Annals of Pure and Applied Logic 28 (1985) 1–12
North-Holland

NARROW BOOLEAN ALGEBRAS

Robert BONNET

U.E.R. de Mathématiques, Université Claude Bernard, 69622-Villeurbanne Cedex, France

Saharon SHELAH

Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel

Communicated by A. Nerode
Received 22 September 1983

Introduction

Call a Boolean algebra λ-*narrow* if every set of pairwise incomparable elements has cardinality less than λ. The Boolean Algebra is *narrow* if it has cardinality λ and is λ-narrow. In 1976, the first author announced that, assuming CH, there is a narrow Boolean Algebra of power $2^{\omega} = \omega_1$. Independently, Berney and Nyikos [4], also proved the same result, using a different argument (they 'use' the Sorgenfrey topology on the real line, which has no countable basis and for which the plane is not normal). In 1979, the second author improved the argument still further and proved, without any assumption, that there is a narrow Boolean Algebra of power $\mathrm{cf}(2^{\omega})$ — the cofinal cardinal of the continuum. In this paper, we present Shelah's improvement. We remark that for every $k \geqslant \omega$, the construction also gives a narrow Boolean Algebra of power k^+ if we assume GCH.

1. On narrow Boolean algebras

1.1. Let $\boldsymbol{E} = \langle E, \leqslant \rangle$ be a partial ordering. A set of pairwise incomparable elements of E is called an *antichain*. Let $\mu \leqslant \lambda$ be two cardinals. A partial ordering of cardinality λ is said to be μ-*narrow* (or more simply *narrow* if $\mu = \lambda$) whenever every antichain of the ordering is of cardinality $<\mu$. In particular, we have the notion of *narrow Boolean algebra* (in this case we must remark that an antichain is not necessarily a set of pairwise disjoint elements).

Subsequently, we denote by κ the cofinal cardinal of 2^{ω}.

1.2. We will prove:

Theorem 1. *There is a narrow Boolean algebra of cardinality* $\kappa = \mathrm{cf}(2^{\omega})$.

In fact, we will prove that there is a subset P of the real line $\mathbb{R}$, which is κ-dense (that means each non-empty open interval of $\mathbb{R}$, contains κ points of P) and such that the Boolean algebra $B=B(P)$, which is of cardinality κ, is narrow.

A Boolean algebra B of cardinality $\lambda \geqslant \omega$, is said to be homogeneous in cardinality whenever for every $0 \neq a \in B$, the Boolean algebra Ba (defined on the set of $t \in B$ verifying $t \subset a$) is of cardinality λ too. For instance, the above Boolean algebra $B=B(P)$ is homogeneous in cardinality.

Now, we will recall a result of J. Baumgartner. Let B be a narrow and homogeneous in cardinality Boolean algebra of cardinality $\lambda \geqslant \omega$, then B has only one increasing one-to-one function f from B into itself, namely the identity. Otherwise let us suppose $f(a) \neq a$ for some a.

First case: $a \not\subset f(a)$. Let $c=a \cap (f(a))' = a-f(a)$ (here d' is the complement of d in B). We have $c \neq 0$ and $c \cap f(c)=0$. The set of $x \cup (f(c)-f(x))$ for $x \subset c$ is an antichain of cardinality λ: contradiction.

Second case: $a \subset f(a)$ and $a \neq f(a)$. Let $b=f(a)'=1-f(a)$. We must have $0 \neq b \not\subset f(b)$ and $f(b) \neq b$, and thus we conclude as in the first case. Otherwise, we can assume that b and $f(b)$ are comparable and so $b \subset f(b)$. Let $d=a \cup b$. We have

$$1=(1-b) \cup f(b)=f(a) \cup f(b) \subset f(d)$$

and thus $f(1)=1=f(d)$, since $d \subset 1$, and we obtain a contradiction (f is one-to-one, and $d \neq 1$).

Remark. From B is a narrow interval Boolean algebra of cardinality κ, and from Theorem 5.7 of Mati Rubin [7], it follows that every subalgebra of B, of cardinality κ, contains an interval subalgebra of cardinality κ too. So we obtain another proof of Theorem 5.3 of [3].

1.3. Let S be a set and $n \geqslant 2$ be an integer. A subset A of S^n is said to be *good* whenever every element $\bar{a}=\langle a_k \rangle_{k<n}$ of A verifies $a_i \neq a_j$ for $0 \leqslant i<j<n$.

Now, let $\boldsymbol{C}=\langle C, \leqslant \rangle$ be a chain. For every $n \geqslant 2$ and $\varepsilon \in \{-1,+1\}^n$, we denote by $\leqslant_\varepsilon$ the order relation on C^n defined by $\bar{x}=\langle x_k \rangle_{k<n} \leqslant_\varepsilon \bar{y}=\langle y_k \rangle_{k<n}$ whenever:

either $\varepsilon(k)=+1$ and then $x_k \leqslant y_k$,

or $\varepsilon(k)=-1$ and then $y_k \leqslant x_k$.

For instance, if $\varepsilon(k)=+1$, for every k, then $\leqslant_\varepsilon$ is the usual product order.

Definition. A chain $\boldsymbol{C}=\langle C, \leqslant \rangle$, of cardinality $\lambda \geqslant \omega_1$, is said to be hyper-rigid whenever for every $n \geqslant 2$ and every $\varepsilon \in \{-1,+1\}^n$, every good antichain of $\langle C^n, \leqslant_\varepsilon \rangle$ has cardinality $<\lambda$.

