ISRAEL JOURNAL OF MATHEMATICS, Vol. 55, No. 2, 1986

THE SPECTRUM PROBLEM III: UNIVERSAL THEORIES†

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Dedicated to Professor Abraham Robinson

ABSTRACT

We solve the classification problem and essentially the spectrum problem for universal theories (see [6] for discussion of the meaning of this). We first solve it for T such that if M_1, M_2 elementarily extend M_0 and are independent over it, then over $M_0 \cup M_1$ there is a prime model. This generalizes [2]. This was subsequently used and generalized for countable first order theories. (This will appear in [5].) But note that there the theory is countable and in the case of structure the model is prime over a non-forking tree of models; here the model is generated by the union (and the T not necessarily countable). The universality is used in

THEOREM. *If T is stable and complete then either* (A) *for every $M_l < M$ ($l = 0, 1, 2$) models of T, if $M_0 \subseteq M_1 M_2$, $\{M_1, M_2\}$ is independent over M_0 (i.e. $\mathrm{tp}(M_1, M_2)$ is finitely satisfiable in M_0), then the submodel of M which $M_1 \cup M_2$ generates is an elementary submodel of M, or* (B) *there is an unstable theory extending the universal part of T (we can replace universal by Σ_2 and slightly more).*

CONCLUSION. For any universal T: *Either* (a) for every model M of T there is a tree I with $\leqq \omega$ levels and submodels N_η ($\eta \in I$) of power $\leqq 2^{|T|}$ (by [5], just $\leqq |T|$) such that (i) M is generated by $\bigcup_{\eta \in I} N_\eta$, (ii) $\eta \lhd \nu \Rightarrow N_\eta \subseteq N_\nu$, (iii) if ν is an immediate successor of η then $\mathrm{tp}(N_\nu, \bigcup\{N_\rho : \rho \in I,\ \nu \not\leqq \rho\})$ is finitely satisfiable in N_η (note that asking this just for quantifier-free formulas is enough). *Or* (b) for every cardinal $\lambda > |T|$, there are 2^λ non-isomorphic models for power λ.

† The author would like to thank John Baldwin for the interesting talks in September 1980 which led to §3 of this work, Rami Grossberg for various corrections, and the BSF and NSF for their partial support.

This paper was originally intended to appear in the Proceedings of the Model Theory Year at the Institute for Advanced Studies, The Hebrew University of Jerusalem, September 1980 – August 1981, published in Isr. J. Math., Vol. 49, Nos. 1-3, 1984.

Received September 1, 1982 and in revised form May 31, 1985

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Convention. We use some convention and definitions freely; $\mathfrak{C}$ is the "monster" quite saturated model (of T). $\mathrm{Cb}(p)$ is the canonical base (see III §7).

When referring to [1], we do not write [1] usually (no confusion arises as we have to use a chapter number (latin numeral)).

In Sections 0–3, $\mathrm{Th}(\mathfrak{C})$ is assumed to be stable. $<$ is elementary submodel.

§0. Canonization

This section can be avoided, if you avoid 1.5A (but not 1.5), 1.8 and 2.3(1) which are not used later.

Here we quote some facts from the new edition of the author's book [5] using the definition from there (see Definition V 4.5). So $\mathfrak{C}$ is a quite saturated model of a complete first-order stable T.

0.1. CLAIM. ($\mathfrak{C}^{\mathrm{eq}}$) *Suppose r is regular and* $\mathrm{stp}(\bar{a}, A)$ *is not orthogonal to r. Then*:

(1) *There is $E \in FE^m(\mathrm{acl}\, A)$ $(m = l(\bar{a}))$ such that* $\mathrm{stp}(\bar{c}/E, \bar{b})$ *is* $\mathrm{cl}^2(r)$*-simple but not orthogonal to r.*

(2) *Moreover* $\mathrm{stp}(\bar{c}/E, A)$ *contains a formula* $\vartheta(y, \bar{b})$ *that is* $\mathrm{cl}^2(r)$*-simple.*

(3) *Also, for every* $\bar{b}'$ *realizing* $\mathrm{tp}(\bar{b}, \emptyset)$, $\vartheta(y, \bar{b}')$ *is* $\mathrm{cl}^3(r)$*-simple.*

0.1A. Fact. There is $\varphi(x, \bar{c})$ which is $\mathrm{cl}^3(r)$-regular but has completions not orthogonal to r.

Let $\varphi(x, \bar{c})$ be any formula which has completions not orthogonal to r, but has no extension of smaller rank $(R^1(-, L, \infty))$ with this property.

Clearly such $\varphi(x, \bar{c})$ exists. Now any such $\varphi(x, \bar{c})$ is $\mathrm{cl}^2(r)$-regular — this follows by

$\oplus$ for every complete stationary p, $\varphi(x, \bar{c}) \in p$: if $R(p, L, \infty) < R(\varphi(x, \bar{c}), L, \infty)$ then p is orthogonal to r; hence if p forks over $\bar{c}$ then p is orthogonal to r.

0.1B. Fact. Every complete stationary $\mathscr{P}$-regular p not orthogonal to $r \in \mathscr{P}$ is regular.

0.1C. Fact. There is $\varphi(x, \bar{c})$ as in Fact 0.1A, such that for every $\bar{c}'$ realizing $\mathrm{tp}(\bar{c}, \emptyset)$, $\varphi(x, \bar{c}')$ is $\mathrm{cl}^2(r)$-regular.

0.2. Claim. $(\mathfrak{C}^{\mathrm{eq}})$ *If* $\mathrm{tp}(\bar{a}, A)$ *is not orthogonal to some trivial regular type* r *then for some* $b \in \mathrm{act}(A \cup \bar{a}) - \mathrm{acl}(A)$, $\mathrm{tp}(b, A)$ *is* $\mathrm{cl}^3(\{r\})$*-simple and* $w_r(b, A) = 1$. *If* $\mathrm{tp}(\bar{a}, A)$ *is regular we can replace simple by regular, and if* T *is superstable then some* $\varphi \in \mathrm{tp}(b, A)$ *is* $\mathrm{cl}^3(\{r\})$*-simple* [*regular*].

§1. On Strong Elementary Submodels

Hypothesis. T is superstable.

1.1. Definition. (1) We say that $M \subseteq_s N$ if $M \subseteq N$ and for every $\bar{a} \in M$, $\bar{b} \in N$ there is $\bar{b}' \in M$ realizing $\mathrm{tp}(\bar{b}, \bar{a})$. We define $M \subseteq_s A$, $B \subseteq_s A$ similarly.

(2) We say that $M \subseteq_a N$ if $M \subseteq N$, and for every $\bar{a} \in M$, $\bar{b} \in N$ there is $\bar{b}' \in M$ realizing $\mathrm{stp}(\bar{b}, \bar{a})$. We define $M \subseteq_a A$, $B \subseteq_a A$ similarly.

1.2. Claim. (1) *If* $A \subseteq B \subseteq C$, $A \subseteq_s C$ *then* $A \subseteq_s B$.

(2) *Let* $A \subseteq_s B_l$ *for* $l < n$, $\bar{a} \in A$, $\bar{b}_l \in B_l$, $\models \varphi[\bar{b}_0, \ldots, \bar{b}_{n-1}, \bar{a}]$ *and* $\{B_l : l < n\}$ *is independent over* A. *Then there are* $\bar{b}'_0, \ldots, \bar{b}'_{n-1} \in A$, *such that*: $\models \varphi[\bar{b}'_0, \ldots, \bar{b}'_{n-1}, \bar{a}]$ *and* $\bar{b}'_l$ *realizes* $\mathrm{tp}(\bar{b}_l, \bar{a})$ (*and in fact it realizes* $\mathrm{tp}(\bar{b}_l, \bar{a} \cup \bigcup_{m<l} \bar{b}'_m)$ *and* $\{\bar{b}'_0, \ldots, \bar{b}'_{n-1}\}$ *is independent over some* $\bar{a}'$, $\bar{a} \subseteq \bar{a}' \subseteq A$).

(3) *If* $N \subseteq N_1 \subseteq M$, $N \subseteq_a A$, M *is* $\mathbf{F}^l_{\aleph_0}$*-atomic over* $N_1 \cup A$, $\{A, N_1\}$ *is independent over* N. *Then* $N_1 \subseteq_s M$ (*in fact* $N_1 \subseteq_a M$ *holds,* M *a set suffices*).

(4) *If* $A \subseteq B \subseteq C$, $A \subseteq_a C$ *then* $A \subseteq_a B$.

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(5) *If* $A \subseteq_a B_i$ $(i < \alpha)$ *and* $\{B_i : i < \alpha\}$ *is independent over* A, *then* $A \subseteq_a \bigcup_{i<\alpha} B_i$.

(6) *If* $A \subseteq_a B$ *then* $A \subseteq_s B$.

PROOF. (1), (4), (5), (6) are easy.

(2) We can find a finite $\bar a'$, $\bar a \subseteq \bar a' \subseteq A$, such that $\mathrm{tp}(\bar b_0 {}^\wedge \cdots {}^\wedge \bar b_{n-1}, A)$ does not fork over $\bar a'$. Now we define by induction on $i < \omega$, $\langle \bar b^i_l : l < n\rangle$, such that $\bar b^i_l \in A$. For a given i we define $\bar b^i_l$ by induction on $l : \bar b^i_l$ realizes $\mathrm{tp}(\bar b_l, \bar a' \cup \bigcup_{j<i} \bigcup_{m<n} \bar b^j_m \cup \bigcup_{m<l} \bar b^i_m)$ and $\bar b^i_l \in A$. We can easily prove by induction on $k < \omega$ that $\{\bar b^i_l : ni + l < k\}$ is independent over $\bar a'$. As in the proof of [1; Ch. II, 2.17, p. 38, Ch. III, 2.13, pp. 98] we finish.

