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A Partition Theorem for Scattered Order Types

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If φ is a scattered order type, and μ is a cardinal, then there exists a scattered order type ψ such that $\psi \to [\varphi]^1_{\mu,\aleph_0}$ holds.

In this note we prove a Ramsey-type statement on scattered order types. A trivial fact on ordinals implies the following statement. If μ is an infinite cardinal, then $\mu^+ \to (\mu^+)^1_\mu$. It is less trivial but still easy to show that, if φ is an order type, and μ is a cardinal, then there is some order type ψ such that $\psi \to (\varphi)^1_\mu$ holds. We can say that these results show that the classes of ordinals and order types are both Ramsey classes in the natural sense: given a target element and a cardinal for the number of colours, there is another element of the class which, when coloured with the required number of colours, always has a monocoloured copy of the target. We can wonder which other classes have similar Ramsey properties. A natural, and well-investigated, class in between is the class of *scattered order types*. For this class, the Ramsey property fails for the following well-known and simple reason. There is some scattered order type ψ such that, for every scattered φ, we have $\varphi \not\to [\psi]^1_\omega$. See Lemma 1.

In this paper we show that this is the most that can be proved in the negative direction, that is, for every scattered order type φ and cardinal μ there exists a scattered order type ψ such that $\psi \to [\varphi]^1_{\mu,\omega}$ holds.

In a further paper we will prove the corresponding variant of the Erdős–Rado theorem, that is, for any scattered order type φ, natural number r, and cardinal μ, there is a scattered order type ψ such that $\psi \to [\varphi]^r_{\mu,\omega}$ holds.

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We notice, however, that the full Ramsey result can be proved if the number of colours is finite, that is, for every scattered order type φ and natural number n, there is a scattered order type ψ such that $\psi \to (\varphi)^1_n$ holds. Then ψ is simply the n-fold lexicographic product of φ, $\psi = \varphi \times \cdots \times \varphi$. An inductive argument gives the Ramsey property. In fact, the lexicographic μth power of any φ has the Ramsey property with μ colours and φ as the target type. What is specific about scattered types is that, if $|\varphi| \geqslant 2$, then the lexicographic μth power of φ is no longer scattered for $\mu \geqslant \omega$.

Notation. We use the standard axiomatic set theory notation. If φ, ψ are order types, then $\varphi \leqslant \psi$ denotes that there is an order-preserving embedding of φ into ψ, that is, every ordered set of order type ψ has a subset of order type φ. If φ is an order type, then φ^* denotes the reverse order type, that is, if φ is the order type of $(S, <)$, then φ^* is the order type of $(S, >)$. Here ω is the ordinal of the set of natural numbers, $(\mathbb{N}, <)$, and η is the order type of the set of rational numbers, $(\mathbb{Q}, <)$.

If φ, ψ are order types, and μ is a cardinal, then $\varphi \to (\psi)^1_\mu$ denotes the following statement. If $(S, <)$ is an ordered set of order type φ and $f : S \to \mu$, then for some $i < \mu$ the subset $f^{-1}(i)$ contains a subset of order type ψ. That is, if a set of order type φ is coloured with μ colours, then there is a monochromatic ψ. If the statement does not hold, we cross the arrow: $\varphi \not\to (\psi)^1_\mu$.

If φ, ψ are order types, and λ, μ are cardinals, then $\varphi \to [\psi]^1_{\lambda,\mu}$ denotes the following statement. If $(S, <)$ is an ordered set of order type φ and $f : S \to \lambda$, then there is a subset $X \subseteq \lambda$ of cardinality μ such that the set $\{x \in S : f(x) \in X\}$ contains a subset of order type ψ. Again, crossing the arrow denotes the negation of the statement: $\varphi \not\to [\psi]^1_{\lambda,\mu}$. Notice that $\varphi \to (\psi)^1_\mu$ is equivalent to $\varphi \to [\psi]^1_{\mu,1}$.

If φ, ψ are order types, and μ is a cardinal, then $\varphi \not\to [\psi]^1_\mu$ denotes the following statement. If $(S, <)$ is an ordered set of order type φ, then there is a function $f : S \to \mu$ such that, on every subset of S of order type ψ, f assumes every value. If the statement fails, that is, we have a positive statement for every function $f : S \to \mu$, then we do not cross the arrow: $\varphi \to [\psi]^1_\mu$.

The order type φ is *scattered* if and only if $\eta \not\leqslant \varphi$. Hausdorff proved that the class of scattered order types is exactly the smallest class containing 0, 1, and closed under well-ordered and reverse well-ordered sums (see [1], [2], [3]).