Let $\boldsymbol{C}$ be a λ-dense hyper-rigid chain of cardinality λ. Let $\boldsymbol{C}'$ be a subchain of

$\boldsymbol{C}$ and f be a one-to-one monotonic function from $\boldsymbol{C}'$ into $\boldsymbol{C}$. Then the set $N(f)$ of $t \in C'$ verifying $f(t) \neq t$ has cardinality $<\lambda$ (and so $\boldsymbol{C}$ is rigid).

The main theorems are:

Theorem 2. *There is a κ-dense hyper-rigid subchain of the real line $\mathbb{R}$.*

Theorem 3. *If P is a κ-dense hyper-rigid subchain of $\mathbb{R}$, then the interval algebra $B(P)$ is narrow and of cardinality κ. Moreover $B(P)$ is homogeneous in cardinality.*

2. Proof of the Theorem 3

2.1. Let P be a κ-dense subchain of $\mathbb{R}$. Let us suppose there is an antichain A of $B(P)$, of cardinality κ.

We use the notations, and notions introduced in Section 1 of [3, p. 345]. For each $U_i \in A$, we have

$$U_i = \bigcup_{0 \leq k < l(i)} [a^i_{2k}, a^i_{2k+1}[$$

in which $a^i_l \in \{-\infty\} \cup P \cup \{+\infty\}$ and $l(i)$ is chosen as small as possible.

Now, there are a subset $R(A)$ of A, of cardinality κ (called a *residual subset of* A), an integer $m \geq 1$, a finite strictly increasing sequence $\langle r_1, r'_1, \ldots, r_{2m+1}, r'_{2m+1}\rangle$ of rational numbers (called the separative sequence of $R(A)$), a none-empty subset $\rho_{\mathrm{R}}(A)$ of $\{0, 1, \ldots, 2m+1\}$, satisfying the following properties:

(i) $l(i) = m$ for every $U_i \in R(A)$.

(ii) If $U_i \in R(A)$, then

$$a^i_0 < r_1 < r'_1 < a^i_1 < r_2 < \cdots < a^i_{2m} < r_{2m+1} < r'_{2m+1} < a^i_{2m+1}.$$

(iii) For every $k \leq 2m+1$ and $k \notin \rho_{\mathrm{R}}(A)$, we have $a^i_k = a^j_k$ for every U_i and U_j in $R(A)$.

(iv) For every $k \in \rho_{\mathrm{R}}(A)$, we have $a^i_k \neq a^j_k$ for distinct elements U_i and U_j in $R(A)$.

2.2. Now, let $\rho_{\mathrm{R}}(A) = \{k_0, k_1, \ldots, k_{p-1}\}$, with $k_0 < k_1 < \cdots < k_{p-1}$. Let us define $\varepsilon \in \{-1, +1\}^p$ by $\varepsilon(l) = +1$ iff k_l is odd and $\varepsilon(l) = -1$ if not. For each $U_i \in R(A)$, let $\bar{c}^i = \langle c^i_l\rangle_{l<m}$ with $c^i_l = a^i_{k_l}$ for $l < p$. So $U_i \subset U_j$ iff $\bar{c}^i \leq_\varepsilon \bar{c}^j$. Consequently the set C of $\bar{c}^i$ for $U_i \in R(A)$ is an antichain of $\langle P^p, \leq_\varepsilon\rangle$ of cardinality κ.

3. How to begin the proof of the Theorem 2

3.1. In the following, we denote by P a subset of $\mathbb{R}$, by $n \geq 2$ an integer and by ε an element of $\{-1, +1\}^n$.

Definition. A subset A of P^n is said to be *separate*, whenever there is a sequence $\langle r_0, r_1, \ldots, r_{2n-1}\rangle$ of rational numbers verifying:

(1) If $\bar{a} = \langle a_k\rangle_{k<n} \in A$, then

$$r_0 < a_0 < r_1 < r_2 < a_1 < r_3 < \cdots < a_{n-2} < r_{2n-3} < r_{2n-2} < a_{n-1} < r_{2n-1}.$$

(2) If $\bar{a} = \langle a_k\rangle_{k<n}$ and $\bar{b} = \langle b_k\rangle_{k<n}$ are distinct elements of A, then $a_k \neq b_k$ for every $k<n$.

Let $I_k =]r_{2k}, r_{2k+1}[\subset \mathbb{R}$ for $k<n$. Then the I_k's are pairwise disjoint and $a_k \in I_k$ for every $\bar{a} = \langle a_k\rangle_{k<n}$ in A.

3.2. Now, let $n \geqslant 2$ and $0 \leqslant l < n$ be given. Let A be a separate subset of $\mathbb{R}^n$. For each $\bar{a} = \langle a_k\rangle_{k<n}$, we put

$$\bar{a}[l] = \langle a_0, a_1, \ldots, a_{l-1}, a_{l+1}, \ldots, a_{n-1}\rangle,$$

i.e. $\bar{a}[l] = \langle a_k[l]\rangle_{k<n-1}$ where $a_k[l] = a_k$ for $k \leqslant l-1$ and $a_k[l] = a_{k+1}$ for $l \leqslant k < n-1$. We denote by $A[l]$ the set of $\bar{a}[l]$, for $a \in A$. So $A[l] \subset \mathbb{R}^{n-1}$. Let A_l be the set of a_l for $\bar{a} = \langle a_k\rangle_{k<n}$ in A.

The function ψ_l from A onto $A[l]$, defined by $\psi_l(\bar{a}) = \bar{a}[l]$, is one-to-one (according A is separate) and thus we define a one-to-one function π_l from $A[l]$ onto A_l by $\pi_l(\bar{c}) = a_l$ iff $\bar{c} = \bar{a}[l]$ and $\bar{a} = \langle a_k\rangle_{k<n}$ (that is to say, p_l denoting the lth projection from $\mathbb{R}^n$ onto $\mathbb{R}$, we have $\pi_l = p_l \cdot \psi_l^{-1}$). We must remark that *we can interpret $A \subset \mathbb{R}^n$ as the graph of the function π_l* (up to an isomorphism of indexes $0, 1, \ldots, n-1$, which translates l at the last place).

