ISRAEL JOURNAL OF MATHEMATICS, Vol. 60, No. 2, 1987

ON RECONSTRUCTING SEPARABLE REDUCED p-GROUPS WITH A GIVEN SOCLE†

BY
SAHARON SHELAH
The Hebrew University of Jerusalem, Jerusalem, Israel; and Simon Fraser University, Burnaby, British Columbia, Canada

ABSTRACT

Let $\bar{B}^*$ be a separable reduced (abelian) p-group which is torsion complete. We ask whether for $G \subseteq_{\mathrm{pr}} \bar{B}^*$ there is $H \subseteq_{\mathrm{pr}} \bar{B}^*$, $H[p] = G[p]$, H not isomorphic to G. If G is the sum of cyclic groups or is torsion complete, the answer is easily no. For other G, we prove that the answer is yes assuming G.C.H. Even without G.C.H. the answer is yes if the density character of G is equal to $\mathrm{Min}_{n<\omega} | p^n G |$, i.e.,

$$\operatorname*{Min}_{n<\omega} | p^n G | = \operatorname*{Min}_{m} \sum_{n>m} |(p^n G)[p]/(p^{n+1}G)[p]|.$$

Of course, instead of two non-isomorphic we can get many, but we do not deal much with this.

NOTATION. A group will mean here an abelian group.

We assume knowledge on separable reduced p-groups from Fuchs [F].

GROUP THEORETIC NOTATION

p	a fixed prime (natural number)
B^*	a fixed p-group which is the sum of cyclic groups
$\bar{B}^*$	the torsion completion of B^*
A, B, C, G, H	subgroups of $\bar{B}^*$
$G \subseteq H$	G is a subgroup of H

† Partially supported by NSERC Grant A3040.

Part A arose from the conference in Udine, Italy, April 1984, in answer to a question of Cutler. Part B was written in Vancouver, summer 1985. For complementary consistency results, see Mekler and Shelah [Mk–Sh].

Received September 15, 1986

$G \subseteq_{\mathrm{pr}} H$	G is a pure subgroup of H
$G[p]$	(the socle of G) is $\{x \in G : px = 0\}$
x, y, z	elements of $\bar{B}^*$
$\langle A, B\rangle^g$	the subgroup of $\bar{B}^*$ which $A \cup B$ generates
	We usually say "sum of ... groups" instead of "direct sum of . . . groups".
$A + B$	$\langle A, B\rangle^g$
$A \oplus B$	is the direct sum

Explanation of Proof. We have two kinds of H (st. psf and direct); they usually exist and (except for the sum of cyclics) are contradictory. However, each has various obvious variants (and we can mix them, e.g., in order to get many non-isomorphic H's).

OTHER NOTATION

a, b, c, d, e	integers
n, m, k, l, r	natural numbers
$i, j, \alpha, \beta, \gamma, \delta, \varepsilon$	ordinals
$\lambda, \mu, \kappa, \chi$	cardinals
$\langle x_0, \ldots, x_n\rangle$	means a sequence
$H(\lambda)$	the family of sets with transitive closure of cardinality $<\lambda$; we do not distinguish strictly between this set and the model $(H(\lambda), \varepsilon)$
$\prec$	elementary submodel
lg	"the length of"

Part A

Section 1

1.1. DEFINITION. (1) For $A \subseteq \bar{B}^*$, $\lambda_n(A)$ is the dimension of $p^nA/p^{n+1}A$ as a vector space over $\mathbf{Z}/p\mathbf{Z}$. Similarly for groups isomorphic to such A.

(2) $\lambda^*(A) = \mathrm{Min}_n \sum_{n<m<\omega} \lambda_m(A)$.

(3) We call A wide if $|A| + \aleph_0 = \lambda^*(A) + \aleph_0$.

(4) We say (B, A) is wide if: $A + B/A$ is wide and $A \subseteq_{\mathrm{pr}} A + B$.

(5) We say (B, A) is very wide if there is $C = \langle t_i^m : i < \lambda,\ n < \omega\rangle$ and $B_1, A \subseteq B_1 \subseteq B + A$, s.t. A, B_1 are pure subgroups of $B + A$, $p^{n+1}t_i^n = 0 \neq p^n t_i^n$, $\{t_i^n : n, i\}$ free (see 1.2), and $B = B_1 \oplus C$ and $(\exists\, m)\lambda \geqq |p^m(B_1/A)|$. We say A is very wide if $(A, \{0\})$ is.

1.2. Definition. (1) A subset of A, $\{x_i : i<\alpha\}$ is free if $\Sigma e_i x_i = 0 \Rightarrow \bigwedge_i e_i x_i = 0$.

(2) A basis of A is a maximal free subset of $A - pA$ (it is a basis of a subgroup of A which is dense, and called basic). But for $A \subseteq \bar{B}^*[p]$ we sometimes use a basis of A as a vector space over $\mathbf{Z}/p\mathbf{Z}$.

(3) $\mathrm{ht}_A(x) = \sup\{n : (\exists y \in A) p^n y = x\}$; if $A = \bar{B}^*$ we omit it. The function ht_A defines a norm on A; we know (see [F]): $\bar{B}^*$ is the closure of any basic subgroup. Note that $A \subseteq_{\mathrm{pr}} \mathrm{B} = \mathrm{ht}_A \leqq \mathrm{ht}_B$.

(4) $A^{\mathrm{cl}}_B = \mathrm{cl}_B(A) = \{x \in B : \bigwedge_n (\exists y \in A) \mathrm{ht}(x-y) \geqq n\}$ (when $A \subseteq B$). If $B = \bar{B}^*$ we omit it. If $X \subseteq B$, $X^{\mathrm{cl}}_B = (\langle X \rangle^{\mathrm{g}})^{\mathrm{cl}}_B$.

1.3. Theorem (see, e.g., Fuchs [F]).

(1) *Every A has a basis.*

(2) *A bounded pure subgroup of A is a direct summand of A.*

1.4. Fact. Suppose $\bar{B}^* \in H(\lambda)$. If $B \subseteq_{\mathrm{pr}} A + B$, $N \prec (H(\lambda), \in)$, $A \in N$, $B \in N$ *then* $B \subseteq_{\mathrm{pr}} B + A \cap N \subseteq_{\mathrm{pr}} B + A$.

1.5. Definition. Let $\{t_i : i < i(*)\} \subseteq A$, $x \in A$, $a_i \in \mathbf{Z}$, we say $x = \Sigma_{i<i(*)} a_i t_i$ if, for every m, $\{i : \mathrm{ht}_A(a_i t_i) < m\}$ is finite and $\mathrm{ht}_A(x - \Sigma\{a_i t_i : \mathrm{ht}_A(a_i t_i) \leqq m\}) \geqq m$.

1.6. Claim. (1) If $\{t_i : i < i(*)\} \subseteq A$ then $\Sigma_{i<i(*)} a_i t_i$ (in A) has at most one value, and if $A \subseteq_{\mathrm{cl}} \bar{B}^*$ [see 2.1(2)] and $\Sigma_i a_i t_i$ satisfies the condition above then the sum has exactly one value.

(2) If $\{t_i : i < i(*)\}$ is independent in A, each x has at most one representation.

(3) If $\{t_i : i < i(*)\}$ is the basis of A *then* each $x \in A$ has one and only one representation, the canonical one.

Section 2

2.1. Definition. (1) $B \subseteq_{\mathrm{sp}} A$ means $B \subseteq_{\mathrm{pr}} A$ and $(B)^{\mathrm{cl}}_A \subseteq B + A[p]$.

(2) $B \subseteq_{\mathrm{cl}} A$ means $B \subseteq_{\mathrm{pr}} A$ and $(B)^{\mathrm{cl}}_A = B$.

2.2. Fact. (1) If $B \subseteq_{\mathrm{pr}} A \subseteq_{\mathrm{pr}} \bar{B}^*$ *then*:

$$B \subseteq_{\mathrm{sp}} A \quad \text{iff } B^{\mathrm{cl}} \cap A \subseteq B + A[p].$$

(2) If $B \subseteq_{\mathrm{cl}} A$ then A/B can be embedded into $\bar{B}^*$ (so we can apply to it appropriate properties).

(3) $\subseteq_{\mathrm{pr}}$ is transitive as well as $\subseteq_{\mathrm{cl}}$.

2.3. Claim. (1) If A_i is $\subseteq_{\mathrm{pr}}$-increasing (for $i < \alpha$), $A_0 \subseteq_{\mathrm{sp}} A_i$ for each i, then $A_0 \subseteq_{\mathrm{sp}} \bigcup_{i<\alpha} A_i$.

(2) For every A, $\{0\} \subseteq_{\mathrm{sp}} A$.

(3) For every $A, A \subseteq_{\mathrm{sp}} A$.

(4) If $A \subseteq_{\mathrm{sp}} C, A \subseteq B \subseteq C$ *then* $A \subseteq_{\mathrm{sp}} B$.

(5) $A \subseteq_{\mathrm{cl}} B \Rightarrow A \subseteq_{\mathrm{sp}} B \Rightarrow A \subseteq_{\mathrm{pr}} B$.

(6) $A \subseteq_{\mathrm{sp}} B \subseteq_{\mathrm{cl}} C$ then $A \subseteq_{\mathrm{sp}} C$.

2.4. Definition. (1) We call $A \subseteq \bar{B}^*$ st. psf. (strongly pseudo-free) *if*, when for every λ large enough (so $\bar{B}^* \in H(\lambda)$) for some $\bar{x} \in H(\lambda)$), *if* $k < \omega$, $N_0, N_1, \ldots, N_{k-1}$ are elementary submodels of $H(\lambda)$, $\bar{x}$ belongs to each N_l, $\bigwedge_{l<m<k} N_l \in N_m$ *then* $(\bigcup_{l<k}(N_l \cap A)) \subseteq_{\mathrm{sp}} A$.

