x_\mu : \mu \neq v \rangle$  for some  $n < r_\gamma^{p(0)}$ . Without loss of generality, we can assume (passing to a subset of  $S'$ ) that

( $\dagger\dagger$ ) there exists  $\tau$  such that for all  $\delta \in S'$ ,  $\delta > \tau$  and every ordinal  $< \delta$  which occurs in  $p_\delta$  is less than  $\tau$ ;  $\{\text{dom}(p_\delta) : \delta \in S'\}$  forms a  $\Delta$ -system, whose root we denote  $C$  (i.e.,  $\text{dom}(p_{\delta_1}) \cap \text{dom}(p_{\delta_2}) = C = \{0, \mu_1, \dots, \mu_d\}$  for all  $\delta_1 \neq \delta_2$  in  $S'$ );  $r_\delta$  ( $=r^*$ ) and  $r'_{\delta,j}$  ( $=r'_j$ ) are independent of  $\delta$ . Moreover, for every  $j \in C$ ,  $\{\text{dom}(p_\delta(j)) : \delta \in S'\}$  forms a  $\Delta$ -system and for all  $\delta_1 \neq \delta_2$  in  $S'$ ,  $p_{\delta_1}(j)$  and  $p_{\delta_2}(j)$  agree on  $\text{dom}(p_{\delta_1}(j)) \cap \text{dom}(p_{\delta_2}(j))$ , so  $p_{\delta_1}(j) \upharpoonright \delta_1 = p_{\delta_2}(j) \upharpoonright \delta_2$ . Also,  $\text{dom}(p_\delta(0)) \cap \delta$  and  $p_\delta(0) \upharpoonright (\text{dom}(p_\delta(0)) \cap \delta)$  are independent of  $\delta$ . Finally,  $\eta_\delta^{p_\delta(0)}(n)$  ( $=\zeta_n$ ),  $g^{p_\delta(0)}(\delta, n)$  ( $=g(n)$ ) and  $u^{p_\delta(0)}(\delta, n)$  are independent of  $\delta$  for each  $n < r^*$ .

Let  $p^*$  denote the “heart” of  $\{p_\delta : \delta \in S'\}$ ; that is,  $\text{dom}(p^*) = C$  and for all  $\mu \in C$ ,  $\text{dom}(p^*(\mu)) = \text{dom}(p_{\delta_1}(\mu)) \cap \text{dom}(p_{\delta_2}(\mu))$  ( $=C_\mu$ , say) for  $\delta_1 \neq \delta_2 \in S'$ ; and  $p^*(\mu) = p_{\delta_1}(\mu) \upharpoonright C_\mu = p_{\delta_2}(\mu) \upharpoonright C_\mu$ .

The conditions in ( $\dagger\dagger$ ) insure that if  $\delta_1 < \delta_2$  are members of  $S'$  such that every ordinal which occurs in  $p_{\delta_1}$  is  $< \delta_2$ , then  $p_{\delta_1}$  and  $p_{\delta_2}$  are almost compatible; however, there may be problems in determining a value for  $p_{\delta_\ell}(j)(x_{\zeta_n})$  for  $r'_j \leq n < r^*$  (independent of  $\ell=1, 2$ ); it is because of these that the following argument is necessary.

We can assume that  $r^* \geq m$  and that for all  $\delta \in S'$   $\delta \notin \text{dom}(p^*(0))$ . Choose  $M \geq t$  such that  $g(n) \leq M$  for all  $n < r^*$ . Let

$$N = 2^{(r^*+1)M}.$$

To obtain a contradiction, it suffices to prove that  $p^*$  forces:

( $\nabla$ )  $A/Z_{m,t}[S]$  is a group of cardinality  $\leq N^d$ .

This is a contradiction since  $p^* \geq q^*$ . If  $p^*$  does not force ( $\nabla$ ), then there is a finite subset  $\Theta$  of  $\omega_1$  and a condition  $p^{**} \geq p^*$  such that  $p^{**}$  forces

( $\nabla\nabla$ )  $(\langle x_v : v \in \Theta \rangle + Z_{m,t}[S])/Z_{m,t}[S]$  has cardinality  $> N^d$ .

(Note that it follows from (1.3) that  $A/p^t A$  is generated by  $\langle x_v : v \in \Theta \rangle$ .) We can assume that if  $v$  occurs in  $p^{**}$ , then  $v \in \Theta$ . Let  $T$  be the subset of  $\langle x_v : v \in \Theta \rangle$

composed of all elements of the form  $\sum_{v \in \Theta} c_v x_v + x_\beta$  where  $0 \leq c_v < 2^t$ . Let  $\theta = 2^{|\Theta|t}$ ; so  $T$  has  $\theta > N^d$  elements; list them as  $\{\tau_\ell: \ell < \theta\}$ . Now choose elements  $\{\delta_\ell: \ell < \theta\}$  of  $S'$  listed in increasing order and such that the smallest,  $\delta_0$ , is larger than  $\max \Theta$  and such that every ordinal which occurs in  $p_{\delta_\ell}$  is  $< \delta_{\ell+1}$ . Moreover, we can choose them so that for any  $\ell < \theta$ , the “common part” of  $p_{\delta_\ell}$  and  $p^{**}$  is  $p^*$ ; that is,  $\text{dom}(p_{\delta_\ell}) \cap \text{dom}(p^{**}) = C = \text{dom}(p^*)$  and for all  $\mu \in C$ ,  $\text{dom}(p_{\delta_\ell}(\mu)) \cap \text{dom}(p^{**}(\mu)) = \text{dom}(p^*(\mu))$ . (So, in particular,  $\delta_\ell \notin \text{dom}(p^{**}(0))$ .)