Lemma 1. *If S is an ordered set with the scattered order type φ, then there is some $f : S \to \omega$ such that $f^{-1}(n)$ has no subset of order type $(\omega^* + \omega)^n$. Therefore, $\varphi \not\to (\psi)^1_\omega$, where $\psi = 1 + (\omega^* + \omega) + (\omega^* + \omega)^2 \cdots$.*

Proof. The second statement obviously follows from the first one. In order to prove the first statement, using Hausdorff's characterization of scattered order types, it suffices to show it for $(S, <)$ which is the well-ordered sum of the ordered sets $\{(S_i, <) : i < \alpha\}$, and we have the required function $f_i : S_i \to \omega$ for every $i < \alpha$.

Define $f : S \to \omega$ by $f(x) = f_i(x) + 1$ when $i < \alpha$ is the unique ordinal such that $x \in S_i$. If we now have a set of order type $(\omega^* + \omega)^{n+1}$ in colour $n + 1$, then all but finitely many of the ω^* copies of $(\omega^* + \omega)^n$ on its left-hand side must be in the same S_i, of colour n, which contradicts the assumption on f_i. $\square$

Before proceeding to our main theorem we need to show a technical result.

In what follows, for an ordinal λ, we let $\mathrm{FS}(\lambda)$ denote the set of all finite decreasing sequences from λ, that is, an element $\mathbf{s}$ is of the form $\mathbf{s} = s(0)s(1)\cdots s(n-1)$ with $\lambda > s(0) > s(1) > \cdots > s(n-1)$. Here $n = |\mathbf{s}|$ is the *length* of $\mathbf{s}$. The extension of the string $\mathbf{s}$ with one ordinal γ is denoted by $\mathbf{s}\gamma$. We therefore identify the finite subsets of λ with decreasingly ordered strings.

If α is an ordinal, then an *α-tree* is a system of ordinals $\{x(\mathbf{s}) : \mathbf{s} \in \mathrm{FS}(\alpha)\}$ with the following properties:

$$x(\mathbf{s}\gamma) < x(\mathbf{s}\gamma') < x(\mathbf{s}) \qquad \text{for } \gamma < \gamma' < \min(\mathbf{s}).$$

Theorem 2. *Assume that α is an ordinal and μ is a cardinal. Set $\lambda = (|\alpha|^{\mu^{\aleph_0}})^+$. Assume that $F : \mathrm{FS}(\lambda^+) \to \mu$. Then there exist an α-tree $\{x(\mathbf{s}) : \mathbf{s} \in \mathrm{FS}(\alpha)\}$ and a function $c : \omega \to \mu$, such that*

$$F(x(s(0)), x(s(0)s(1)), \ldots, x(s(0)s(1)\cdots s(n))) = c(n)$$

holds for every element $\mathbf{s} = s(0)s(1)\cdots s(n)$ of length $n+1$ of the tree.

Proof. We define, for every $\mathbf{s} \in \mathrm{FS}(\alpha)$ and for every function $c : \omega \to \mu$, a rank $r_c(\mathbf{s})$ as follows. Assume that $\mathbf{s} = s(0)s(1)\cdots s(n-1)$. $r_c(\mathbf{s}) = -1$ if, for some $0 \leqslant i < n$, we have $F(s(0)s(1)\cdots s(i)) \neq c(i)$. Otherwise, we declare that $r_c(\mathbf{s}) \geqslant 0$. Then we define by induction on ξ when $r_c(\mathbf{s}) \geqslant \xi$ holds: we set $r_c(\mathbf{s}) \geqslant \xi$ if and only if, for every $v < \xi$, we have

$$\lambda \leqslant \mathrm{tp}(\{\gamma < \min(\mathbf{s}) : r_c(\mathbf{s}\gamma) \geqslant v\}).$$

Naturally, $r_c(\mathbf{s}) = \xi$ holds if $r_c(\mathbf{s}) \geqslant \xi$ but $r_c(\mathbf{s}) \geqslant \xi + 1$ is not true.