(2) For $B \subseteq A + B \subseteq \bar{B}^*$ we define "(A, B) is st. psf." similarly only in the end

$$B + \left(\bigcup_{l<k} N_l \cap A \right) \subseteq_{\mathrm{sp}} B + A.$$

2.5. Remark. $(A, \{0\})$ is st. psf. iff A is st. psf.

2.6. Lemma. *Suppose* $G \subseteq_{\mathrm{pr}} \bar{B}^*$, *and G is very wide. Then there is H such that*: $H \subseteq_{\mathrm{pr}} \bar{B}^*$, $H[p] = G[p]$ *and is* st. psf.

2.6A. Remark. We can have (H, B) very wide, $B \subseteq_{\mathrm{cl}} G$ and get $H \subseteq_{\mathrm{pr}} \bar{B}^*$, $H[p] = G[p]$, $B \subseteq H$, and (H, B) is st. psf.

Pf: So $G = B_1 \oplus B_2 \oplus B_3$, B_3 bounded,

$$B_2 = \bigoplus_{\substack{i<\lambda \\ n<\omega}} \langle s_i^n \rangle^g,$$

$\langle s_i^n \rangle^g$ cyclic of order p^{n+1}, and $|B_1| \leqq \lambda$. We can forget B_3 for notational simplicity.

Let $\{t_i^n : n < \omega, i < \lambda_n\}$ be a basis of B_1 ($p^{n+1} t_i^n = 0 \neq p^n t_i^n$). Choose a basis I for $B_1[p]$ extending $\{p^n t_i^n : n < \omega, i < \lambda_n\}$ (as a vector space over $\mathbf{Z}/p\mathbf{Z}$),

$$I = \left\{ \sum_{(n,i)\in w_\alpha} a^\alpha_{(n,i)} p^n t_i^n : \alpha < \alpha(*) \right\} \cup \{ p^n t_i^n : n < \omega, i < \lambda_n \}$$

and $\alpha(*) \leqq \lambda$.

We now define H:

$$H = \langle t_i^n : n, i\rangle^{g} + \langle s_i^n : i < \lambda, n < \omega\rangle^{g}$$

$$+ \Big\langle \sum_{\substack{(n,i)\in w_\alpha \\ n \geq m}} a^\alpha_{(n,i)}\, p^{n-m} t_i^n + \sum_{n\geq m} p^{n-m+1} s_\alpha^n : n < \omega, \alpha < \alpha(*)\Big\rangle^{g}.$$

So $\{t_i^n, s_i^n : n, i\}$ is a basis of H. Clearly H is as required, but we shall check. We leave "$H[p] = G[p]$, $H \subseteq_{\mathrm{pr}} \bar{B}^*$" to the reader.

Let μ be regular large enough, so $\bar{B}^*$, B^*, B_1, B_2, B_3, B and $\langle t_i^n : n, i\rangle$, $\langle s_i^n : n, i\rangle$ belong to $\boldsymbol{H}(\mu)$. Let

$$\bar{V} = \Big\langle \bar{B}^*, G, H, \langle t_i^n : n, i\rangle, \langle s_i^n : n, i\rangle,$$

$$\Big\langle\Big\langle \sum_{(n,i)\in w_\alpha} a^\alpha_{(n,i)}\, p^{n-m} t_i^n : \alpha\langle\alpha(*)\rangle : m < \omega\Big\rangle, B_1, B_2, B_3\Big\rangle.$$

Suppose $k < \omega$, for $l < k$ $N_l \prec (\boldsymbol{H}(\mu), \in)$, $\bar{V} \in N_l$, and $N_l \in N_m$ for $l < m < k$. We shall show

$$(*) \qquad \Big\langle \bigcup_{l<k} (H \cap N_l)\Big\rangle^{g} \subseteq_{\mathrm{sp}} H.$$

The purity is easy: use 1.4(1) (inductively on k). So suppose $x \in H$, so there are m, $\alpha_0 < \cdots < \alpha_{r_0 - 1} < \alpha(*)$, $r_0, r_1, r_2 < \omega$ and $b_q^l \in \mathbf{Z}$ such that:

$$x = \sum_{q<r_0} b_q^0 \Big(\sum_{(n,i)\in w_{\alpha_q}} a^\alpha_{(n,i)}\, p^{n-m} t_i^n + \sum_{n\geq m} p^{n-m+1} s^n_{\alpha_q}\Big) + \sum_{q<r_1} b_q^1 t_{i_q}^{n^1(q)} + \sum_{q<r_2} b_q^2 s_{j_q}^{n^2(q)}$$

as we can increase m, w.l.o.g. $(n^1(q_1), i_{q_1}) \notin w_{\alpha_{q_0}}$, $(n^2(q_2), j_{q_2}) \notin w_{\alpha_{q_0}}$ for any $q_0 < r_0$, $q_1 < r_1$, $q_2 < r_2$.

Let $x \in (\bigcup_{l<k}(H \cap N_l))^{\mathrm{cl}}$. We want to prove $x \in (\bigcup_{l<k}(H \cap N_l)) + H[p]$. We can replace x by $x - x'$ if $x' \in \langle \bigcup_{l<k}(H \cap N_l)\rangle^{g}$.† So w.l.o.g. $\alpha_q \notin \bigcup_{l<k} N_l$.

† As $\bar{V} \in N_l$, and as obviously $\{m : m < \omega\} \subseteq N$, for $i < \lambda$ clearly $[i \in N \Rightarrow s_i^n \in N_l]$, $[i \in N \Rightarrow t_i^n \in N_l]$ and if $\alpha < \alpha(*)$,

$$\Big[\alpha \in N_l \Rightarrow \sum_{\substack{(n,i)\in w_\alpha \\ n\geq m}} a^\alpha_{(n,i)}\, p^{n-m} t_i^n \in N_l\Big]$$

and even $[i \in N_l \cap \lambda \Rightarrow \langle s_i^n : n < \omega\rangle \in N_l]$ hence $[i \in N_l \cap \lambda \Rightarrow \Sigma_{n\geq m}\, p^{n-m+1} s_i^n \in N_l]$. Of course $[z \in N_l \cap H, b \in \mathbf{Z} \Rightarrow bz \in N_l \cap H]$.

Also $i_q \notin \bigcup_{l<k} N_l$, $j_q \notin \bigcup_{l<k} N_l$. So necessarily $r_1 = 0 = r_2$,† p^m divides b_q^0 (for $q < r_0$);‡ so $x \in B[p]$ and we finish.

2.7. Lemma. *Suppose $B \subseteq_{\mathrm{cl}} G$, $G \subseteq_{\mathrm{pr}} \bar{B}^*$, and (G, B) is wide and for no $C \subseteq G$, $|C| < \lambda^*(G)$, is $G/(B + C)_G^{\mathrm{cl}}$ torsion complete.*

Then *there is $H \subseteq_{\mathrm{pr}} \bar{B}^*$, $B \subseteq H$, $H[p] = G[p]$, and (H, B) is* st. psf.

Proof. Let $\{t_i^n : i < \xi_n, n < \omega\}$ be a basis of G s.t. $\langle t_i^n : i < \zeta_n, n < \omega \rangle$ is the basis of B (so

$$B \oplus \bigoplus_{\substack{n<\omega \\ \zeta_n \leqq i < \xi_n}} \langle t_i^n \rangle^g$$

exists and is $\subseteq_{\mathrm{pr}} G$, $p_i^{n+1} t_i^n = 0 \neq p^n t_i^n$). Let $\lambda_n = |\xi_n - \zeta_n|$, $\lambda(*) = \Sigma_{n \geqq m} \lambda_n$ for every m large enough. But for every m

$$(\exists G')\left[B \subseteq G' \subseteq G \wedge G = G' \oplus \bigoplus_{\substack{n \geqq m \\ \zeta_n \leqq i < \xi_n}} \langle t_i^n \rangle\right]$$

so w.l.o.g. $\lambda(*) = \Sigma_{n<\omega} \lambda_n$.

Let $\{t_i^n : \zeta_n \leqq i < \xi_n, n < \omega\} \cup \{\Sigma_{(n,i)} a_{(n,i)}^\alpha p^n t_i^n : \alpha < \alpha(*) \leqq \lambda(*)\}$ be a basis of $G[p]$ over $B[p]$ (as vector spaces over $\mathbf{Z}/p\mathbf{Z}$) (so $a_{(n,i)}^\alpha \in \mathbf{Z}$, $w_\alpha \stackrel{\mathrm{def}}{=} \{(n, i) : a_{(n,i)}^\alpha \neq 0\}$ countable etc.) w.l.o.g. $0 \leqq a_{(n,i)}^\alpha < p$.

Let, for $z = \Sigma a_{(n,i)} t_i^n \in G^{\mathrm{cl}}$, dom $z = \{t_i^n : a_{(n,i)} t_i^n \neq 0\}$. We define, by induction on $\alpha < \alpha(*)$, H_α, W_α, y_α^n $(n < \omega)$, w_α, v_α s.t.

(a) H_α is increasing continuous,

(b) $B \subseteq H_\alpha \subseteq_{\mathrm{pr}} \bar{B}^*$,

† As $x \in (\bigcup_l (H \cap N_l))^{\mathrm{cl}}$ and the w.l.o.g. above and as $N_l \cap H \subseteq (\langle t_i^n : t_i^n \in N_l \rangle^g)_H^{\mathrm{cl}}$, clearly (by the w.l.o.g. above) $t_{i_q}^{n^1(q)} \in \bigcup N_l$, $t_{j_q}^{n^2(q)} \in \bigcup N_l$ for $q < r_1$, $q < r_2$ resp. as $\{t_i^n, s_\alpha^m : n, m < \omega, i < \lambda_n, \alpha < \lambda\}$ is a basis of H.