(3) Let $\bar a \in N_1$, $\bar b \in M$ and we should find $\bar b' \in N_1$ realizing $\mathrm{stp}(\bar b, \bar a)$. As M is $\mathbf{F}^t_{\aleph_0}$-atomic over $N_1 \cup A$, there are $\bar b_1 \in N_1$, $\bar b_2 \in A$ such that $\vDash \psi[\bar b, \bar b_1, \bar b_2]$ and $\psi(\bar x, \bar b_1 \bar b_2) \vdash \mathrm{tp}(\bar b, N_1 \cup A)$; w.l.o.g. $\bar a \subseteq b_1$. For some $\bar b_0 \in N$, $\mathrm{tp}(\bar b_l, N)$ does not fork over $\bar b_0$ for $l = 1, 2$ and remember $\{\bar b_1, \bar b_2\}$ is independent over N.

Now choose $\bar b'_2 \in N$ which realizes $\mathrm{stp}(\bar b_2, \bar b_0)$ (possible as $N \subseteq_a A$). As $\mathrm{tp}(\bar b_1, A)$ does not fork over $\bar b_0$, $\bar b'_2$ realizes $\mathrm{stp}(\bar b_2, \bar b_0 \cup \bar b_1)$. So $\bar b_0 {}^\wedge \bar b_1 {}^\wedge \bar b_2$, $\bar b_0 {}^\wedge \bar b_1 {}^\wedge \bar b'_2$ realizes the same type, hence $\vDash (\exists \bar x)\psi(\bar x, \bar b_1, \bar b'_2)$, and letting $\bar b'$ realize $\psi(\bar x, \bar b_1, \bar b'_2)$, $\bar b_0 {}^\wedge \bar b_1 {}^\wedge \bar b_2 {}^\wedge \bar b$, $\bar b_0 {}^\wedge \bar b_1 {}^\wedge \bar b'_2 {}^\wedge \bar b'$ realizes the same type. But we can choose $\bar b' \in N_1$. So there is $\bar b' \in N_1$ realizing $\mathrm{tp}(\bar b, \bar a)$ (as $\bar a \subseteq \bar b_1$). In fact $\bar b'$ realizes $\mathrm{stp}(\bar b, \bar a)$. For every $E \in FE(\bar b_1)$, E is a formula over N_1 (as it is almost over $\bar b_1$) hence $\mathrm{tp}(\bar b {}^\wedge \bar b_2, N_1) \vdash E(\bar x, \bar y; \bar b, \bar b_2)$. But for every $\Theta(\bar x, \bar y) \in \mathrm{tp}(\bar b {}^\wedge \bar b_2, N_1)$, $\mathrm{tp}(\bar b, N_1 \cup \bar b_2) \vdash \Theta(\bar x, \bar b_2)$, hence $\psi(\bar x, \bar b_1, \bar b_2) \vdash \Theta(\bar x, \bar b_2)$. We can conclude that $\psi(\bar x, \bar b_1, \bar b'_2) \vdash E(\bar x, \bar b'_2; \bar b, \bar b_2)$ [as $\Theta(\bar x, \bar y) \overset{\mathrm{def}}{=} E(\bar x, \bar y; \bar b, \bar b_2)$ is almost over $\bar b_1$ and $\bar b'_2$ realizes $\mathrm{stp}(\bar b_2, \bar b_1 {}^\wedge \bar b_0)$, clearly $\psi(\bar x, \bar b_1, \bar b'_2) \vdash \Theta(\bar x, \bar b'_2)$ hence $\psi(\bar x, \bar b_1, \bar b'_2) \vdash E(\bar x, \bar b'_2; \bar b, \bar b_2)$], hence $\vDash E(\bar b', \bar b'_2; \bar b, \bar b_2)$. As this holds for every $E \in FE(\bar b_1)$ clearly $\bar b' {}^\wedge \bar b'_2$ realizes $\mathrm{stp}(\bar b {}^\wedge \bar b_2, \bar b_1)$, hence $\bar b'$ realizes $\mathrm{stp}(\bar b, \bar a)$.

1.3. LEMMA. *Suppose* $N \subseteq A$, $N \subseteq_a M$, $\{M, A\}$ *is independent over* N:

(1) *If* $\mathrm{tp}(\bar a, A)$ *is orthogonal to* N *or is* $\mathbf{F}^c_{\aleph_0}$*-isolated (for* $\mathbf{F}^c_{\aleph_0}$*-isolation see Definition* IV 2.7), *then* $\mathrm{tp}(\bar a, A) \vdash \mathrm{tp}(\bar a, A \cup M)$.

(2) *If for each* $i < \alpha$, $\mathrm{tp}(\bar a_i, A \cup \bigcup_{j<i} \bar a_j)$ *is orthogonal to* N *or* $\mathbf{F}^c_{\aleph_0}$*-isolated (for* $\mathbf{F}^c_{\aleph_0}$*-isolation see Definition IV, 2.7) then*

$$\mathrm{tp}_*(\{\bar a_i : i < \alpha\}, A) \vdash \mathrm{tp}_*(\{\bar a_i : < \alpha\}, M \cup A).$$

(3) *In* (1), (2) *we can replace* M *by any* B *(if* $N \subseteq_a B$*)*.

PROOF. (1) Trivial (for the second case note $\mathrm{dcl}(M \cup A) \cap \mathrm{acl}(A) = \mathrm{dcl}(A)$).

(2) Follows from (1) by induction on α.

(3) The same proof.

1.4. CLAIM. (1) (in $\mathfrak{C}^{eq}$) *Let* $x \in \{t, a\}$. *If* p *is an* m*-type over* A, r *is a regular type*, $\operatorname{stp}(\bar{a}, A)$ *is not orthogonal to* r, r *extends* p, p *is an* $\mathbf{F}^x_\mu$*-type*, $A = |N|$, N *is* $\mathbf{F}^x_\mu$*-compact, then* $\operatorname{stp}_*(B, A)$ *is not orthogonal to* r *where* $B = \operatorname{Cb}(\operatorname{tp}(\bar{a}, A \cup p(\mathfrak{C}^{eq})))$.

(2) *Suppose* $\bar{b} \in M$, $\bar{a} \notin M$, $\operatorname{tp}(\bar{a}, M)$ *is not orthogonal to some type* q *to which* $\psi(\bar{x}, \bar{b})$ *belongs. Then for every model* N *including* $M \cup \bar{a}$, $\psi(N, \bar{b}) \neq \psi(M, \bar{b})$.

REMARK. The most interesting cases of (1) are $\mathbf{F}^x_\mu = \mathbf{F}^t_{\aleph_0}$, p finite (so N is just a model) and $\kappa = \kappa_r(T)$, $|\operatorname{Dom} p| < \kappa$ (even for any stable T).

PROOF. (1) We shall prove later:

1.4A. FACT. For every $\bar{d}$

$$\operatorname{tp}(\bar{a}, \bar{d} \cup A \cup p(\mathfrak{C}^{eq})) \text{ does not fork over } A \cup B$$
$$\text{whenever } \operatorname{tp}(\bar{a}, A \cup \bar{d}) \text{ does not fork over } A.$$

Let $\lambda > |T| + |A| + \|M\|$ be regular, let N_0 be an $\mathbf{F}^a_\lambda$-saturated model, $A \subseteq N_0$; and w.l.o.g. r is a type over N_0 and $\operatorname{tp}_*(N_0, A \cup \bar{a})$ does not fork over A.

By 1.4A, $\operatorname{tp}(\bar{a}, N_0 \cup p(\mathfrak{C}^{eq}))$ does not fork over $N \cup B$ hence over $N_0 \cup B$. Let $p \subseteq r_0 \in S^m(N_0)$, r_0 regular not orthogonal to r (r_0 exists, of course). Let N_1 be $\mathbf{F}^a_\lambda$-prime over $N_0 \cup \bar{a}$, hence it is $\mathbf{F}^a_\lambda$-atomic over $N_0 \cup \bar{a}$ hence over $N_0 \cup \bar{a} \cup B$ (note that $B \subseteq N_1$, as $B \subseteq \operatorname{acl}(N \cup \bar{a})$). As $\operatorname{tp}_*(\bar{a}, N_0 \cup p(\mathfrak{C}^{eq}))$ does not fork over $N_0 \cup B$ clearly $\operatorname{tp}_*(\bar{a}, N_0 \cup p(N_1))$ does not fork over $N_0 \cup B$ hence (by III, 0.1) $\operatorname{tp}_*(p(N_1), N_0 \cup B \cup \bar{a})$ does not fork over $N_0 \cup B$. By IV, 4.3 it is easy to see that $p(N_1)$ is a $\mathbf{F}^a_\lambda$-atomic over $N_0 \cup B$ ($B \subseteq N_1$ as noted above). So if 1.4(1)'s conclusion fails, as for every $\bar{b} \in B$, as $\operatorname{tp}(\bar{b}, N_0)$ is a stationarization of $\operatorname{stp}(\bar{b}, A)$, clearly it is orthogonal to r.

Note $\operatorname{tp}(\bar{b}, N_0)$ is orthogonal to r_0.

So $\operatorname{stp}_*(B, N_0)$ is orthogonal to r_0 and $p(N_1)$ is $\mathbf{F}^a_\lambda$-atomic over $N_0 \cup B$.

Hence if N_2 is $\mathbf{F}^a_\lambda$-prime over $N_0 \cup B$, r_0 is not realized in N_2 and for every $\bar{c} \in p(N_1)$, $\operatorname{tp}(\bar{c}, N_0 \cup B)$ is $\mathbf{F}^a_\lambda$-isolated hence is realized in N_2. We can conclude that (as p is over N_0) N_1 does not realize r_0. But as $\operatorname{tp}(\bar{a}, N_0)$ is not orthogonal to r_0, r_0 is realized in N_1, and $p \subseteq r_0$, contradiction.

PROOF OF FACT 1.4A. Why does it hold? As we have assumed $\operatorname{tp}(\bar{a}, A \cup \bar{d})$ does not fork over A, also $\operatorname{tp}(\bar{d}, A \cup \bar{a})$ does not fork over A, hence [as

$B \subseteq \mathrm{acl}(A \cup \bar{a})]$ $\mathrm{tp}(\bar{d}, A \cup \bar{a} \cup B)$ does not fork over A, hence [as A is a model] is finitely satisfiable in A. Suppose the conclusion of 1.4A fails, i.e., for some $\bar{e} \in A \cup \bar{d} \cup p(\mathfrak{C}^{eq})$, and φ, $\models \varphi[\bar{a}, \bar{e}]$, but $\varphi(x, \bar{e})$ forks over $A \cup B$. W.l.o.g. for some finite Δ, $R(\varphi(\bar{x}, \bar{e}), \Delta, \aleph_0) < R(\mathrm{tp}(\bar{a}, A), \Delta, \aleph_0)$.