Assume first that, for some function $c : \omega \to \mu$, we have $r_c(\emptyset) \geqslant \alpha$. In this case we can select the α-tree as required in the theorem, with the additional property that

$$r_c(x(s(0)), x(s(0)s(1)), \ldots, x(s(0)s(1)\cdots s(n))) \geqslant s(n).$$

To show this we have to show that, if we are given an $\mathbf{s}$ with $r_c(\mathbf{s}) \geqslant \beta$, then we can select the ordinals $\{x_\gamma : \gamma < \beta\}$ with $x_\gamma < x_{\gamma'} < \min(\mathbf{s})$ for $\gamma < \gamma' < \beta$ and with $r_c(\mathbf{s}x_\gamma) \geqslant \gamma$ for $\gamma < \beta$. To this end, we let δ_γ be the supremum of the first λ ordinals x with the property that $r_c(\mathbf{s}x) \geqslant \gamma$. Notice that $\delta_{\gamma'} \leqslant \delta_\gamma$ for $\gamma' < \gamma$ and the cofinality of is δ_γ is λ. We are going to select by transfinite recursion the elements $x_\gamma < \delta_\gamma$ as required. At step γ we have the elements $\{x_{\gamma'} : \gamma' < \gamma\}$ selected, and as $\sup(\{x_{\gamma'} : \gamma' < \gamma\}) \leqslant \sup(\{\delta_{\gamma'} : \gamma' < \gamma\}) \leqslant \delta_\gamma$ we have $\sup(\{x_{\gamma'} : \gamma' < \gamma\}) < \delta_\gamma$ and so we can choose x_γ.

Assume now that for every function $c : \omega \to \mu$ we have $r_c(\emptyset) < \alpha$.

In this case we construct by induction on $0 \leqslant n < \omega$ the ordinals

$$\{x(n, \gamma, s) : \gamma < \lambda^+, s : k \to \lambda, k \leqslant n\},$$

the ordinals $d(n) < \mu$, and for every $c : \omega \to \mu$, the values $-1 \leqslant \xi(n, c) < \alpha$ with the following properties:

$$x(n, \gamma, s\tau) < x(n, \gamma, s\tau') < x(n, \gamma, s)(1 \leqslant |s| < n, \tau < \tau' < \min(s)), \tag{1}$$

$$\gamma < x(n, \gamma, s). \tag{2}$$

Finally, if $\gamma < \lambda^+$, $s : n \to \lambda$, $1 \leqslant k \leqslant n$, and we set $y_i = x(n, \gamma, s|i)$, then

$$F(y_0, \ldots, y_k) = d(k) \tag{3}$$

and

$$r_c(y_0, \ldots, y_k) = \xi(k, c) \tag{4}$$

hold for every $c : \omega \to \mu$.

Initially, we select λ^+ ordinals $x(0, \gamma, \emptyset)$ $(\gamma < \lambda^+)$ such that the value $F(x(0, \gamma, \emptyset))$ is the same (let this be $d(0)$), and for every $c : \omega \to \mu$ the value $r_c(x(0, \gamma, \emptyset))$ is the same (this will be $\xi(0, c)$). This is possible by the pigeon hole principle, counting possibilities.

Assume that we have the result for some value n and we have the corresponding system $\{x(n, \gamma, s) : \gamma < \lambda^+, s : k \to \lambda, k \leqslant n\}$ with $\gamma < x(n, \gamma, s)$. Thinning out this system, and re-indexing, we can achieve $\gamma + \lambda < x(n + 1, \gamma, s)$.

We can define $x(n + 1, \gamma, s\tau) < x(n, \gamma, s)$ for $\tau < \lambda$ satisfying (1) and (2). Thinning and re-indexing, we can modify this system so that, if we set $y_i = x(n + 1, \gamma, s|i)$ for $i \leqslant n + 1$, then $F(y_0, \ldots, y_{n+1}) = d(n + 1)$ and $r_c(y_0, \ldots, y_{n+1}) = \xi(s, c)$ hold for every $s : n \to \lambda$, $c : \omega \to \mu$, that is, the colour and the rank do not depend on the last value.

Repeating this, again thinning and re-indexing, we find that the value of $r_c(y_0, \ldots, y_{n+1})$ depends only on c, so it is a value $\xi(n + 1, c)$, as claimed.

For the above function $d : \omega \to \mu$ we have that

$$\xi(0, d) > \xi(1, d) > \cdots$$

a contradiction. □

In order to handle scattered order types we represent them.

If α is an ordinal then let $H(\alpha)$ be the set of all functions $f : \alpha \to \{-1, 0, 1\}$ for which the set $D(f) = \{\beta < \alpha : f(\beta) \neq 0\}$ is finite. Order $H(\alpha)$ as follows: $f < f'$ if and only if $f(\beta) < f'(\beta)$ holds for the largest β with $f(\beta) \neq f'(\beta)$. This clearly orders $H(\alpha)$.