‡ Suppose p^m does not divides $b_{q_0}^0$, then $b_q^0 p^{n-m+1} s_{\alpha_{q_0}}^n \neq 0$. By the choice of $\langle s_\alpha^m : m < \omega, \alpha < \lambda \rangle$, $s_{\alpha_{q_0}}^n$ (for $n \geqq m$) does not appear anywhere else and is

$$\sum_{q<r_0} b_q^0 \left(\sum_{(m,i) \in w_q, n \geqq m} a_{(n,i)}^\alpha p^{n-m} t_i^n + \sum_{n \geqq m} p_{w_q}^{n-m+1\,n} \right),$$

hence appears in the cannonical expression for x. Let us choose $m(*) < \omega$ (so that $m(*) > m$, and $p^{m(*)} B_3 = 0$). So there is $x^* \in (\bigcup_l (H \cap N_l))^{\mathrm{cl}}$, $x - x^*$ divisible by $p^{m(*)}$. So $s_{\alpha_q}^{m(*)}$ appear in the canonical representation of x^* by the basis $\{t_i^n : s_i^n : n, i\}$. But $x^* = \Sigma x_l$, $x_l \in H \cap^0 N_l$, each x_l has a representation by $\{t_i^n, s_i^n : n, i\} \cap N_l$. So necessarily $s_{\alpha_{q_0}}^{m(*)}$ belongs to some N_l, hence $\alpha_{q_0} \in N_l$ for some l, contradiction.

(c) $H_\alpha = \langle B \cup \{t_i^n : t_i^n \in W_\alpha\} \cup \{y_\alpha^n : n < \omega, \beta < \alpha\}\rangle^g$,

(d) W_α is increasing continuous, $|W_{\alpha+1} - W_\alpha| \leqq \aleph_0$, $W_\alpha \subseteq \{t_i^n : n < \omega, i < \xi_n\}$,

(e) $\Sigma_{(n,i)\in w_\alpha} a_{(n,i)}^\alpha p^n t_i^n \in H_{\alpha+1}$,

(f) $H_\alpha[p] \subseteq G$,

(g) $W_0 = \{t_i^n : n < \omega, i < \zeta_n\}$,

(h) $\operatorname{dom} y_\alpha^n \subseteq W_{\alpha+1}$,

(i) $y_\alpha^m = \Sigma_{(n,i)\in w_\alpha} a_{(n,i)}^\alpha p^{n-m} t_i^n$ for $m = 0$,

(j) $p y_\alpha^{n+1} - y_\alpha^n \in \langle t_i^n : (n,i) \in W_{\alpha+1}\rangle^g$,

(k) for $n > 0$, $\operatorname{dom} y_\alpha^n - W_\alpha$ is infinite,

(l) for $n > 0$, $y_\alpha^n \notin \langle B, \langle t_i^n : t_i^n \in W_\alpha^1\rangle^g\rangle^g + G$.

For $\alpha = 0$: $H_\alpha = B$, $W_\alpha = \{t_i^n : n < \omega, i < \zeta_n\}$.

For α *limit*: $H_\alpha = \bigcup_{\beta<\alpha} H_\beta$, $W_\alpha = \bigcup_{\beta<\alpha} W_\beta$.

For $\alpha + 1$: Let $W'_\alpha = W_\alpha \cup \{t_i^n : (n,i) \in w_\alpha\}$. By hypothesis $G/(B + \langle t_i^n : t_i^n \in W_\alpha\rangle)_G^{\mathrm{cl}}$ is not torsion complete.

So there is a countable $v_\alpha \subseteq \{(n,i) : n < \omega, i < \xi_n\}$ and b_i^n, $0 \leqq b_i^n < p$ (for $(n,i) \in v_\alpha$), such that:

$$\sum_{(n,i)\in v_\alpha} b_i^n p^n t_i^n \notin (\langle B + \{t_i^n : t_i^n \in W'_\alpha\}\rangle^g)^{\mathrm{cl}} + G$$

(and is well defined). W.lo.g. v_α is disjoint to W'_α.

Let

$$y_\alpha^m = \sum_{\substack{n \geqq m \\ (n,i)\in v_\alpha}} b_i^n p^{n-m+1} t_i^n + \sum_{\substack{(n,i)\in w_\alpha \\ n \geqq m}} a_i^n p^{n-m} t_i^n$$

and

$$W_{\alpha+1} = \{t_i^n : (n,i) \in v_\alpha\} \cup W_\alpha.$$

It is easy to check that this works, $H \subseteq_{\mathrm{pr}} \bar{B}^*$ and $H[p] = G[p]$. Let us show that H is st. psf.

Suppose $k < \omega$, μ regular large enough, for $l < k$, $N_l \prec (H(\mu), \in)$, $N_l \in N_m$ for $l < m < k$ and B_1, B_2, B_3, B, G, $\langle t_i^n : n, i < \xi_n, n < \omega\rangle$, $\langle \Sigma_{(n,i)\in w_\alpha} a_{(n,i)}^\alpha p^{n-m} t_i^n : \alpha < \alpha(*)\rangle$, $\langle \Sigma_{(n,i)\in v_\alpha} b_{(n,i)}^\alpha p^{n-m} t_i^n : \alpha < \alpha(*)\rangle$, etc. belongs to each N_l.

We want

$$(*) \qquad \left\langle B \cup \bigcup_{l<k} (H \cap N_l)\right\rangle^g \subseteq_{\mathrm{sp}} H.$$

The purity is easy: use 1.4(1).

Suppose $x \in \langle B \cup \bigcup_{l<k} H \cap N_l \rangle_H^{\mathrm{cl}}$ ($\subseteq H$). So let, for some m,

$$x = y + \sum_{q<r_0} c_q y_{\alpha_q}^m + \sum_{q<r_1} c^q t_{i_q}^{n[q]}$$

(where $r_0, r_1 < \omega$, c_q, $c^q \in \mathbf{Z}$, $i_q = i(q)$ and $y \in B$), w.l.o.g. $t_{i_{q_1}}^{n(q_1)} \notin \operatorname{dom} y_q^m$ and $i_{q_1} \geqq \zeta$ for $q < r_0$, $q_1 < r_1$ (as we can increase m).

We want to show

$$x \in \left\langle B \cup \bigcup_{l<k} (H \cap N_l) \right\rangle^{g} + H[p]$$

so we can replace x by $x - x'$ if $x' \in B \cup \bigcup_{l<k} (H \cap N_l)$. So $i_q \in N_l \Rightarrow (n(q), i_q) \in N_l \Rightarrow t_{i_q}^{n(q)} \in H \cap N_l \Rightarrow$ we can replace x by $x - c^q t_{i(q)}^{n(q)}$. So w.l.o.g. for $q < r_1$, $i_q \notin N_l$. However for any $z \in \bar{B}^*$

$$z \in B \cup \bigcup_l (H \cap N_l) \Rightarrow \operatorname{dom} z \subseteq \left\{ t_i^n : i < \zeta_n \text{ or } i < \xi_n \text{ and } i \in \bigcup_{l<k} N_l \right\}$$

hence $z \in \langle B \cup \bigcup_l H \cap N_l \rangle_G^{\mathrm{cl}} \Rightarrow \operatorname{dom} z \subseteq \{ t_i^n : i < \zeta_n \text{ or } i < \xi_n,\ i \in \bigcup N_l \}$. We can assume $\bigwedge_{q<r_1} n(q) < m$ (as we can increase m).

So as $x \in \langle B \cup \bigcup_l H \cap N_l \rangle_G^{\mathrm{cl}}$, and $t_{i_q}^{n(q)} \in \operatorname{dom} x$, and $i_q \notin \bigcup_l N_l$ necessarily $c^q t_{i_q}^{n(q)} = 0$, so really $r_1 = 0$.

Also if $\alpha_q \in N_l$, $y_{\alpha_q}^m \in N_l \cap G$, so we can replace x by $x - c_q y_{\alpha_q}^m$. So w.l.o.g. $\bigwedge_{q<r_0} (\alpha_q \notin N_l)$.

If there is q s.t. $p c_q y_{\alpha_q}^m \neq 0$, w.l.o.g. $\alpha_0 < \alpha_1 \cdots < \alpha_{r_0 - 1}$, and let $q = q(*)$ be a maximal s.t. $p c_q y_{\alpha_q}^m \neq 0$.

So $q(*) < q < r_0 \Rightarrow p c_q y_{\alpha_q}^m = 0 \Rightarrow c_q y_{\alpha_q}^m \in H[p] = G[p]$. As $\langle v_\alpha : \alpha < \alpha(*) \rangle \in N_l$, v_α not a subset of (and even disjoint to) $\bigcup_{\beta<\alpha} v_\iota$, clearly

$$v_\alpha \cap N_l \neq \emptyset \Leftrightarrow V_\alpha \subseteq N_l \Leftrightarrow y_\alpha^n \in N_l.$$

So as $\operatorname{dom} x \subseteq (\bigcup_l (N_l \cap \{ t_i^n : i < n, i \}) \cup W_0$, clearly $v_{\alpha_{q(*)}} \cap \operatorname{dom} x = \emptyset$; now computing formal sums, looking outside $W_{\alpha_{q(*)}}$ we easily get for some $m \geqq 1$

$$y_{\alpha_{q(*)}}^m \notin \langle B \cup \{ t_i^n : t_i^n \in W'_{\alpha_{q(*)}} \} \rangle^{\mathrm{cl}} H[p]$$

and so to

$$y_{\alpha_{q(*)}}^1 \notin \langle B \cup \{ t_i^n : t_i^n \in W_{\alpha_{q(*)}} \} \rangle^{\mathrm{cl}} + H[p].$$

Hence there is no $q < r_0$, $p c_q y_{\alpha_q}^m \neq 0$ so $x \in H[p]$ and we finish.

We can note also

2.8. Definition. $G \subseteq_{\mathrm{pr}} \bar{B}^*$ is called direct if G has a base $\{t_i^n : n < \omega, i < \lambda_n\}$ s.t. for every $x \in pG$ there is $y \in G$, $py = x$ and $\mathrm{dom}\, x = \mathrm{dom}\, y$. We define similarly $B \subseteq_{\mathrm{cl}} G$ when (G, B) is direct: if G/B is.

2.9. Claim. If $G \subseteq_{\mathrm{pr}} \bar{B}^*$, *then* there is $H \subseteq_{\mathrm{pr}} \bar{B}^*$, $G[p] = H[p]$ and H is direct.