Choose new ordinals  $\alpha_\ell$  for  $-1 \leq \ell < \theta$  such that

$$\alpha_{-1} < \alpha_0 < \delta_0 < \alpha_1 < \delta_1 < \dots < \delta_\ell < \alpha_{\ell+1} < \delta_{\ell+1} < \dots$$

Moreover, we make the choice so that for all  $\ell$ ,  $\alpha_\ell$  is larger than any ordinal  $< \delta_\ell$  which occurs in any  $p_{\delta_k}$ . There is a condition  $q_0 \in Q_0$  which extends  $p^{**}(0)$  and each  $p_{\delta_\ell}(0)$  ( $\ell < \theta$ ) such that  $q_0$  forces for all  $\ell < \theta$ :

$$\eta_{\delta_\ell}(r^*) = \alpha_{-1}; \quad \eta_{\delta_\ell}(r^* + 1) = \alpha_\ell; \quad u(\delta_\ell, r^*) = \tau_\ell \quad \text{and} \quad g(\delta_\ell, r^*) = t.$$

This  $q_0$  will force versions of (1.5) and (1.7) for the  $\delta_\ell$ . Also choose  $q_0$  to force values for  $\eta_\gamma(r^*)$  for any  $\gamma \in \bigcup_{\ell < \theta} (\text{dom}(p_{\delta_\ell}(0)) - \{\delta_\ell\})$  so that (1.5) and (1.7) hold for  $p_{\delta_\ell}(\mu) \cup p_{\delta_{\ell'}}(\mu)$  for any  $\ell, \ell' < \theta$  and any  $\mu \in C$ . We claim that

(IV.1) There is a subset  $W$  of  $\{0, \dots, \theta - 1\}$  of size at least  $\theta \cdot N^{-d}$  and a condition  $q \in P_{\omega_2}$  which extends  $p^{**}$  and  $p_{\delta_\ell}$  for every  $\ell \in W$  and satisfies  $q(0) = q_0$

Assuming (IV.1), let us deduce a contradiction, which will prove that  $p^*$  forces  $(\nabla)$ . Work in a generic extension  $V[G']$  such that  $q \in G'$ . For  $\ell_1 \neq \ell_2$  in  $W$  we have  $\tau_{\ell_1} - \tau_{\ell_2} \in \langle x_v: v \in \Theta \rangle \cap Z_{m,t}[S]$  because

$$z_{\delta_{\ell_1}, r^*} - z_{\delta_{\ell_2}, r^*} = \tau_{\ell_1} - \tau_{\ell_2} + 2^t a$$

for some  $a \in A$  and, letting  $e = \sum_{n=m}^{r^*-1} g(n)$ ,

$$2^e (z_{\delta_{\ell_1}, r^*} - z_{\delta_{\ell_2}, r^*}) = z_{\delta_{\ell_1}, m} - z_{\delta_{\ell_2}, m} \in Z_m[S],$$

so since  $Z_m[S]$  is pure-closed,  $\tau_{\ell_1} - \tau_{\ell_2} + 2^t a \in Z_m[S]$ , and hence  $\tau_{\ell_1} - \tau_{\ell_2} \in Z_{m,t}[S]$ . Therefore  $(\langle x_v: v \in \Theta \rangle + Z_{m,t}[S]) / Z_{m,t}[S]$  has cardinality at most

$$2^{|\Theta|t} / |W| = \theta / |W| \leq N^d,$$

which is a contradiction of the choice of  $p^{**}$ .

In order to prove (IV.1) we define inductively, for  $1 \leq n \leq d + 1$ , a condition  $q_n \in P_{\mu_n}$  (where  $\mu_n$  is as in the enumeration of  $C$  for  $n \leq d$  and  $\mu_{d+1} = \omega_2$ ) such that  $q_n \geq p^{**} \upharpoonright \mu_n$  and for  $n' < n$ ,  $q_n \upharpoonright \mu_{n'} \geq q_{n'}$ . We also define a subset  $W_n$  of  $W$  of size at least  $\theta \cdot N^{-(n-1)}$  such that for each  $\ell \in W_n$ ,  $q_n \geq p_{\delta_\ell} \upharpoonright \mu_n$ . (So in the end we let  $q = q_{d+1}$  and  $W = W_{d+1}$ .)

To begin, let  $W_1 = \{0, \dots, \theta - 1\}$  and let  $q_1$  be any common extension of  $q_0$  and the  $p_{\delta_\ell} \upharpoonright \mu_1$ . (There is no problem finding such an extension.) Suppose now that  $q_n$  and  $W_n$  have been defined for some  $n \geq 1$ . Choose  $\tilde{q}_n \geq q_n$  in  $P_{\mu_n}$  such that  $\tilde{q}_n$  decides for all  $\ell \in W_n$  the value of  $\psi_{\mu_n}(w_{\gamma,k})$  for all  $\gamma \in \text{dom}(p_{\delta_\ell}(0))$  and all  $k \leq r^*$ . For each  $\ell \in W_n$  fix  $s_\ell \in V$  such that  $s_\ell \in Q_{\mu_n}$  extends  $p_{\delta_\ell}(\mu_n)$  and satisfies



$\text{num}(s_\ell, \delta_\ell) = r^* + 1$ ,  $s_\ell(x_{z_{-1}}) < 2^M$ , and  $s_\ell(x_{z_k}) < 2^M$  for all  $k < r^*$ . (This is possible by the proof of Proposition 1.3 since we only need to find solutions to Eqs. (1.2) modulo  $2^M$  since  $g^{p_{\delta_\ell}(0)}(\delta_\ell, k) \leq M$  for  $k \leq r^*$ .)

Define an equivalence relation  $\equiv_n$  on  $W_n$  by:  $\ell_1 \equiv_n \ell_2$  iff  $s_{\ell_1} \cup s_{\ell_2}$  is a function. By choice of  $M$  and  $N$ , there is an equivalence class,  $W_{n+1}$ , of size at least  $|W_n|/N$ . For  $\ell \in W_{n+1}$  we can define a common extension  $q_{n+1}(\mu_n)$  of  $p^{**}(\mu_n)$  and the  $p_{\delta_\ell}(\mu_n)$  and let  $q_{n+1} \upharpoonright \mu_n = \tilde{q}_n$ . This completes the inductive construction.  $\square$

For any abelian group  $H$ , let  $v(H)$  be the Chase radical of  $H$ : the intersection of the kernels of all homomorphisms of  $H$  into an  $\aleph_1$ -free group (cf. [2]). Then  $H/v(H)$  is  $\aleph_1$ -free ([2, Proposition 1.2], [4, p. 290]). Let  $\text{cl}(Z_m[S])$  be defined by:  $\text{cl}(Z_m[S])/Z_m[S] = v(A/Z_m[S])$ , so in particular  $A/\text{cl}(Z_m[S])$  is  $\aleph_1$ -free. Notice also that every homomorphism from  $A$  to  $\mathbb{Z}$  is determined on  $\text{cl}(Z_m[S])$  by its values on  $\{z_{\delta, m} + K : \delta \in S\}$ .