We know that for some large enough finite κ, $R[\varphi(\bar{x}, \bar{e}), \Delta, \kappa] = R[\varphi(\bar{x}, \bar{e}), \Delta, \aleph_0]$, $R[\mathrm{tp}(\bar{a}, A \cup B), \Delta, \kappa] = R[\mathrm{tp}(\bar{a}, A \cup B), \Delta, \aleph_0]$. W.l.o.g. for every $\bar{e}'$, $R[\varphi(\bar{x}, \bar{e}'), \Delta, \kappa] < R[\varphi(\mathrm{tp}(\bar{a}, A \cup B), \Delta, \kappa]$. W.l.o.g. $\bar{e} = \bar{d} \,{}^\wedge \bar{a}' \,{}^\wedge \bar{e}_1 \,{}^\wedge \cdots {}^\wedge \bar{e}_n$ where $\bar{a}' \subseteq A$, $\bar{e}_l \in p(\mathfrak{C}^{eq})$; so $\models \varphi(\bar{a}, \bar{d}, \bar{a}', \bar{e}_1, \ldots, \bar{e}_m)$ and p w.l.o.g. is a singleton or closed under conjunction. So $\bar{d}$ realizes the type

$$q = \left\{ (\exists \bar{z}_1, \ldots, \bar{z}_m) \left[\varphi(\bar{a}, \bar{x}, \bar{a}', \bar{a}_1, \ldots, \bar{z}_m) \wedge \bigwedge_{l=1}^{m} \Theta(\bar{z}_l) \right] : \Theta(\bar{z}) \in p \right\}.$$

As q does not fork over A, q an F^x_μ-type, some $\bar{d}' \in A$ realizes q. So for some $\bar{e}'_l \in p(\mathfrak{C}^{eq}) \models \varphi[\bar{a}, \bar{d}', \bar{a}', \bar{e}'_1, \ldots]$. But this is a contradiction as

$$R[\mathrm{tp}(\bar{a}, A \cup p(\mathfrak{C}^{eq}), \Delta, \aleph_0] = R[\mathrm{tp}(\bar{a}, A \cup B), \Delta, \aleph_0]$$

$$= R[\mathrm{tp}(\bar{a}, A \cup B), \Delta, \kappa] > R[\varphi(x, \bar{d}', \bar{a}', \bar{e}'_1, \ldots), \Delta, \kappa].$$

Proof of 1.4(2). By (1) with $(p = \{\varphi(\bar{x}, \bar{b})\}, A = M)$. We know that $\mathrm{Cb}(\mathrm{tp}(\bar{a}, M \cup \varphi(\mathfrak{C}, \bar{b}))$ is not contained in M, but it is contained in $\mathrm{acl}(M \cup \varphi(N, \bar{b}))$. Hence $\varphi(N, \bar{b}) \not\subseteq M$.

1.5. Lemma. *Assume $N \subseteq_a M'$, $N \subseteq M' \subseteq M$ and $m < \omega$. Suppose $\bar{c} \in M$, $\bar{c} \notin M'$, $l(\bar{c}) = m$ and $R[\mathrm{tp}(\bar{c}, M'), L, \infty]$ is minimal (under this condition). If $\mathrm{tp}(\bar{c}, M')$ is not orthogonal to N* then *there is $\bar{c}' \in M$, $l(\bar{c}') = l(\bar{c})$, $\bar{c}' \notin M'$, $\mathrm{tp}(\bar{c}', M')$ does not fork over M and $R[\mathrm{tp}(\bar{c}', M'), L, \infty] = R[\mathrm{tp}(\bar{c}, M), L, \infty]$.*

Proof. Let $\psi(\bar{x}, \bar{b}) \in \mathrm{tp}(\bar{c}, M')$, $R^m[\psi(\bar{x}, \bar{b}), L, \infty] = R^m[\mathrm{tp}(\bar{c}, M'), L, \infty]$ hence $\mathrm{tp}(\bar{c}, M')$ does not fork over $\bar{b}$. We work in $\mathfrak{C}^{eq}$. We can choose $\bar{a} \in N$, such that $\mathrm{tp}(\bar{b} \,{}^\wedge \bar{c}, N)$ does not fork over $\bar{a}$ and $\mathrm{stp}(\bar{c}, \bar{b})$ is not orthogonal to $\bar{a}$ (as $\mathrm{tp}(\bar{c}, M')$, $\mathrm{stp}(\bar{c}, \bar{b})$ are parallel, $\mathrm{stp}(\bar{c}, \bar{b})$ is not orthogonal to some $q \in \bigcup_n S^n(N)$ and w.l.o.g. q does not fork over $\bar{a}$). Now as $N \subseteq_a M'$ there is a sequence $\bar{b}' \in N$ realizing $\mathrm{stp}(\bar{b}, \bar{a})$ and for some $\bar{c}'$, $\bar{b}' \,{}^\wedge \bar{c}'$ realizes $\mathrm{stp}(\bar{b} \,{}^\wedge \bar{c}, \bar{a})$. By V, 3.5 $\mathrm{stp}(\bar{c}, \bar{b} \cup \bar{a})$, $\mathrm{stp}(\bar{c}', \bar{b}' \cup \bar{a})$ are not orthogonal (note that $\mathrm{stp}(\bar{c}, \bar{b} \cup \bar{a})$, $\mathrm{stp}(\bar{c}, \bar{b})$ and $\mathrm{stp}(\bar{c}, M')$ are parallel). So $\mathrm{tp}(\bar{c}, M')$ is not orthogonal to some type to which $\psi(\bar{x}, \bar{b}')$ belongs, hence by 1.4(2) there is $\bar{c}'' \in M$, $\bar{c}'' \notin M'$, $\models \psi[\bar{c}'', \bar{b}']$. So

$$R[\mathrm{tp}(\bar{c}'', M'), L, \infty] \leqq R[\psi(\bar{x}, \bar{b}'), L, \infty] = R[\psi(\bar{x}, \bar{b}), L, \infty] = R[\mathrm{tp}(\bar{c}, M'), L, \infty].$$

By the hypothesis that $R[\mathrm{tp}(\bar{c}, M'), L, \infty]$ is minimal equality holds, hence $\mathrm{tp}(\bar{c}'', M')$ does not fork over N. So $\bar{c}''$ is as required.

1.5A. LEMMA. *Suppose* $N \subseteq_s M$ *and* $N \subseteq M' \subseteq M$ *and* $m < \omega$. *Suppose* $\bar{c} \in M$, $\bar{c} \not\subseteq M$, $l(\bar{c}) = m$, $\mathrm{tp}(\bar{c}, M')$ *is not orthogonal to* N *and* $R^m[\mathrm{tp}(\bar{c}, M'), L, \infty]$ *is minimal (under the previous constraints).*

Then there is $\bar{c}' \in M$, $\bar{c}' \not\subseteq M'$, $l(\bar{c}) = l(\bar{c}')$, $\mathrm{tp}(\bar{c}', M')$ *does not fork over* N *and*

$$R^m[\mathrm{tp}(\bar{c}', M'), L, \infty] = R^m[\mathrm{tp}(\bar{c}, M'), L, \infty].$$

REMARK. We can wave this lemma if in the decomposition theorems we omit 2.3(1).

PROOF. Let $\varphi(\bar{x}, \bar{b}) \in \mathrm{tp}(\bar{c}, M')$, $R^m[\varphi(\bar{x}, \bar{b}), L, \infty] = R^m[\mathrm{tp}(\bar{c}, M'), L, \infty]$. We work in $\mathfrak{C}^{eq}$. Let r^* be a regular type not orthogonal to N and not orthogonal to $\mathrm{tp}(\bar{c}, M')$, with $R^1[r^*, L, \infty]$ minimal. By 0.1(1) (maybe replacing $\bar{b}$ by $\bar{b}' \subseteq \mathrm{acl}\, \bar{b}$) there is a formula $E = E(\bar{x}, \bar{y}, \bar{b})$ where $\bar{b} \in M'$, $\mathrm{tp}(\bar{c}, M')$ does not fork over $\bar{b}$, such that $\mathrm{stp}(\bar{c}/E, \bar{b})$ is $\mathrm{cl}^3(r^*)$-simple not orthogonal to r. Moreover (see 0.1(2), (3)):

$\bigoplus$ $\mathrm{stp}(\bar{c}/E, \bar{b})$ contains a formula $\Theta(y, \bar{b})$ which is $\mathrm{cl}^3(r^*)$-simple. Moreover, $\mathrm{tp}(\bar{b}, \emptyset) = \mathrm{tp}(\bar{b}', \emptyset)$ implies $\Theta(x, \bar{b}')$ is $\mathrm{cl}^3(r^*)$-simple.

Let $\varphi_0(y, \bar{b}) = (\exists \bar{x})[\varphi(\bar{x}, \bar{b}) \wedge \bar{x}/E = y \wedge \Theta(y, \bar{b})]$. We shall prove

$(*)$ there are $\bar{b}' \in N$ realizing $\mathrm{tp}(\bar{b}, \emptyset)$, and $c' \in M - M'$, $\models \varphi_0[c', \bar{b}']$, $\mathrm{tp}(c', M')$ not orthogonal to N (in fact, to r^*).

If $(*)$ holds, there is $\bar{c}'' \in M$ such that $\varphi[\bar{c}'', \bar{b}'] \wedge \bar{c}''/E = c' \wedge \Theta(c', \bar{b})$. Now $\mathrm{tp}(\bar{c}'', M')$ is not orthogonal to N as $c' \in \mathrm{acl}(N \cup \{\bar{c}''\})$, $\mathrm{tp}(c', M')$ is not orthogonal to N. Now $R^m[\mathrm{tp}(\bar{c}'', M'), L, \infty] \leqq R^m[\varphi(\bar{x}, \bar{b}'), L, \infty] = R^m[\varphi(\bar{x}, \bar{b}), L, \infty]$. Equality holds by the hypothesis "$R^m[\mathrm{tp}(\bar{c}, M'), L, \infty]$ is minimal". ($\bar{c}'' \not\subseteq M'$ follows from: $\mathrm{tp}(\bar{c}'', M')$ is not orthogonal to N.) Hence $\mathrm{tp}(\bar{c}'', M')$ does not fork over $\bar{b}' \subseteq N$, so we get our conclusion.

