1

Some Notes on Iterated Forcing With $2^{\aleph_0} > \aleph_2$

SAHARON SHELAH*

Introduction   By Solovay and Tenenbaum ([7]) and Martin and Solovay ([3]) we can iterate c.c.c. forcing with finite support. There have been many works on iterating more general kinds of forcings adding reals (e.g., [4]), getting generalizations of MA, and so on, but we were usually restricted to $2^{\aleph_0} = \aleph_2$. Note only this is a defect per se, but there are statements that we think are inde- *pendent but which follow from* $2^{\aleph_0} \leq \aleph_2$.

Some time ago Groszek and Jech (in [2]) got $2^{\aleph_0} > \aleph_2 + MA$ for a family of forcing wider than c.c.c. but for $\aleph_1$ dense sets only.

In Section 1 we generalize RCS iteration to κ-RS iteration.

In Section 2 we combine from [4], X, XII (i.e., RS iteration and some properness and semicompleteness) with Gitik's definition of order ([1]). (He uses Easton support, each Q ($\{2\}$, κ_i) -complete where for important i, $\kappa_i = i$. His main aim was properties of the club filter on inaccessible: precipitousness and approximation to saturation.)

In Section 3 we get MA-like consequences (strongest-from supercompact). In Section 4 we get that, e.g., for Sacks forcing (though not included), and in the models we naturally get, for every $\aleph_1$ dense subset there is a directed set intersecting all of them.

In Section 5 we solve the second Abraham problem.

The main result was announced (somewhat inaccurately) in [6].

1 On κ-revised support iteration   We redo [4], Ch. X, Section 1, with "$< \kappa$" instead countable.

Remarks 1.0:

   (1) Now if $P_1 = P_0 * Q_0$, q_1 a P_1-name, $G_0 \subseteq P_0$ generic over V, then in $V[G_0]$, q_1 can be naturally interpreted as a Q_0-name, called q_1/G_0,

which has a P_0-name q_1/G_0, or q_1/P_0; but usually we do not care to make those fine distinctions.

(2) Using $\bar{Q} = \langle P_i, Q_i : i < \alpha \rangle$, P_α will mean $R\mathrm{Lim}\,\bar{Q}$ (see Definition 1.2).

(3) If D is a filter on a set J, $D \in V$, $V \subseteq V^\dagger$ (e.g., $V^\dagger = V[G]$) then in an abuse of notation, D will denote also the filter it generates (on J) in $V^\dagger$.

(4) D_κ is the closed unbounded filter on κ.

Definition 1.1   We define the following notions by *simultaneous induction on* α:

(A) $\bar{Q} = \langle P_i, Q_i : i < \alpha \rangle$ is a κ-RS iteration (RS stands for revised support)

(B) a $\bar{Q}$-named ordinal (or $[j, \alpha)$-ordinal)

(C) a $\bar{Q}$-named atomic condition (or $[j, \alpha)$-condition), and we define $q \upharpoonright \xi$, $q \upharpoonright \{\xi\}$ for a $\bar{Q}$-named atomic $[j, \alpha)$-condition q and ordinal ξ.

(D) the κ-RS limit of $\bar{Q}$, $R\mathrm{Lim}_\kappa\,\bar{Q}$ which satisfies $P_i <_\circ \tilde{R}\mathrm{Lim}_\kappa\,\bar{Q}$ for every $i < \kappa$ and we define $p \upharpoonright \beta$ for $p \in R\mathrm{Lim}_\kappa\,\bar{Q}$, $\beta < \alpha$. (We may omit κ.)

(A) We *define "$\bar{Q}$ is a κ-RS iteration"*

$\alpha = 0$: no condition.

α is limit: $\bar{Q} = \langle P_i, Q_i : i < \alpha \rangle$ is a κ-RS iteration iff for every $\beta < \alpha$, $\bar{Q} \upharpoonright \beta$ is one.

$\alpha = \beta + 1$: $\bar{Q}$ is an RCS iteration iff $\bar{Q} \upharpoonright \beta$ is one, $P_\beta = R\mathrm{Lim}_\kappa(\bar{Q} \upharpoonright \beta)$, and Q_β is a P_β-name of a forcing notion.

(B) We define: ζ is a $\bar{Q}$-named $[j, \beta)$-ordinal above r. It means $r \in \bigcup_{i < \gamma} P_i$ (where $\gamma = \mathrm{Min}\{\beta, l(\bar{Q})\}$) and ζ is a function such that:

(1) $\mathrm{Dom}(\zeta)$ is a subset of $\bigcup \{P_i : i < \gamma\}$

(2) for every $q \in \mathrm{Dom}(\zeta)$ for some i, $\{q, r\} \subseteq P_i$ and $P_i \models r \leq q$.

(3) for every $q_1, q_2 \in \mathrm{Dom}(\zeta)$, if for some $i < \alpha$ $\{q_1, q_2\} \subseteq P_i$ and in P_i they are compatible *then* $\zeta(q_1) = \zeta(q_2)$.

(4) *if* $q \in \mathrm{Dom}(\zeta)$, $q \in \bigcup_{i < \alpha} P_i$ and $i = i(q)$ is the minimal i such that $q \in P_i$ *then* $\zeta(q)$ is an ordinal $\geq i, j$ but $< \gamma, \beta$.

We define "ζ is a $\bar{Q}$-named ordinal above r" as "ζ is a $\bar{Q}$-named $[0, l(\bar{Q}))$ ordinal above r". We omit "above r" when $r = \varnothing$ (i.e., we omit demand (2)).

(C) We say "q is a $\bar{Q}$-named atomic $[j, \alpha)$-condition above r" if

(1) q is a pair of functions (ζ_q, cnd_q) with a common domain $D = D_q$:

(2) cnd_q satisfies (1) and (3) above and:

(3) ζ_q is a $(\bar{Q} \upharpoonright \alpha)$ − named $[j, \alpha)$ − ordinal above r

(4) for $p \in D_q$, $cnd_q(p)$ is a $P_{\zeta_q(p)}$-name of a member of $Q_{\zeta_q(p)}$. We omit "$[j, \alpha)$-" when $j = 0$, $\alpha = \ell(\bar{Q})$ and we omit "above r" when $r = \varnothing$. If $l(\bar{Q}) > \alpha$ we mean $\bar{Q} \upharpoonright \alpha$. We define $q \upharpoonright \xi$ as $(\zeta_q \upharpoonright D_1, cnd_q \upharpoonright D_1)$ where $D_1 = \{p \in D_q : \zeta_q(p) < \xi\}$. We define $q \upharpoonright \{\xi\}$ as $(\zeta_q \upharpoonright D_2, cnd_q \upharpoonright D_2)$ where $D_2 = \{p \in D_q : \zeta_q(p) = \xi\}$.

(D) We define $R\mathrm{Lim}_\kappa\,\bar{Q}$ as follows:

if $\alpha = 0$: $R\mathrm{Lim}_\kappa\,\bar{Q}$ is trivial forcing with just one condition, $\varnothing$.

if $\alpha > 0$: we call q an atomic condition of $R\mathrm{Lim}_\kappa \bar{Q}$, if it is a $\bar{Q}$-named atomic condition.

The set of conditions in $R\mathrm{Lim}_\kappa \bar{Q}$ is

> $\{p: p$ a set of λ atomic conditions for some $\lambda < \kappa$; and for every $\beta < \alpha$, $p \upharpoonright \beta =^{dsf} \{r \upharpoonright \beta: r \in p\} \in P_\beta$, and $p \upharpoonright \beta \Vdash_{P_\beta}$ "the set $\{r \upharpoonright \{\beta\}: r \in p\}$ has an upper bound in Q_β"$\}$.

We define $p \upharpoonright \beta = \{r \upharpoonright \beta: r \in p\}$.

The order is inclusion.

Now we have to show $P_\beta <_\circ R\mathrm{Lim}_\kappa \bar{Q}$ (for $\beta < \alpha$). Note that any $\bar{Q}$-named $[j, \beta]$-ordinal (or condition) is a $\bar{Q}$-named $[j, \alpha]$-ordinal (or condition), and see Claim 1.4(1) below.

Remark 1.1A: Note that for the sake of 1.5(3) we allow κ to be not a cardinal and then we really use $|\kappa|^+$.

Remark 1.1B: We can obviously define $\bar{Q}$-named sets; but for conditions (and ordinals for them) we want to avoid the vicious circle of using names which are interpreted only after forcing with them below.

Definition 1.2

(1) Suppose $\bar{Q}$ is a κ-RS iteration, ζ is a $\bar{Q}$-named $[j, \alpha]$-ordinal above r, $\beta \leq \alpha$, $r \in G \in \mathrm{Gen}(\bar{Q})$ (see Definition (3) below). We define $\zeta[G]$ by:

(i) $\zeta[G] = i$ if for some $\gamma \leq \beta > \alpha$ and $p \in Dom(\zeta) \cap G_\gamma$ we have $\zeta(p) = i$.

(ii) otherwise (i.e., $G \cap D_\zeta = \phi$ or $r \notin G$) $\zeta[G]$ is not defined.

For a $\bar{Q}$-named $[j, \alpha]$-condition above r, q, we defined $q[G]$ similarly.

(2) We denote the set of $G \subseteq \bigcup_{i<\alpha} P_{i+1}$ such that $G \cap P_{i+1}$ is generic over V for each $i < \alpha$ by $\mathrm{Gen}(\bar{Q})$.

(3) For ζ a $\bar{Q}$-named $[j, \alpha]$-ordinal (above r) and $q \in \bigcup_\alpha P_i$ let $q \Vdash_{\bar{Q}}$ "$\zeta = \xi$" if for every $G \in \mathrm{Gen}(\bar{Q})$ such that $r \in G$: $q \in G \Rightarrow \zeta[G] = \xi$.

Remark 1.3: From where is G taken in (2), (3)? e.g., V is a countable model of set theory, G taken from the "true" universe.

Now we point out some properties of κ-RS iteration.