**Corollary 2.2.** *In  $V[G]$ , for all  $m \in \omega$  and stationary  $S \subseteq E$ ,  $A/\text{cl}(Z_m[S])$  is a finite rank free group.*

**Proof.** If not, then since  $A/\text{cl}(Z_m[S])$  is  $\aleph_1$ -free, it contains a free pure subgroup of countably infinite rank. Let  $\{a_n + \text{cl}(Z_m[S]) : n \in \omega\}$  be a basis of such a subgroup. For any  $n \neq m$ ,  $a_n + Z_{m,1}[S] \neq a_m + Z_{m,1}[S]$  since 2 does not divide  $a_n - a_m \pmod{Z_m[S]}$  (or even  $\pmod{\text{cl}(Z_m[S])}$ ). Therefore,  $\{a_n + Z_{m,1}[S] : n \in \omega\}$  is an infinite subset of  $A/Z_{m,1}[S]$ , which contradicts Proposition 2.1.  $\square$

For the next corollary we will need the following:

**Lemma 2.3.** *The Chase radical,  $v(H)$ , of a torsion-free group  $H$  is absolute for generic extensions.*

**Proof.** We give an absolute construction of  $v(H)$  using the fact that a torsion-free group is  $\aleph_1$ -free if and only if every finite rank subgroup is finitely generated (cf. [5, Theorem 19.1]), that is, if and only if the pure closure of every finitely generated subgroup is finitely generated. For any group  $H'$ , let  $\mu(H')$  be the sum of all finite rank subgroups  $G$  of  $H'$  which are not free but are such that every subgroup of  $G$  of smaller rank is free; it is easy to see that for such  $G$ ,  $v(G) = G$  and hence  $\mu(H') \subseteq v(H')$ . Moreover, the definition of  $\mu(H')$  is absolute. Now define  $v_\beta(H)$  by induction:  $v_0(H) = 0$ ,  $v_{\alpha+1}(H)/v_\alpha(H) = \mu(H/v_\alpha(H))$ , and for limit ordinals  $\beta$ ,  $v_\beta(H) = \bigcup_{\alpha < \beta} v_\alpha(H)$ . It follows by induction that  $v_\alpha(H) \subseteq v(H)$  for all  $\alpha \leq \omega_1$ . We claim that  $v(H) = v_{\omega_1}(H)$ ; it suffices to prove that  $H/v_{\omega_1}(H)$  is  $\aleph_1$ -free. If not, then there is a finite rank subgroup of  $H/v_{\omega_1}(H)$  which is not finitely generated. We can choose one,  $G$ , of minimal rank, so all of its subgroups of smaller rank are free; say  $G$  is the pure closure of  $\{a_1 + v_{\omega_1}(H), \dots, a_n + v_{\omega_1}(H)\}$ ; but then for some  $\alpha < \omega_1$ , the pure closure of  $\{a_1 + v_\alpha(H), \dots, a_n + v_\alpha(H)\}$  is not free, but still has the property that every subgroup of smaller rank is free; hence  $\{a_1, \dots, a_n\} \subseteq v_{\alpha+1}(H)$ , which is a contradiction.  $\square$

**Corollary 2.4.** *If  $h \in \text{Hom}(A, \mathbb{Z})^{V[G]}$  and for some  $i \in \omega_2$ ,  $m \in \omega$  and some stationary set  $S \in V[G_i]$ , the sequence  $(h(z_{\delta,m} + K): \delta \in S)$  belongs to  $V[G_i]$ , then  $h$  belongs to  $V[G_i]$ .*

**Proof.** Suppose  $h$ ,  $S$  and  $m$  are as in the hypotheses. First we claim that  $h \upharpoonright Z_m[S]$  belongs to  $V[G_i]$ . Indeed, we can define  $h \upharpoonright Z_m[S]$  in  $V[G_i]$  as follows:  $h(a) = k$  if for some  $n \neq 0$ ,  $na$  belongs to the subgroup generated by  $\{z_{\delta,m} + K: \delta \in S\}$  and  $h(na) = nk$ ; and otherwise  $h(a) = \zeta$  for some fixed  $\zeta \notin \mathbb{Z}$ . In fact, the second case never occurs because  $h(a)$  is defined in  $V[G]$ . Next, we claim that  $h \upharpoonright \text{cl}(Z_m[S])$  belongs to  $V[G_i]$ . The proof is similar in principle, using the inductive construction of  $\text{cl}(Z_m[S])$  given by the proof of Lemma 2.3. But then by Corollary 2.2,  $h$  is determined by only finitely many more values, so also  $h$  belongs to  $V[G_i]$ .  $\square$

For the next corollary, we introduce some notation that will be used in Section 4. Let  $\varphi_i \in V[G_{i+1}]$  denote the generic function for  $Q_i$ ; thus  $\varphi_i$  is a homomorphism:  $F \rightarrow \mathbb{Z}$  extending  $\psi_i: K \rightarrow \mathbb{Z}$ , where  $\psi_i$  is the interpretation in  $V[G_i]$  of the name  $\dot{\psi}_i$ . The canonical map  $\text{Hom}(K, \mathbb{Z}) \rightarrow \text{Ext}(A, \mathbb{Z})$  sends  $\psi_i$  to a short exact sequence

$$\mathcal{E}_i: 0 \rightarrow \mathbb{Z} \xrightarrow{\iota} B_i \xrightarrow{\pi} A \rightarrow 0$$

and there is a commuting diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \xrightarrow{\iota'} & F & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow \psi_i & & \downarrow \sigma_i & & \downarrow 1_A & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\iota} & B_i & \xrightarrow{\pi} & A & \longrightarrow & 0 \end{array}$$

where  $\iota$  and  $\iota'$  are inclusion maps. Moreover, for all  $z \in F$ ,  $(\pi \circ \sigma_i)(z) = z + K \in A$ . Then  $\varphi_i$  gives rise to a splitting  $\rho_i \in \text{Hom}(B_i, \mathbb{Z})^{V[G_{i+1}]}$  defined by  $\rho_i(\sigma_i(z)) = \varphi_i(z)$ . Thus  $\rho_i \circ \iota = 1_{\mathbb{Z}}$  (the identity on  $\mathbb{Z}$ ) and also  $\rho_i \circ \sigma_i = \varphi_i$ .