Proof. Let $\{t_i^n : n < \omega,\ i < \lambda_n\}$ be a base of G. Now every $x \in G$, $(\mathrm{ht}(x) = m(x) + 1)$ has a unique representation $x = \Sigma_{n \geqq m(x)}\, a^x_{(n,i)}\, p^{n-m(x)} t_i^n$, $(\forall n)(\exists^{<\aleph_0} i)\ a^x_{(n,i)} \neq 0$; w.l.o.g. $0 \leqq a^x_{(n,i)} < p^{m(*)+1}$, $\mathrm{dom}\, x = \{t_i^n : a^x_{(n,i)} \neq 0\}$.

Let $\{x^\alpha + \oplus_{n,i} \langle p^n t_i^n \rangle^g : \alpha < \alpha^*\}$ be a basis of $G[p]/\oplus_{n,i} \langle p^n t_i^n \rangle^g$ (so $m(x^\alpha) = 0$) (as a vector space over $\mathbf{Z}/p\mathbf{Z}$). Let H be the subgroup of $\bar{B}^*$-generated by

$$\{t_i^n : n, i\} \cup \left\{ \sum_{\substack{n \geqq m \\ (n,i) \in w_\alpha}} a^x_{(n,i)} p^{n-m} t_i^n : \alpha < \alpha^*,\ m < \omega \right\}.$$

2.10. Claim. Suppose $G_1 \subseteq_{\mathrm{pr}} G \subseteq_{\mathrm{pr}} \bar{B}^*$, $H_1 \subseteq_{\mathrm{pr}} \bar{B}^*$, $H_1[p] = G_1[p]$, $[H_1 \cap G^{\mathrm{cl}} \subseteq G_1^{\mathrm{cl}}]$. Then there is H, $H_1 \subseteq H \subseteq \bar{B}^*$, $H[p] = G[p]$, $H \cap H_1^{\mathrm{cl}} = H_1$, and (H, H_1) is direct.

Proof. Let $\{t_i^n : n < \omega,\ i < \zeta_n\}$ be a basis of G_1, $\{t_i^n : n < \omega,\ i < \xi_n\}$ be a basis of G. Let $\{\Sigma_{(n,i) \in w_\alpha} a^\alpha_{(n,i)}\, p^n t_i^n + G_1[p] \oplus \bigoplus_{n,i} \langle p^n t_i^n \rangle^g : \alpha < \alpha(*)\}$ be a basis of $G[p]/G_1[p] + \bigoplus_{(n,i)} \langle p^n t_i^n \rangle^g$. Let H be

$$H_1 + \langle t_i^n : n < \omega,\ \zeta_n \leqq i < \xi_n \rangle^g + \left\langle \sum_{(n,i) \in w_\alpha n \geqq m} a^\alpha_{(n,i)}\, p^{n-m} t_i^n : i < \alpha(*),\ m < \omega \right\rangle^g.$$

2.11. Remark. (1) We can prove that if H is direct and not the sum of cyclics, *then* H is not st. psf. This is really the content of 6.1.

(2) Note that if H, G are pure subgroups of $\bar{B}^*$, $H[p] = G[p]$ *then* H is the sum of cyclics iff G is the sum of cyclics.

Section 3

Context U is a fixed set (we shall deal with subsets of it) and $\mathbf{F}$ a family of pairs of subsets of it; we write $A/B \in \mathbf{F}$ or say "A/B is free" or "A is free over B when $(A, B) \in \mathbf{F}$. χ will be a fixed cardinal.

Convention. Adding a superscript $+$ to an axiom means that whenever

"$A/B \in \mathrm{F}$" or its negation appears in the assumption, then we demand B to be free over $\varnothing$.

Ax I**: If A/B is free, and $A^* \subseteq A$, *then* A^*/B is free.

Ax II: (a) A/B is free iff $A \cup B/B$ is free.

(b)$_\mu$ A/B is free *if* $|B| < \mu$, $A \subseteq B$.

Ax III: If A/B, B/C are free and $C \subseteq B \subseteq A$ then A/C is free.

Ax IV$_{\kappa,\mu}$: If A_i $(i < \lambda)$ is increasing, for $i < \gamma < \lambda$, $A_\gamma / \bigcup_{j<i} A_j \cup B$ is free, $\lambda < \kappa$, $|\bigcup_{i<\lambda} A_i| < \mu$ then $\bigcup_{i<\lambda} A_i/B$ is free. (IV$_\mu$ will mean IV$_{\mu,\mu}$ and IV means IV$_\infty$.)

3.1. Definition. We say "for the χ-majority of $X \subseteq A$, $P(X)$" if there is an algebra A with universe A and χ functions, such that any $X \subseteq A$ closed under those functions satisfies P. We can replace $X \subseteq A$ by $X \in \mathscr{P}(A)$ or $X \in \mathscr{P}_{<\lambda}(A)$: alternatively we say $\{X \subseteq A : P(A)\}$ is a χ-majority.

Ax VI: If A is free over $B \cup C$, then for the χ-majority of $X \subseteq A \cup B \cup C$, $A \cap X/(B \cap X) \cup C$ is free.

Ax VII: If A is free over B, then for the χ-majority of $X \subseteq A \cup B$, $A/(A \cap X) \cup B$ is free.

Convention. (1) We are always assuming Ax II$_\lambda$, III, IV$_\lambda$, VI, VII; others will be assumed explicitly, except when we mention some of them but not others.

(2) Ax II$_\lambda$ means II(a) + II(b)$_\lambda$.

(3) Ax II(b) means Ax II(b)$_\mu$ for every μ, and Ax II means II(a) + II(b). Similarly for the other axioms.

3.2. Definition. A/B is κ-free if: $\kappa > \chi$ and for the χ-majority of $X \subseteq A \cup B$ which has power $< \kappa$, $A \cap X/B$ is free *or* $\kappa \leqq \chi$ and $[A' \subseteq A \wedge |A'| < \kappa \Rightarrow A'/B$ is free$]$.

3.2A. Remark. Note that if Ax I** holds, then A/B is κ-free iff for every $A' \subseteq A$ of cardinality $< \kappa$, A'/B is free (so the distinction between the two cases disappears). It can easily be shown (see [Sh 1]) that:

3.3. Claim. (1) [Ax II (a), (b)$_\lambda$, III, IV$_{\lambda^+}$, VI, VII and $\lambda > \chi$]. Suppose $A = \bigcup_{i<\lambda} A_i$, A_i increasing continuous, $|A_i| < \lambda$, λ regular uncountable, then A/B is free *iff* for some closed unbounded set $C \subseteq \lambda$, $C \cup \{0\} = \{\delta_i : i < \lambda\}$, δ_i increasing and $A_{\delta_{i+1}}/A_{\delta_i} \cup B$ is free for each i *iff* A/B is λ-free and $\{i : A/A_i \cup B$ is λ-free$\}$ contains a closed unbounded subset of λ.

(2) If $|A| = \lambda$ we can omit II$_{\mu^+}$.

Also by [Sh 1]:

3.4. CLAIM. [Ax I**, II(a), III, IV$_{\mu^+}$, VI, VII]. If A/B is λ-free, $\chi < \mu < \lambda$, *then* for every $A' \subseteq A$, $|A| < \mu$ there is A'', $A' \subseteq A''$, $|A''| \leqq |A'| + \chi$, A''/B is free and $A/A'' \cup B$ is λ-free.

3.5. DEFINITION. $E^{\kappa}_{\kappa}(A)$ is the filter on $\mathscr{P}_{\leqq\kappa}(A)$ generated by the sets

$$\left\{ \bigcup_{i \leqq \kappa} A_i : A_i \subseteq A,\ |A_i| < \kappa,\ F(\langle A_j : j \leqq i \rangle) \subseteq A_{i+1} \right\}$$

where $F : {}^{\kappa >}[\mathscr{P}_{<\kappa}(A)] \to P_{<\kappa}(A)$ (we use κ regular $\geqq \aleph_1$).

3.6. THEOREM [(Shelah) Ax II(a), III, IV$_{\lambda^+}$, VI, VII]. *Suppose* $|A| = \lambda$, λ *is singular* $> \chi$, $\lambda = \Sigma_{i < \mathrm{cf}\,\lambda} \lambda_i$, λ_i *increasing continuous. Then* A/B *is free for* **F** *iff* A/B *is* λ*-free iff, for every* i, $\{X \in \mathscr{P}_{\leqq \lambda_i^+}(A) : X/B \text{ free}\} \neq \varnothing \bmod E^{\lambda_i^+}_{\lambda_i^+}(A)$.

3.7. REMARK. The theorem was proved with more axioms (I*, V) in [Sh 1], then the author eliminates I* and this is represented in [BD]. Later (see [Sh 2]) the author found a simpler proof and both new parts avoid Ax V. Hodges includes in [H] a representation of this proof in a different, but equivalent, axiomatic treatment. Lately we note that Ax III is not needed.

Section 4

4.1. DEFINITION. $U^{sc} = \bar{B}^*$. $F^{sc} = \{(B, A) : B + A = A \oplus \bigoplus_{(n,i) \in J} \langle t_i^n \rangle^g\}$ (equivalently: $B + A/A$ is the sum of cyclic p-groups).

Really we should have written

$$F^{sc} = \left\{ (X, Y) : \langle X \cup Y \rangle^g = \langle Y \rangle^g + \bigoplus_{(n,i) \in J} \langle t_i^n \rangle^g \right\}$$

but as we have only countably many functions in U^{sc}, this has no consequence.

4.2. DEFINITION. $U^{sp} = \bar{B}^*$.

$$F^{sp} = \{(B, A) : (B, A) \text{ is st. psf.}\}.$$

REMARK. If $A, B \in N_l$, $N_l \prec (H(\chi), \in)$ then $A + B \cap N_l = A + (A + B) \cap N_l$.

4.3. LEMMA. (1) F^{sc} *satisfies* Ax I**, II, III, IV, VI, VII.

(2) *If* $A/B \notin F^{sc}$, $|\langle A \cup B\rangle^g/\langle B\rangle|^g$ *is countable*, then *there is* $x \in \langle A\rangle^g$, $x + B$ *divisible by* p^n *for every* n, *equivalently* $x \in B^{cl}_{A+B}$.