Claim 1.4: Let $\bar{Q} = \langle P_i Q_i: i < \alpha \rangle$ be a κ-RS iteration, $P_\alpha = R\mathrm{lim}_\kappa \bar{Q}$.

(1) *If* $\beta < \alpha$ *then:* $P_\beta \subseteq P_\alpha$; *for* $p_1, p_2 \in P_\beta$, $P_\beta \vDash p_1 \leq p_2$ *iff* $P_\alpha \vDash p_1 \leq p_2$: *and* $P_\beta <_\circ P_\alpha$. *Moreover, if* $q \in P_\beta$, $p \in P_\alpha$, *then* q, p *are compatible iff* $q, p \upharpoonright \beta$ *are compatible.*

(2) *If* ζ *is a* $\bar{Q}$-named $[j, \alpha]$-ordinal G, $G' \in \mathrm{Gen}(\bar{Q})$ $G \cap P_\xi = G' \cap P_\xi$ *and* $\zeta[G] = \xi$ *then* $\zeta[G'] = \xi$; *hence we write* $\zeta[G \cap P_\xi] = \xi$.

(3) *If* β, γ *are* $\bar{Q}$-named $[j, l(\bar{Q})]$-ordinals, then $\mathrm{Max}\{\beta, \gamma\}$ *(defined naturally) is a* $\bar{Q}$-named $[j, l(\bar{Q})]$-ordinal.

(4) *If* $\alpha = \beta_0 + 1$, *in Definition 1.1(D), in defining the set of elements of* P_α *we can restrict ourselves to* $\beta = \beta_0$. *Also in such a case,* $P_\alpha =$

$P_{\beta_0} * Q_{\beta_0}$ (essentially). More exactly, $\{p \cup \{q\} : p \in P_{\beta_0},\ q$ a P_{β_0}-name of a member of $Q_{\beta_0}\}$ is a dense subset of P_α, and the order $p_1 \cup \{q_1\} \le p_2 \cup \{q_2\}$ iff $p_1 \le p_2$, $p_2 \Vdash q_1 \le q_2$ is equivalent to that of P_α; i.e., we get the same Boolean algebra.

(5) The following set is dense in P_α: $\{p \in P_\alpha$; for every $\beta < \alpha$, if $r_1, r_2 \in p$, then $\Vdash_{P_\beta}$ "if $r_1 \restriction \{\beta\} \ne \varnothing$, $r_2 \restriction \{\beta\} \ne \varnothing$ then they are equal"$\}$.

(6) $|P_\alpha \le (\Sigma_{i<\alpha} 2^{P_i})^\kappa$, for limit α.

(7) If $\Vdash_{P_i}$ "$|Q_i| \le \lambda$", α a cardinal, then $|P_{i+1}| \le 2^{|P_i|} + \lambda$ (assuming, e.g., that the set of elements of G is λ).

Proof: By induction on α.

Lemma 1.5 *The Iteration Lemma*
(1) *Suppose F is a function, then for every ordinal α there is one and only one κ-RS-iteration $\bar{Q} = \langle P_i, Q_i : i < \alpha^\dagger \rangle$, such that:*
(a) *for every i, $Q_i = F(Q \restriction i)$,*
(b) *$\alpha^\dagger \le \alpha$,*
(c) *either $\alpha^\dagger = \alpha$ or $F(\bar{Q})$ is not an $(\mathrm{RLim}_\kappa \bar{Q})$-name of a forcing notion.*
(2) *Suppose $\bar{Q}$ is a κ-RS-iteration, $\alpha = l(\bar{Q})$, $\beta < \alpha$, $G_\beta \subseteq P_\beta$ is generic over V. Then in $V[G_\beta]$, $\bar{Q}/G_\beta = \langle P_i/G_\beta, Q_i : \beta \le i < \kappa \rangle$ is a κ-RS-iteration and $\mathrm{RLim}_\kappa \bar{Q} = P_\beta * (\mathrm{RLim}\, \bar{Q}/G_\beta)$ (essentially).*
(3) *The Associative Law: If $\alpha_\xi (\xi \le \xi(0))$ is increasing and continuous, $\alpha_0 = 0$; $\bar{Q} = \langle P_i, Q_i : i < \alpha_{\xi(0)} \rangle$ is a κ-RS-iteration, $P_{\xi(0)} = \mathrm{RLim}_\kappa \bar{Q}$; then so are $\langle P_{\alpha(\xi)}, P_{\alpha(\xi+1)}/P_{\alpha(\xi)} : \xi < \xi(0) \rangle$ and $\langle P_i/P_{\alpha(\xi)}, Q_i : \alpha(\xi) \le i < \alpha(\xi + 1) \rangle$; and vice versa.*

Remark 1.5A: In (3) we can use α_ξ's which are names.

Proof: (1) Easy.
(2) Pedantically, we should formalize the assertion as follows:

(*) There is a function F (= a definable class) such that for every κ-RS-iteration $\bar{Q}$ and $l(\bar{Q}) = \alpha$, and $\beta < \alpha$, $F_0(\bar{Q}, \beta)$ is a P_β-name of $\bar{Q}^\dagger$ such that:
(a) $\Vdash_{P_\beta}$ "$\bar{Q}^\dagger$ is a κ-RS-iteration of length $\alpha - \beta$".
(b) $P_\beta * (\mathrm{RLim}_\kappa \bar{Q}^\dagger)$ is equivalent to $P_\alpha = \mathrm{RLim}_\kappa \bar{Q}$, by $F_1(\bar{Q}, \beta)$ (i.e., $F_1(\bar{q}, \beta)$ is an isomorphism between the corresponding completions to Boolean algebras)
(c) if $\beta \le \gamma \le \alpha \Vdash_{P_\beta}$ "$F_0(\bar{Q} \restriction \gamma, \beta) = F(\bar{Q}, \beta) \restriction (\gamma - \beta)$" and $F_1(\bar{Q}, \beta)$ extends $F_1(\bar{Q} \restriction \gamma, \beta)$ and $F_1(\bar{Q} \restriction \gamma, \beta)$ transfer the P_γ-name Q_γ to a P_β-name of a $(\mathrm{RLim}_\kappa(\bar{Q}^\dagger \restriction (\gamma - \beta))$-name of $Q^\dagger_{\gamma - \beta}$ (where $Q^\dagger_{\gamma - \beta} = \langle Q^\dagger_{\beta + i} : i < \gamma - \beta \rangle$).

The proof is the induction on α, and there are no special problems.
(3) Again, pedantically the formulation is

(**) For $\bar{Q}$ is an RCS-iteration, $l(\bar{Q}) = \alpha_{\xi(0)}$, $\bar{\alpha} = \langle \alpha_\xi : \xi \le \xi(0) \rangle$ increasing continuous, $F_3(\bar{Q}, \bar{\alpha})$ is a κ-RS-iteration $\bar{Q}^\dagger$ of length $\alpha_{\xi(0)}$ such that
(a) $F_4(\bar{Q}, \bar{\alpha})$ is an equivalence of the forcing notions $\mathrm{RLim}_\kappa \bar{Q}$. $\mathrm{RLim}_\kappa \bar{Q}^\dagger$.

(b) $F_3(\bar{Q} \upharpoonright \alpha_\xi, \alpha \upharpoonright (\zeta + 1)) = F_3(\bar{Q}), \bar{\alpha}) \upharpoonright \zeta$

(c) $Q_\xi^\dagger$ is the image by $F_4(\bar{Q} \upharpoonright \alpha_\xi, \bar{\alpha} \upharpoonright (\xi + 1))$ of the $P_{\alpha_\xi} = R\mathrm{Lim}_\kappa(\bar{Q} \upharpoonright \alpha_\xi)$-name $F_0(\bar{Q} \upharpoonright \alpha_{\xi+1}, \alpha_\xi)$.

The proof again poses no special problems.

Claim 1.6: Suppose we add in Definition 1.1(B) also:

(5) if α is inaccessible, and for some $\beta < \alpha$ for every γ satisfying $\beta \leq \gamma < \alpha$, $\Vdash_{P_\beta}$ "$|P_\gamma/P_\beta| < \alpha$" then $(\exists \beta < \alpha)$ [Dom $\zeta \subseteq P_\beta$].

Then nothing changes in the above (only we have to prove everything by simultaneous induction on α), and if λ is an inaccessible cardinal $> \alpha$ and $|P_i| < \lambda$ for every $i < \lambda$ and $\bar{Q} = \langle P_i, Q_i \colon i < \lambda \rangle$ is a κ-RS iteration, then

(1) every $\bar{Q}$-named ordinal is in fact a $(\bar{Q} \upharpoonright i)$-named ordinal for some $i < \alpha$,
(2) like (1) for $\bar{Q}$-named conditions.
(3) $P_\kappa = \bigcup_{i<\kappa} P_i$.
(4) if κ is a Mahlo cardinal then P_λ satisfies the λ-c.c. (in a strong way).

2 The κ-finitary revised support

We deal with forcing notions Q satisfying:

Definition 2.1   Let γ be an ordinal, $S \subseteq \{2\} \cup \{\lambda \colon \lambda \text{ a regular cardinal}\}$. Now Q satisfies $(S, \gamma) - Pr_1$ if
    (i) $Q = (|Q|, \leq, \leq_0)$
    (ii) as a forcing $Q = (|Q|, \leq)$
    (iii) $\leq_0$ is a partial order
    (iv) $[p \leq_0 q \Rightarrow p \leq q]$
    (v) *for every* cardinal $\kappa \in S$ and Q-name τ, such that $\Vdash_Q$ "$\tau \in \kappa$" and $p \in Q$ *for some* $q \in Q$, $l \in \kappa$, $p \leq_0 q$ and $q \Vdash_Q$ "if $\kappa = 2$, $\tau = l$ and if $\kappa \geq \aleph_0$, $\tau \leq l$"
    (vi) for each $q \in Q$ in the following game player I has a winning strategy: for $i < \gamma$ player I chooses $p_{2i} \in Q$ such that $q \leq_0 p_{2i}$, $\wedge \bigwedge_{j<2i} p_j \leq_0 p_{2i}$ and then player II chooses $p_{2i+1} \in Q$, $p_{2i} \leq_0 p_{2i+1}$.
    Player I loses if he has sometimes no legal move which can occur in limit stages only.
    Let $(S, \gamma) - Pr_1^-$ means $(\{\kappa\}, \gamma) - Pr_1$ for every $\kappa \in S$.