3.3. Now, let $\leqslant_\varepsilon^l$ be the order relation on $\mathbb{R}^{n-1}$, defined by

$$\bar{a}' = \langle a'_k\rangle_{k<n-1} \leqslant_\varepsilon^l \bar{b}' = \langle b'_k\rangle_{k<n-1}$$

whenever for every $k<l$, if $\varepsilon(k) = +1$, then $a'_k \leqslant b'_k$, and if $\varepsilon(k) = -1$, then $a'_k \geqslant b'_k$; and for every k verifying $l \leqslant k < n-1$, if $\varepsilon(k+1) = +1$ then $a'_k \leqslant b'_k$ and, if $\varepsilon(k+1) = -1$ then $a'_k \geqslant b'_k$. Moreover, we define the order $\leqslant_{\varepsilon(l)}$ on $\mathbb{R}$ by $u \leqslant_{\varepsilon(l)} v$ iff either $\varepsilon(l) = +1$ and $u \leqslant v$, or $\varepsilon(l) = -1$ and $u \geqslant v$.

The interest of these orders are given by the following remark:

Note. We have $\bar{a} \leqslant_\varepsilon \bar{b}$ in A iff $\psi_l(\bar{a}) = \bar{a}[l] \leqslant_\varepsilon^l \bar{b}[l] = \psi_l(\bar{b})$ in $A[l]$ and $a_l \leqslant_{\varepsilon(l)} b_l$.

So, we have:

Proposition. *Let A be a separate subset of $\mathbb{R}^n$. The following properties are equivalent:*

(i) *A is an antichain of $\langle \mathbb{R}^n, \leqslant_\varepsilon\rangle$.*

(ii) *π_l is decreasing from $\langle A[l], \leqslant_\varepsilon^l\rangle$ onto $\langle A_l, \leqslant_{\varepsilon(l)}\rangle$ for some $l<n$.*

(iii) *π_l is decreasing from $\langle A[l], \leqslant_\varepsilon^l\rangle$ onto $\langle A_l, \leqslant_{\varepsilon(l)}\rangle$ for every $l<n$.*

3.4. Definition. A subset A of $\mathbb{R}^n$ is said to be a *nice antichain* of $\mathbb{R}^n$, whenever

A satisfies the following conditions:

(i) A is a good and separate set.

(ii) A is an antichain of cardinality κ.

(iii) For every $l<n$, the subset $A[l]$ is κ-narrow in $\langle \mathbb{R}^{n-1}, \leqslant_\varepsilon^l \rangle$, i.e. has no antichain of cardinality κ.

Now, we will prove the following:

Proposition. *Let P be a κ-dense subchain of $\mathbb{R}$. The following properties are equivalent*:

(i) *P is hyper-rigid.*

(ii) *For $n\geqslant 2$ and $\varepsilon\in\{-1,+1\}^n$, there is no nice antichain in $\langle P^n, \leqslant_\varepsilon\rangle$.*

Proof. Trivially (i) implies (ii). Conversely, let us assume P is not hyper-rigid. So let $n\geqslant 2$ be the smallest integer such that for some $\varepsilon\in\{-1,+1\}^n$, the ordered set $\langle P^n, \leqslant_\varepsilon\rangle$ contains a good antichain A of cardinality κ. We choose such ε and A. According to the choice of n, ε and A, we can construct a nice antichain A' of $\langle P^n, \leqslant_\varepsilon\rangle$, included in A, and of cardinality κ.

First Stage. We can assume that $a_0<a_1<\cdots<a_{n-1}$ in $\mathbb{R}$ for every $\bar a=\langle a_k\rangle_{k<n}$. Indeed, since A is a good set, for every $\bar a$ in A, we have $a_{\sigma(0)}<a_{\sigma(1)}<\cdots< a_{\sigma(n-1)}$ for some unique permutation σ (which depends on $\bar a$) of indexes. The set A having κ elements, then for some subset A'' of A, of cardinality κ, we have the same permutation σ for all $\bar a\in A$. Without loss of generality, we assume $A''=A$ and σ is the identity.

Second Stage. We apply the method which appears in [3, §1.9, pp. 345–346] to construct $A'=R(A)$.

Step 2.1. For each $\bar a=\langle a_k\rangle_{k<n}$ in A, we choose a strictly increasing sequence $\langle r_k\rangle_{k<2n}$ (depending on $\bar a$) of rational numbers such that $r_{2k}<a_k<r_{2k+1}$ for every $k<n$. According A is of cardinality κ (which is a regular cardinal $\geqslant\omega_1$) and the set of possible sequences $\langle r_k\rangle_{k<2n}$ is countable, for some subset A_0 of A, of cardinality κ, we have, for every $\bar a\in A_0$, the same choice of the sequence $\langle r_k\rangle_{k<2n}$ (so which does not depend of $\bar a\in A_0$).

Step 2.2. I claim the set S_0 of a_0 for $\bar a=\langle a_k\rangle_{k<n}$ in A_0 is of cardinality κ. Otherwise S_0 is of cardinality $<\kappa$. According to κ is regular and A_0 is of cardinality κ, there are t_0 in S_0 and a subset A'' of A_0, of cardinality κ verifying $a_0=t_0$ for $\bar a=\langle a_k\rangle_{k<n}$ in A''. Obviously $A''[0]$ in $\langle \mathbb{R}^{n-1}, \leqslant_\varepsilon^0\rangle$, which is isomorphic to A'' in $\langle \mathbb{R}^n, \leqslant_\varepsilon\rangle$, is a good antichain of cardinality κ. We obtain a contradiction with the minimality of n.