**Corollary 2.5.** *If  $f \in \text{Hom}(B_i, \mathbb{Z})^{V[G]}$  and for some  $m \in \omega$  and some stationary set  $S \in V[G_i]$ , the sequence  $(f(\sigma_i(z_{\delta,m})): \delta \in S)$  belongs to  $V[G_i]$ , then  $f$  belongs to  $V[G_i]$ .*

**Proof.** If  $Z'$  is defined to be the pure subgroup of  $B_i$  generated by  $\{\sigma_i(z_{\delta,m}): \delta \in S\} \cup \{i(1)\}$  and  $C'$  is such that  $v(B_i/Z') = C'/Z'$ , then  $\pi$  induces an isomorphism of  $C'/\text{rge}(\iota)$  with  $\text{cl}(Z_m[S])$ . Hence  $B_i/C' \cong A/\text{cl}(Z_m[S])$  is finite rank free; therefore, arguing as in Corollary 2.4,  $f$  belongs to  $V[G_i]$ .  $\square$

### 3. Proof of (II)

We divide the proof of (II) into three cases according to the cofinality of  $\beta$ . The case of cofinality  $\omega_2$  (i.e.,  $\beta = \omega_2$ ) is trivial since any function from  $A$  (which has cardinality  $\aleph_1$ ) to  $\mathbb{Z}$  must belong to  $V[G_i]$  for some  $i < \beta$ .

Let  $\dot{h}$  be a  $P_\beta$ -name and  $p \in G_\beta$  such that

$$p \Vdash_{P_\beta} \dot{h} \in \text{Hom}(A, \mathbb{Z}).$$

Then for each  $\delta \in E$  there is  $p_\delta \in G_\beta$  and  $k_\delta \in \mathbb{Z}$  such that  $p_\delta \geq p$  and  $p_\delta \Vdash_{P_\beta} \dot{h}(z_{\delta,0} + K) = k_\delta$ .

Suppose that the cofinality of  $\beta$  is  $\omega$ , and fix an increasing sequence  $(\beta_n: n \in \omega)$  whose sup is  $\beta$ . Then there is  $n \in \omega$  and a stationary subset  $S_1$  of  $E$ , belonging to  $V[G_\beta]$ , such that for  $\delta \in S_1$ ,  $p_\delta \in G_{\beta_n}$ . Without loss of generality there is  $p^* \in G_{\beta_n}$  such that  $p^*$  forces

$$S = \{\delta \in E: \exists p_\delta \in \dot{G}_{\beta_n} \text{ and } k_\delta \in \mathbb{Z} \text{ s.t. } p_\delta \Vdash_{P_\beta} \dot{h}(z_{\delta,0} + K) = k_\delta\} \text{ is stationary.}$$

Then  $(h(z_{\delta,0} + K): \delta \in S)$  belongs to  $V[G_{\beta_n}]$ , so  $h \in \text{Hom}(A, \mathbb{Z})^{V[G_{\beta_n}]}$ , by Corollary 2.4.

The final, and hardest, case is when the cofinality of  $\beta$  is  $\omega_1$ . Fix an increasing continuous sequence  $(\beta_\nu: \nu < \omega_1)$  whose sup is  $\beta$ . Then there is  $\nu \in \omega_1$  and a stationary subset  $S$  of  $\omega_1$  such that for  $\delta \in S$ ,  $p_\delta \upharpoonright \beta_\delta \in G_{\beta_\nu}$ .

For any  $t \geq 1$ ,  $(Z_{0,t}[S \setminus \alpha]: \alpha < \omega_1)$  is a non-increasing sequence of groups. Since the groups  $A/Z_{0,t}[S \setminus \alpha]$  are finite, it follows that there is a countable ordinal  $\alpha_t$  such that for  $\gamma, \alpha \geq \alpha_t$ ,  $Z_{0,t}[S \setminus \alpha] = Z_{0,t}[S \setminus \gamma]$ . Therefore, there is a countable ordinal  $\alpha_*$  and a countable subset  $Y$  of  $A$  such that for all  $t \geq 1$ ,  $\alpha_t \leq \alpha_*$ , and  $Y$  contains representatives of all the elements of  $A/Z_{0,t}[S \setminus \alpha_*]$ . Increasing  $\nu$  if necessary, we can assume that we can compute  $h(y)$  in  $V[G_{\beta_\nu}]$  for all  $y \in Y$ .

We claim that  $h$  belongs to  $V[G_{\beta_\nu}]$ . In pursuit of a contradiction, suppose that there are  $a \in A$ , conditions  $q_1, q_2 \in P_\beta/G_{\beta_\nu}$  and integers  $c_1 \neq c_2$  such that  $q_\ell \Vdash_{P_\beta} \dot{h}(a) = c_\ell$  for  $\ell = 1, 2$ . Choose  $t$  sufficiently large such that  $2^t$  does not divide  $c_2 - c_1$  and choose  $\mu < \omega_1$  such that  $q_1, q_2 \in P_{\beta_\mu}$ . For some  $y \in Y$ ,  $a - y \in Z_{0,t}[S \setminus \alpha_*] = Z_{0,t}[S \setminus \beta_\mu]$ . Thus

$$a - y = z + 2^t a'$$

for some  $a' \in A$  and  $z$  in the pure closure of the subgroup generated by  $\{z_{\delta_j,0}: j = 1, \dots, n\}$  for some  $\delta_1, \dots, \delta_n \in S \setminus \beta_\mu$ . For  $\ell = 1, 2$  there is an upper-bound  $r_\ell \in P_\beta$  of  $\{q_\ell, p_{\delta_1}, \dots, p_{\delta_n}\}$ . Then  $r_1$  and  $r_2$  force the same value,  $b$ , to  $h(z)$  (because they are both  $\geq p_{\delta_j}$  for  $j = 1, \dots, n$ ) and the same value,  $k$ , to  $h(y)$  (because it is determined in  $V[G_{\beta_\nu}]$ ). Therefore

$$r_\ell \Vdash 2^t \text{ divides } c_\ell - k - b.$$

So for  $\ell = 1, 2$ , the integer  $c_\ell - k - b$  is divisible by  $2^t$ . But this contradicts the choice of  $t$ .