Fact 2.2:
    (1) If $\kappa < \gamma_1$, $\gamma_2 < \kappa^+$ then $(S, \gamma_1) - Pr_1$ is equivalent to $(S, \gamma_2) - Pr_1$.
    (2) If $\kappa + 1 \leq \gamma < \kappa^+$ and $\square_\kappa$ (i.e., there is a sequence $\langle C_\delta \colon \delta < \kappa^+ \rangle$, $C_\delta \subseteq \delta$ closed unbounded) $[\delta_1 \in C_\delta, \delta_1 = \sup \delta_1 \cap C_\delta \to C_{\delta_1} = C_\delta \cap \delta_1]$ and Q satisfies $(S, \gamma) - Pr_1$ then Q satisfies $(S, \kappa^+) - Pr_1$.
    (3) If Q satisfies $(S, \gamma) - Pr_1$, $\lambda \leq \gamma$, and $\lambda \in S$ then in V^Q λ is still a regular cardinal and when $\lambda = 2$, Q does not add bounded subsets to γ.

(4) If Q satisfies $(S, \gamma) - Pr_1$, $\lambda \in S$, λ regular, and for every regular μ, $\gamma \le \mu < \lambda \Rightarrow \Vdash_Q$ "μ is not regular" (e.g., $[\gamma, \lambda)$ contains no regular cardinal) then λ is regular in V^Q.

Proof: Straightforward.

Definition 2.3 $(S, <\kappa) - Pr_1$, will mean $(S, \gamma) - Pr_1$ for every $\gamma < \kappa$.
Fact 2.4: The following three conditions on forcing notion Q, a set $S \subseteq \{2\} \cup \{\lambda: \lambda$ a regular cardinal$\}$ and regular ordinal κ are equivalent:
  (a) there is $Q' = (Q', \le, \le_0)$ such that $(Q', \le)$, $(Q, \le)$ are equivalent and Q' satisfies $(S, \kappa) - Pr_1$.
  (b) for each $p \in Q$, in the following game (which last κ moves) player II has a winning strategy:
       in the ith move player I chooses $\lambda_i \in S$ and a Q-name τ_i of an ordinal $< \lambda_i$ then player II chooses an ordinal $\alpha_i < \lambda_i$.
       In the end player II wins if for every $\alpha < \kappa$ there is $p_\alpha \in Q$, $p \le p_\alpha$ such that for every $i < \alpha$ $p_\alpha \Vdash$ "either $\lambda_1 = 2_i$, $\tau_i = \alpha_i$ or $\lambda_i \ge \aleph_0$ $\tau_i < \alpha_i$".
  (c) like (a) but moreover $(Q, \le_0)$ is κ-complete.

Proof: (c) $\Rightarrow$ (a): trivial.

Proof: (a) $\Rightarrow$ (b): Choose $q \in Q'$ which is above p. We describe a winning strategy for player II: he plays on the side a play (for q) of the game from 2.1 (vi) where he uses a winning strategy (whose existence in guaranteed by (a)). In step i of the play (for 4.2(b)) he already has the initial segment $\langle p_j: j < 2i \rangle$ of the play for 2.1(vi). If player II plays λ_i, τ_i in the actual game, he plays $p_{2i} \in Q'$ in the simulated play by the winning strategy of player I there and then he chooses p_{2i+1}, $p_{2i} \le_0 p_{2i+1} \in Q'$, which forced the required α_i (exists by 2.1(v)) and then plays α_i in the actual play.

Proof: (b) $\Rightarrow$ (c): Find winning strategy for player II in the game from 2.9(b). We define Q': $Q' = \{(p, \langle \lambda_i, \tau_i, \alpha_i: i < \xi \rangle): p \in Q$, and $\langle \lambda_i, \tau_i, \alpha_i: i < \alpha \rangle$ is an initial segment of a play of the game from 2.4(b) for p in which II uses his winning strategy.
   The order $\le_0$ is:

$$(p, \langle \lambda_i, \tau_i, \alpha_i: i < \xi \rangle) \le_0 (p', \langle \lambda_i', \tau_i', \alpha_i': i < \xi' \rangle)$$

iff (both are in Q') and

$$Q \vDash p = p', \xi \le \xi', \text{ and for } i < \xi$$
$$\lambda_i = \lambda_i', \tau_i = \tau_i', \alpha_i = \alpha_i'$$

and the order $\le$ on Q' is

$$(p, \langle \lambda_i, \tau_i, \alpha_i: i < \xi \rangle) \le (p', \langle \lambda_i', \tau_i', \alpha_i': i < \xi' \rangle)$$

iff (both are in Q' and) $Q \vDash p \le p'$. Moreover, $p' \Vdash_Q$ "$\lambda_i = 2$, $\tau_i = \alpha_i$ or $\lambda_i \ge \aleph_0$, $\tau_i < \alpha_i$" for $i < \xi$.
   The checking is easy.

Definition 2.5

   (1) Let $\mathrm{Gen}(\bar{Q}) = \Big\{ G : G \subseteq \bigcup_{i<\alpha} P_i$ is directed, $G \cap P_i$ generic over V for $i < \alpha \Big\}$. Let $\mathrm{Gen}^r(\bar{Q}) = \Big\{ G :$ for some (set) forcing notion P^*, $\bigwedge_{i<\alpha} P_i <_\circ P^*$ and $G^* \subseteq P^*$ generic over V and $G = G^* \cap \bigcup_{i<\alpha} P_i \Big\}$.

   (2) If $\bar{Q} = \langle P_i : i < \alpha \rangle$ or $\bar{Q} = \langle P_i, Q_i : i < \alpha \rangle$ P_i is $<_\circ$-increasing we define a $\bar{Q}$-name τ almost as we define $\Big(\bigcup_{i<\alpha} P_i\Big)$-names, but we do not use maximal antichains of $\bigcup_{i<\alpha} P_i$, $G \subseteq \bigcup_{i<\alpha} P_i$:

   $(*)$ τ is a function, $\mathrm{Dom}(\tau) \subseteq \bigcup_{i<\alpha} P_i$ and every directed $G \in \mathrm{Gen}^r(\bar{Q})$, $\tau[G]$ is defined iff $\mathrm{Dom}(\tau) \cap G \neq \varnothing$ and then $\tau[G] \in V[G]$ [where "every $G\ldots$" is taken? e.g., V is countable, G any set from the true universe] and τ is definable with parameters from V (so τ is really a first-order formula with the variable G and parameters from V).

   (3) For $p \in \bar{Q}$ $\Big($i.e., $p \in \bigcup_{i<\alpha} P_i\Big)$, $\bar{Q}$-names $\tau_0, \ldots, \tau_{n-1}$, and (first-order) formula ψ let $p \Vdash_{\bar{Q}} \psi(\tau_0, \ldots \tau_{n-1})$ means that for every directed $G \in \mathrm{Gen}^r(\bar{G})$, with $p \in G$, $V[G] \vDash \psi(\tau_0[G], \ldots, \tau_{n-1}[G])$.

   (4) A $\bar{Q}$-named $[j, \beta)$-ordinal ζ is a $\bar{Q}$-name ζ such that if $\zeta[G] = \xi$ then $j \leq \xi < \beta$ and $(\exists p \in G \cap P_{\xi \cap \alpha})\, p \Vdash_{\bar{Q}} "\zeta = \xi"$ (where $\alpha = l(\bar{Q})$). If we omit "$[j, \beta)$" we mean $[0, l(\bar{Q}))$.

Remark 2.5A: We can restrict in the definition of $\mathrm{Gen}^r(\bar{Q})$ to P^* in some class K, and get a K-variant of our notions.

Fact 2.6:

   (1) For $\bar{Q}$ as above and $\bar{Q}$-named $[j, \beta)$-ordinal ζ and $p \in \bigcup_{i<\alpha} P_i$ there are ξ, q and q_1 such that $p \leq q$, $q \Vdash_{\bar{Q}} "q_1 \in \mathcal{G}"$, $q_1 \in P_\xi$, $\xi < \alpha$, and $q_1 \Vdash_{\bar{Q}} "\zeta = \xi"$ or $q \Vdash_{\bar{Q}}$, "ζ is not defined".

   (2) For $\bar{Q}$ as above, and ζ, ξ $\bar{Q}$-named $[j, \beta)$-ordinals, also $\mathrm{Min}\{\zeta, \xi\}$, $\max\{\zeta, \xi\}$ (naturally defined) are $\bar{Q}$-named $[j, \beta)$-ordinals.

   (3) For $\bar{Q}$ as above and $\bar{Q}$-named ordinals $\xi_1, \ldots, \xi_n$ and $p \in \bigcup_{i<\alpha} P_i$ there are $\zeta < \alpha$ and $q_0 \in P_\zeta$, $p \leq q$, $q \Vdash_{\bar{Q}} "\zeta = \mathrm{Max}\{\xi_1, \ldots, \xi_n\}"$. Similarly for Min.

Definition 2.7   We define and prove by induction on α the following simultaneously:

   (A) $\bar{Q} = \langle P_i, Q_i : i < \alpha \rangle$ is a κ-Sp_2-iteration.

   (B) A $\bar{Q}$-named atomic condition q (or $[j, \beta)$-condition, $\beta \leq \alpha$) and we define $q \upharpoonright \xi$, $q \upharpoonright \{\xi\}$ for a $\bar{Q}$-named atomic condition q and ordinal $\xi < \alpha$ (or $\bar{Q}$-named ordinal ξ).