Step 2.3. For each $t\in S_0$, we choose only one $\bar a=\langle a_k\rangle_{k<n}$ in A_0 verifying $a_0=t$. So we define a subset A_1 of A_0, of cardinality κ, and A_1 verifies: if $\bar a=\langle a_k\rangle\neq\bar b=\langle b_k\rangle$ in A_1, then $a_0\neq b_0$.

Step 2.4. Replacing A_0 by A_1 and the index 0 by the index 1, we construct a subset A_2 of A_1 of cardinality κ, such that if $\bar a\neq\bar b$ in A_2, then $a_1\neq b_1$ (and moreover $a_0 \neq b_0$). Repeating n times, we obtain a subset $A' = A_n \subset A$ of cardinality κ and which verifies the condition: if $\bar{a} = \langle a_k \rangle_{k<n} \neq \bar{b} = \langle b_k \rangle_{k<n}$ in A', then $a_k \neq b_k$ for every $k < n$.

4. How to continue the proof

4.1. We begin by a nice theorem, proved independently by E. Corominas and S. Shelah [unpublished], and which is useful in 4.1 of [3], and in 6.4.

Theorem. *Let λ be an infinite cardinal and let $\mu = \mathrm{cf}(\lambda)$ be its cofinal cardinality. Let $\langle E, \leqslant \rangle$ be a partially ordered set of cardinality λ. Let us assume there is a subset D of E, of cardinality $<\mu$, which is dense in E, in the following meaning:*

(1) *If $x < y$ in E, then $x \leqslant d \leqslant y$ for some $d \in D$.*

(2) *For every $x \in E$, there are d_1 and d_2 in D verifying $d_1 \leqslant x \leqslant d_2$.*

Then, for every subset F of E, of cardinality λ, the subordered set $\langle F, \leqslant \rangle$ contains, either an antichain of cardinality λ, or a chain order-isomorphic to the rational chain Q.

Proof. Let F be given. Let G be a subset of F of cardinality λ. Let $G_* = G \cup D$. For each $x \in G_*$ we will define $\varepsilon_x = \langle \varepsilon_x^-, \varepsilon_x^+ \rangle$, where ε_x^-, ε_x^+ belong to $\{0, 1\}$ in the following way:

- Let G_x^- be the set of $t \in G$ verifying $t \leqslant x$. So $\varepsilon_x^- = 0$ iff G_x^- is of cardinality $<\lambda$ (i.e. G_x^- is small), and $\varepsilon_x^- = 1$ otherwise.
- Let G_x^+ be the set of $t \in G$ verifying $t \geqslant x$. So $\varepsilon_x^+ = 0$ iff G_x^+ is of cardinality $<\lambda$, and $\varepsilon_x^+ = 1$ otherwise.

Now let G_*^- be the initial interval of G_* generated by the $d \in D$ verifying $\varepsilon_d^- = 0$, i.e. $x \in G_*^-$ iff $\varepsilon_d^- = 0$ for some $d \in D$ verifying $x \leqslant d$. Dually let G_*^+ be the set of $x \in G$ such that $\varepsilon_d^+ = 0$ for some $d \in D$ verifying $x \geqslant d$ (G_*^+ is a final interval of G_*). According to the hypothesis the sets G_*^+, G_*^- and thus $G_*^+ \cup G_*^-$ are of cardinality $<\lambda$. Consequently $N(G) = G - (G_*^+ \cup G_*^-)$ is of cardinality λ. We must remark that for $x \in G$, the value $\varepsilon_x = \langle \varepsilon_x^-, \varepsilon_x^+ \rangle$ depends on the set, but for $x \in N(G)$, the value ε_x, computed in G or in $N(G)$, are identical.

Case 1. Assume that for every subset G of F, of cardinality λ, there is $x(G) \in N(G)$ verifying $\varepsilon_{x(G)} = \langle 1, 1 \rangle$. Then F contains a chain order-isomorphic to Q. Indeed let $N_{1/2} = N(F)$ and $a(1/2) \in N_{1/2}$ such that $\varepsilon_{a(1/2)} = \langle 1, 1 \rangle$. Now we define $F_{1/4} = \{x \in N_{1/2};\ x \leqslant a(1/2)\}$ and $F_{3/4} = \{x \in N_{1/2}; x \geqslant a(1/2)\}$. Now let $a(1/4) \in N_{1/4} = N(F_{1/4})$ and $a(3/4) \in N_{3/4} = N(F_{3/4})$ verifying $\varepsilon_{a(1/4)} = \langle 1, 1 \rangle = \varepsilon_{a(3/4)}$. Continuing this process (at the next stage, choose $a(1/8)$, $a(3/8)$, $a(5/8)$ and $a(7/8)$ we define a sequence $(a(r))_{r \in \Delta}$ where Δ is the dyadic chain verifying $r_1 < r_2$ iff $a(r_1) < a(r_2)$.

Case 2. Assume that for some subset G of F, of cardinality λ, we have $\varepsilon_x \neq \langle 1, 1 \rangle$ for every $x \in G$. Then $N(G)$, and thus F, contains an antichain of cardinality λ. It is sufficient to prove that subchains of $N(G)$ have at most two

Sh:210

elements. First let x and y be elements of $N(G)$. If $x<y$, then $\varepsilon_x=\langle 0,1\rangle$ and $\varepsilon_y=\langle 1,0\rangle$. Indeed let $d\in D$ verifying $x\leqslant d\leqslant y$. As x, y, d are not elements of $G_*^-\cup G_*^+$, we have $\varepsilon_d^-=1=\varepsilon_d^+$ (note that $d\notin G$) and thus $\varepsilon_y^-=1=\varepsilon_x^+$. We conclude, since $\varepsilon_x=\langle 1,1\rangle=\varepsilon_y$ are impossible.