#### 4. Proof of (III)

We continue with the notation from the end of Section 2; so  $\mathcal{E}_i \in V[G_i]$ . Suppose that  $\mathcal{E}_i$  represents a torsion element of  $\text{Ext}(A, \mathbb{Z})$ , of order  $e \geq 1$ , that is, there is a homomorphism  $g_i: B_i \rightarrow \mathbb{Z}$  such that  $g_i \upharpoonright \mathbb{Z} = e1_{\mathbb{Z}}$ , or more precisely,  $g_i \circ \iota = e1_{\mathbb{Z}}$ .

(We consider the zero element to be torsion of order 1.) Then  $e\rho_i - g_i$  is a homomorphism from  $B_i$  to  $\mathbb{Z}$  which is identically 0 on  $\mathbb{Z}$ , so it induces a homomorphism  $\theta_i \in \text{Hom}(A, \mathbb{Z})^{V[G_{i+1}]}$  (that is,  $\theta_i \circ \pi = e\rho_i - g_i$ ) which is a new element of  $A^*$  — that is, it is not in  $V[G_i]$ . To prove (III) it will suffice to prove that if there is an element  $h$  of  $A^*$  which is in  $V[G_{i+1}]$  but not in  $V[G_i]$ , then  $\mathcal{E}_i$  is torsion, and in that case  $h$  is an integral multiple of  $\theta_i$  modulo  $(A^*)^{V[G_i]}$ .

Given such an  $h$ , let  $h' = h \circ \pi : B_i \rightarrow \mathbb{Z}$ . Clearly,  $h' \in V[G_{i+1}] - V[G_i]$ . We claim that:

(III.1) For some integer  $c$ ,  $h' - c\rho_i$  belongs to  $V[G_i]$ .

Let us see first why this Claim implies the desired conclusion. Note that  $c \neq 0$  since  $h'$  does not belong to  $V[G_i]$ . Since  $(h' - c\rho_i) \upharpoonright \mathbb{Z} = -c1_{\mathbb{Z}}$ , we conclude that in  $V[G_i]$ ,  $\mathcal{E}_i$  is torsion, of order  $e$  dividing  $-c$ ; let  $g_i \in \text{Hom}(B_i, \mathbb{Z})^{V[G_i]}$  such that  $g_i \upharpoonright \mathbb{Z} = e1_{\mathbb{Z}}$ . Let  $\theta_i$  be induced by  $e\rho_i - g_i$ , as above. Say  $c = ne$ ; then  $h' - c\rho_i + ng_i$  belongs to  $V[G_i]$  and is identically 0 on  $\mathbb{Z}$  so it induces a homomorphism  $f \in \text{Hom}(A, \mathbb{Z})^{V[G_i]}$ . By composing both sides with  $\pi$  one sees that  $h = n\theta_i + f$ .

We shall now work on the proof of (III.1). Let  $F'$  be the subgroup of  $F$  generated by  $\{x_\nu : \nu < \omega_1\}$ . We work in  $V[G_i]$ . For any countable ordinal  $\alpha \in \omega_1 - E$ , define

$$\mathcal{Q}_{i,\alpha} = \{q \in \mathcal{Q}_i : z_{\delta,n} \in \text{dom}(q) \Rightarrow \delta < \alpha \text{ and } x_\nu \in \text{dom}(q) \Rightarrow \nu < \alpha\}.$$

Then  $\mathcal{Q}_{i,\alpha}$  is a complete subforcing of  $\mathcal{Q}_i$ . In particular,

$$V[G_{i+1}] = V[G_i][G_{i+1,\alpha}][H_{i+1,\alpha}],$$

where  $G_{i+1,\alpha}$  is  $\mathcal{Q}_{i,\alpha}$ -generic over  $V[G_i]$  and  $H_{i+1,\alpha}$  is  $\mathcal{Q}_i/G_{i+1,\alpha}$ -generic over  $V[G_i][G_{i+1,\alpha}]$ . We claim:

(III.2) There is a countable ordinal  $\alpha \in \omega_1 - E$  such that in  $V[G_i][G_{i+1,\alpha}]$  there is an assignment to every  $y \in F'$  of a function  $\xi_y : \mathbb{Z} \rightarrow \mathbb{Z}$  such that for all  $y \in F' \Vdash_{\mathcal{Q}_i/G_{i+1,\alpha}} \dot{h}(y + K) = \xi_y(\dot{\varphi}_i(y))$ .

Let us see first why this implies (III.1). First we assert that the following consequence of (III.2) holds in  $V[G_i][G_{i+1,\alpha}]$ :

(III.2.1) There is an integer  $c$  such that for every  $k \in \mathbb{Z}$ , and every  $\beta \geq \alpha$ ,  $\xi_{x_\beta}(k) - \xi_{x_\beta}(0) = kc$ .