Lemma 3. *The order type of* $(H(\alpha), <)$ *is scattered.*

Proof. Assume that the mapping $q \to f_q$ is an order-preserving injection for $q \in \mathbb{Q}$. Let $\beta < \alpha$ be the *least* ordinal that occurs as the largest ordinal where f_q, $f_{q'}$ differ, for some $q < q'$. Now choose the rational numbers q'', q''' with $q < q'' < q''' < q'$. Then all four functions f_q,$f_{q'}$, $f_{q''}$,$f_{q'''}$ agree above β, and some two at β, too, a contradiction. □

Lemma 4. *Every scattered order type can be embedded into some* $(H(\alpha), <)$.

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Proof. Using Hausdorff's characterization, it suffices to show that if some order types can be so represented then any well-ordered and reverse well-ordered sum of them can also be so represented. For this, it suffices to show that the antilexicographic products $H(\alpha) \times \beta$ and $H(\alpha) \times \beta^*$ can be embedded into $H(\alpha + \beta)$. Indeed, if we map the pair (f, γ) to the function g which is f restricted to α and in the interval $[\alpha, \alpha + \beta)$ is everywhere zero except at $\alpha + \gamma$ where it is 1, then this is the required embedding for $H(\alpha) \times \beta$. For the other case we use extensions that assume -1 at exactly one place. □

Given an α-tree $\{x(\mathbf{s}) : \mathbf{s} \in \mathrm{FS}(\alpha)\} \subseteq \lambda^+$, we define an injection $\Phi : H(\alpha) \to H(\lambda^+)$ as follows. If $f \in H(\alpha)$, $D(f) = \{\beta_0, \dots, \beta_n\}$ in decreasing enumeration, then set $\gamma_j = x(\{\beta_j, \dots, \beta_0\})$ for $0 \leqslant j \leqslant n$. Now $\Phi(f) = g$ where $D(g) = \{\gamma_0, \dots, \gamma_n\}$ and $g(\gamma_j) = f(\gamma_j)$.

Lemma 5. *This mapping $\Phi : H(\alpha) \to H(\lambda^+)$ is order-preserving.*

Proof. Assume that $f, f' \in H(\alpha)$, $D(f) = \{\beta_0, \dots, \beta_n\}$, $D(f') = \{\beta'_0, \dots, \beta'_m\}$ in decreasing enumeration. Let r be the largest index such that $\beta_i = \beta'_i$ and $f(\beta_i) = f'(\beta_i)$ hold for $i < r$. For some β we have $f(\beta) < f'(\beta)$, where either $\beta = \beta_r = \beta'_r$ or $\beta = \beta_r \notin D(f')$ or $\beta = \beta'_r \notin D(f)$.

Set $\gamma_j = x(\{\beta_j, \dots, \beta_0\})$ for $j < r$ and $\gamma = x(\{\beta, \beta_{j-1}, \dots, \beta_0\})$. Then the functions $\Phi(f)$ and $\Phi(f')$ agree above γ and $\Phi(f)(\gamma) < \Phi(f')(\gamma)$, and we are done. □

Theorem 6. *If φ is a scattered order type, and μ is a cardinal, then there exists a scattered order type ψ satisfying*

$$\psi \to [\varphi]^1_{\mu, \aleph_0}.$$

Proof. By Lemmas 3 and 4, it suffices to show that, if α is an ordinal, and μ is a cardinal, then for some λ the ordered set $(H(\lambda^+), <)$ has the property that, for every colouring with μ colours, there is a subset isomorphic to $(H(\alpha), <)$ that is coloured with only countably many colours.

Select λ as in Theorem 2. Assume that $G : (H(\lambda^+), <) \to \mu$ is a colouring. Let F be the following colouring of $\mathrm{FS}(\lambda^+)$. If $\mathbf{s} = s(0)s(1)\cdots s(n-1)$ is an element of it, let $F(\mathbf{s})$ be the following function defined on $\{-1, 1\} \times \cdots \{-1, 1\}$: $F(i_0, \dots, i_{n-1}) = G(f)$, where f is the function with $D(f) = \mathbf{s}$ and $f(s(j)) = i_j$.

Notice that this is a colouring with μ colours. By Theorem 2 there is an α-tree $\{x(\mathbf{s}) : \mathbf{s} \in \mathrm{FS}(\alpha)\}$ such that

$$F(x(s(0)), x(s(0)s(1)), \dots, x(s(0)s(1)\cdots s(n))) = c(n)$$

holds for some function c.

If we now consider the corresponding mapping $\Phi : H(\alpha) \to H(\lambda^+)$, then it gives a subset of $(H(\lambda^+), <)$ isomorphic to $(H(\alpha), <)$ getting only μ colours. □

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