Now we shall prove $(*)$. For notational simplicity let E be the equality, and $\varphi(\bar{x}, \bar{b}) \vdash \Theta(\bar{x}, \bar{b})$. Choose $\bar{a} \in N$ such that $\mathrm{tp}(\bar{b}, N)$ does not fork over $\bar{a}$. We have assumed that r^* (hence $\mathrm{tp}(\bar{c}, M')$) is not orthogonal to N, hence it is not orthogonal to some $r \in S^*(N)$. Also, let $\bar{d}_0$ realize r, $\mathrm{tp}(\bar{d}_0, M)$ does not fork over N. W.l.o.g. r does not fork over $\bar{a}$.

Now there are $\bar{b}_n$, $\bar{c}_n$ $(n < \omega)$ such that $\bar{b}_n \frown \bar{c}_n$ realizes $\mathrm{stp}(\bar{b} \frown \bar{c}, N)$, $\{\bar{b}_n \frown \bar{c}_n : 0 < n < \omega\}$ is independent over $(M \cup \bar{d}_0, N)$, $\bar{b}_0 \frown \bar{c}_0 = \bar{b} \frown \bar{c}$ (so $\{\bar{b}_n \frown \bar{c}_n : n < \omega\}$ is independent over N). Note that $\Theta(x, \bar{b}_n)$ is $\mathrm{cl}^3(r)$-simple.

For each n let $\{\bar{c}_{n,i} : i < \omega\}$ be a family of sequences realizing $\mathrm{stp}(\bar{c}_n, \bar{b}_n \cup N)$, independent over $N \cup \bar{b}_n$. Let $\{\bar{d}^i : i < \omega\}$ be a family of sequences realizing

$\text{tp}(\bar d_0, N)$ independent over $(N \cup \bigcup_n \bar b_n \cup \bigcup_{n,i} \bar c_{n,i}, N)$. By V 2.7 for some k, l $\text{tp}(\bar d^0 {}^\wedge \cdots {}^\wedge \bar d^l, N \cup \bar c_n)$, $\text{tp}(\bar c_{n,0} {}^\wedge \cdots {}^\wedge \bar c_{n,k}, N \cup \bar c_n)$ are not weakly orthogonal. Now w.l.o.g. $l = 0$.

Now by the proof of V 4.11 there is d_1, $d_1 \in \text{acl}(N \cup \bar d_0) - N$, $\text{tp}(d_1, N)$ not orthogonal to r^* and some $n < \omega$, and there is a formula $\psi(x, \bar a)$ such that

(a) $\models \psi[d_1, \bar a]$;

(b) $d_1 \in \text{acl}(\bar a \cup \bigcup_{l<n} \bar b_l \cup \bigcup_{l<n, i<i(l)} \bar c'_{l,i})$ where $\bar c'_{l,i}$ realizes $\text{stp}(\bar c_l, \bar b_l)$, $i(l) < \omega$ (for $l < n$).

(c) $\psi(\mathfrak{C}^{eq}, \bar a) \subseteq \text{acl}(\bar a \cup \bigcup_{l<n} \bar b_l \cup \bigcup_{l<n} \varphi(\mathfrak{C}^{eq}, \bar b_l))$.

By 1.2(2) there are $\bar b'_l \in N$ $(l < n)$ realizing $\text{tp}(\bar b_l, \bar a)$ such that

(d) $\psi(\mathfrak{C}^{eq}, \bar a) \subseteq \text{acl}(\bar a \cup \bigcup_{l<n} \bar b'_l \cup \bigcup_{l<n} \varphi(\mathfrak{C}^{eq}, \bar b'_l))$.

This is not exactly a first order property, but if it holds then some first order formula witnesses it, by compactness. Note that $\Theta(\bar x, \bar b'_l)$ is $\text{cl}^3(r^*)$-simple (as $\text{tp}(b'_l, \emptyset) = \text{tp}(\bar b, \emptyset)$). Remember $\text{tp}(d_1, N)$ is not orthogonal to r^*, $\psi(x, \bar a) \in \text{tp}(d_1, N)$. Easily $\text{tp}(\bar c, M' \cup \psi(\mathfrak{C}^{eq}))$ does not fork over $\bar b \cup N \cup \psi(\mathfrak{C}^{eq})$ hence $\text{Cb}(\text{tp}(\bar c, M' \cup \psi(\mathfrak{C}^{eq}))) = \text{Cb}(\text{tp}(\bar c, \bar b \cup N \cup \psi(\mathfrak{C}^{eq})))$ and the type of this set over $M' \cup \psi(\mathfrak{C}^{eq})$ does not fork over $\bar b \cup N \cup \psi(\mathfrak{C}^{eq})$.

Hence by 1.4(1) in $\text{Cb}(\text{tp}(\bar c, \bar b \cup N \cup \psi(\mathfrak{C}^{eq})))$ there is an element $d_2 \notin \text{acl}(N \cup \bar b)$ such that $\text{tp}(d_2, N \cup \bar b)$ is not orthogonal to r^*. As $\text{tp}(\bar c, M')$ does not fork over $\bar b$, clearly $\text{tp}(d_2, M')$ does not fork over $\bar b \cup N$. So as $d_2 \notin \text{acl}(\bar b \cup N)$ also $d_2 \notin M'$ remembering that

$$\text{Cb}(\text{tp}(\bar c, \bar b \cup N \cup \psi(\mathfrak{C}^{eq}, \bar a))) \subseteq \text{acl}(\bar b \cup N \cup \psi(\mathfrak{C}^{eq}, \bar a)).$$

Clearly $d_2 \in M$.

Now there are $\bar c'_{l,i} \in M$, $d_2 \in \text{acl}(\bar b \cup N \cup \{\bar c'_{l,i} : l, i\})$, $\models \varphi[\bar c_{l,i} \bar b'_l]$ (this is by (d)). Now by $\bigoplus$, if $\text{tp}(\bar c'_{l,i}, M')$ is orthogonal to r^*, then $\text{tp}(\bar c'_{l,i}, M' \bigcup \{c_{k,j} : k < l < i$ or $k = l, j < i\})$ is also orthogonal to r^*. If this holds for every l, i then $\text{tp}_*(\{\bar c_{l,i} : l, i\}, M')$ is orthogonal to r^*. Hence $\text{tp}(d_2, M')$ is orthogonal to r^*, contradiction. Now if $\text{tp}(\bar c'_{l,i}, M')$ is not orthogonal to r^*, $(*)$ holds, and we are finished.

1.6. Lemma. *Suppose $N \subseteq_s M$ and $N \subseteq M' \subseteq M$, and $m < \omega$. Suppose $\bar c \in M$, $\bar c \notin M'$, $l(\bar c) = m$, $\text{tp}(\bar c, M')$ is not orthogonal to N and $R[\text{tp}(\bar c, M'), L, \infty]$ is minimal.*

If $\text{tp}(\bar c, M')$ does not fork over N then it is regular.

Remark. We can omit 1.6 if in §2 we waive the regularity, i.e., omit 2.2(b) and 2.4(3).

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PROOF. Let $\bar{b} \in N$, $\operatorname{tp}(\bar{c}, M')$ does not fork over $\bar{b}$. We can choose ψ such that $\models \psi[\bar{c}, \bar{b}]$, $R[\psi(\bar{x}, \bar{b}), L, \infty] = R[\operatorname{tp}(\bar{c}, M'), L, \infty]$. Let $\bar{c}_n$ realize $\operatorname{stp}(\bar{c}, \bar{b})$ for $n < \omega$, $\{\bar{c}_n : < \omega\}$ independent over $(M, \bar{b})$. If $\operatorname{stp}(\bar{c}, \bar{b})$ is not regular then there are m and $\bar{c}^*_j$ $(j < m)$ and n such that $\bar{c}^*_j$ realizes $\operatorname{stp}(\bar{c}, \bar{b})$, $\operatorname{tp}(\bar{c}^*_j, \bar{b} \cup \{\bar{c}_i : i \leqq n\})$ forks over $\bar{b}$ and $\operatorname{tp}(\bar{c}, \bar{b} \cup \{\bar{c}_i : i \leqq n\} \cup \{c^*_j : j < m\})$ forks over $\bar{b}$. For each $j < m$ for some $n_j \leqq n$, $\operatorname{tp}(\bar{c}_{n_j}, \bar{b} \cup \{\bar{c}_i : i < n_j\} \cup \{\bar{c}^*_j\})$ fork over $\bar{b}$ and remember $\operatorname{tp}(\bar{c}, \bar{b} \cup \{\bar{c}_l : l \leqq n\} \cup \{\bar{c}^* : j < m\})$ forks over $\bar{b}$. By III 1.2(2), III 2.6(2), II 3.7 for some finite Δ, k

$$R[\operatorname{tp}(\bar{c}_{n_j}, \bar{b} \cup \{\bar{c}_i : i < n_j\} \cup \{\bar{c}^*_j\}), \Delta, k] < R[\operatorname{tp}(\bar{c}, \bar{b}), \Delta, k]$$

and

$$R[\operatorname{tp}(\bar{c}, \bar{b}), \Delta, k] = R[\operatorname{tp}(\bar{c}, \bar{b}), \Delta, \aleph_0] = R[\operatorname{stp}(\bar{c}, \bar{b}), \Delta, \aleph_0]$$

and

$$R[\operatorname{tp}(\bar{c}, \bar{b} \cup \{\bar{c}_i : i \leqq n\} \cup \{\bar{c}^*_j : j < m\}), \Delta, k] < R[\operatorname{tp}(\bar{c}, \bar{b}), \Delta, k].$$

These properties for fixed $\bar{c}$, $\bar{b}$ are expressed by first-order formulas, i.e. there are formulas which $\bar{b}$, $\bar{c}$, $\bar{c}_i$, $\bar{c}^*_j$ satisfy and imply this (see II 2.19). So by 1.2(2) we can define $\bar{c}'_0, \ldots, \bar{c}'_n \in N$ and then define $\bar{c}^{**}_j \in M$ $(j < m)$ such that

(i) $\bar{c}'_0, \ldots, \bar{c}'_n$ realizes $\operatorname{tp}(\bar{c}, \bar{b})$.