Secondly let us suppose there are a, b, c in $N(G)$ verifying $a<b<c$. From the above remark we obtain $\langle 0,1\rangle=\varepsilon_b=\langle 1,0\rangle$ which is a contradiction.

4.2. In this paragraph $p\geqslant 2$ and $\eta\in\{-1,+1\}^p$ given. Without loss of generality, we assume $\eta(k)=+1$ for every k, and $\leqslant_\eta$ is the usual cartesian order $\leqslant$. Now let f be a partial function from $\mathbb{R}^p$ into $\mathbb{R}$. We denote by $\mathrm{Dom}(f)$ its domain of definition. For two such functions f and g, we denote by $f\subset g$ the order relation "g is an extension of f", i.e. $\mathrm{Dom}(f)\subset\mathrm{Dom}(g)$ and $f(t)=g(t)$ for $t\in\mathrm{Dom}(f)$.

4.2.1. Let B be a subset of $\mathbb{R}^p$ and f be an increasing function from $\langle B,\leqslant\rangle$ into $\langle\mathbb{R},\leqslant\rangle$. Let $\bar b\in\mathbb{R}^p$ (which is not necessarily in B). We define:

$$f(\bar b^-)=\sup\{f(\bar t);\ \bar t\in B \text{ and } \bar t\leqslant\bar b\},$$
$$f(\bar b^+)=\mathrm{Inf}\{f(\bar t);\ \bar t\in B \text{ and } \bar t\geqslant\bar b\}.$$

In fact $f(\bar b^-)=-\infty$ and $f(\bar b^+)=+\infty$ are possible (whenever the corresponding sets are empty) and we have always $f(\bar b^-)\leqslant f(\bar b^+)$, according to f is increasing. If $f(\bar b^-)<f(\bar b^+)$, then we say that $\bar b$ is a *jump* of f.

4.2.2. Let D be a countable subset of $\mathbb{R}^p$, and let f be an increasing function from $\langle D,\leqslant\rangle$ into $\langle\mathbb{R},\leqslant\rangle$. An element $\bar b\in\mathbb{R}^p$ is said to be a *good point for f* whenever either $\bar b\in D$, or $f(\bar b^-)=f(\bar b^+)$, i.e. we have no jump in $\bar b$. We denote by $D^*=G(D)$ the set of good points of f. We define a function $G(f)=f^*$ (denoted also by f_D^*) in the following way: if $\bar b\in D$, then $f^*(\bar b)=f(\bar b)$; and if $\bar b\in D^*-D$, then $f^*(\bar b)=f(\bar b^-)=f(\bar b^+)$.

Obviously f^* is increasing. Indeed, for instance, for $\bar u\leqslant\bar v$ in D^*, if $\bar u\in D$ and $\bar v\in D^*-D$, then $f^*(\bar u)=f(\bar u)\leqslant f(\bar v^-)=f^*(\bar v)$, and if $\bar u\in D^*-D$ and $\bar v\in D^*-D$, then $f(\bar u^-)\leqslant f(\bar v^-)$ and $f(\bar u^+)\leqslant f(\bar v^+)$ and so $f^*(\bar u)\leqslant f^*(\bar v)$.

Moreover f^* is the greatest extension of f, uniquely defined by the function f (that is to say if $\bar c\notin D^*$, for an increasing extension of f, we can choose many values, since $\bar c\notin D$ and $\bar c$ is a jump of f). The increasing function is said to be the *entire extension of f*.

Dually, if f is a decreasing function from $\langle B,\leqslant\rangle$ into $\langle\mathbb{R},\leqslant\rangle$, then f has an entire extension f^*, which is decreasing too.

4.3. Now, we recall, that $\mathbb{R}^n$ is a normed space, and so a topological space, whenever we put

$$\|a\|=\mathrm{Max}\{|a_k|;\ k<n\}$$

where $\bar{a} = \langle a_k \rangle_{k<n}$. For this usual topology, $\mathbb{R}^n$ has a countable base. Consequently every infinite subset S of $\mathbb{R}^n$ contains a countable dense subset H, i.e. $S \subset \bar{H}$, that is to say every point of S is the limit of a sequence of elements of H. Now we will prove:

Proposition. *Let $n \geqslant 2$ and $\varepsilon \in \{-1, +1\}^n$ be given. Let A be a nice antichain of $\langle \mathbb{R}^n, \leqslant_\varepsilon \rangle$. Let $l < n$ be given. Let D_l be a countable topologically dense subset of $A[l]$. Let π_l be the canonical decreasing function from $\langle A[l], \leqslant_\varepsilon^l \rangle$ onto $\langle A_l, \leqslant_{\varepsilon(l)} \rangle$, and φ_l be its restriction onto D_l. Let D_l^* be the domain of the entire extension φ_l^* of φ. With these notations the set $A[l] - D_l^*$ is of cardinality $<\kappa$.*