   (C) If q is a $\bar{Q}$-named $[j, \beta)$-atomic condition, $\xi < \alpha$, then $q \upharpoonright \xi$ is a $(\bar{Q} \upharpoonright \xi)$-named $[j, \mathrm{Min}[\beta, \xi))$-condition and $q \upharpoonright \{\xi\}$ is a P_ξ-name of a member of Q_ξ or undefined (and then it is assigned the value $\varnothing$, the minimal member of Q_ξ similarly for ξ).

(D) The $\kappa - Sp_2$-limit of $\bar{Q}$, Sp_2-$\mathrm{Lim}_\kappa \bar{Q}$, and $p \restriction \xi$ for $p \in Sp_2$-$\mathrm{Lim}_\kappa \bar{Q}$, ξ an ordinal $\le \alpha$ (or $\bar{Q}$-named ordinal).

(E) $P_\beta <_\circ Sp_2 \mathrm{Lim}_\kappa \bar{Q}$ (if $\bar{Q} = \langle P_i, Q_i : i < \alpha \rangle$ is a κ-Sp_2-iteration, $\beta < \alpha$, P_i, Q satisfying (i)-(iv) of Definition 1.2). In fact $P_\beta \subseteq Sp_2$-$\mathrm{Lim}_\kappa \bar{Q}$ (as models with two partial orders, even compatibility is preserved) and $q \in P_\beta$, $p \in Sp_2 \mathrm{Lim}_\kappa \bar{Q}$ are compatible iff q, $p \restriction \beta$ are in P_β.

Proof:

(A) $\bar{Q} = \langle P_i, Q_i : i < \alpha \rangle$ is a κ-Sp_2-iteration if $\bar{Q} \restriction \beta$ is a κ-Sp_2-iteration for $\beta < \alpha$, and if $\alpha = \beta + 1$ then $P_\beta = Sp_2 \mathrm{Lim}_\alpha (\bar{Q} \restriction \beta)$ and Q_β is a P_β-name of a forcing notion as in Definition 2.1(i)-(iv).

(B) We say q is a $\bar{Q}$-named atomic $[j, \beta]$-condition when: q is a $\bar{Q}$-name, and for some $\zeta = \zeta_q$ a $\bar{Q}$-named $[j, \beta]$-ordinal $\Vdash_{\bar{Q}}$ "ζ has a value iff q has, and if they have then $\zeta < \mathrm{Min}(\beta, l(\bar{Q}))$, $q \in Q_\zeta$". Now $q \restriction \xi$ will have a value iff ζ_q has a value $< \xi$ and then its value is the value of q. Lastly, $q \restriction \{\xi\}$ will have a value iff ζ_q has value ξ and then its value is the value of q (similarly for ξ).

(C) Left to the reader.

(D) We are defining $Sp_2 \mathrm{Lim}_\kappa \bar{Q}$. It is a triple $P_\alpha = (|P_\alpha|, \le, \le_0)$ where

(a) $|P_\alpha| = \{\{q_i : i < i(*)\}; i(*) < \kappa$, each q_i is a $\bar{Q}$-named atomic condition, and for every $\xi < \alpha$, $\Vdash_{P_\xi}$ "$\{q_i^e \restriction [\xi] : i < i^*\}$ has an $\le_0$-upper bound in Q_ξ"$\}$.

(b) $P_\alpha \vDash p_1 \le_0 p_2$ iff for every $\zeta < \alpha$ $\Vdash_{P_\zeta} \{q_i^l \restriction \{\zeta\} : i < i^l(*)\}$ are equal for $l = 1, 2$ or for some $i < i^2(*)$ for every $j_1 < i^1(*) \vDash q_{\gamma_1}^1 Q_\zeta \vDash q_{j_1} \le_0 q_i^2$ where $p_l = \{q_i^l : i < i^l(*)\}$

(c) $P_\alpha \vDash p^1 \le p^2$ iff:

(i) for every $\zeta < \alpha$ $(p^2 \restriction \zeta)$ $\Vdash_{P_\zeta}$ "$p^1 \restriction \{\zeta\}$, $p^2 \restriction \{\zeta\}$ are equal as subsets of Q_ζ (remember (F)) or for some $i < i^2(*)$ for every $j < i^1(*)$ $\Vdash_{P_\zeta}$" $Q_\zeta \vDash q_j^1 \le q_i^2$" where $p^l = \{q_i^l : i < i^l(*)\}$

(ii) for some $n < \omega$ and $\bar{Q}$-named ordinals $\xi_1, \ldots, \xi_n$ for each $\zeta < l(\bar{Q})$: $p_2 \restriction \Vdash_{\bar{Q}}$ "if $\zeta \notin \{\xi_1, \ldots, \xi_n\}$ then for some $r \in p_2$, $\zeta_r[G] = \zeta$ and for every $s \in p_1$ $[\zeta_r = \zeta \Rightarrow s \le_0 r]$". We then say: $p_1 \le p_2$ over $\{\xi_1, \ldots, \xi_n\}$.

Remark: We could use names for n too, but as it is finite this is not necessary.

Now for $\xi \le \alpha$, and $p \in Sp_2 \mathrm{Lim}_\kappa \bar{Q}$, let us define

$$p \restriction \xi = \{r \restriction \xi : r \in p\}$$
$$p \restriction \{\xi\} = \{r \restriction \{\xi\} : r \in p\}.$$

Proof of (E): Let us check Definition 2.1 for $P_\alpha =_{df} Sp_2 \mathrm{Lim}_\kappa \bar{Q}$:

$\le^{P_\alpha}$ *is a partial order:* Suppose $p_0 \le p_1 \le p_2$. Let n^l, $\xi_0^l, \ldots, \xi_n^l$ appear in the definition of $p_l \le p_{l+1}$. Let $n = n^0 + n^1$, and

$$\zeta_\ell = \begin{cases} \xi_\ell^0 & \text{if } l < n^0 \\ \xi_{l-n^0}^1 & \text{if } l \ge n^0. \end{cases}$$

Now $\Vdash_{\bar{Q}} p_l \restriction \{\zeta_\ell\} \le p_{l+1} \restriction \{\zeta_\ell\}$, "hence $\Vdash_{\bar{Q}} p_0 \restriction \{\zeta_\ell\} \le p_2 \restriction \{\zeta_\ell\}$".

Also $\Vdash_{\bar{Q}}$ "if $\zeta \notin \{\zeta_0, \ldots, \zeta_{n+1}\}$ then $p_0 \upharpoonright \{\zeta\} \leq_0 p_1 \upharpoonright \{\zeta\} \leq_0 p_2 \upharpoonright \{\zeta\}$". So we finish.

$\leq_0$ *is a partial order:* As in I.

$p \leq_0 q \Rightarrow p \leq q$: By the definition; easy.

So in Definition 2.1, (i), (ii), (iii), and (iv) hold. We leave the checking of the rest to the reader.

Remark 2.8: This is a combination of [4], X with the recent Gitik ([2]) (which uses Easton support, each Q is $(\{2\}, \kappa_i)$-complete, where for the important i's $\kappa_i = i$: as his aim was mainly cardinals which remain inaccessible).

Lemma 2.9   *Suppose γ is an ordinal and $\bar{Q} = \langle P_i, Q_i : i < \alpha \rangle$ is a κ-Sp_2-iteration.*

(1)   *if $p \leq q$ in $P_\alpha = Sp_2 \operatorname{Lim}_\kappa \bar{Q}$ then for some n ordinals $\xi_1 < \ldots, < \xi_n$, $r \in P_\alpha$, $q \leq r$, and $p \leq r$ above $\{\xi_1, \ldots, \xi_n\}$.*

(2)   *If γ is successor cardinal (or not a cardinal) then the parallel of 1.4, 1.5, 1.6 holds.*

(3)   *If κ is inaccessible but $\Vdash_{P_i}$ "κ is a regular cardinal" for each $i < \alpha$ then the parallel of 1.4, 1.5, 1.6 holds.*

Proof: Left to the reader.

Lemma 2.10   *Suppose $\bar{Q} = \langle P_i, Q_i : i < \alpha \rangle$ is a κ-Sp_2-iteration, $\kappa > \aleph_0$ a regular cardinal, $S \subseteq \{2\} \cup \{\mu : \aleph_0 \leq \mu \leq \kappa, \mu$ regular$\}$ and each Q_i (in V^{P_i}), has $(S, < \kappa) - Pr_1$, then:*

(1) $P_\alpha = Sp_2$-$\operatorname{Lim}_\kappa \bar{Q}$ *has $(S, < \kappa) - Pr_1$, and if each Q_α has $(S, \kappa) - Pr_1$ then P_α has it.*

(2)   *If $\kappa \in S$ and $cf(\alpha) = \kappa$ then $\bigcup_{i < \alpha} P_i$ is dense in P_α.*

(3)   *If $\kappa \in S$, α strongly inaccessible, $\alpha > |P_i| + \kappa$ for $i < \alpha$ then P_α satisfies the α-chain condition (in a strong sense).*

(4)   *If each Q_i has a power of $\leq \chi$, then P_α has a dense subset of power $\leq (|\alpha| + \chi)^{<\chi}$.*

(5)   *If $|Q_i| \leq \chi$, $\chi^{<\chi} = \chi$, $l(\bar{Q}) = \chi^+$ then $\bar{Q}$ satisfies the χ^+-c.c.*

(6)   *If $S = \{\kappa\}$, (1) works even for $(S, \kappa) - Pr$ which is defined as the game definition of semiproperness; i.e., using Fact 2.4(b) with winning means:*

$$\bigwedge_\alpha (\exists p_\alpha) p_\alpha \Vdash \operatorname*{Sup}_{i < \alpha} \tau_i \leq \operatorname*{Sup}_{i < \alpha} \alpha_i$$

Proof:

(1) Let us check Definition 2.1. Now (i)–(iv) hold by 2.7.

For (v) let $\mu \in S$, $\Vdash_{P_\alpha}$ "$\tau < \mu$", $p \in P_\alpha$. For simplicity $\mu \neq 2$. We define by induction on n p_n, $p = p^0$, $p^n \leq_0 p^{n+1}$. For each n let $\{\xi_i^n : i < \gamma_n < \kappa\}$ be the domain of p^n (i.e., $\{\zeta_r : r \in p^n\}$) and define by induction on $i < \gamma_n$ p_i^n, $p_0^n = p_n$. p_i^n is $\leq_0$-increasing (in i).