(ii) $\{\bar{c}'_0, \ldots, \bar{c}'_n\}$ is independent over $\bar{b}$.

(iii) $\psi(\bar{c}^{**}_j, \bar{b})$ for $(j < m)$.

(iv) $R^m[\operatorname{tp}(\bar{c}_{nj}, \bar{b} \cup \{\bar{c}'_l : l \leqq n_j\} \cup \{\bar{c}^{**}_j : j < m\}, \Delta, k] < R^m[\operatorname{stp}(\bar{c}, \bar{b}), \Delta, k]$.

(v) $R^m[\operatorname{tp}(\bar{c}, \bar{b} \cup \{\bar{c}'_l : l \leqq n\} \cup \{\bar{c}^{**}_j : j < m\}, \Delta, k] < R^m[\operatorname{stp}(\bar{c}, \bar{b}), \Delta, k]$.

By (v), $\operatorname{tp}(\bar{c}, \bar{b} \cup \{\bar{c}'_l : l \leqq n\} \cup \{\bar{c}^{**}_j : j < m\}) \subseteq \operatorname{tp}(\bar{c}, M' \cup \{\bar{c}^{**}_j : j < m\})$ forks over $\bar{b}$. But $\operatorname{tp}(\bar{c}, M')$ does not fork over $\bar{b}$, hence $\bar{c}^{**}_{j(0)} \notin M'$ for some $j(0)$. As $\operatorname{tp}(\bar{c}, M')$, $\operatorname{tp}(\bar{c}^{**}_{j(0)}, M')$ are not orthogonal, and the first does not fork over N, the second is not orthogonal to N. For notational simplicity assume $n = n_{j(0)}$.

By (iv) (and (i)) $\operatorname{tp}(\bar{c}'_n, \bar{b} \cup \{\bar{c}'_l : l < n\} \cup \{\bar{c}^{**}_{j(0)}\})$ forks over $\bar{b}$.

(Note that as $\bar{c}'_n$, $\bar{c}$ realizes the same type over $\bar{b}$,

$$R^m[\operatorname{ktp}(\bar{c}'_n, \bar{b}), \Delta, k] = R^m[\operatorname{stp}(\bar{c}'_n, \bar{b}), \Delta, k] = R^m[\operatorname{tp}(\bar{c}, \bar{b}), \Delta, k]).$$

Hence by (ii) $\operatorname{tp}(\bar{c}'_n, \bar{b} \cup \{\bar{c}'_l : l < n\} \cup \{\bar{c}^{**}_{j(0)}\})$ forks over $\bar{b} \cup \{\bar{c}'_l : l < n\}$. Hence $\operatorname{tp}(\bar{c}^{**}_{j(0)}, \bar{b} \cup \{c'_l : l \leqq n\})$ forks over $\bar{b} \cup \{\bar{c}'_l : l < n\}$ and hence over $\bar{b}$. So

$$R[\operatorname{tp}(\bar{c}^{**}_{j(0)}, M'), L, \infty] \leqq R[\operatorname{tp}(\bar{c}^{**}_{j(0)}, \bar{b} \cup \{\bar{c}'_l : l \leqq n\}), L, \infty] < R^m[\operatorname{tp}(\bar{c}^{**}_{j(0)}, \bar{b}), L, \infty]$$
$$\leqq R^m[\psi(\bar{x}, \bar{b}), L, \infty].$$

This contradicts the minimality of the rank of $\varphi(x, \bar{b})$. Hence stp $(\bar{c}, \bar{b})$ is regular, and we are finished.

1.7. CLAIM. (1) *For any* $A \subseteq M$ *there is* $N \subseteq_a M$, *such that* $A \subseteq N$, $\|N\| \leqq A + \lambda(T)$.

(2) *For any* $A \subseteq M$ *there is* $N \subseteq_s M$ *such that* $a \subseteq N$, $\|N\| \leqq |A| + |D(T)|$.

PROOF. Trivial.

REMARK. We can replace M by B. The next lemma will not be used in the sequal.

1.8. LEMMA. ($\mathfrak{C}^{eq}$) *Suppose* $N \subseteq A \subseteq M$, $N \subseteq_a M$, $\bar{a} \in M$, $\bar{a} \notin A$, $r =$ stp $(\bar{a}, A)$ *is regular and trivial, but not orthogonal to* N. *Then there is a* $\bar{a}' \in M$ *such that* stp $(\bar{a}', A)$ *is regular but not orthogonal to* stp $(\bar{a}, A)$ *and does not fork over* N.

PROOF. W.l.o.g. $A = \operatorname{acl} A$. By 0.2 there is $b \in \operatorname{acl}(A \cup \bar{a}) - \operatorname{acl}(A)$ (hence $b \in M$) such that: tp (b, A) is regular but not orthogonal to stp $(\bar{a}, A)$ (and hence it is trivial, too) and some $\varphi(x, \bar{c}_0) \in$ tp (b, A) is $\mathrm{cl}^3(r)$-regular. W.l.o.g. tp $(\bar{b}, A)$ does not fork over $\bar{c}_0$. Choose $\bar{d} \in N$ such that tp $(\bar{c}_0{}^\wedge\langle b\rangle, N)$ does not fork over $\bar{d}$. Now as $N \subseteq_a M$ we can choose $\bar{c}_0'$, and $b' \in N$ such that stp $(\bar{c}_0'{}^\wedge\langle b'\rangle, \text{dbar}) \equiv$ stp $(\bar{c}_0{}^\wedge\langle b\rangle, \bar{d})$. By V. 3.4, stp $(b, \bar{c}_0)$, stp $(b', \bar{c}_0')$ are not orthogonal. By [2] 5.11 (or more elaborately [5] X 7.1) there is b'' realizing stp $(b', \bar{c}_0')$ such that tp $(b, \bar{c}_0 \cup \bar{c}_0' \cup \bar{d} \cup \{b''\})$ forks over $\bar{c}_0 \cup \bar{c}_0' \cup \bar{d}$. Hence easily there is $b^* \in M$ satisfying $\varphi(x, \bar{c}_0')$ such that tp $(b, \bar{c}_0, \bar{c}_0' \cup \bar{d} \cup \{b^*\})$ forks over $\bar{c}_0 \cup \bar{c}_0' \cup \bar{d}$.

Hence tp $(b, A \cup \{b^*\})$ forks over A, and thus tp (b^*, A) is not orthogonal to tp (b, A) (and tp $(\bar{a}, A)$). As $\varphi(x, \bar{c}_0)$ is $\mathrm{cl}^3(r)$-regular, $\varphi(x, \bar{c}_0')$ is $\mathrm{cl}^3(r)$-regular also. So as tp (b^*, A) is not orthogonal to r, it is regular, also it does not fork over $\bar{c}_0'$ hence over N.

§2. A prime atomic model over stable amalgamation is enough

HYPOTHESIS. T is superstable, and *if* $\{M_1, M_2\}$ is independent over M, $M \subseteq M_l$ $(l = 1, 2)$ *then* there is a model N, $\mathbf{F}^l_{\aleph_0}$-prime and $\mathbf{F}^l_{\aleph_0}$-atomic over $M_1 \cup M_2$.

2.1. LEMMA. *Suppose* $\{N_\eta : \eta \in I\}$ *is a non-forking tree* (see III or [2] 3.2). *Then there is a model* N, *which is* $\mathbf{F}^l_{\aleph_0}$-*prime and* $\mathbf{F}^l_{\aleph_0}$-*atomic over* $\bigcup_{\eta \in I} N_\eta$.

PROOF. Let $I = \{\eta_\alpha : \alpha < |I|\}$ be such that for every α and $k < l(\eta_\alpha)$,

$\eta_\alpha \restriction k \in \{\eta_\beta : \beta < \alpha\}$. We define by induction, on $\alpha > 0$, M_α such that

$$(*)\quad \begin{cases} (1) & M_\alpha \text{ is } \mathbf{F}^t_{\aleph_0}\text{-prime and } \mathbf{F}^t_{\aleph_0}\text{-atomic over } \bigcup_{\beta<\alpha} N_{\eta_\beta} \\ & \text{and even over } M_\gamma \cup \bigcup_{\beta<\alpha} N_{\eta_\beta} \text{ for each } \gamma < \alpha. \\ (2) & M_i \ (i \leqq \alpha) \text{ is increasing continuous.} \end{cases}$$

For $\alpha = 1$, let $M_\alpha = N_{\eta_0}$; for α a limit take the union, and for $\alpha = \beta + 1$ use the hypothesis (M, M_1, M_2, N there correspond to $N_{\eta_\alpha} \restriction (l(\eta_\alpha) - 1)$, N_{η_α}, M_β, M_α here). Why does this work? Note that $(\bigcup_{\gamma<\alpha} N_{\eta_\gamma}, \bigcup_{\gamma<|I|} N_{\eta_\gamma})$ satisfies the Tarski–Vaught condition (see 3.2A below). Hence if M_α is $\mathbf{F}^t_{\aleph_0}$-prime and $\mathbf{F}^t_{\aleph_0}$-atomic over $\bigcup_{\beta<\alpha} N_{\eta_\beta}$, then necessarily $\{M_\alpha, \bigcup_{\gamma<|I|} N_{\eta_\gamma}\}$ is independent over $\bigcup_{\beta<\alpha} N_{\eta_\beta}$, and if $\bigcup_{\alpha<|I|} N_{\eta_\gamma} \subseteq N$, F an embedding of M_α into N, $F \restriction \bigcup_{\beta<\alpha} N_{\eta_\beta} =$ the identity, then $F \cup G$ is an elementary embedding, where G is the identity map of $\bigcup_{\gamma<|I|} N_{\eta_\gamma}$.