Proof. W.l.o.g., we assume $\varepsilon(k) = +1$ for every k. Let $S = A[l] - D_l^*$. We must prove that S is of cardinality $<\kappa$. Let $\bar{c} \in A[l]$. We recall that $\bar{c} = \bar{a}[l]$ for a unique $\bar{a} = \langle a_k \rangle_{k<n}$ in A, and we have $\pi_l(\bar{c})^* a_l$. Let us suppose $\bar{c} \in A[l] - D_l^*$. We have $\bar{c} \notin D_l$. So, recalling that π_l and thus φ_l are decreasing, the real numbers:

$$\varphi_l(\bar{c}^-) = \inf\{\varphi_l(\bar{v});\ \bar{v} \in D_l \text{ and } \bar{v} < \bar{c}\},$$

$$\varphi_l(\bar{c}^+) = \sup\{\varphi_l(\bar{v});\ \bar{v} \in D_l \text{ and } \bar{v} > \bar{c}\},$$

$$\pi_l(\bar{c}^-) = \inf\{\pi_l(\bar{v});\ \bar{v} \in A[l] \text{ and } \bar{v} < \bar{c}\},$$

$$\pi_l(\bar{c}^+) = \sup\{\pi_l(\bar{v});\ \bar{v} \in A[l] \text{ and } \bar{v} > \bar{c}\}$$

verifying $\varphi_l(\bar{c}^-) > \varphi_l(\bar{c}^+)$, and

$$\varphi_l(\bar{c}^-) \geqslant \pi_l(\bar{c}^-) \geqslant \pi_l(\bar{c}^+) \geqslant \varphi_l(\bar{c}^+).$$

Now, let S_- (resp. S_0, S_+) be the set $\bar{c} \in S$ such that $\varphi_l(\bar{c}^-) > \pi_l(\bar{c}^-)$ (resp. $\pi_l(\bar{c}^-) > \pi_l(\bar{c}^+)$, $\pi_l(\bar{c}^+) > \varphi_l(\bar{c}^+)$). Obviously $S = S_- \cup S_0 \cup S_+$. So it is sufficient to prove that S_-, S_0 and S_+ are of cardinality $<\kappa$.

First stage. S_0 is of cardinality $<\kappa$. Otherwise for each $\bar{u} \in S_0$, let $r(u)$ be rational verifying $\pi_l(\bar{u}^-) > r(\bar{u}) > \pi_l(\bar{u}^+)$. We construct a rational r and a subset S' of S_0, of cardinality κ, such that $r(\bar{u}) = r$ for every $\bar{u} \in S'$. The chains of $\langle S', \leqslant \rangle$ have at most two elements. Otherwise for $\bar{u} < \bar{v} < \bar{w}$ in S', we have

$$r = r(\bar{u}) > \pi_l(\bar{u}^+) \geqslant \pi_l(\bar{v}) \geqslant \pi_l(\bar{w}^-) > r(\bar{w}) = r.$$

Contradiction. Accordingly $A[l]$ is not κ-narrow, we obtain a contradiction.

Second stage. S_- is of cardinality $<\kappa$. Let $\bar{c} \in S_-$. We have $\varphi_l(\bar{c}^-) > \pi_l(\bar{c}^-)$, i.e. for some $\bar{d} \in A[l]$, we have $\varphi_l(\bar{c}^-) > \pi_l(\bar{d}) \geqslant \pi_l(\bar{c}^-)$. Let $\bar{d} = \langle d_k \rangle_{k<n-1}$ and $\bar{c} = \langle c_k \rangle_{k<n-1}$. So $\bar{d} < \bar{c}$ is equivalent to $d_k < c_k$ for every k. Let $I_k =]d_k, c_k[\subset \mathbb{R}$ and $U(\bar{d}, \bar{c})$ be the product $\prod_{k<n-1} I_k$ of the I_k's for $k < n-1$. So $U(\bar{d}, \bar{c})$ is a non-empty set of $\mathbb{R}^{n-1}$ and $U(\bar{d}, \bar{c}) \cap A[l]$ is empty (since $\bar{d} < \bar{x} < \bar{c}$ for every $\bar{x} \in U(\bar{d}, \bar{c})$ and $U(\bar{d}, \bar{c}) \cap D_l$ is empty). Let $\bar{r}(\bar{c})$ be an element of $U(\bar{d}, \bar{c}) \cap Q^{n-1}$. Now, we assume S_- is of cardinality κ. Consequently let $\bar{r} \in Q^{n-1}$ and S'_- be a subset of S_-, of cardinality κ verifying $r = \bar{r}(\bar{c})$ for every $\bar{c} \in S'_-$. I claim that S'_- is an antichain (and thus accordingly $A[l]$ is not κ-narrow, we obtain a contradiction). Let $\bar{u}$, $\bar{v}$ be distinct elements of S'_-. Let us suppose $\bar{u}<\bar{v}$. We have

$$\bar{r}=\bar{r}(\bar{v})=\bar{r}(\bar{u})<\bar{u}<\bar{v}$$

and thus $\bar{u}\in U(\bar{r}(\bar{v}),\bar{v})\cap A[l]$, which is a contradiction.

Third stage. S_+ is of cardinality $<\kappa$ (the proof is identical with the one we have used in the second stage).

5. How to conclude

5.0. Let $p\geqslant 1$, $\eta\in\{-1,+1\}^p$ and $\theta\in\{-1,+1\}$ be given. For a countable subset D of $\mathbb{R}^p$, we denote by $M(p,\eta,\theta,D)$ the set of all decreasing functions from $\langle D,\leqslant_\eta\rangle$ into $\langle\mathbb{R},\leqslant_\theta\rangle$ ($\leqslant_{+1}$ is the usual order on $\mathbb{R}$ and $\leqslant_{-1}$ its converse). Let $M^*(p,\eta,\theta,D)$ be the set of entire extensions f^* of $f\in M(p,\eta,\theta,D)$ (see 4.2.2). Obviously $M(p,\eta,\theta,D)$ and thus $M^*(p,\eta,\theta,D)$ are of cardinality $(2^\omega)^\omega=2^\omega$. Now, let $M^*(p,\eta,\theta)$ be the union of the $M^*(p,\eta,\theta,D)$ for every countable set D of $\mathbb{R}$. According $\{-1,+1\}^p$ and $\{-1,+1\}$ are finite, the union $M^*(p)$ of $M^*(p,\eta,\theta)$ is of cardinality 2^ω, and thus the union M^* of all $M^*(p)$ for $1\leqslant p<\omega$ is of cardinality 2^ω. Now, for $f\in M^*$, we denote by $n(f)$ the unique integer verifying $f\in M^*(n(f))$.