If p_i^n is defined let (writing a little inaccurately) $G \subseteq P_{\xi_i^n + 1}$ be generic over V. In $V[G]$ if there are $\alpha_i^n < \mu$, $r \in P_\alpha$, $r \upharpoonright (\xi_i^n + 1) \in G$, $p_i^n \leq_0 r$, such that $r \Vdash_{P_\alpha/G}$ "$\tau \leq \alpha_i^n$", let $r_i^n[G]$ be like that; otherwise, let $r_i^n = p_i^n$. So r_i^n, α_i^n are $P_{\xi_i^n + 1}$-names. Now in $V[G \cap P_{\xi_i^n}]$, $Q_{\xi_i^n}$ is a forcing notion, α_i^n a name of

an ordinal $< \mu$; hence there are $\beta_i^n < \mu$, q_i^n, $p_i^n \upharpoonright \{\xi_i^n\} \leq_0 q_i^n \in Q_{\xi_i^n}$, $V[G \cap P_{\xi_i^n}] \models "q_i^n \Vdash_{Q_{\xi_i^n}} "\alpha_i^n \leq \beta_i^n"$. So β_i^n is a $P_{\xi_i^n}$-name, q_i^n a $\bar{Q}$-named atomic condition. Now define p_{i+1}^n as $p_{i+1}^n = p_i^n \cup r_i^n \upharpoonright [\xi_i^n + 1, \alpha) \cup \{q_i^n\}$.

We have an obvious flaw—why is there a limit for $p_i^n (i < \delta)$? (or $p^n (n < \omega)$). For this, use (v) of Definition 2.1, i.e., increase p_{i+1}^n albeit according to the winning strategy. Now p_{n+1} will be $_0\geq p_{\gamma_n}^n$ according to the strategy too.

So there is p^*, $p^n \leq_0 p^*$ for each n. Dom $p^* = \bigcup_{n < \omega}$ Dom p_n. We claim that for some $\alpha < \mu$, $p^* \Vdash_{P_\alpha} "\tau \leq \alpha"$. If not, let $q \in P_\alpha$, $q \geq p^*$, and $\beta < \mu$ be such that $q \Vdash_{P_\alpha} "\tau = \beta"$. So by 2.9(3) w.l.o.g. $q \geq p^*$ above some $\{\xi_0, \ldots, \xi_{n-1}\}$, $\xi_0 < \ldots < \xi_{n-1}$. Choose such number n, and ordinals $\xi_l (l < n)$ with minimal ξ_{n-1} (or $n = 0$ is best of all). If $n > 0$, w.l.o.g. for some $m < \omega$ $q \upharpoonright \xi_{n-1}$ $\Vdash_{P_{\xi_{n-1}}} "\xi_n \in$ Dom $p^m"$ and we get contr. to the choice of p^{m+1}

(vi) is left to the reader.

(2), (3) are left to the reader.

(4), (5) Like [4], Ch. III x.x, use only names which are hereditarily $< \kappa$.

Definition 2.10   We define Sp_3 iteration $\bar{Q}$ and Sp_3 Lim$_\kappa$ $\bar{Q}$ like κ-SP_2 with only one change: *instead $p \in P_i$ being of cardinality $< \kappa$, we require:*

(*)   for every $p \in P_\alpha$, $\lambda \leq l(\bar{Q})$ which is strongly inaccessible, and $(\forall i < \kappa) [|P_i| < \lambda] \Vdash_{\bar{Q} \upharpoonright \lambda}$ "the domain of $p \upharpoonright \lambda$ is bounded below λ". Hence, for each $\lambda \bigcup_{i < \lambda} P_i$ is dense in P_λ.

Claim 2.11: The parallel of Definition 2.10 holds.

3   We can get from the lemma of preservation of forcing with $(S, \gamma) - Pr_1$ by κ-Sp_2 iteration (and on the λ-c.c. for then) Martin-like axioms. We list below some variations.

Notation 3.1: Reasonable choices for S are
(1) $S_\kappa^0 = UR \, Car_{\leq \kappa} = \{\mu : \mu \text{ a regular cardinal}, \aleph_0 < \mu \leq \kappa\}$
(2) $S_\kappa^1 = RCar_{\leq \kappa} = \{\mu : \mu \text{ a regular cardinal}, \aleph_0 \leq \mu \leq \kappa\}$
(3) $S_\kappa^2 = \{2\} \cup Car_{\leq \kappa}$
(4) If we write "$< \kappa$" instead $\leq \kappa$ (and $S_{<\kappa}^l$ instead S_κ^l) the meaning should be clear.

Fact 3.2: Suppose the forcing notion P satisfies $(S, \gamma) - Pr_1$
(1) If $2 \in S$ then P does not add any bounded subset of γ.
(2) If μ is regular, and $\lambda_i (i < \mu)$ are regular, and $\{\mu\} \cup \{\lambda_i : i < \mu\} \subseteq S$, D is a uniform ultrafilter on μ, $\theta = cf\left(\prod_{i<\mu} \lambda_i/D\right)$ (λ_i-as an ordered set) then P satisfies $(S \cup \{\theta\}, \gamma') - Pr_1$ whenever $\mu\gamma' \leq \mu$. (We can do this for all such θs simultaneously.)
(3) If $\lambda \in S$ is regular, $\mu < \gamma$ then for every $f : \mu \to \lambda$ from V^P for some $g : \mu \to \lambda$ from V for every $\alpha < \mu$, $f(\alpha) < g(\alpha)$.

Claim 3.3: Suppose $MA_{<\kappa}$ holds $\Big($*i.e., for every P satisfying the $\aleph_1$-c.c. and*

dense $D_i \subseteq P$ (for $i < \alpha < \kappa$) there is a directed $G \subseteq Q$, $\bigwedge_{i<\kappa} G \cap D_i \neq \varnothing$). Then the following forcing notions have expansions (by $\leq_0$) having the $(U \ RCar, \kappa) - Pr_1^0$.

(1) Silver forcing: $\{(w, A): w \subseteq \omega$ finite, $A \subseteq \omega$ infinite$\}$
    $(w_1, A_1) \leq (w_2, A_2)$ iff $w_1 \subseteq w_2 \subseteq w_1 \cup A_1$, $A_2 \subseteq A_1$.
(2) The forcing from [5], Section 2 (changed suitably).

Proof: (1) Let P' be the set of (w, A, B) satisfying: $w \subseteq \omega$ finite, $B \subseteq \omega$ infinite, $B \subseteq A \subseteq \omega$, with the order

$$(w_1, A_1 B_1) \leq (w_2, A_2, B_2) \ iff \ (w_1, A_1) \leq (w_2, A_2)$$
$$\text{and } B_2 \subseteq^* B_1 \text{ (i.e., } B_2 - B_1 \text{ finite)}$$

$$(w_1, A_1, B_1) \leq_0 (w_2, A_2, B_2) \ if \ w_1 = w_2$$
$$A_1 = A_2$$
$$B_2 \subseteq * B_1.$$

Let us check Definition 2.1: (i)–(iv) easy.
Note that $\{(w, A, A): (w, A, A) \in P\}$ is dense in P.
(iv) Let $\mu > \aleph_0$ be a regular cardinal, τ a P'-name, $\Vdash_P$ "$\tau < \mu$. Let $p = (w, A, B)$ be given. Choose by induction on $i < \omega$, n_i, A_i such that

(a) $A_0 = B(\subseteq A)$
(b) $n_i = \text{Min } A_i$
(c) $A_{i+1} \subseteq A_i - \{n_i\}$
(d) for every $u \subseteq \{0, 1, 2, \ldots, n_i\}$ for some $\alpha_{i,u} < \mu$, (u, A_{i+1}, A_{i+1})
    $\Vdash_{P'}$ "$\tau = \alpha_{i+1}$" or for no $B \subseteq \omega$ and $\alpha < \mu (u, B, B) \Vdash$ "$\tau = \alpha_{i,u}$".
There is no problem to do this, now $q =_{df} (w, A, \{n_i: i < \omega\})$ satisfies:
(e) $p \leq q \in P'$ and even $p \leq_0 q$.
(f) $q \Vdash_{P'}$ "" $\tau \in \{\alpha_{i,u}: i < \omega, u \subseteq \{0, 1, 2, \ldots, n_i\}\}$.

So q is as required.
(v): Suppose $p_i (i < \gamma)$ is $\leq_0$-increasing so $p_i = (w, A, B_i)$ $B_i \subseteq A$, B_i is $*$-decreasing. It is well known that for $\gamma < \kappa$, $MA_{<\kappa}$ implies the existence of an infinite $B \subseteq \omega$, $(\forall i < \gamma) B \subseteq * B_i$.

Claim 3.4: The following forcing notions have the $(U \ RCar, \kappa) - Pr_1$:
(1) $\aleph_1$-c.c.
(2) κ-complete
(3) $\{f: f$ a function from A to $\{0, 1\}$, $A \subseteq \omega$, $A = \phi \mod D\}$ where D is a filter on ω, containing the co-finite sets, such that if $A_i \in D$ for $i < i^* < \kappa$ then for some $B \in D \bigwedge_{i<i^*} B \subseteq *A_i$

Discussion 3.5: Let $\kappa < \lambda$, λ regular. Each of the following gives rise naturally to a generalized MA, stronger as λ is demanded to be a larger cardinal (so if λ is supercompact we get parallels to PFA).

12   SAHARON SHELAH

Case I: We use $\bar{Q}$ of length λ, a κ-SP_2 iteration, $\Vdash_{P_i}$ "$|Q_i| < \lambda$", each Q_i having $(S_\kappa^l, \kappa) - Pr_1^-$.