2.2. The Atomic Decomposition Lemma (in $\mathfrak{C}^{\mathrm{eq}}$). *Suppose T is superstable with the dop. Then for any pair of models $N_1 \subseteq_a M$, there are elements $a_i \in M$ $(i < \alpha)$ and models M_i such that*:

(a) $N_1 \subseteq_a M_i \subseteq_a M$;

(b) $\mathrm{tp}(a_i, N_1)$ *is regular*;

(c) $|M_i| = N_1 \cup \{a_i\} \cup \{b_{i,\alpha} : \alpha < \alpha_i\}$, *for every* α, $b_{i,\alpha} \notin A_{i,\alpha}$ *and one of the following occurs* (*letting* $A_{i,\alpha} = N_1 \cup \{a_i\} \cup \{b_{i,\beta} : \beta < \alpha\}$):

(c1) $\mathrm{tp}(b_{i,\alpha}, A_{i,\alpha})$ *is* $\mathbf{F}^c_{\aleph_0}$-*isolated*,

(c2) $\mathrm{tp}(b_{i,\alpha}, A_{i,\alpha})$ *is orthogonal to* N_1;

(d) *for no* $b \in M - M_i$ *is* $\mathrm{tp}(b, M_i)$ *orthogonal to* N_1;

(e) M *is* $\mathbf{F}^t_{\aleph_0}$-*prime and* $\mathbf{F}^t_{\aleph_0}$-*atomic over* $\bigcup_{i<\alpha} M_i$ (*and* $\mathbf{F}^t_{\aleph_0}$-*minimal*);

(f) $\{M_i : i < \alpha\}$ *is independent over* N_1.

Proof. Let $\mathbf{I} = \{a_i : i < \alpha^*\}$ be a maximal subset of M independent over N_1, of elements realizing regular types of N_1 and for each $i < \alpha^*$ define $b_{i,\alpha}$, M_i, α_i as required in (c), $b_{i,\alpha} \in M - A_{i,\alpha}$ and α_i is maximal. So (b), (c) hold trivially.

Why is $|N_1| \cup \{a_i\} \cup \{b_{i,\alpha} : \alpha < \alpha_i\}$ the universe of a submodel (elementary, of course)? See IV 2.21. Now (d) follows from the choice of α_i.

Clearly (f) follows by 1.3(2) and (a) by 1.2(3) provided that (e) holds. Apply 2.1. Let M' be $\mathbf{F}^t_{\aleph_0}$-prime $\mathbf{F}^t_{\aleph_0}$-atomic model over $\bigcup_{i<\alpha} M_i$. So w.l.o.g. (3) $M' \subseteq M$.

The only missing point is $M' = M$.

If not, there is $c \in M - M'$, $R[\mathrm{tp}(c, M'), L, \infty]$ is minimal. Then by 1.5: $\mathrm{tp}(c, M')$ is orthogonal to N_1, or $\mathrm{tp}(c, M')$ does not fork over N_1, hence by 1.6

$\mathrm{tp}(c, N_1)$ is regular. The latter case contradicts the maximality of **I**. In the former case, we can find N_1^*, M_i^* such that: $N_1 \subseteq N_1^*$, $M_i \subseteq M_i^*$ are $\mathbf{F}^a_{\aleph_0}$-saturated, $\mathrm{tp}(N_1^*, \bigcup_{i<\alpha} M_i)$ does not fork over N_1, $\mathrm{tp}(M_i^*, \bigcup_{j\neq i} M_j^* \cup N_1^*)$ does not fork over $N_1^* \cup M_i$. By 3.3 (next section) the pair $(\bigcup_{i<\alpha} M_i, \bigcup_{i<\alpha} M_i^*)$ satisfies the Tarski–Vaught condition.

Let $\mathrm{tp}(c, M')$ not fork over some $\bar a \in M'$, then $\mathrm{tp}(\bar a\ \bigcup_{i<\alpha} M_i)$ is $\mathbf{F}^l_{\aleph_0}$-isolated, hence $\mathrm{tp}(\bar a, \bigcup_{i<\alpha} M_i^*)$ is $\mathbf{F}^l_{\aleph_0}$-isolated. By [2] §2 (as T does not have the dop) for some i, $\mathrm{tp}(c, M')$ (equivalently, $\mathrm{stp}(c, \bar a)$) is not orthogonal to M_i^*, hence (as $\{M_i^*, \bar a\}$ is independent over M_i), the type is not orthogonal to M_i. For notational simplicity let $i = 0$. As $M_0 \subseteq_a M'$ (by 1.2(3), as mentioned above), we can apply 1.5 and find $c' \in M - M'$, $\mathrm{tp}(c', M')$ does not fork over M_0. If $\mathrm{tp}(c', M_0)$ is not orthogonal to N_1, we can get a contradiction to the maximality of **I**. If $\mathrm{tp}(c', M_0)$ is orthogonal to N_1 we get a contradiction to the choice of M_0.

2.2A. Assertion. We can add the demand: for each i, $\mathrm{tp}(b_{i,\alpha}, A_{i,\alpha})$ is orthogonal to every trivial regular type not orthogonal to N_1.

Proof. The only problem is when A_{i,α_i} is not the universe of an M_i. Let $\alpha = \alpha_i$. If not there is a formula $\varphi(x, \bar b)$, $\bar b \in A_{i,\alpha}$, $\models (\exists x)\varphi(x, \bar b)$ but for no $c \in A_{i,\alpha}$, $\models \varphi(c, \bar b)$. Choose such $\varphi(x, \bar b)$ with minimal $R[\varphi(x, \bar b), L, \infty]$, hence every q, $\varphi(x, \bar b) \in q \in S^1(A_{i,\alpha})$ is $\mathbf{F}^c_{\aleph_0}$-isolated. Let $c \in M$, $\models \varphi[c, \bar b]$, so $\mathrm{tp}(c, A_{i,\alpha})$ is $\mathbf{F}^c_{\aleph_0}$-isolated. So necessarily some trivial regular r is not orthogonal to $\mathrm{stp}(c, A_{i,\alpha})$ and not orthogonal to N_1. We can find $\bar d \in N_1$ such that $\mathrm{tp}(\bar b\,{}^\wedge\langle c\rangle, N_1)$ does not fork over $\bar d$, and r is not orthogonal to $\bar d$. As $N_1 \subseteq_a M$ there are $\bar b', c' \in N_1$ such that $\bar b'\,{}^\wedge\langle c'\rangle$ realizes $\mathrm{stp}(\bar b\,{}^\wedge\langle c\rangle, \bar d)$. It is easy to see that r is not orthogonal to $\mathrm{stp}(c, \bar b')$. So by [2] 5.11 or [5] X 7.1 there is c'' realizing $\mathrm{stp}(c, \bar d \cup \bar b \cup \bar b')$ such that $\mathrm{tp}(c'', \bar d \cup \bar b \cup \bar b' \cup \{c'\})$ fork over $\bar d \cup \bar b \cup \bar b'$. Hence $\mathrm{tp}(c', \bar d \cup \bar b \cup \bar b' \cup \{c''\})$ fork over $\bar d \cup \bar b \cup \bar b'$, so some $c^* \in M$, $\models \varphi(c^*, \bar b)$ and $\mathrm{tp}(c', \bar d \cup \bar b \cup b' \cup \{c^*\})$ fork over $\bar d \cup \bar b \cup \bar b'$ hence $\mathrm{tp}(c^*, \bar d \cup \bar b \cup \bar b' \cup \{c'\})$ fork over $\bar d \cup \bar b \cup \bar b'$. As $\bar d \cup \bar b \cup \bar b' \cup \{c'\} \subseteq A_{i,\alpha}$, we get a contradiction to the choice of $\varphi(x, \bar b)$.

Remark. We have essentially used (and proved):

2.2B. Fact. Suppose $A \subseteq B \subseteq M$, and $\mathrm{tp}(\bar a, A)$ has an extension over B which forks over A. Then for every $\varphi(x, \bar b) \in \mathrm{tp}(\bar a, A)$ there is $\bar a' \in |M|$ such that $\models \varphi[\bar a', \bar b]$ and $\mathrm{tp}(\bar a', B)$ forks over A.

2.3. Claim. *In* 2.2, 2.2A:

(1) *If* $N_{2,i} \subseteq M_i$, $N_1 \cup \{a_i\} \subseteq N_{2,i}$ *then* $N_{2,i} <_{N_1} M_i$.

(2) *If* $p \in S^m(M_i)$ *is orthogonal to* N_1 *and* $\mathrm{Dp}(\mathrm{tp}(a_i, N_i))$ *is* $<\infty$, *then* $\mathrm{Dp}(p) < \mathrm{Dp}(\mathrm{tp}(a_i, N_i))$.

(3) *If* $\bar{a} \in M_i$, $\mathrm{Dp}(\mathrm{tp}(a_i, N_1)) < \infty$, $N_{2,i}$ *as in* (1) *then*

$$\mathrm{Dp}(\mathrm{tp}(\bar{a}, N_{2,i})) < \mathrm{Dp}(\mathrm{tp}(a_i, N_1)).$$

PROOF. (1) By 1.5, 1.6 and I's maximality.

(2) Let N_1^* be $\mathbf{F}^a_{\aleph_0}$-saturated, $N_1 \subseteq N_1^*$, $\{N_1^*, M_i\}$ independent over N_1. Now for every $\alpha < \alpha_i$ (see 2.2) $\mathrm{tp}(b_{i,\alpha}, A_{i,\alpha})$ is orthogonal to every regular $p \in S^m(N_1^*)$ with depth $\geqq 1$. [If p is orthogonal to N_1 then it is orthogonal to any type over N_1^*. Suppose p is not orthogonal to N_1, then by 2.2A, if $\mathrm{tp}(b_{i,\alpha}, A_{i,\alpha})$ is not orthogonal to p then p is not trivial. We finish remembering that by [2] 5.10, as T does not have the dop, any regular type of depth >0 is trivial.] So clearly $\mathrm{tp}_*(M_i, N_1 \cup \{a_i\})$, is orthogonal to every regular complete type of N_1^* of depth >0. We can find $\mathbf{F}^a_{\aleph_\alpha}$-saturated M_i^*, $N_1^* \cup M_i \subseteq M_i^*$ with this property. We can apply [2] 3.2; so the conclusion of 2.3(2), (3) follows.