5.1. Let $M^*=\bigcup\{M^*_\alpha;\alpha<\kappa\}$, where κ is the cofinal cardinal of 2^ω (and thus κ is an initial regular ordinal). Each M^*_α is of cardinality $<2^\omega$ and the M^*_α's are increasing w.r.t. inclusion, i.e. $M^*_\alpha\subset M^*_\beta$ for $\alpha<\beta<\kappa$.

Now let $(I_\alpha)_{\alpha<\kappa}$ be an enumeration of non-empty open intervals $]r',r''[$, determined by rationals $r'<r''$, each interval being repeated κ times.

We will construct P as a set of $x_\alpha\in\mathbb{R}$, for $\alpha<\kappa$. For this let $x_0\in\mathbb{R}$. Let $\beta<\kappa$. Let us suppose the x_α's, for $\alpha<\beta$ to be constructed. We denote by P_β the set of x_α for $\alpha<\beta$, which is of cardinality $<\kappa$. Let T_β be the set of $f(\bar{a})\in\mathbb{R}$, for every $f\in M^*_\beta$ and $\bar{a}\in P_\beta^{n(f)}$. Obviously T_β and thus $P_\beta\cup T_\beta$, is of cardinality $<2^\omega$. Let us choose $x_\beta\in I_\beta-(P_\beta\cup T_\beta)$.

5.2. Let P be the set of the x_α's for $\alpha<\kappa$. Obviously P is κ-dense (since $x_\alpha\in I_\alpha$ and each interval appears κ times). To prove P is hyper-rigid let us suppose (according to Proposition 3.4), that there are $n\geqslant 2$, $\varepsilon\in\{-1,+1\}^n$ and a nice antichain A of $\langle P^n,\leqslant_\varepsilon\rangle$ of cardinality κ. Let n and A be chosen as above. For each $\bar{a}=\langle a_k\rangle_{k<n}$ in A, we have $a_i\neq a_j$ for $i\neq j$ and $a_i\in P$, i.e. $a_i=x_{\alpha(i)}$. So $\bar{a}=\langle x_{\alpha(k)}\rangle_{k<n}$. We define the index $q(\bar{a})$ in the following way: $q(\bar{a})=k$ iff $\alpha(k)$ is the greatest ordinal of the set $\{\alpha(0),\alpha(1),\ldots,\alpha(n-1)\}$. For instance if $\bar{a}=\langle x_6,x_8,x_3,x_5\rangle$, then $q(\bar{a})=1$. We have $q(\bar{a})<n$, and thus let $l<n$ and A' be a subset of A, of cardinality κ such that $q(\bar{a})=l$ for every $\bar{a}\in A'$. So we can assume $A=A'$. Now, according to 3.2 and 3.3, let π_l be the canonical decreasing function from $\langle A[l],\leqslant^l_\varepsilon\rangle$ onto $\langle A_l,\leqslant_{\varepsilon(l)}\rangle$. Now, put $\varepsilon(l)=\theta\in\{-1,+1\}$, and $\leqslant^l_\varepsilon$

equal to $\leqslant_\eta$, where η is defined by $\eta(k)=\varepsilon(k)$ for $k<l$ and $\eta(k)=\varepsilon(k+1)$ for $l\leqslant k<n-1$. Applying Proposition 4.3, let $f=\varphi_l^*\in M^*$. Let $\alpha<\kappa$ verify $f\in M_\alpha^*$. We must remark that the domain of definition D of the function f is a subset of $\mathbb{R}^{n-1}$ and $A[l]-D$ is of cardinality $<\kappa$. So $A[l]\cap D$ is of cardinality κ. Now, let $\bar{a}=\langle a_k\rangle_{k<n}$ be such that $\bar{a}[l]$ belongs to $(A[l]\cap D)-P_\alpha^{n-1}$. Let γ be such that $a_i=x_{\gamma(i)}$ for every $i<n$. We have:

$$\gamma(i)<\gamma(l)=\beta \quad \text{for every } i\neq l, i<n,$$

$$\alpha<\gamma(k) \quad \text{for some } k<n,$$

and so $\alpha<\beta$. Putting $\bar{b}=\bar{a}[l]$, we have $f\in M_\beta^*$ and $\bar{b}\in P_\beta^{n-1}$, and thus $x_\beta=f(\bar{b})\in T_\beta$, which contradicts the construction of x_β.

6. Application

6.0. Let B be a Boolean algebra. A subset S of B is said to be a well-founded set of generators, whenever: first the Boolean algebra generated by S is B; secondly S had no infinite strictly decreasing sequence (or equivalently, every non-empty subset of S has a minimal element). A Boolean algebra is said to be *well-generated* whenever it has a well-founded set of generators.

6.1. We recall that a Boolean algebra B is said to be *superatomic* (or *scattered*) whenever B verifies one of the equivalent properties:

(i) Every subalgebra is atomic.
(ii) Every quotient algebra is atomic.
(iii) There is no chain in B, order-isomorphic to the rational chain Q.