Now $P_\lambda = \kappa - SP_2 \operatorname{Lim}_\kappa \bar{Q}$ have the $(S_\kappa^l, \kappa) - Pr_1$ by 2.10, so all regular $\mu \le \kappa$ remain regular and usually every $\lambda' \in (\kappa, \lambda)$ is collapsed. But λ is not collapsed if it is strongly inaccessible (by 2.10(3)) and also if $(\forall \chi < \lambda)(\chi^{<\kappa} < \lambda)$ (by 2.10(5)). If $2 \in S_\kappa^Q$, no bounded subset of κ is added.

Case II: Like Case I with $(\kappa + 1) - Sp_2$ iteration $Sp_2 \operatorname{Lim}_{\kappa+1}$ and every $\lambda' \in (\kappa, \lambda)$ is collapsed. Here we need λ to be strongly inaccessible.

Case III: $\bar{Q}$ is Sp_3-iteration, has length κ, $|Q_i| < \kappa$ for $i < \kappa$, κ is strongly inaccessible, and Q_i have $(S, \gamma_i) - Pr_1^-$.

By 2.11 $P_\kappa = Sp_3 \operatorname{Lim} \bar{Q}$ has the κ-c.c. (and $|P_i| < \kappa$ of course). Let $S = \{\mu < \kappa; \mu$ regular and for some i, $\Vdash_{P_i}$ "μ is regular and $\mu \in S_j$, $\mu \le \gamma_j$, for $j > i\}$ then $\Vdash_{P_\alpha}$.

Fact 3.6: Suppose λ is strongly inaccessible, limit of measurables, $\lambda > \kappa$, κ regular. Then for some λ-cc forcing P not adding bounded subsets of κ, $|P| = \lambda$, and $\Vdash_P$ "$2^\kappa = \lambda = \kappa^+$, and for every $A \subseteq \kappa$ there is a countable subset of λ not in $L(A)$.

Proof: We use κ-SP_2-iteration $\langle P_i, Q_i : i < \lambda \rangle$, $|P_i| < \lambda$. For i even: let κ_i be the first measurable $> |P_i|$, (but necessarily $< \lambda$) and τ. Then Q_i is Prikry forcing on κ_i and Q_{i+1} is Levi collapse of κ_i^+ to κ.

4

Lemma 4.1   *Suppose*
   (i)   *R is an $\aleph_1$-complete forcing notion.*
   (ii) *For $r \in R$, $\bar{Q}^r = \langle P_i^r : i \le \alpha_\alpha^r \rangle$, P_i^r is $<_\circ$-increasing in i and if $i \le \alpha^r$ has cofinality ω_1, then every countable subset of $V^{P_\kappa^r}$ belongs to $V^{P_i^r}$ for some $i < \alpha$.*
   (iii) *If $r^1 \le r^2$ then $\bar{Q}^{r^1} \le \bar{Q}^{r^2}$.*
   (iv) *If $r \in R$ and Q is a $P_{\alpha_r}^r$-name of a forcing notion, then for some $r^1 \ge r$*

$$P_{\alpha_\mu + 1}^{r^1} = P_{\alpha_\mu}^r * Q \text{ or } \Vdash_{P_{\alpha_M}^{r^1}} Q \text{ does not satisfy the c.c.c.}$$

   (v)   *If $r^\zeta (\zeta < \delta)$ is increasing, $\delta \le \omega_1$, then for some r*

$$\bigwedge_{\zeta < \delta} r^\zeta \le r \text{ and } \alpha_r = \bigcup_{\zeta < \delta} \alpha_{r^\zeta}.$$

Let $P[G_R]$ be $\cup \{P_i^r : r \in G_R, i \le \alpha_r\}$, so it is an R-name of a forcing notion. *Then* $\Vdash_R [\Vdash_{P[G_R]}$ "for any $\aleph_1$ dense subsets of Sacks forcing, there is a directed subset of Sacks forcing not disjoint to any of them"].

Remark: $Q_{Sacks} = \{\tau : \tau \subseteq^{\omega >} 2$ is closed under initial segments nonempty and $(\forall \eta \in \tau)(\exists v)(\eta < v \wedge v \frown \langle 0 \rangle \in T \wedge v \frown \langle 1 \rangle \in T)$ and $\tau_i \le \tau_2$ if $\tau_2 \subseteq \tau_1$.

Proof: Let D_i be $R*P[G_R]$-name of dense subset of $Q_{Sacks}^{R*P[G_R]}$ for $i < \omega_1$ (Q_{Sacks}^V is Sacks forcing in the universe V).

For a subset E of Sacks forcing let $var(E)$ be $\{(n, T) : T \in E, n < \omega\}$ ordered by $(n_1 T_1) \le (n_2, T_2)$ iff $n_1 \le n_2$, $T_2 \subseteq T_1$, and $T_1 \cap^{n_1 \ge} 2 = T_2 \cap^{n_1 \ge} 2$. We now define by induction on $\zeta \le \omega_1$, $r(\zeta)$, and D_ζ such that:

(a) $r(\zeta) \in R$ is increasing, $\alpha_{r(\zeta)}$-increasing continuous.

(b) D_ζ is a $P^{r(\zeta+1)}_{\alpha_r(\zeta+1)}$-name of a countable subset of Q_{Sacks}.

(c) If $T \in D_\zeta$, $\eta \in T$ then $T_{[\eta]} =_{df} \{v: \eta \frown v \in T\}$ belongs to D_ζ.

(d) If $T_1, T_2 \in D_\zeta$ then $\{\langle \; \rangle, \langle 0 \rangle \frown \eta: \eta \in T_1\}$, $\{\langle \; \rangle, \langle 1 \rangle \frown \eta: \eta \in T_2\}$ and their union belongs to D_ζ.

(e) Let $\xi < \zeta$, then for $T_1 \in D_\xi$ there is $T_2 \in D_\zeta$, $T_1 \geq T_2$ and for $T_2 \in D_\zeta$ there is $T_1 \in D_\xi$, $T_1 \geq T_2$.

(f) If $T \in D_{\zeta+1}$ then for some n for every $\eta \in {}^n 2 \cap T$, $T_{(\eta)} =_{df} \{v \in T: v \leq \eta \text{ or } \eta \leq v\}$ belongs to D_ζ.

(g) Suppose ζ is limit, then $P^{r(\zeta+1)}_{\alpha_r(\zeta)+1} = P^{r(\zeta)}_{\alpha_r(\zeta)} * T_\zeta$, T_ζ is $\left[\text{var} \bigcup_{\xi < \zeta} p_\xi \right]$ if $\zeta < \omega_1$ and T_ζ is $\left[\text{var} \bigcup_{\xi < \zeta} D_\xi \right]^\omega$ if $\zeta = \omega_1$ (the ω-th power, with finite support).

Next the generic subset of T_ζ gives a sequence of length ω of Sacks conditions closing the set of those conditions by $(c) + (d)$ we get D_ζ. We have to prove that T_ζ satisfies the $\aleph_1$-c.c. in $V^{R*P_{\alpha_r(t)}}$: When $\zeta < \omega_1$ this is trivial (as T_ζ is countable). Let $\zeta = \omega_1$. It suffices to prove that $\left[\text{var} \bigcup_{\xi < \zeta} D_\xi \right]^n$ satisfies the $\aleph_1$-c.c. where $n < \omega$. So let I be a $R * P^{r(\zeta)}_{\alpha_r(\zeta)}$ name of a dense subset of $\left[\text{var} \bigcup_{\xi < \zeta} D_\xi \right]^n$. We can find a $\xi < \zeta$, cf $\xi = \aleph_0$ such that $I_\xi = \{x: x \in V^{R*P^{r(\zeta)}_{\alpha_r(\zeta)}}$ and every $p \in R * P^{r(\zeta)}_{\alpha_r(\zeta)}/R * P^{r(\zeta)}_{\alpha_r(\zeta)}$ force x to be in $I\}$ is predense in $\left[\text{var} \bigcup_{\gamma < \xi} D_\gamma \right]^n$ (exists by (e)). Check the rest.

Remark: This argument works for many other forcing notions like Laver.

5

Definition 5.1 Let S be a subset of $\{2\} \cup \{\lambda: \lambda$ is regular cardinal$\}$, D a filter on a cardinal λ (or any other set). For any ordinal γ, we define a game $Gm^*(S, \gamma, D)$. It lasts γ moves. In the i-th move player I choose a cardinal $\lambda \in S$ and function F_i from λ to λ_i and then player II chooses $\alpha_i < \lambda_i$.

Player II wins a play if for every $i < \gamma$,

$$d(\langle \lambda_j, F_j, \alpha_j: j < i \rangle) =_{df} \{\zeta < \lambda: \text{for every } j < i \; [\lambda_i = 2 \Rightarrow F_j(\zeta) = \alpha_i] \\ [\lambda_i > 2 \Rightarrow F_j(\zeta) < \alpha_i\} \neq \varnothing \; mod \; D.$$

Remark 5.1A:

(1) See [4], Chapter X on this.

(2) If not said, otherwise we assume that $\lambda - \{\zeta\} \in D$ for $\zeta < \lambda$.

(3) If D is an ultrafilter on λ, $(|\gamma| + \kappa^+)$-complete for each $\kappa \in S$ then player II has a winning strategy.

Definition 5.2 For **F** a winning strategy for player II in $Gm^*(S, \gamma, D)$, D a filter on λ (we write $\lambda = \lambda(D)$), we define $Q = Q_{F,\lambda} = Q_{F,S,\gamma,D}$, $Q = (|Q|, \leq, \leq_0)$.

Part A: Let $(T, H) \in Q$ iff

(i) T is a nonempty set of finite sequence of ordinals $< \lambda$.

(ii) $\eta \in T \Rightarrow \eta \restriction \ell \in T$, and for some n and η: $T \cap {}^{n \geq} \lambda = \{\eta \restriction \ell : \ell \leq n\}$, $|T \cap {}^{n+1}\lambda| \geq 2$; we denote $\eta = stam(T) = stam(T, H)$ (it is unique).