(3) See the proof of (2).

2.4. THE DECOMPOSITION LEMMA. *Suppose T is superstable without the dop. Then for any model M there is a tree $I(\subseteq^{\omega>}\|M\|)$ and N_η $(\eta \in I)$, a_η $(\eta \in I^-)$ such that*:

(1) $N_\eta \subseteq_a M$ (*hence* $N_\eta < M$);

(2) $N_\eta \subseteq_a \bigcup\{N_\nu; \eta \leqq \nu\}$;

(3) $p_{\eta^\frown\langle i\rangle} = \mathrm{tp}(a_{\eta^\frown\langle i\rangle}, N_\eta)$ *is regular*;

(4) $\mathrm{tp}_*(\bigcup\{N_\nu : \eta^\frown\langle i\rangle \leqq \nu, \nu \in I\} \cup \{N_\nu :$ *not* $\eta^\frown\langle i\rangle \leqq \nu, \nu \in I\})$ *does not fork over* N_η;

(5) $\{a_{\eta^\frown\langle i\rangle} : \eta^\frown\langle i\rangle \in I\}$ *is a maximal subset of M independent of* N_η;

(6) $\mathrm{tp}_*(\bigcup\{N_\nu : \eta^\frown\langle i\rangle \leqq \nu \in I\}, N_{\eta^\frown\langle i\rangle})$ *is orthogonal to* N_η;

(7) *if* $\mathrm{Dp}(p_\eta) < \infty$, $\eta^\frown\langle i\rangle \in I$ *then* $\mathrm{Dp}(p_{\eta^\frown\langle i\rangle}) < \mathrm{Dp}(p_\eta)$.

PROOF. Just combine the proofs of [2] 3.2 and 2.2 (and 2.3(2), (3)).

Now it is no problem to compute the number of non-isomorphic models, as in [2], [3] (using the same depth function).

§3. Universal Theories

3.1. DEFINITION. (1) We call cl a closure operation if:

(i) for every A, $A \subseteq \mathrm{cl}\, A \subseteq \mathrm{acl}\, A$, and for every function symbol F (of $\mathfrak{C}$) and $\bar{a} \in A$, $F^{\mathfrak{C}}(\bar{a}) \in \mathrm{cl}\, A$;

(ii) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}\, A$, and $A \subseteq B$ implies $\mathrm{cl}\, A \subseteq \mathrm{cl}\, B$;

(iii) the property "$a \in \mathrm{cl}\, A$" is preserved by an automorphism of $\mathfrak{C}$.

(2) We call a closure operation cl local if in addition

(iv) for every $b \in \mathrm{cl}\, A$, there are a formula $\varphi(x, \bar{y})$ and a sequence $\bar{a} \in A$ such that $\models \varphi[b, \bar{a}]$ and: $\varphi(b_1, \bar{a}_1)$ implies $b_1 \in \mathrm{cl}(\bar{a}_1)$.

(3) For a set of formulas Φ (of the form $\varphi(\bar{x}, \bar{y})$) let acl_Φ be defined by:

$$\mathrm{acl}^1_\Phi(A) = \bigcup\{\bar{b} : \text{for some } \bar{a} \in A, \text{and } \varphi(\bar{x}, \bar{y}) \in \Phi, \models \varphi[\bar{b}, \bar{a}], \text{ and } \varphi(\bar{x}, \bar{a}) \text{ is an algebraic formula}\} \cup A,$$

$$\mathrm{acl}^0_\Phi(A) = A,$$

$$\mathrm{acl}^{n+1}_\Phi(A) = \mathrm{acl}^1_\Phi(\mathrm{acl}^n_\Phi(A)),$$

$$\mathrm{acl}_\Phi(A) = \bigcup_n \mathrm{acl}^n_\Phi(A).$$

(4) We call cl a Φ-closure operation if it is an operation and $A \subseteq \mathrm{cl}\, A \subseteq \mathrm{acl}_\Phi A$.

3.1A. CLAIM. (1) *Every* acl_Φ *is a local closure operation and is* acl^1_Ψ *for some* Ψ.

(2) *Every local closure operation is a* Φ*-closure operation for some* Φ*, and in Definition* 3.1(2) *there is a* φ *satisfying in addition: for some* n*,* $\models (\forall \bar{y})(\exists^{\leq n} \bar{x})\varphi(\bar{x}, \bar{y})$ *and* $\varphi(\bar{x}, \bar{a}) \vdash \mathrm{tp}(\bar{b}, \bar{a})$.

PROOF. Easy.

3.2. DEFINITION. We call $\langle M_s : s \in I\rangle$ a stable system *if* $M_s \subseteq \mathfrak{C}$, I a family of finite subsets of $\bigcup_{s \in I} s$ closed under subsets,

$$s < t \Rightarrow M_s \subseteq M_t$$

and for every $s \in I$, $\mathrm{tp}_*(M_s, \bigcup_{s \not\subseteq t} M_t)$ does not fork over

$$A_s \stackrel{\mathrm{def}}{=} \cup\{M_t : t \subseteq s, t \neq s\}.$$

We implicitly assume $\mathrm{Th}(\mathfrak{C})$ is stable.

3.3. CLAIM. (1) *If* $I = \{s_\alpha : \alpha < \alpha_0\}$, $[s_\alpha \subseteq s_\beta \Rightarrow \alpha \leqq \beta]$; $M_s < C$, *and* $\mathrm{tp}_*(M_{s_\alpha}, \bigcup_{j < \alpha} M_{s_j})$ *does not fork over* A_{s_α}, then $\langle M_s : s \in I\rangle$ *is a stable system of models*

(2) *If* $\langle M_s : s \in I\rangle$ *is a stable system,* $J \subseteq I$ *and* $s \in I \wedge s \subseteq \cup J \Rightarrow s \in J$, then $(\bigcup_{s \in J} M_s, \bigcup_{s \in I} M_s)$ *satisfies the Tarski–Vaught condition (i.e. if* $\bar{a} \in \bigcup_{s \in J} M_s$, $\bar{b} \in \bigcup_{s \in I} M_s$, $\mathfrak{C} \models \varphi[\bar{a}, \bar{b}]$ *then for some* $\bar{b}' \in \bigcup_{s \in J} M_s$, $\mathfrak{C} \models \varphi[\bar{a}, \bar{b}']$).

(3) *If* $\langle M_s : s \in I\rangle$ *is a stable system,* $N_s < M_s$, $\mathrm{tp}(N_s, \bigcup_{t \subseteq s, t \neq s} M_t)$ *does not fork*

over $\bigcup_{t\subseteq s, t\neq s} N_t$, $[s<t \Rightarrow N_s \subseteq N_t]$ *then* $\langle N_s : s\in I\rangle$ *is a stable system and* $(\bigcup_{s\in I} N_s, \bigcup_{s\in I} M_s)$ *satisfies the Tarski–Vaught condition.*

PROOF. Essentially like [4] 3.5. Since we do not want to assume that the reader is familiar with [4], we prove the claim completely:

3.3A. FACT. If $\langle M_s : s\in I\rangle$ is a stable system $J_0\subseteq I$, $J\subseteq I$, J_0 is closed under subsets then $\mathrm{tp}_*(\bigcup_{t\in J} M_t, \bigcup_{s\in J_0} M_s)$ does not fork over $\bigcup\{M_s : s\in J_0$ and $(\exists t\in J) s\subseteq t\}$.

REMARK. If J is closed under subsets, the last set is $\bigcup\{M_s : s\in J_0\cap J\}$.

PROOF. W.l.o.g. J is closed under subsets, and let $J_1 = J\cap J_0$. We can find a list $\{s_\alpha : \alpha<\alpha^*\}$ of I (and $\alpha_1\leqq\alpha_2\leqq\alpha_3\leqq\alpha$) such that $[s_\alpha\subseteq s_\beta \Rightarrow \alpha\leqq\beta]$, $J_0=\{s_\alpha : \alpha<\alpha_2\}$, $J=\{s_\alpha : \alpha<\alpha_1$, or $\alpha_2\leqq\alpha<\alpha_3\}$. Clearly for $\alpha<\alpha_3$, $\alpha\geqq\alpha_2$, $\mathrm{tp}_*(M_{s_\alpha}, \bigcup_{\beta<\alpha} M_{s_\alpha})$ is included in $\mathrm{tp}_*(M_{s_\alpha}, \bigcup\{M_t : s_\alpha\not\subseteq t, t\in I\})$ hence does not fork over A_{s_α}, but $A_{s_\alpha}\subseteq\bigcup\{M_{s_\beta} : \beta<\alpha_1$ or $\alpha_2\leqq\beta<\alpha\}$. So $\mathrm{tp}_*(M_{s_\alpha}, \bigcup_{\beta<\alpha} M_{s_\beta})$ does not fork over $\bigcup\{M_{s_\beta} : \beta<\alpha_1$, or $\alpha_2\leqq\beta<\alpha\}$. By IV 3.2(1) we can conclude that $\mathrm{tp}_*(\bigcup\{M_{s_\alpha} : \alpha_2\leqq\alpha<\alpha_3$ or $\alpha<\alpha_1\}, \bigcup_{\beta<\alpha_2} M_{s_\beta})$ does not fork over $\bigcup_{\beta<\alpha_1} M_{s_\beta}$, but this is as required.

3.3B. FACT. If $\mathbf{S}=\langle M_s : s\in I\rangle$ is a stable system $\bar a_l\in M_{s(l)}$ $(l<n)$, $t\subseteq\bigcup I$ and $\vDash\varphi[\bar a_0,\dots,\bar a_{n-1}]$ then we can find $\bar a'_l\in M_{s(l)\cap t}$ such that $\vDash\varphi[\bar a'_0,\dots,\bar a'_{n-1}]$ and $s(l)\subseteq t \Rightarrow a'_l = a_l$.