PROOF. Probably well known (anyhow, it is true).

4.4. FACT. (1) F^{sp} satisfies Ax II.

(2) If A_i $(i \leqq \alpha)$ is increasing continuous, $A_i \subseteq_{sp} A_{i+1}$, $(A_i)^{cl}_{A_\alpha} \subseteq A_{i+1}$ then $A_0 \subseteq_{sp} A_\alpha$.

Section 5. λ-sets and λ-systems

5.1. DEFINITION. (1) For a regular uncountable cardinal λ $(> \aleph_0)$ we call S a λ-set if:

(a) S is a set of strictly decreasing sequences of ordinals $< \lambda$.

(b) S is closed under initial segments and is non-empty.

(c) for $\eta \in S$, *if* $W(\eta, S) \overset{\text{def}}{=} \{i : \eta^\wedge\langle i\rangle \in S\}$ is non-empty *then* it is a stationary subset of $\lambda(\eta, S) \overset{\text{def}}{=} \text{Sup}\, W(\eta, S)$ and $\lambda(\eta, S)$ is a regular uncountable cardinal. Also $\lambda(\langle\ \rangle, S) = \lambda$.

We sometimes allow $\lambda = 0$, then the only λ-set is $\{\langle\ \rangle\}$.

(2) For a λ-set S, let S_f (= set of final elements of $S\}$ be $\{\eta \in S : (\forall i)\eta^\wedge\langle i\rangle \notin S\}$ and S_i (= set of initial elements of $S\}$ be $S - S_f$ (so $S_f = \{\eta \in S : \lambda(\eta, S) = 0\}$). Let $k(S)$ be $\lg(n)$ for $\eta \in S_f$ if all $\eta \in S_f$ have the same length.

(3) We call S a (λ, κ)-set if S is a λ-set and $\lambda(\eta, S) > \kappa$ for $\eta \in S_i$.

(4) For λ-sets S^1, S^2 we say $S^1 \leqq S^2$ (S^1 a sub-λ-set of S^2) if $S^1 \subseteq S^2$ and $\lambda(\eta, S^1) = \lambda(\eta, S^2)$ for every $\eta \in S^1$ (so $S^1_i = S^1 \cap S^2_i$). Clearly $\leqq$ is transitive.

(5) We say that "for almost every $\eta \in S[\eta \in S_f]P\ldots$" iff for every $S' \leqq S$ some $\eta \in S'[\eta \in S'_f]$ satisfies P.

(6) For $\eta = \langle\alpha_0, \ldots, \alpha_m\rangle$ let $\eta^+ = \langle\alpha_0, \ldots, \alpha_{m-1}, \alpha_m + 1\rangle$.

5.1A. NOTATION. In this section S will be used to denote λ-sets.

5.1B. REMARK. Sometimes we can change (a) to "$\lambda(\eta \restriction l, S) > \lambda(\eta \restriction m, S)$ for $l < m \leqq \lg(\eta)$", but we found it less useful.

5.2. CLAIM. (1) S is a λ-set, $\eta \in S_i$, then $S^{[\eta]} \overset{\text{def}}{=} \{\nu : \eta^\wedge\nu \in S\}$ is a $\lambda(\eta, S)$-set and $\lambda(\nu, S^{[\eta]}) = \lambda(\eta^\wedge\nu, S)$.

(2) If $\lambda > \aleph_0$ is regular, $W \subseteq \lambda$ is a stationary set and for each $\delta \in W$, S^δ is a λ_δ-set where λ_δ is a cardinal $\leqq \delta$ (possibly $\lambda_\delta = 0$, $S^\delta = \{\langle\ \rangle\}$) *then*

$S \overset{\text{def}}{=} \{\langle\ \rangle\} \cup \{\langle\delta\rangle{}^\wedge\eta : \eta \in S^\delta$ and $\delta \in W\}$ is a λ-set. In this case $\lambda(\langle\delta\rangle{}^\wedge\eta, S) = \lambda(\eta, S^\delta)$ for $\delta \in W$, $\eta \in S^\delta$.

5.3. Claim. (1) If S is a λ-set,. $\lambda(\eta, S) > \kappa$ for every $\eta \in S_i$ (holds always for $\kappa = \aleph_0$) and G is a function from S_f to κ, *then* for some $S^1 \leqq S$ the function G is constant on S^1_f.

(2) If S is a λ-set, κ a regular cardinal $(\forall \eta \in S)(\lambda(\eta, S) \neq \kappa)$ and G is a function from S to κ, *then* for some $S^1 \leqq S$ and $\gamma < \kappa$ for every $\eta \in S^1$, $G(\eta) < \gamma$.

(3) If h is a function from S_f to a set K of regular cardinals and $(\forall \eta \in S_f) \bigwedge_{l<l(\eta)} (\lambda(\eta \restriction l, S) \neq h(\eta))$, and G is a function with domain S_f, $G(\eta) < h(\eta)$, *then* for some $S' \leqq S$, there are ordinals $\alpha_\kappa < \kappa$ for $\kappa \in K$, such that for $\eta \in S'_f$, $G(\eta) < \alpha_{h(\eta)}$.

(4) If h is a function from S_f to ordinals, S a λ-set, *then* there are a λ-set $S' \leqq S$ and k, m, h such that

(i) for every $\eta \in S'_f$, $l(\eta) = k$;

(ii) if $\eta, \nu \in S_f$, $\eta \restriction m = \nu \restriction m$ then $h(\eta) = h(\nu)$;

(iii) if $\eta \restriction m \neq \nu \restriction m$, $\eta \in S_f$, $\nu \in S_f$ but $\eta \restriction l = \nu \restriction l$ for $l < m$, *then* $h(\eta) \neq h(\nu)$; moreover (if $m > 0$)

$$\eta(m-1) < \nu(m-1) \Leftrightarrow h(\eta) < h(\nu).$$

(5) For a given λ-set S and property P the following are equivalent:

(a) for almost every $\eta \in S$, $P(\eta)$;

(b) there are closed unbounded sets C_η of $\lambda(\eta, S)$ such that $(\forall \eta \in S)[\bigwedge_{l<l(\eta)} \eta(l) \in C_{\eta \restriction l} \rightarrow P(\eta)]$.

5.4. Definition. (1) A λ-system is $\mathscr{B} = (B_\eta : \eta \in S_c)$ where:

(a) S is a λ-set, and we let $S_c = \text{com}(S) \overset{\text{def}}{=} \{\eta{}^\wedge\langle i\rangle : \eta \in S_i, i < \lambda(\eta, S)\}$,

(b) $B_{\eta^\wedge\langle i\rangle} \subseteq B_{\eta^\wedge\langle j\rangle}$ when $\eta \in S_i$, $i < j$ are $< \lambda(\eta, S)$,

(c) if δ is a limit ordinal $< \lambda(\eta, S)$ then $B_{\eta^\wedge\langle\delta\rangle} = \bigcup\{B_{\eta^\wedge\langle i\rangle} : i < \delta\}$,

(d) $|B_{\eta^\wedge\langle i\rangle}| < \lambda(\eta, S)$ for $i < \lambda(\eta, \delta)$.

Note: $\eta \in S_c \Rightarrow \eta^+ \in S_c$.

Section 6

6.1. Definition. Assume $A, B, A + B \subseteq_{\text{pr}} \bar{B}^*$ we say that $\mathscr{B} = \langle B_\eta : \eta \in S_c\rangle$ is a λ-witness for (A, B) if:

(a) λ is regular uncountable or $\lambda = 0$,

(b) S is a λ-set,

(c) $\langle B_\eta : \eta \in S_c \rangle$ *a* λ-system and let $B_{\langle\ \rangle} = B$ and for $\eta \in S_c$, $B_\eta \subseteq A$,

(d) $\langle \bigcup_{l \leqq \lg(\eta)} B_{\eta \restriction l} \rangle^g$ is a pure subgroup of $A + B$,

(e) $\langle B_\eta \cap \bigcup_{l < \lg(\eta)} B_{\eta \restriction l} \rangle^g$ a pure subgroup of B_η (eq. of $\langle \bigcup_{l < i(\eta)} B_{\eta \restriction l} \rangle^g$),

(f) for $\eta \in S_f$ there is $x_\eta \in B_\eta$, $x_\eta \notin \langle \bigcup_{l \leqq \lg(n)} B_{\eta \restriction l} \rangle^g + (A + B)[p]$, (equivalently, $px_\eta \notin \langle \bigcup_l B_{\eta \restriction l} \rangle^g$, $x_\eta \in \langle \bigcup_{l \leqq \lg(n)} B_{\eta \restriction l} \rangle^{\mathrm{cl}}$.

6.2. Lemma. *Suppose* A, B, $A + B$ *are pure subgroups of* $\bar{B}^*$. *If there is a* λ*-witness* $\mathscr{B} = \langle B_\eta : \eta \in S_c \rangle \rangle$ *for* (A, B) then $(A, B) \notin F^{\mathrm{sp}}$.

Proof. Suppose $(A, B) \in F^{\mathrm{sp}}$, let μ be regular large enough, $x \in \boldsymbol{H}(\mu)$. We choose by induction on l $\eta_l \in S$ and N_l s.t. (letting $B_{\langle\ \rangle +} = B + A$):

(1) $\eta_0 = \langle\ \ \rangle$, $\lg(\eta_l) = l$, $\eta_l = \eta_{l+1} \restriction l$;

(2) $x \in N_l \prec (\boldsymbol{H}(\mu), \in)$, $N_0, \ldots, N_l \in N_{l+1}$;

(3) $N_l \cap \lambda(\eta_l, S)$ is an ordinal α_l, $\eta_{l+1} = \eta_l{}^\frown \langle \alpha_l \rangle \in S$.

There is no problem to do this.