To see this, let  $\beta, \gamma \geq \alpha$  with  $\beta \neq \gamma$ , and let  $k \in \mathbb{Z}$ . By the proof of Proposition 1.3, there are conditions  $q_1, q_2 \in \mathcal{Q}_i/G_{i+1,\alpha}$  such that

$$q_1 \Vdash \dot{\varphi}_i(x_\beta) = k \wedge \dot{\varphi}_i(x_\gamma) = 0$$

and

$$q_2 \Vdash \dot{\varphi}_i(x_\beta) = 0 \wedge \dot{\varphi}_i(x_\gamma) = k.$$

Let  $y = x_\beta + x_\gamma$ . By (III.2) and the fact that  $\dot{h}$  and  $\dot{\varphi}_i$  are homomorphisms,

$$q_1 \Vdash \xi_{x_\beta}(k) + \xi_{x_\gamma}(0) = \xi_y(k),$$

which implies that  $\check{\xi}_{x_\beta}(k) + \check{\xi}_{x_\gamma}(0) = \check{\xi}_y(k)$  holds in  $V[G_i][G_{i+1,\alpha}]$ . Similarly, reasoning with  $q_2$ , we can conclude that  $\check{\xi}_{x_\gamma}(k) + \check{\xi}_{x_\beta}(0) = \check{\xi}_y(k)$  holds in  $V[G_i][G_{i+1,\alpha}]$ . Thus  $\check{\xi}_{x_\beta}(k) - \check{\xi}_{x_\beta}(0) = \check{\xi}_{x_\gamma}(k) - \check{\xi}_{x_\gamma}(0)$  in  $V[G_i][G_{i+1,\alpha}]$ ; we denote this value by  $\Xi(k)$ . If we can prove that for all  $k$ ,  $\Xi(k) = k\Xi(1)$ , then we can let  $c = \Xi(1)$ . Again, let  $\beta, \gamma \geq \alpha$  with  $\beta \neq \gamma$  and this time let  $y = kx_\beta + x_\gamma$ . Using conditions  $q_3 \Vdash \dot{\phi}_i(x_\beta) = 1 \wedge \dot{\phi}_i(x_\gamma) = 0$  and  $q_2 \Vdash \dot{\phi}_i(x_\beta) = 0 \wedge \dot{\phi}_i(x_\gamma) = k$  we conclude that

$$k\check{\xi}_{x_\beta}(1) + \check{\xi}_{x_\gamma}(0) = k\check{\xi}_{x_\beta}(0) + \check{\xi}_{x_\gamma}(k)$$

from which it follows that  $k\Xi(1) = \Xi(k)$ . This proves (III.2.1)

Now work in  $V[G_{i+1}]$ ; we have

$$h(x_\beta + K) = \check{\xi}_{x_\beta}(\varphi_i(x_\beta)) = c\varphi_i(x_\beta) + \check{\xi}_{x_\beta}(0)$$

for  $\beta \geq \alpha$ . Since  $((h' - c\rho_i) \circ \sigma_i)(x) = h(x + K) - c\varphi_i(x)$  for  $x \in F'$ , it follows that  $h' - c\rho_i \upharpoonright \{\sigma_i(x_\beta) : \beta \geq \alpha\}$  belongs to  $V[G_i][G_{i+1,\alpha}]$ . Moreover, for  $\beta < \alpha$ ,  $\varphi_i(x_\beta)$  is determined in  $V[G_i][G_{i+1,\alpha}]$ , and hence so are  $h(x_\beta + K) = \check{\xi}_{x_\beta}(\varphi_i(x_\beta))$  and  $(h' - c\rho_i)(\sigma_i(x_\beta))$ . Therefore  $h' - c\rho_i$  belongs to  $V[G_i][G_{i+1,\alpha}]$  (since it is determined by its values on  $\{\sigma_i(x_\beta) : \beta \in \omega_1\} \cup \{1(1)\}$ ). Let  $f = h' - c\rho_i$ . For each  $\delta \in E$ , there exist  $p_\delta \in G_{i+1,\alpha}$  and  $k_\delta \in \mathbb{Z}$  such that

$$p_\delta \Vdash_{Q_i} \dot{f}(\sigma_i(z_{\delta,m})) = k_\delta.$$

Since  $Q_{i,\alpha}$  and  $\mathbb{Z}$  are countable, there exist  $p \in G_{i+1,\alpha}$ ,  $k \in \mathbb{Z}$ , and a stationary  $S \in V[G_i]$  such that for  $\delta \in S$ ,  $p \Vdash_{Q_i} \dot{f}(\sigma_i(z_{\delta,m})) = k$ . Then the (constant) sequence  $(f(\sigma_i(z_{\delta,m}))) : \delta \in S$  belongs to  $V[G_i]$ , so by Corollary 2.5,  $f$  belongs to  $V[G_i]$ .

So it remains to prove (III.2). Work in  $V[G_i]$ . Let

$$D_{i,\alpha} = \{q \in Q_i : \forall \delta \in (\text{cont}(q) - \alpha) \forall n \in \omega[(\eta_\delta(n) < \alpha) \Rightarrow x_{\eta_\delta(n)} \in \text{dom}(q)]\}.$$

Then  $D_{i,\alpha}$  is a dense subset of  $Q_i$ . We claim that it is true in  $V[G_i]$  that:

(III.3) there is a countable ordinal  $\alpha \in \omega_1 - E$  such that for every  $y \in F'$ ,  $t, c_1, c_2 \in \mathbb{Z}$ , and  $q_1, q_2 \in D_{i,\alpha}$  with  $q_1 \upharpoonright \alpha = q_2 \upharpoonright \alpha$ , if

$$q_\ell \Vdash_{Q_i} \dot{\phi}_i(y) = t \wedge \dot{h}(y + K) = c_\ell$$

for  $\ell = 1, 2$ , then  $c_1 = c_2$ .

Clearly this implies (III.2). Indeed, we define  $\check{\xi}_y(t)$  to be  $c$  if there is a  $q \in D_{i,\alpha}$  such that  $q \upharpoonright \alpha \in G_{i+1,\alpha}$  and  $q \Vdash_{Q_i/G_{i+1,\alpha}} \dot{\phi}_i(y) = t \wedge \dot{h}(y + K) = c$  and otherwise  $\check{\xi}_y(t) = 0$ . By (III.3),  $\check{\xi}_y(t)$  is well-defined.