Proposition. *Every superatomic Boolean algebra is well-generated.*

Proof. Let B be a superatomic algebra. Now let $K(B)$ be the set of subsets S of B verifying: first $\langle S,\leqslant\rangle$ is well-founded, and secondly the ideal $I(S)$ generated by S, in B, is included into the subalgebra $B(S)$ generated by S. The order $\leqslant$ on $K(B)$, defined by $S_1\leqslant S_2$ whenever S_1 is an initial segment of S_2, is obviously inductive. Thus let G be a maximal element of $K(B)$. Let π be the Boolean homomorphism from B onto $B/I(G)=B^*$. We have $B^*=Z/2$ and so B has G as well-founded set of generators. Otherwise let a be an atom of B^*. Let $b\in B$ verify $\pi(b)=a$. We have $b\notin G$, $G\cup\{b\}=G_1\in K(B)$ and $G\leqslant G_1$. Contradiction.

Now, we will give that for a superatomic Boolean algebra B, we cannot assume some hypothesis on the chains of B. For this, let ω_α be a regular ordinal. So it is a well-ordered chain. Let $B(\omega_\alpha)$ be the algebra of finite unions of intervals of the form $]a_{2l}, a_{2l+1}]$ (we consider this kind of intervals, since $B(\omega_\alpha)$ is exactly the algebra of closed and open subsets of chain $\omega_\alpha+1$ with the interval topology).

Moreover, we recall that ω_α^* denotes the converse chain of ω_α (i.e. $x\leqslant y$ in ω_α^* iff $x\geqslant y$ in ω_α). We don't give the proof of the following result:

Proposition. *Let $\omega_\alpha > \omega$ be a regular cardinal. Then every set of generators of the Boolean algebra $B(\omega_\alpha)$ contains a chain order-isomorphic to ω_α or ω_α^*.*

Now, we give two other examples of well-generated Boolean algebras:

6.2. Proposition. *Every free Boolean algebra is well-generated. Every complete algebra $\mathcal{P}(X)$ of all subsets of a given set X is well-generated.*

The first part is trivial, and the second too if X is finite. Now let us suppose X infinite and let $B = \mathcal{P}(X)$. So B and $B \times B$ are isomorphic. Let $1\ (=X)$ be the unity of B. Let G be the set of $\langle x, 1-x\rangle$ for $x \in B$. So G is an antichain of $B \times B$ and for $\langle u, v\rangle \in B \times B$ we have

$$\langle u, v\rangle = [\langle u, 1-u\rangle \cap \langle 1, 0\rangle] \cup [\langle 0, 1\rangle \cap \langle 1-v, v\rangle].$$

Remarks. (1) In fact, we have proved that $B = \mathcal{P}(X)$ contains an antichain G, which generates B as lattice (we don't use the complement).

(2) Now let n be the cardinal of X and $\omega \leq m < n$ be given. Let $\mathcal{P}_m(X)$ the Boolean algebra of subset Y of X verifying Y or $X - Y$ is of cardinality $<m$. Assuming GCH we can prove $\mathcal{P}_m(X)$ is well-generated.

6.3. Proposition. *Every countable Boolean algebra is well-generated.*

This is consequence of the following result: let B be a Boolean algebra and I be an ideal of B. Let us suppose first I is well-generated, that is to say there is a well-founded subset S of I such that the Boolean algebra generated by S contains I, and secondly the quotient algebra B/I is well-generated. Then B is well-generated. Indeed let π be the canonical homomorphism from B onto B/I, and K be a well-founded set of generators of B/I. For each $x \in K$, let $a_x \in B$ verify $\pi(a_x) = x$. The set S' of all a_x is well-founded and $S \cup S'$ is a well-founded set of generators of B.

Now let B be countable. Let I be the ideal of all $a \in B$ verifying: $[0, a]$ has no subchain order-isomorphic to the rationals. Then either $I = B$, or B/I is the free countable Boolean algebra. So we conclude.

6.4. Proposition. *There is an interval Boolean algebra which is not well-generated.*

Let P be a κ-dense and hyper-rigid subchain of $\mathbb{R}$ (where κ is the cofinal cardinal of 2^ω, so $\kappa > \omega$). Let G be a residual subset (see 2.1) of a set of generators of $B(P)$. We will prove that G contains a chain order-isomorphic to the rational chain. For this we remark we can assume $Q \subset P$. Let $E = B(Q) \cup G$. So we can assume that E verifies the hypothesis of Theorem 4.1 and as G is narrow and of cardinality κ, we conclude.

Acknowledgments

We wish to thank Donald Monk, for reading the different original manuscripts and for his useful comments, the referee for all his useful remarks, and Eric Milner for his invitation of Robert Bonnet to visit the University of Calgary, and for discussions during the preparation of this final version.

References

[1] R. Bonnet, Problem in Boolean algebra, European Meeting of A.S.L. (Clermont-Ferrand, 1975), J. Symbolic Logic 42 (1977) 115.

[2] R. Bonnet, Sur les Algèbres de Boole rigides, Thèse, Lyon, Université Claude Bernard (1978).

[3] R. Bonnet, Very strongly rigid Boolean algebras, continuum discrete set condition, countable antichain condition (I), Algebra Universalis 11 (1980) 341–364.

[4] E.S. Berney and P.J. Nyikos, The length and breadth of Boolean algebras, Preprint.

[5] J.G. Baumgartner, Chains and antichains in $\mathcal{P}(\omega)$, J. Symbolic Logic 45 (1980) 85–92.

[6] J.G. Baumgartner and P. Komjath, Boolean algebras in which every chain and every antichain is countable, Fund. Math. 111 (1981) 125–133.

[7] M. Rubin. A Boolean algebra with few subalgebras, interval Boolean algebras and retractiveness, Trans. A.M.S. 278 (1) (1983) 65–89.