So for some $k < 0$, $\eta_k \in S_f$. We prove, by induction on $l = 0, \ldots, k$,

(*) (a) $\langle B \cup \bigcup_{i \leqq l} (N_i \cap A) \rangle^g \subseteq_{\mathrm{pr}} A + B$,

(b) $\langle B \cup \bigcup_{i \leqq l} (N_i \cap A) \rangle^g \cap \langle B_{\eta_0} \cup \cdots \cup B_{\eta_{l-1}} \cup B_{(\eta_l^+)} \rangle^g$
$= \langle B_{\eta_0} \cup \cdots \cup B_{\eta_{l-1}} \cup B_{\eta_l} \rangle^g$.

For (*) (a), use 1.4(1). For (*) (b), look at (3) above.† For $l = k$, we get (as $px_{\eta_k} \in B_{(\eta_k^+)} - \langle \bigcup_{l \leqq k} B_{\eta_k \restriction l} \rangle^g$ that

$$px_{\eta_k} \notin \langle B_{\eta_0} \cup \cdots \cup B_{\eta_k} \rangle^g.$$

On the other hand:

$$x_\eta \in \left\langle B \cup \bigcup_{l \leqq k} B_{\eta_k \restriction l} \right\rangle^{\mathrm{cl}} = \langle B_{\eta_0} \cup \cdots \cup B_{\eta_k} \rangle^{\mathrm{cl}} \subseteq \left\langle B \cup \bigcup_{l \leqq k} (N_l \cap A) \right\rangle^{\mathrm{cl}}.$$

So x_η show that

$$\left\langle B \cup \bigcup_{l \leqq k} (A \cap N_l) \right\rangle^g \not\subseteq_{\mathrm{sp}} A + B.$$

†We prove by induction on l. For $l = 0$, check. Suppose $x \in \langle B \cup \bigcup_{i \leqq l} (N_i \cap A) \rangle^g$ and $x \in \langle B_{\eta_0} \cup \cdots \cup B_{\eta_l - 1} \cup B_{(\eta_l^+)} \rangle^g$. So for some $y \in B$, $x_i \in N_i \cap A$ we have $x = y + \Sigma x_i$. As

$$x \in \langle B_{\eta_0} \cup \cdots \cup B_{\eta_l - 1} \cup B_{(\eta_l^+)} \rangle^g \subseteq \langle B_{\eta_0} \cup \cdots \cup B_{\eta_l - 1} \cup B_{(\eta \restriction_{l-1})} \rangle^g$$

hence for some $z_0 \in N_l \cap B_{\eta_0}, \ldots, z_{l-1} \in N_l \cap B_{(\eta \restriction_{l-1})}$, $\zeta_l \in N_l \cap B_{(\eta \restriction_{l-1})}$, $x_l = \Sigma z_i$. Now $x = \Sigma x_l'$ where $x_l' = x_i + z_i$ if $i < l$, $x_l' = z_l$. However $N_l \cap B_{(\eta \restriction_{l-1})} = B_{\eta_l}$ so $x_l' \in B_{\eta_l}$. As $l > 0$ we can use the induction hypothesis on l for $x - x_l'$.

Hence this shows (A, B) is not st. psf.

Section 7

7.1. CLAIM. Suppose $(A, B) \notin F^{sc}$, $B \subseteq_{pr} A \subseteq_{pr} \bar{B}^*$ then there is A_1, $B \subseteq A_1$, $A_1[p] = A[p]$ and (A, B) has a witness.

PROOF. As $(A, B) \notin F^{sc}$, by 4.3 and [Sh 2] there is $\langle B_\eta : \eta \in S_c \rangle$ s.t.

(a) $\lambda = 0$ or λ a regular uncountable cardinal,

(b) S is a λ-set,

(c) $\langle B_\eta : \eta \in S_c \rangle$ is a λ-system, we let $B_{\langle\ \rangle} = B$ and $B_\eta \subseteq A$,

(d) $\langle \bigcup_{l \leqq \lg(\eta)} B_{\eta \restriction l} \rangle^g \subseteq_{pr} A$,

(e) $B_\eta \cap \langle \bigcup_{l \leqq \lg(\eta)} B_{\eta \restriction l} \rangle^g \subseteq_{pr} B_\eta$, $\langle \bigcup_{l \leqq \lg(\eta)} B_{\eta \restriction l} \rangle^g$,

(f) for $\eta \in S_f$ there is $x_\eta \in B_{\eta^+}$, $x_\eta \notin \langle \bigcup_{l \leqq \lg(\eta)} B_{\eta \restriction l} \rangle^g$, $x_\eta \in \langle \bigcup_{l \leqq \lg(\eta)} B_{\eta \restriction l} \rangle^{cl}$.

Let $D_\eta = B_\eta[p]$. Easily (by (e)):

$$(*) \qquad \left\langle \bigcup_{l \leqq \lg(\eta)} B_{\eta \restriction l} \right\rangle^g [p] = \left\langle \bigcup_{l \leqq \lg(\eta)} D_{\eta \restriction l} \right\rangle^g .$$

We now define E_η for $\eta \in S_c$ by induction (with the order: inclusion on $\bigcup_{l \leqq \lg(\eta)} B_{\eta \restriction l}$) s.t. (letting $E_{\langle\ \rangle} = B$)

(A) $\langle \bigcup_{l \leqq \lg(\eta)} E_{\eta \restriction l} \rangle^g \subseteq_{pr} \bar{B}^*$,

(B) $\langle \bigcup_{l \leqq \lg(\eta)} E_{\eta \restriction l} \rangle^g [p] = \langle \bigcup_{l \leqq \lg(\eta)} D_{\eta \restriction l} \rangle^g$,

(C) $\langle E_\eta : \eta \in S_c \rangle$ will be a λ-system (set $E_{\langle\ \rangle} = D_{\langle\ \rangle}$),

(D) if $\eta \in S_c$ *then* $E_{\eta^+} / \langle \bigcup_{l \leqq \lg(\eta)} E_{\eta \restriction l} \rangle^g$ has an element x of height infinite and order p^2.

In limit stages and in the first stage, there are no problems. Dealing with ν successor necessarily $\nu = \eta^+$, η of maximal length. Defining B_{η^+}, if $\eta \notin S$ use 2.10. If $\eta \in S_l$ w.l.o.g. $px_\eta \in \langle \bigcup_{l \leqq \lg(\eta)} B_{\eta \restriction l} \rangle^g$ so by purity there is $x'_\eta \in B_\eta \cap \langle \bigcup_{l \leqq \lg(\eta)} B_{\eta \restriction l} \rangle^g$, $px'_\eta = px_\eta$ so w.l.o.g. $px_\eta = 0$ hence $x_\eta \in D_{\eta^+}$. So for some $t_n \in \langle \bigcup_{l \leqq \lg(\eta)} B_{\eta \restriction l} \rangle^g$, $\mathrm{ht}(x_\eta - \Sigma_{m<n} t_m) \geqq n$, so w.l.o.g. $pt_m = 0$ so $\mathrm{ht}(t_m) \geqq m$.

Now when $E_{\eta \restriction l}$ $(l \leqq l(\eta))$ are defined, choose $s_n \in \langle \bigcup_{l \leqq \lg(\eta)} E_{\eta \restriction l} \rangle^g$, $p^n s_n = t_n$, and let $B'_{\eta^+} = \langle \Sigma_{n \geqq m} p^{n-m} s_n : m < \omega \rangle^g$ and complete as before (using 2.10).

7.2. CONCLUSION. If $\lambda = \min |p^n G|$, $G \subseteq_{pr} \bar{B}^*$ is not the sum of cyclics, G is not torsion complete and even for no $A \subseteq_{pr} G$, $|A| < \lambda$, is G/A^{cl}_G is torsion complete, *then* there is $H \subseteq_{pr} \bar{B}^*$, $H[p] = G[p]$, H, G are not isomorphic.

PROOF. By 2.7, there is H_1, $H_1[p] = G[p]$, $H_1 \subseteq_{pr} \bar{B}^*$, $(H_1, \{0\}) \in F^{sp}$. By 7.1, there is $H_2 \subseteq_{pr} \bar{B}^*$, $H_2[p] = H[p]$, H_2 has a witness.

By 6.2, $(H_2, \{0\}) \notin F^{sp}$ together H_1, H_2 are not isomorphic so G is not isomorphic to H_1 or to H_2 (or to both).

Part B

Section 8

8.1. LEMMA. *Suppose* $G \subseteq_{pr} \bar{B}^*$, $\bigwedge_{n<\omega}[\lambda_n(G) \leqq \lambda^*(G)]$, $\lambda^*(G) < |G|$, *moreover* $2^{\lambda^*(G)} < 2^{|G|}$ *and* $G \neq G^{cl}$. *Then Conclusion* 7.2 *holds* (*we really have* $2^{|G|}$ *non-isomorphic ones*).

PROOF. We can find $x^* \in G^{cl} - G$, hence $x^* \in G^{cl} - G$, $x^* \neq 0 = px^*$. Let $\{t_i^n : i < \lambda_n(G), n < \omega\}$ be a basis of G ($p^{n+1}t_i^n = 0 \neq p^n t_i^n$). Let $G_0 = \langle t_i^n : i < \lambda_n(G), n < \omega\rangle^g$.

Let $\{s_i : i < i(*)\}$ be a maximal subset of $G[p^2]$ s.t. $\Sigma_i e_i s_i \in G[p] + G_0$ implies $e_i s_i \in G[p] + G_0$ (for each i). Clearly $|i(*)| = |G|$.

For $T \subseteq \{i : i < i(*)\}$, let

$$s_i^T = \begin{cases} s_i, & i \notin T, \\ s_i + x^*, & i \in T; \end{cases}$$

$$A_T \stackrel{\text{def}}{=} C_0[p^2] + G[p] + \langle s_i^T : c < i(*)\rangle^g.$$

As in 2.9, there is $H_T \subseteq_{pr} \bar{B}^*$, $H_T[p^2] = A_T$ hence $H_T[p] = G[p]$.

It suffices to prove that no $(2^{\lambda^*(G)})^+$ of the groups H_T are isomorphic.