**Proof of (III.3).** The proof is by contradiction and uses some of the methods of the proof of Proposition 2.1. So suppose that for every  $\alpha \in \omega_1 - E$  there are  $y^\alpha \in F'$ ,  $t^\alpha, c_1^\alpha, c_2^\alpha \in \mathbb{Z}$ , and  $q_1^\alpha, q_2^\alpha \in D_{i,\alpha}$  such that  $q_1^\alpha \upharpoonright \alpha = q_2^\alpha \upharpoonright \alpha$  and  $q_\ell^\alpha \Vdash_{Q_i} \dot{\phi}_i(y^\alpha) = t^\alpha \wedge \dot{h}''(y) = c_\ell^\alpha$  where  $c_1^\alpha \neq c_2^\alpha$  for  $\ell = 1, 2$ . Then, by Fodor's Lemma and counting, there is a  $p_0 \in G_i$ ,  $t, c_1, c_2 \in \mathbb{Z}$ ,  $\tilde{q} \in V$  and names  $\dot{S}, \dot{q}_1^\alpha, \dot{y}^\alpha$  such that

$$p_0 \Vdash_{P_i} \dot{S} \text{ is a stationary subset of } \omega_1 - E \text{ s.t. for all } \alpha \in \dot{S}, \\ t^\alpha = t, c_1^\alpha = c_1, c_2^\alpha = c_2 \text{ and } \dot{q}_1^\alpha \upharpoonright \alpha = \dot{q}_2^\alpha \upharpoonright \alpha = \tilde{q}$$

and moreover such that  $p_0$  forces the names to be a counterexample to (III.3), as above.

There is a stationary subset  $S' \subseteq \omega_1 - E$  such that for every  $\alpha \in S'$  there is a condition  $p_\alpha \geq p_0$  in  $P_i$  which forces  $\alpha \in \dot{S}$  and forces values (elements of  $V$ ) to  $\dot{q}_\ell^\alpha$  and to  $\dot{y}^\alpha$ . Moreover, we can suppose that the  $p_\alpha \cup \{(i, q_\ell^\alpha)\} \in P_{i+1}$  ( $\ell = 1, 2$ ) are as in  $(\dagger)$  [cf. proof of Proposition 2.1] and that  $\{p_\alpha: \alpha \in S'\}$  is as in  $(\dagger\dagger)$  [with  $\alpha$  in place of  $\delta$ , but since  $\alpha \notin E$ , the last sentence does not apply]. Let  $p^*$  be the heart of  $\{p_\alpha: \alpha \in S'\}$ . We can also assume that  $\{q_\ell^\alpha: \alpha \in S'\}$  forms a  $\Delta$ -system with heart  $\tilde{q}$  (for  $\ell = 1, 2$ ).

For each  $\delta \in E$ , there is  $\hat{p}_\delta \in P_{i+1}$  and  $k_\delta \in \mathbb{Z}$  such that  $\hat{p}_\delta \upharpoonright i \geq p^*$ ,  $\hat{p}_\delta(i) \geq \tilde{q}$  and  $\hat{p}_\delta \Vdash \dot{h}(z_{\delta,0}) = k_\delta$ . There is a stationary  $\hat{S} \subseteq E$  such that  $\{\hat{p}_\delta: \delta \in \hat{S}\}$  satisfies  $(\dagger)$  and  $(\dagger\dagger)$ ; in particular,  $\hat{r} = r_{\hat{p}_\delta(0)}$  for  $\delta \in \hat{S}$  and  $\eta_{\hat{p}_\delta(0)}(n)$ ,  $g^{\hat{p}_\delta(0)}(\delta, n) = g(n)$  and  $u^{\hat{p}_\delta(0)}(\delta, n)$  are independent of  $\delta$  for each  $n < \hat{r}$ . Moreover, we can assume that there is  $k \in \mathbb{Z}$  such that  $k_\delta = k$  for all  $\delta \in \hat{S}$ . Let  $\hat{p}^*$  be the heart of  $\{\hat{p}_\delta: \delta \in \hat{S}\}$  (so  $\hat{p}^* \geq p^* \cup \{(i, \tilde{q})\}$ ).

Choose  $m$  such that  $2^m$  does not divide  $c_1 - c_2$ . Let  $M \geq \max(\{g(n): n < \hat{r}\} \cup \{m\})$  and let

$$N = 2^{1+(\hat{r}+1)M(d+1)}$$

(where  $d$  is the size of the domain of  $\hat{p}^* - \{0\}$ ). Choose

$$\alpha_0 < \cdots < \alpha_{N-1} < \gamma < \delta_0 < \cdots < \delta_{N-1},$$

where  $\alpha_j \in S'$ ,  $\delta_j \in \hat{S}$ , every ordinal which occurs in  $\hat{p}^*$  is  $< \alpha_0$ , and for all  $j \leq N-1$  every ordinal which occurs in  $p_{\alpha_j}$  or in  $q_\ell^{\alpha_j}$  ( $\ell = 1, 2$ ) is less than  $\alpha_{j+1}$  (where  $\alpha_N$  is taken to be  $\gamma$ ); and for all  $j < N-1$ , every ordinal which occurs in  $p_{\delta_j}$  is less than  $\delta_{j+1}$ . Then there is a condition  $q_0 \in \mathcal{Q}_0$  which extends  $\hat{p}^*(0)$  and each  $p_{\alpha_j}(0)$  and  $p_{\delta_j}(0)$  such that  $q_0$  forces for all  $j < N$ :

$$\eta_{\delta_j}(\hat{r}) = \gamma; \quad \eta_{\delta_j}(\hat{r} + 1) = \delta_{j-1} + 1; \quad u(\delta_j, \hat{r}) = y^{\alpha_j} \quad \text{and} \quad g(\delta_j, \hat{r}) = 2^m,$$

where  $\delta_{-1} = \gamma + 1$ .