(iii) H is a function, $T - \{stam(T) \restriction \ell : \ell < lg(stam(T))\} \subseteq dom\, H \subseteq {}^{\omega >}\lambda$.

(iv) for each $\eta \in Dom\, H$, $H(\eta)$ is a proper initial segment of a play of the game $Gm^*(S, \gamma, D)$ in which player II use his strategy **F** so $H(\eta) = \langle \lambda_i^{H(\eta)}, F_i^{H(\eta)}, \alpha_i^{H(\eta)} : i < i^{H(\eta)} \rangle$, and $i^{H(\eta)} < \gamma$.

(v) for $\eta \in T$, $d(H(\eta)) = \{\zeta < \lambda : \eta \,\widehat{}\, \langle \zeta \rangle \in T\}$.

Part B: $(T_1, H_1) \leq (T_2, H_2)$ (where both belong to Q) *iff* $T_2 \subseteq T_1$ and for each $\eta \in T_2$, if $stam(T_2) \leq \eta$ then $H_1(\eta)$ is an initial segment of $H_1(\eta)$.

Part C: $(T_1, H_1) \leq_0 (T_2, H_2)$ (where both belong to Q) *if* $(T_1, H_1) \leq (T_2, H_2)$ and $stam(T_1) = stam(T_2)$.

Remark 5.2A: (1) So if $(T, H) \in Q_{\mathbf{F}, \lambda}$ and **F**, $S(\gamma, D)$ are as above, $\eta \in T$, $\eta \geq stam(T)$ *then* $d(H(\eta)) \neq \varnothing \mod D$.

(2) We could restrict H to T in (iii).

Notation 5.2B: For $p = (T, H) \in Q_{\mathbf{F}, \lambda}$ and $\eta \in T$ let $p^{[\eta]} = (T^{[\eta]}, H)$, $T^{[\eta]} = \{\nu \in T : \nu \leq \eta \text{ or } \eta \leq \nu\}$. Clearly $p \leq p^{[\eta]} \in Q_{\mathbf{F}, \lambda}$.

Lemma 5.3   If $Q = Q_{\mathbf{F}, S, \gamma, D}$, D *a uniform filter on* $\lambda(D)$ *then* $\Vdash_Q$ cf $\lambda(D) = \aleph_0$.

Proof: Let $\eta_Q = \bigcup \{stam(p) : p \in \underset{\sim}{G}_Q\}$.

Clearly if $(T_\ell, H_\ell) \in \underset{\sim}{G}_Q$ for $\ell = 1, 2$ then for some $(T, H) \in \underset{\sim}{G}_Q$, $(T_\ell, H_\ell) \leq (T, H)$; hence $stam(T_\ell) \leq stam(T)$, hence $stam(T_1, H_1) \cup stam(T_1, H_2)$ is in ${}^{\omega >}\lambda$. Hence η_Q is a sequence of ordinals of length $\leq \omega$. It has length ω, as for every $p = (T, H) \in Q$, and n, there is $\eta \in T \cap {}^n\lambda$, hence $p \leq p^{[\eta]} \in Q$ (see 5.2B), and $p^{[\eta]} \Vdash$ "$lg(\eta_Q) \geq n$" because $\eta \leq stam(p^{[\eta]})$ and for every $q \in Q$, $q \Vdash_Q$ "$stam(q) \leq \eta_Q$". Obviously, $\Vdash_Q$ "$Rang(\eta_Q) \subseteq \lambda$". Why $\Vdash_Q$ sup $Rang(\eta_Q) = \lambda$? Because for every $(T, H) \in Q$ and $\alpha < \lambda$, letting $\eta =_{df} stam(\tilde{T})$, clearly $d(H(\eta)) \neq \varnothing \mod D$ (see Definition 5.12) but D is uniform, hence there is $\beta \in d(H(\eta))$, $\beta > \alpha$, so $\eta \,\widehat{}\, \langle \beta \rangle \in T$, and $(T, H) \leq (T, H)^{[\eta \,\widehat{}\, \langle \beta \rangle]} \in Q$, $(T, H)^{[\eta \,\widehat{}\, \langle \beta \rangle]} \Vdash_Q$ "$\eta \,\widehat{}\, \langle \beta \rangle \leq \eta_Q$" hence $(T, H)^{[\eta \,\widehat{}\, \langle \beta \rangle]} \Vdash$ "sup $Rang(\eta_Q) \geq \beta$", as $\alpha < \beta$ we finish.

Lemma 5.4   If S, γ, D *are as in Definition 5.1,* $\aleph_0 \notin S$, **F** *a winning strategy of player II in* $Gm^*(S, \gamma, D)$, cf $\gamma > \aleph_0$, *then* Q *satisfies* $(S, cf\, \gamma) - Pr_1$ *(see Definition 2.1).*

Proof: In Definition 2.1, parts (i), (ii), (iii), (iv), (vi) are clear. So let us check (v). Let $\kappa \in S$, $\underset{\sim}{\tau}$ be a Q-name, $\Vdash_Q$ "$\underset{\sim}{\tau} \in \kappa$" and $p = (T, H) \in Q$. We define by induction on n, $p_n = (T_n, H_n)$ such that:

(i) $p_0 = p$, $p_n \leq_0 p_{n+1}$, $T_n \cap {}^{n >}\lambda = T_{n+1} \cap {}^{n >}\lambda$

(ii) if $\eta \in T_n \cap {}^n\lambda$, and there are q, α satisfying

(*)   "$p_n^{[\eta]} \leq_0 q \in Q$, $\alpha < \kappa$, $q \Vdash$ "if $\kappa = 2$, $\underset{\sim}{\tau} = \alpha$, if $\kappa \geq \aleph_0$, $\underset{\sim}{\tau} < \alpha$"

then $p_{n+1}^{[\eta]}$, α_η satisfying this.

(iii) if $\eta \in T_{n+1} \cap {}^n\lambda$ and there are q, β satisfying

(*) $p_{n+1}^{[\eta]} \leq_0 q \in Q$, and for every r, $\beta < \kappa$,

$[q \leq_0 r \in Q \to \neg(\exists r_1)(r \leq r_1 \in Q \wedge r_1 \Vdash$ if $\kappa = 2$, $\underline{\tau} = \beta$, if $\kappa \geq \aleph_0$, $\underline{\tau} < \beta'']$

then $p_{n+1}^{[\eta]}$ satisfies (*).

Let p_ω be the limit of $\langle p_n: n < \omega \rangle$, i.e., $p_\omega = (T_\omega, H_\omega)$, $T_\omega = \bigcap_{n < \omega} T_n$, $H_\omega(\eta)$

is the limit of the sequences $H_n(\eta)$ (for $\eta \in T_\omega - \{stam(T) \restriction \ell: \ell\}$). It is well defined as $cf(\gamma) > \aleph_0$.

Now for each $\eta \in T_\omega$, $H_\omega(\eta)$ is a proper initial segment of a play of the game $Gm^*(S, \gamma, D)$, and it lasts $i^{H_\omega(\eta)}$ moves. Player I could choose in his $i^{H_\omega(\eta)}$-th move the cardinal κ and the function $f_\eta: \lambda \to \kappa$,

$$f_\eta(\zeta) = \begin{cases} \alpha_{\eta^\wedge\langle\zeta\rangle} & \text{if defined (which is } < \kappa) \\ 0 & \text{otherwise.} \end{cases}$$

So, for some β_η, $H_\omega(\eta) {}^\frown \langle \alpha, f_\eta, \beta_\eta \rangle$ is also a proper initial segment of a play of $Gm^*(S, \gamma, D)$ in which player II use the strategy **F**. So there is $p_{\omega+1} = (T_{\omega+1}, H_{\omega+1}) \in Q$, $p_\omega \leq_0 p_{\omega+1}$, and for each $\eta \in T_{\omega+1} - \{\nu: \nu < stam(T)\}$, $H_{\omega+1}(\eta) = H_\omega(\eta) {}^\frown \langle \kappa, f_\eta, \beta_\eta \rangle$.

We can easily show

Fact 5.4A: If $p = (T, H) \in Q$, $\kappa \in S$, $f: T \to \kappa$, *then* for some $p_1 = (T_1, H_1) \in Q$, $p \leq p_1$, and for every $\eta \in T_1$, $[\kappa = 2 \wedge f \restriction Suc_{T_1}(\eta)$ is constant$]$ or $[\kappa \geq \aleph_1 \wedge f \restriction Suc_{T_1}(\eta)$ is bounded below $\kappa]$.

[*Proof:* Define by induction r^n, $p \leq_0 r^n \leq_0 r^{n+1} \in Q$, r^{n+1} satisfies the conclusion of 5.4A for η of length n, now any $r^\omega \in Q$, $(\forall n) r^n \leq_0 r^\omega$ is as required].

Fact 5.4B: If $p = (T, H) \in Q$, $A \subseteq T$ then there is $p_1 = (T_1, H_1) \in Q$, $p \leq_0 p_1$ and for every $\eta \in T_1$, and $k < \omega$:

$$(\exists \nu \in A)\, [\nu \in T_1 \wedge \eta \leq \nu \wedge lg(f) = k] \to$$
$$(\forall q)\, [q \in Q \wedge p_1^{[\eta]} \leq_0 q \to (\exists \nu \in A)(\nu \in q \wedge \eta \leq \nu \wedge lg(\nu) = k)]$$

[*Proof:* Define by induction on n r^n, $p \leq_0 r^n \leq_0 r^{n+1} \in Q$, r^{n+1} satisfies the conclusion of 5.4B for η of length $\leq n$ and $k \leq n$. Now any $r^\omega \in Q$, $(\forall n) r^n \leq_0 r^\omega$ is as required.]