Suppose $\{H_{T_i} : i < (2^{\lambda^*(G)})^+\}$ are isomorphic, $T_i \neq T_j$ for $i \neq j$. Let $h_i : H_{T_i} \to H_{T_0}$ be an isomorphism. For some $i \neq j$, $h_i \restriction G[p^2] = h_j \restriction G_0[p^2]$. So $h_j^{-1}h_i : H_{T_i} \to H_{Tj}$ is the identity on $G_0[p^2]$. Choose $\gamma \in T_i \equiv \gamma \notin T_j$. Now $s_\gamma^{T_i}$ is necessarily sent to itself being the limit of a ω-sequence from $G_0[p^2]$. But $s_j^{T_i} - s_j^{T_i} \neq x^r$ which is not in H_{T_j}, a contradiction.

8.2. LEMMA. *Suppose* $G \subseteq_{pr} \bar{B}^*$, $(G, \{0\}) \notin F^{sc}$, $\lambda = \mathrm{Min}_n |p^n G|$, $B \subseteq_{pr} G$, $|B| < \lambda$, G/B_G^{cl} *is torsion complete of power* λ, then *there is* $H \subseteq_{pr} \bar{B}^*$, $H[p] = G[p]$, $H \not\cong G$ *provided G.C.H. holds* (*or at least* $[\mu < \lambda \Rightarrow 2^\mu \leqq \lambda]$).

PROOF. Let $\lambda = |p^{n(*)}G|$, so for some G_1, G_2, $G = G_1 \oplus G_2$, $p^{n(*)}G_2 = 0$, $|G_1| = \lambda$, w.l.o.g. $B \subseteq_{pr} G_1$, $|B| < \lambda$, G/B_G^{cl} torsion complete.

As G is not torsion complete there is $x \in B^{cl} - G$, hence $x^* \in B^{cl} - G$, $px^* = 0 \neq x^*$. Let $\{t_i^n : n < \omega, i < \xi_n\}$ be a basis of G (t_i^n of order p^{n+1}) where $\{t_i^n : n < \omega, i < \zeta_n\}$ is a basis of B.

We can find infinite $v \subseteq \omega$ s.t. $\langle |\xi_n - \zeta_n| : n \in v\rangle$ is non-decreasing, $\Pi_{n\in v}|\xi_n - \zeta_n| = \lambda$, $v = \{n_l : l < \omega\}$, $n_i < n_{i+1}$, $n(l) = n_l$. Let $\kappa_l = |\xi_{n_l} - \zeta_{n_l}|$, and let $h_n : \Pi_{l \leq n}\kappa_l \to \{i : \zeta_{n(l)} \leqq i < \xi_{n(l)}\}$ be one to one. For $\eta \in \Pi_{l<\omega}\kappa_l$ let

$$y^m_\eta = \sum_{\substack{n \geqq m \\ n \leqq \lg(\eta)}} p^{n(l)-m} t^{n(l)}_{h(\eta \restriction n)}.$$

For some $s^0_\eta \in (B)^{\mathrm{cl}}_G$

$$z^0_\eta = y^m_\eta + s^0_\eta \in G.$$

Let $\{x_i : i < i(*)\} \subseteq G[p]$ be s.t. $\{z^0_\eta : \eta \in \Pi_{l<\omega}\kappa_l\} \cup \{x_\gamma : \gamma < \gamma(*)\}$ is a basis of $G[p]/B[p] \oplus \bigoplus_{(n,i)} \langle t^n_i \rangle^g$.

Let $s_{0\eta} = \Sigma_{(n,i)\in w_\eta} a^\eta_{(n,i)} p^n t^n_i$, $w_\eta \subseteq \{(n,i) : i < \zeta_n,\ n < \omega\}$, w.l.o.g. $x^* = \Sigma\, c_n p^n t^n_0$. For $S \subseteq \Pi_{i<\omega}\kappa_i$ let H_S be generated by

$$B \cup \left\{ y^m_\eta + \sum_{(n,i)\in w_\eta} a^\eta_{(n,i)} p^{n-m} t^n_i : \eta \in S, m < \omega \right\}$$

$$\cup \left\{ y^m_\eta + \sum_{\substack{(n,i)\in w_\eta \\ n \geqq m}} a^\eta_{(n,i)} p^{n-m} t^n_i + \sum_{n \geqq m} c_n p^{n-m} t^n_0 : \eta \notin S,\ \eta \in \prod_{l<\omega}\kappa_l,\ m < \omega \right\}$$

$$\cup \left\{ \sum_{\substack{n \geqq m \\ (n,i)}} b^\gamma_{(n,i)} p^{n-m} t^n_i : m < \omega, \gamma < \gamma(*) \right\}$$

where $x_\gamma = \Sigma b^\gamma_{(n,i)} p^{n-m} t^n_i$.

For every S this is o.k.

Case α: $\lambda^{|B|} = \lambda$

In this case

8.2A. FACT. We can find $\langle g_\eta : \eta \in \Pi_{n<\omega}\kappa_n \rangle$, g_η a function from $B \cup \{t^{n(l)}_{h(\eta\restriction n)} : n < \omega\}$ into g such that for every function $g : B \cup \{t^{n(l)}_{h(v)} : v \in \bigcup_{n<\omega}\Pi_{l<n}\kappa_l\}$ into G for some $\eta \in \Pi_{l<\omega}\kappa_l$, $g_\eta \subseteq g$.

Pf: Like [Sh 3, VIII, 2.6].

Now we can choose S as follows: for each $\eta \in \Pi_{l<\omega}\kappa_l$, the truth value of "$\eta \in S$" is determined such that no isomorphism from H_S onto G extending g_η exists. This is easily done, and clearly sufficient.

Case β: λ strong limit singular. Necessarily $\mathrm{cf}\,\lambda > \aleph_0$. We use [Sh 4, 2.5] and do the obvious things.

8.2A. REMARK. We may be tempted to use in case (α) $\lambda = \lambda^{\aleph_0}$ (instead of $\lambda = \lambda^{<\lambda}$), but by [Mk–Sh] 5.3 this is problematic.

8.3. REMARK. If λ is regular, $\{\delta < \lambda : \mathrm{cf}\,\delta = \aleph_0\}$ is not "small" (for definition and references see [G–S]), we can get the result.

If λ is singular, $|B| < \mu < \kappa < \lambda \leqq 2^\mu$, $\{\delta < \kappa : \mathrm{cf}\,\delta = \omega\}$ not small, we can still get the result (see [Sh 5, XIV, §1]).

8.4. FACT. We can weaken the hypothesis in 8.1: $G \subseteq_{\mathrm{pr}} \bar{B}^*$ is not the sum of cyclics and is not torsion complete, $\lambda = \mathrm{Min}_{n<\omega} |p^n G| > \lambda^*(G)$, $\{t_i^n : n < \omega,\ i < \xi_n\} \subseteq G$, a base,

$$\kappa_n < \kappa_{n+1} < \omega \quad \text{for } n < \omega,$$

$$\kappa_n \leqq \kappa_{n+1} \leqq \lambda^*(G),$$

$h_n : \Pi_{l<n} \kappa_l \to \xi_n$ one to one, and for $\eta \in \Pi_{n<\omega} \kappa_n$ there is $x_\eta \in G[p]$, $x_\eta = \Sigma\, a_i^n p^n t_i^n$,

$$\{t_i^n : a_i^n p^n t_i^n \neq 0\} \cap \left\{ t_{h(\nu)}^n : \nu \in \prod_{l<n} \kappa_l,\ n < \omega \right\} \subseteq \{t_{h_n(\eta \restriction n)}^n : n < \omega\}$$

and is infinite.

PROOF. The same proof essentially as 8.2 (really $\{t_i^n : a_i^n p^n t_i^n \neq 0\} \cap \{t_{h_n(\eta\restriction n)}^n : n < \omega\}$ is infinite, $\kappa_n > \aleph_0$ suffice).

8.5. CONCLUSION. (1) (G.C.H.) If $G \subseteq_{\mathrm{pr}} \bar{B}^*$ is not s.c. nor torsion complete, *then* there is $H \subseteq_{\mathrm{pr}} \bar{B}^*$, $H[p] = G[p]$, H, G not isomorphic.

(2) Instead of G.C.H., "$(\forall \lambda)$ $\{\delta < \lambda^+ : \mathrm{cf}\,\lambda = \aleph_0\}$ is not small" is enough.

PROOF. (1) W.l.o.g. $\lambda_n(G) \leqq \lambda^*(G)$ for each n. [Two possibilities:

(A) all non-isomorphism pf work if we say not "isomorphic even if we add a bound p-group".

(B) $\exists\, n(*)$, $\forall n \geqq n(*)$, $\lambda_n(G) \leqq \lambda^*$ and make $p^{n(*)}G$, $p^{n(*)}H$ non-isomorphic. Now the proof is just using 7.2, 8.1, 8.2 — they cover all cases.]

(2) For this observe

(A) If $\mathrm{Min}_n |p^n G| \geqq \mu > \mathrm{Min}_m \Sigma_{n>m} \lambda_n(G)$, μ regular, *then* there is $H \subseteq_{\mathrm{cl}} G$, $|p^n H| \geqq \mu$, $\mathrm{Min}_m(\Sigma_{n>m} \lambda_n(H))$ has confinality ω [prove by induction on $\mathrm{Min}_m(\Sigma_{n>m} \lambda_n(H))$].

(B) The proof of 8.1 gives: if $\mu \overset{\text{def}}{=} \mathrm{Min}_n |p^n G|$, $H \subseteq_{\mathrm{cl}} G$, $\kappa \overset{\text{def}}{=} \mathrm{Min}_m(\Sigma_{n>m} \lambda_n(H))$, $\mu^\kappa < 2^{|H|}$, *then* the conclusion of 7.2 holds (we get really $2^{|H|}$ non-isomorphic ones).

REMARK. We cannot just omit $G \subset \check{H}$ by [Mk–SH] §6.