As in the proof of Proposition 2.1, there is a condition  $q' \in P_i$  and a subset  $W'$  of  $N$  of size  $\geq 2^{1+(\hat{r}+1)M}$  such that  $q'(0) = q_0$ ,  $q' \geq p_{\alpha_j}$  for all  $j \leq N-1$ , and  $q' \geq p_{\delta_j} \upharpoonright i$  for  $j \in W'$ . Repeating the argument one more time and using the facts that  $q_1^{\alpha_j}$  and  $q_2^{\alpha_j}$  force the same value to  $\varphi(y^{\alpha_j})$  and that  $q_1^{\alpha_j} \upharpoonright \alpha_j = q_2^{\alpha_j} \upharpoonright \alpha_j = \tilde{q}$ , there is a subset  $W = \{j, j_o\}$  of  $W'$  such that for any function  $f: W \rightarrow \{1, 2\}$  there is a condition  $q_* \in P_{i+1}$  such that  $q_* \upharpoonright i = q'$  and  $q_*(i)$  is an upper bound of  $\{p_{\delta_j}(i), p_{\delta_{j_o}}(i)\} \cup \{q_1^{\alpha_j}, q_2^{\alpha_{j_o}}\}$ . In a generic extension  $V[G']$  where  $q_* \in G'$  we have (since  $h(z_{\delta_j,0} + K) = h(z_{\delta_{j_o},0} + K) = k$  and  $g(\delta, n)$  and  $u(\delta, n)$  are independent of  $\delta \in \hat{S}$  for  $n < \hat{r}$ ) that  $2^m$  divides

$$h(u(\delta_j, \hat{r})) - h(u(\delta_{j_o}, \hat{r})) = h(y^{\alpha_j} + K) - h(y^{\alpha_{j_o}} + K) = c_1 - c_2,$$

which is a contradiction of the choice of  $m$ . This proves (III.3) and thus finally completes the proof of Theorem 1.5.  $\square$

## 5. Theorem 0.2

To prove Theorem 0.2 we use a variant of the iterated forcing that is described in Section 1. Let  $\mathcal{Q}_0$  and  $\mathcal{Q}_\psi$  be as defined there. We shall use a finite support iteration  $P' = \langle P'_i, \dot{Q}_i : 0 \leq i < \omega_2 \rangle$ ; the  $\dot{Q}_i$  are defined inductively. We consider an enumeration, as before, of names  $\{\dot{\psi}_i : i < \omega_2\}$  for functions from  $K$  to  $\mathbb{Z}$ . In  $V^{P'_i}$  we define

$$\dot{Q}_i = \begin{cases} \{0\} & \text{if the s.e.s } \mathcal{E}_i \text{ is torsion,} \\ \mathcal{Q}_{\dot{\psi}_i} & \text{otherwise.} \end{cases}$$

We claim that if  $G$  is  $P'$ -generic, then in  $V[G]$  (i)  $\text{Ext}(A, \mathbb{Z})$  is torsion and (ii)  $\text{Hom}(A, \mathbb{Z}) = 0$ .

To see why (i) holds, consider  $\psi \in \text{Hom}(K, \mathbb{Z})$ . For some  $i \in \omega_2$ ,  $\dot{\psi}_i$  is a name for  $\psi$ . In  $V[G_i]$  either  $\psi$  represents a torsion element of  $\text{Ext}(A, \mathbb{Z})$  or else, by construction, in  $V[G_{i+1}]$   $\psi = \varphi|K$  for some  $\varphi \in \text{Hom}(F, \mathbb{Z})$ , in which case  $\psi$  represents the zero element of  $\text{Ext}(A, \mathbb{Z})$ .

To prove (ii), it suffices to show for all  $i \in \omega_2$  that if  $h \in \text{Hom}(A, \mathbb{Z})^{V[G_{i+1}]}$ , then  $h \in \text{Hom}(A, \mathbb{Z})^{V[G_i]}$ . If not, then  $\dot{Q}_i \neq \{0\}$ ; but then by the arguments in Section 4 it follows that  $\mathcal{E}_i$  is torsion, so  $\dot{Q}_i = \{0\}$ , a contradiction.

## 6. Co-Moore spaces

Following [7] we call a topological space  $X$  a *co-Moore space of type*  $(G, n)$ , where  $n \geq 1$ , if its reduced integral cohomology groups satisfy

$$\tilde{H}^i(X, \mathbb{Z}) = \begin{cases} G & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

For  $n \geq 2$ , application of the Universal Coefficient Theorem shows that

(♣) there exist  $B_1$  and  $B_2$  such that  $G \cong \text{Hom}(B_1, \mathbb{Z}) \oplus \text{Ext}(B_2, \mathbb{Z})$  where  $\text{Ext}(B_1, \mathbb{Z}) = 0 = \text{Hom}(B_2, \mathbb{Z})$ .

Conversely, if  $G$  satisfies (♣), then there is a co-Moore space of type  $(G, n)$  for any  $n \geq 2$  (cf. [7, Theorem 5, 6]). A sufficient condition for  $G$  to be of the form (♣) is that  $G = D \oplus C$  where  $C$  is compact and  $D$  is isomorphic to a direct product of copies of  $\mathbb{Z}$  ([7, Theorem 5]). In a model of ZFC where every W-group is free, this condition is necessary (cf. [7, Theorem 3(a)] and [3, Theorem 2.20]); in particular, the (torsion-free) rank of  $C$  is of the form  $2^\mu$  for some infinite cardinal  $\mu$ . However, as a consequence of our proofs we have:

**Corollary 6.1.** *It is consistent with ZFC +  $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$  that there is a group  $A$  of cardinality  $\aleph_1$  such that  $\text{Hom}(A, \mathbb{Z}) = 0$  but  $\text{Ext}(A, \mathbb{Z})$  does not admit a compact topology.*

**Corollary 6.2.** *It is consistent with  $ZFC + 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$  that for any  $n \geq 2$  there is a co-Moore space of type  $(F, n)$  where  $F$  is the free abelian group of rank  $\aleph_2$ .*

**Corollary 6.3.** *It is consistent with  $ZFC + 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$  that for any  $n \geq 2$  there is a co-Moore space of type  $(C, n)$  for some uncountable torsion divisible group  $C$ .*

Compare Corollary 6.3 with [6, 2.5 and 2.6]. The conclusions of the corollaries are not provable in ZFC. Moreover, by an easy modification we can replace  $\aleph_2$  in the corollaries by any regular cardinal greater than  $\aleph_1$ . (Note that by [1, Theorem 5.6],  $\text{Hom}(B_1, \mathbb{Z})$  cannot be the free group of rank  $\aleph_1$  if  $\text{Ext}(B_1, \mathbb{Z}) = 0$ .)

### Note added in Proof

The authors have recently obtained consistency results with GCH analogous to Theorems 0.2 and Theorem 0.1, and in fact, stronger versions of Theorem 0.2, which give new possible co-Moore spaces of type  $(C, n)$  and where  $C$  is divisible.

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