Let $A = \{\eta \in T_{\omega+1}: \alpha_\eta$ well defined$\}$, and let q, $p_{\omega+1} \leq q \in Q$ be as in 5.4B. Now for every $\eta \in T^q$ there is $r \in Q$, $q^{[\eta]} \leq r$, and r force a value for $\underline{\tau}$. So $stam(r) \in A$ (as $p_\omega \leq q$, see the definition of the p_η's), and $p_\omega^{[stam\, r]}$ force a value to $\underline{\tau}$; hence, $q^{[stam\, r]}$ does, and let k_η be $lg(stam\, r)$ for such r with minimal $lg(stam(r))$. So by 5.4B,

(*)   For every $\eta \in T^q$, and r, $q^{[\eta]} \leq_0 r \in Q$, for some $\nu \in q^{[\eta]}$, $\eta \leq \nu$, $lg(\nu) = k_\eta$, and $\nu \in A$.

Now for each q_1, $q \leq_0 q_1 \in Q$, $\eta \in T^{q_1}$ we can, by k_η applications of 5.4A, get an ordinal $\alpha < \kappa$ and q_2, $q_1^{[\eta]} \leq_0 q_2$, and

(*) $(\forall q_3 \in Q)\, [q_2 \leq_0 q_3 \to (\exists \nu \in A)(\nu \in T^{q_3} \wedge lg(\nu) = k_\eta \wedge \alpha_\nu \leq \alpha)]$ (or if $\kappa = 2$, $\alpha_\nu = \alpha$).

But this shows that β_η is defined for every $\eta \in T^q$. Finishing alternatively by repeated application of 5.4A we can define by induction on n, $q(n) \in Q$, $q(0) = q$, $q(n) \leq_0 q(n+1)$ and β_η^n for $\eta \in T^{q(n)}$ such that:

(a) $\beta_\eta^0 = \beta_\eta$

(b) when $\kappa \geq \aleph_0$: $\eta \frown \langle \zeta \rangle \in T_{n+1} \Rightarrow \beta_\eta^{n+1} \geq \beta_{\eta \frown \langle \zeta \rangle}^n$

(c) when $\kappa = 2$: $\eta \frown \langle \zeta \rangle \in T_{n+1} \Rightarrow \beta_\eta^{n+1} = \beta_{\eta \frown \langle \zeta \rangle}^n$.

Let $q_\omega \in Q$ be such that $q_n \leq_0 q_\omega$ for $n < \omega$.

Now if $\kappa > \aleph_0$ (is regular), we claim

$$q_\omega \Vdash_Q \tau \leq \bigcup_{n < \omega} \beta_{\langle \rangle}^n$$

Clearly $p \leq_0 q_\omega \in Q$, $\bigcup_{n < \omega} \beta_{\langle \rangle}^n < \kappa$ so this suffices. Why does this hold? If not, then for some q', $q_\omega \leq q' \in Q$, $q' \Vdash_Q$ "$\tau \geq \bigcup_n \beta_{\langle \rangle}^n$. Let $\eta = stam(q')$, so $\eta \in T^q$, and $\alpha_\eta \omega$ is well defined, and as $p_\omega^{[\eta]} \leq_0 (q')^{[\eta]}$, $\alpha_\eta > \bigcup_n \beta_{\langle \rangle}^n$. But as $\eta \in \bigcap_{n < \omega} T^{q(n)}$, $\beta_{\langle \rangle}^{lg(n)} \geq \beta_\eta$, and we get a contradiction.

If $\kappa = 2$, we note just that if $\eta \in T^{q(1)}$, $\beta_\eta = \beta_\eta^0 = \beta_\eta^1$.

Lemma 5.5   *Suppose $\bar{Q} = \langle P_i, Q_i : i < \lambda \rangle$ is a κ-Sp_2-iteration, $|P_i| < \lambda$ for $i < \lambda$, each Q_i has $(S, < \kappa) - Pr_1$ and $(S, \sigma) - Pr_1$ $\sigma \leq \kappa$ regular, $S \subseteq \{2\} \cup \{\theta : \theta$ regular uncountable $\leq \kappa\}$ and in V, D is a normal ultrafilter on λ (so λ is a measurable cardinal). Then $\Vdash_{P_\lambda}$ "player II wins $Gm^*(S, \kappa, D)$".*

Remark: Also for κ-Sp_3.

Proof: Let $A = \{\mu < \lambda : (\forall i < \mu)|P_i| < \mu, \mu$ strongly inaccessible $> \kappa\}$.

Let $G_\lambda \subseteq P_\lambda$ be generic over V, $G_\alpha = G \cap P_\alpha$.

W.l.o.g. player I choose P_λ-names of functions and cardinals in S. Now we work in V and describe player II's strategy there. For each $\mu \in A$ the forcing notion P_λ / P_μ has $(S, \sigma) - Pr_2$; hence, player II has a winning strategy $F(P_\lambda / G_\mu) \in V[G_\kappa]$, so $F(P_\lambda / G_\mu)$ is a P_κ-name, $\langle F(P_\lambda / G_\mu) : \mu \rangle$ a P_λ-name. Let us describe a winning strategy for player II.

So in the ith move player I chooses $\theta_i \in S$ and $f_i : \lambda \to \theta_i$. Player II chooses in his i-th move not only $\alpha_i < \theta_i$ but also $A_i, f_i, \gamma_i, \langle \langle p_j^\mu : j \leq i \rangle : \mu \in A_i \rangle$ such that γ_i is an ordinal $< \lambda$,

(1) $j < i \Rightarrow \gamma_j < \gamma_i$.

(2) $A_i \in D$, $A_i \in V$, $A_i \subseteq \bigcap_{j < i} A_j$ and $A_\delta = \bigcap_{j < \delta} A_j$

(3) $\Vdash$ "$f_i : \lambda \to \theta_i$, $\theta_i \in S$".

(4) for $\mu \in A_i$,

$$\langle p_j^\mu : j \leq 2i + 2 \rangle$$

is a P_κ-name of an initial segment of a play as in (vi) of 2.1, for the forcing P_λ / G_κ, $p_{2j+1}^\mu \Vdash_{P_\lambda / G_\mu}$ "$f_i(\mu) = \alpha_i$ if $\theta_i = 2$, $f_i(\mu) < \alpha_i^\mu$ if $\theta_i \geq \aleph_0$", α_i^μ a P_{α_i}-name.

In the i-th stage clearly $A_i^0 =_{df} \bigcap_{j < i} A_j \cap A$ is in D, and let $\gamma_i^0 = \sup_{j < i} \gamma_j$, so $\gamma_i^0 < \lambda$ and choose $\gamma_\mu^1 \in (\gamma_\mu^0, \lambda)$ such that θ_i is a $P_{\gamma_\mu^1}$-name. For every $\mu \in A$, $\mu > \gamma'$, we can define P_μ-names p_{2i}^μ, p_{2i+1}^μ, α_i^μ such that:

(a) $\Vdash_{P_\mu}$ "$\langle p_j^\mu : j < 2i + 2\rangle$ is an initial segment of a play as in (v) of 2.1 for P_λ / P_μ in which player II uses his winning strategy $F(P_\lambda / G_\mu)$.

(b) $p_{2i+1}^\mu \Vdash_{P_\lambda / P_\mu}$ "$f_i(\mu) = \alpha_i^\mu$ if $\theta_i = 2$, $f_i(\mu) < \alpha_i^\mu$ if $\theta_i \geq \aleph_0$".

Now α_i^μ is a P_μ-name of an original $< \kappa \leq \mu$, it is $P_{\beta[\mu]}$-name for some $\beta[\mu] < \mu$ (as P_μ satisfies the μ-c.c. see 2.x). By the normality of the ultrafilter D, on some $A_i^1 \subseteq A_i^0$, $\beta[\mu] = \beta_i$ for every $\mu \in A_i^1$. Let $\gamma_i = \gamma_n^1 + \beta_i$.

Easily for each $i < \sigma$, $\Vdash_{P_\lambda}$ "$\{\mu \in A_i : p_{2i+1}^\mu \in G_\lambda\} \neq \varnothing \ mod \ D$", so we finish.

Now we can solve the second Abraham problem.

Conclusion 5.6: Suppose λ is strongly inaccessible $\{\mu < \lambda : \mu$ measurable$\}$ is stationary, $\kappa < \lambda$, $S \subseteq \{2\} \cup \{\theta : \theta \leq \kappa$ regular uncountable$\}$. Then for some forcing notion P: $|P| = \lambda$, P satisfies λ-c.c. and $(S, < \kappa) - Pr_1$ (and $(S, \kappa) - Pr_1$, if we want), and $\Vdash_P$ "$\lambda = |\kappa|^+$" (so $\Vdash_{P_\lambda} 2^{|\kappa|} = \lambda$) in V^P: and: for every $A \subseteq \lambda$, for some $\delta < \lambda$, there is a countable set $\alpha \subseteq \delta$, which is not in $V[A \cap \delta]$, we can also get suitable axiom (see 3.5).

Remark 5.6A: We can also prove (by the same forcing) the consistency of $D_\lambda + \{\delta < \lambda : cf \ \delta = \aleph_0\}$ is precipitous: if in addition there is a normal ultrafilter on λ concentrates on measurables.

REFERENCES

[1] Gitik, M., "Changing cofinalities and the non-stationary ideal," a preprint, first version, Fall, 1984.

[2] Groszek, M. and T. Jech, announcement.

[3] Martin, D. and R. M. Solovay, "Internal Cohen extensions," *Annals of Mathematical Logic*, vol. 2 (1970), pp. 143–173.

[4] Shelah, S., "Proper Forcing," Springer Verlag Lecture Notes, vol. 840 (1982).

[5] Shelah, S., "On cardinal invariants of the continuum," *Proceedings of the 6/83 Boulder Conference in Set Theory*, eds. J. Baumgartner, D. Martin and S. Shelah, *Contemporary Mathematics*, vol. 31 (1984), pp. 183–207.

[6] Shelah, S., Abstracts of AMS (1984).

[7] Solovay, R. M. and Tenenbaum, "Iterated Cohen extensions and Souslin's problem," *Annals of Mathematics*, vol. 94 (1971), pp. 201–245.

Institute of Mathematics
The Hebrew University
Jerusalem, Israel

and

Departments of Mathematics and Computer Science
University of Michigan
Ann Arbor, MI 48109