Section 9

9.1. REMARK. An alternative definition of "H is direct" is: if $\bar{B}^* \in H(\mu)$, $N_l \prec (H(\mu), \in)$, $\bigwedge_{l<m} N_l \in N_m$ *then* $\langle \bigcup_{l<k} (N_l \cap H) \rangle_H^{\mathrm{cl}} \subseteq_{\mathrm{pr}} H$ (similarly for "(H, H_1) is direct").

9.2. THEOREM. If G.C.H., $G \subseteq_{\mathrm{pr}} \bar{B}^*$, λ is regular, $(\forall K)[K \subseteq G \wedge |K| < \lambda \rightarrow G/K$ not sum of cyclic], G not torsion complete, *then* there are $\geqq 2^\lambda$ pairwise non-isomorphic groups H, $H \subseteq_{\mathrm{pr}} \bar{B}^*$, $H[p] = G[p]$.

REMARK. (1) Under $V = L$ we can get rid of "λ regular". We should correct case (B) as in 8.2's proof. It is enough that $\{\delta < \lambda^+ : \mathrm{cf}\,\delta = \omega\}$ is not small for every λ.

(2) By 9.2 and compactness for singular, if in 9.2 λ is singular, the number is $\geqq {}^{\lambda>}2$.

PROOF. W.l.o.g. $|G| = \mathrm{Min}_{n<\omega} |p^n G|$. Clearly there is $G_1 \subseteq_{\mathrm{pr}} G$, $|G_1| \geqq \lambda$, $(\forall K)[K \subseteq G_1 \wedge |K| < |G_1| \Rightarrow G_1/K$ not sum of cyclic].

By applying suitably compactness for singular, we get $\mu \overset{\text{def}}{=} |G_1| \leqq |G|$ is a regular cardinal.

Case A: For some $H \subseteq_{\mathrm{cl}} G$, $|H| < |G|$, G/H is torsion complete and of power $|G|$.

The desired conclusion follows by [Sh 4] and the proof of 8.2.

Case B: For some $H \subseteq_{\mathrm{pr}} G$, $|H| < |G|$, $|(H)_G^{\mathrm{cl}}| = |G|$ or even just $|H| < |(H)_G^{\mathrm{cl}}| \leqq |G| \leqq 2^{|H|}$. Then use the proof of 8.5(2) (or 8.1).

* * * * *

Let $\lambda \overset{\text{def}}{=} \mathrm{Min}\{|K| : G/K$ is sum of cyclic, $K \subseteq_{\mathrm{pr}} G\}$. So if λ is not strong limit singular, we can assume that $2^\mu \geqq \lambda$.

Case C: Not case A, not case B.

OBSERVATION. W.l.o.g. $K \subseteq G \wedge |K| < \mu \Rightarrow |(K)_G^{\mathrm{cl}}| < \mu$.

Really "$\Rightarrow |(K)^{cl}_G| \leqq \mu$" suffices, and this follows by GCH. For trying to weaken the assumption GCH, note the following. If $2^\mu \geqq \lambda$, as not case B, $K \subseteq_{pr} G \wedge |K| \leqq \mu \Rightarrow |(K)^{cl}_G| \leqq \lambda$, so w.l.o.g. $G_1 \subseteq_{cl} G$.

If λ is strong limit singular $\mu_1 \overset{\text{def}}{=} (2^\mu)^+ < \lambda$ and $[K \subseteq_{pr} G \wedge |K| \leqq \mu_1 \Rightarrow |(K)^{cl}_G| \leqq \lambda]$. So if for some $G_2 \subseteq_{pr} G$, $|G_2| = \mu_1$, $(\forall K)[K \subseteq_{pr} G_2 \wedge |K| < \mu \rightarrow G_2/K$ not sum of cyclic], we finish. Otherwise there is a minimal $\mu_2 \geqq \mu$,

$$G_3 \subseteq_{pr} G, \quad |G_3| = \mu_2, \quad (\forall K)[K \subseteq_{pr} G_3 \wedge |K| < \mu_2 \rightarrow G/K \text{ not sum of cyclic}].$$

By 1.x μ_2 is regular, and easily $[K \subseteq_{pr} G \wedge |K| < \mu_2 \Rightarrow |(K)^{cl}_G| < \mu_2]$, so we can use μ_2 instead μ.

If λ is not strong limit we have assumed $2^\mu \leqq \lambda$, and by not case B, $[K \subseteq G \wedge |K| \leqq \mu \Rightarrow |(K)^{cl}_G| \leqq \mu]$. Trying to replace μ by $\mu_2 \overset{\text{def}}{=} \mu^+$ we succeed in the previous case except when $\mu = \lambda$. By then "not case B" gives the conclusion.

Observation. W.l.o.g. if $\mu < |G|$: (i) $(\forall K \subseteq_{pr} G_1)[|K| < \mu \rightarrow G_1/(K)^{cl}_G$ is not torsion complete] and (ii) $G_1 \subseteq_{cl} G$.

Proof of the Observation. Define by induction on $\zeta \leqq \mu$, G_1^ζ, s.t.

(a) $G_1^\zeta \subseteq G$, $|G_1^\zeta| \leqq \mu$,

(b) G_1^ζ is increasing continuous (in ζ),

(c) $G_1^0 = G_1$ is not s.c. (hence G_1 will not be),

(d) $G_1^{3\zeta+1} = (G_1^{3\zeta})^{cl}_G$,

(e) $G_1^{3\zeta+2} \subseteq_{pr} G$,

(f) $G_1^{3\zeta+3} \subseteq_{pr} G$, $G_1^{3\zeta+3}/G_1^{3\zeta+2}$ is not bounded.

Note: $G_1^\delta \subseteq_{pr} G_1$. Now replace G_1 by G_1^μ.

Observation. W.l.o.g. $\mu = |G| \Rightarrow G_1 = G$, hence (i), (ii) alone hold by $\neg$ case A, $\neg$ case B (so (i), (ii) always hold).

Let $\langle B_\eta : \eta \in S_c \rangle$ be a μ-system satisfying (a)–(f) from the proof of 7.1 with $B = B_{\langle\ \rangle} = \{0\}$, $B_{\langle\ \rangle} = \bigcup_{\alpha<\mu} B_{\langle\alpha\rangle} = G_1$ and

(g) $\bigcup\{B_{\eta\hat{}\langle i\rangle} : i < \lambda, (\eta, S)\} \subseteq B_{\eta^+}$,

(h) $G/B_{\langle\alpha\rangle}$ is $\lambda(\langle\alpha\rangle, S) - F^{sc}$-free, $B_{\langle\alpha+1\rangle} \subseteq_{cl} G$. By [Sh 2] w.l.o.g. there is $m(*)$ s.t. for every $\eta \in S_f$, $\mathrm{cf}[\eta(0)] = \lambda(\eta \restriction m(*), S)$.

Let $\{t_i^n : n < \omega, i < \mu\}$ be a basis of G_1, and w.l.o.g. for $\alpha \in W(\langle\ \rangle, S)$, α is divisible by $|\alpha|$ and $\{t_i^n : n < \omega, i < \alpha\}$ is a basis of $B_{\langle\ \rangle}$, and there are $U_\eta \subseteq \{t_i^n : n < \omega, i < \mu\}$ for $\eta \in S$ s.t. U_η is a basis of $B_\eta / \bigcup_{l<\lg(\eta)} B_{\eta\restriction l}$. Now for each $\delta \in W^* = \{\alpha < \mu : \alpha \in W(\langle\ \rangle, S),\ \alpha = \sup \alpha \cap W(\langle\ \rangle, S)\}$ choose a

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closed unbounded $C_\delta \subseteq \delta \cap W(\langle\quad\rangle, S)$ of order type cf δ. We can assume that $(\alpha) W^*$ is a set of inacessible, $\lambda(\langle\delta\rangle, S) = \delta$ or $(\beta) \delta \in W^* \Rightarrow \mathrm{cf}\,\delta = \kappa_1$, $(\kappa_1 < \lambda(\langle\quad\rangle, S))$, $\lambda(\langle\delta\rangle, S) = \kappa_2$.

In case (α) we know $\{\delta \in W^* : W^* \cap \delta$ is not stationary in $\delta\}$ is stationary so w.l.o.g. $(*)$ for $\delta \in W^*$, $W^* \cap \delta$ is not stationary in δ, hence w.l.o.g. each C_δ is disjoint to W^*.

We shall define for every $W \subseteq W^*$ a group $H^W \subseteq_{\mathrm{pr}} B^*$, $H^W[p] = G[p]$ s.t.: (D_λ — the club filter) $W_1 \neq W_2 \bmod D_\lambda$ implies $H^{W^1} \not\cong H^{W^2}$.

We now define E^W_η for $\eta \in S_c$ as in the proof of 7.1 but

if $\alpha \notin W$, we define $E^W_{\langle\alpha+1\rangle}$ as in the proof of 2.7,

if $\alpha \in W$, we want to define $E^W_\eta(\langle\alpha\rangle \in \eta \in S)$ as in the proof of 7.1, however we have a problem wanting to reconstruct W/D_μ from H^W. We do not want that what we do for (α) will spoil what we have done for any $\beta < \alpha$, $\beta \notin W$.

Assume first that $m(*) = 1$; w.l.o.g.

$(***)$ for every α $\{t^n_i : n < \omega,\ i < \gamma_\alpha\}$ is a basis of B_α, $\gamma_\alpha + \gamma_\alpha < \gamma_{\alpha+1}$; for every i:

$$\langle t^n_j : j \leqq i,\ n < \omega\rangle^{\mathrm{cl}}_{B^*} + \langle t^n_{j+l} : j < n,\ l < \omega\rangle^{g} \not\supseteq \langle t^n_j : j < i + \omega,\ n < \omega\rangle^{\mathrm{cl}}_G$$

and say z_i witness it, $pz_i = 0$.

Now building $E^W_{\langle\alpha\rangle\hat{}\eta}$ we make them direct over $B_{\langle\alpha\rangle}$, but we use z_i essentially like in 2.7.

The case $m(*) > 1$ is more complicated — we should imitate [Sh 2].

Completing the definition of H^W after $\langle E^W_\eta; \eta \in S\rangle$ was defined, is as in 2.7.

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