PROOF. W.l.o.g. $s\subseteq s(l)\Rightarrow s\in\{s(m) : m<l\}$. We prove it by induction on n. For $n=0$ there is nothing to prove, and for $n=1$ note $M_{s(l)\cap t}$ is an elementary submodel $M_{s(l)}$. So suppose we have proved for n and we shall prove for $n+1$, i.e. for given $\bar a_l\in M_{s(l)}$ $(l<n+1)$, $t\subseteq\bigcup I$ and φ. W.l.o.g. the $s(l)$ $(l\leqq n)$ are distinct and $s(n)\not\subseteq s(l)$ for $l<n$. We concentrate on the case $s(n)\not\subseteq t$. As $\mathrm{tp}(\bar a_{s(n)}, \bigcup_{l<n} M_{s(l)})$ does not fork over $A^{\mathbf{S}}_{s(n)}$ clearly $\varphi(\bar a_0,\dots,\bar a_{n-1},\bar x)$ does not fork over $A^{\mathbf{S}}_{s(n)}$ hence is realized in every model which includes $A^{\mathbf{S}}_{s(n)}$. So for some type $p=p(\bar x_i)_{i<\alpha}$ over $A^{\mathbf{S}}_{s(n)}$ (infinitely many variables) $p(\bar x_0,\dots,\bar x_i,\dots)\vdash\bigvee_{i\in\alpha}\varphi(\bar a_0,\dots,\bar a_{n-1},\bar x_i)$. So for some $\bar b\subseteq A^{\mathbf{S}}_{s(n)}$ and $\psi=\psi(\bar x_0,\bar x_1,\dots,\bar x_k,\bar b)$

$$\vDash(\exists\bar x_0, x_1,\dots,x_k)\psi(\bar x_0,\dots,\bar x_k,\bar b),$$

$$\psi(\bar x_0,\dots,\bar x_k,\bar b)\vdash\bigvee_{i\leqq k}\varphi(\bar a_0,\dots,\bar a_{n-1},\bar x_i).$$

As $\bar b\subset A^{\mathbf{S}}_{s(n)}$ $(\forall s\subset s(n))\,[s\in\{s(l) : l<n\}]$, w.l.o.g. $\bar b=\bar b_0{}^\frown b_1{}^\frown\cdots{}^\frown\bar b_{n-1}$, $\bar b_l\subseteq M_{s(l)}$, $[s(l)\not\subseteq s(n)\Rightarrow\bar b_l$ empty]. Now apply the induction hypothesis to $\bar a_l{}^\frown\bar b_l\in$

$M_{s(l)}$ (for $l < n$) and the formula

$$(\exists \bar{x}_0, \dots, \bar{x}_k)\psi(\bar{x}_0, \dots, \bar{x}_k, \bar{b}_0, \dots, \bar{b}_{n-1}) \wedge (\forall \bar{x}_0, \dots, \bar{x}_k)$$

$$\left[\psi(\bar{x}_0, \dots, \bar{x}_k, \bar{b}_0, \dots, \bar{b}_{n-1}) \rightarrow \bigvee_{i \leq k} \varphi(\bar{a}_0, \dots, \bar{a}_{n-1}, \bar{x}_i)\right].$$

So there are $\bar{a}'_l \,\hat{}\, \bar{b}'_l \in M_{s(l) \cap t}$ $(l < n)$ satisfying the above formula (and as in 3.3B). Now clearly $\bar{b}'_0 \,\hat{}\, \bar{b}'_1 \,\hat{}\, \cdots \,\hat{}\, \bar{b}'_{n-1} \subseteq A^{s}_{s(n) \cap t}$, hence there are $\bar{c}_0, \dots, \bar{c}_k \in M_{s(n) \cap t}$ such that $\models \psi[\bar{c}_0, \dots, \bar{c}_k, \bar{b}'_0, \dots, \bar{b}'_{n-1}]$. So for some $i \leq k \models \varphi[\bar{a}'_0, \dots, \bar{a}'_{n-1}, \bar{c}_i]$. So $\bar{a}'_0, \dots, \bar{a}'_{n-1}, \bar{a}'_{n-1}, \bar{a}'_n \overset{\text{def}}{=} \bar{c}_i$ are as required.

Proof of 3.3. (1) An exercise in non-forking.

(2) Follows from Fact 3.3B.

(3) First we prove that $\langle N_s : s \in I\rangle$ is a stable system. For every s, $\text{tp}_*(M_s, \cup\{M_t : M_t \in I, s \not\subseteq t\})$ does not fork over $\cup\{M_t : t \subseteq s, t \neq s\}$, hence (as $N_s \subseteq M_s$) also $\text{tp}_*(N_s, \cup\{M_t : t \in I, s \not\subseteq t\})$ does not fork over $\cup\{M_t : t \subseteq s, t \neq s\}$. But $\text{tp}_*(N_s, \cup\{M_t : t \subseteq s, t \neq s\})$ does not fork over $\cup\{N_t : t \subseteq s, t \neq s\}$. So by III 0.1, $\text{tp}_*(N_s, \cup\{M_t : t \in I, s \not\subseteq t\})$ does not fork over $\cup\{N_t : t \subseteq s, t \neq s\}$. As $N_t \subseteq M_t$, by monotonicity of non-forking we get the stability of the system ($s \subseteq t \Rightarrow N_s \subseteq N_t$ was assumed, and we know I is as required).

The Tarski–Vaught condition follows by Fact 3.3B and the following fact.

Let $j \notin \cup I$, $J = I \cup \{s \cup \{j\} : s \in I\}$, and $N_{s \cup \{j\}} = M_s$.

3.3C. Fact. $\langle N_s ; s \in J\rangle$ is a stable system (J, N_s as above).

Proof. Let s_α $(\alpha < \alpha_0)$ be as in 3.3(1), and define t_α $(\alpha < 2\alpha_0)$ by: $t_{2\alpha} = s_\alpha$, $t_{2\alpha+1} = s_\alpha \cup \{j\}$. Clearly $J = \{t_\alpha : \alpha < 2\alpha_0\}$ and $t_\alpha \subseteq t_\beta \Rightarrow \alpha \leq \beta$. Now use 3.3(1): For α even $(= 2\beta)$ remember we have proved $\text{tp}_*(N_{s_\beta}, \cup\{M_s ; s \in I, s_\beta \not\subseteq s\})$ does not fork over $\cup\{N_s : s \subseteq s_\beta, s \neq s_\beta\}$ and this is what we need. For α odd $(= 2\beta + 1)$ remember $\text{tp}_*(M_{s_\beta}, \bigcup_{\gamma<\beta} M_{s_\gamma})$ does not fork over $\cup\{M_s : s \subseteq s_\beta, s \neq s_\beta\})$. As $N_{s_\beta} \subseteq M_{s_\beta}$, by III 0.1 this gives $\text{tp}_*(N_{t_\alpha}, \bigcup_{\gamma<\alpha} N_{t_\gamma}) = \text{tp}_*(M_{s_\beta}, \bigcup_{\gamma<\beta} M_{s_\gamma} \cup N_{s_\beta})$ does not fork over $\cup\{M_s : s \subseteq s_\beta, s \neq s_\beta\} \cup N_{s_\beta} = \cup\{N_s : s \subseteq t_\alpha, s \neq t_\alpha\}$, and this is what we need.

3.3D. Fact. Suppose $\langle M_s : s \in I\rangle$ is a stable system, and each M_s is κ-compact, $a_{i,j} \in M_{s(i)}$ for $i < \alpha$, $j < j_\alpha$, $\Sigma_\alpha j_\alpha < \kappa$, and p is a set of $< \kappa$ formulas, in the variables $x_{i,j}$ $(i < \alpha, j < j_\alpha)$ satisfied by the assignment $x_{i,j} \mapsto a_{i,j}$. If $t \subseteq \cup I$ (not necessarily finite) then we can find $a'_{i,j} \in M_{s(i) \cap t} [s(i) \subseteq t \Rightarrow a'_{i,j} = a_{i,j}]$ such that p is satisfied by the assignment $x_{i,j} \mapsto a'_{i,j}$.

Proof. Use Fact 3.3B, and the observation: a set of formulas (not necessarily over M) of power $< \kappa$ finitely satisfiable in a κ-compact model, is satisfiable in it.

3.3E. FACT. Suppose $\langle M_s : s \in I \rangle$ is a stable system, $t \subseteq \bigcup I$, each M_s is $\mathbf{F}^a_\kappa$-saturated and $a_{i,j} \in M_{s(i)}$ for $i < \alpha$, $j < j_i$, $\Sigma_{i<\alpha} j_i < \kappa$, *then* we can find $a'_{i,j} \in M_{s(i) \cap t}$, such that

$$\mathrm{stp}_*(\langle a'_{i,j} : i < \alpha, j < j_\alpha \rangle, \{a_{i,j} : s(i) \subseteq t\}) \equiv \mathrm{stp}_*(\langle a_{i,j} : i < \alpha, j < j_\alpha \rangle, \{a_{i,j} : s(i) \subseteq t\}).$$

PROOF. Left to the reader.

3.3F. FACT. If $\mathbf{S} = \langle M_s : S \in I \rangle$ is a stable system, then for any $t \in I$, $\mathrm{tp}_*(M_t, \bigcup \{M_s : t \not\subseteq s\})$ is definable over $\overset{s}{A}_t$.

This follows from 3.3(2).

3.4. THE MAIN THEOREM. *Suppose T is stable and* cl *is a Σ_1-closure operation; then at least one of the following holds:*

(A) *If $M_0 < M_1$, M_2, $\{M_1, M_2\}$ independent over M_0, then* $\mathrm{cl}(M_1 \cup M_2) < \mathfrak{C}$.

(B) *There is a set $A = \mathrm{cl}\, A$ such that the theory of $\mathfrak{C} \restriction A$ is unstable ($\mathfrak{C} \restriction A$ is the model $\mathfrak{C}$ restricted to the set A, which by Definition* 3.1(1) *is closed under functions*); *moreover, the theory of $\mathfrak{C} \restriction A$ has the independence property* (*see* [1] II §4). *In fact, we can have $A = \mathrm{cl}(\bigcup_{i<i_0} M_i)$, $i_0 \neq 0$.*

REMARKS. (1) If cl is as above, by 3.5(2) we can assume that in a counterexample to (A), M_1, M_2 are isomorphic over M_0, hence get rid of the predicate P in the proof.

(2) Really we do not need the order $<$ of L, but then we have to work a little more. It is also quite reasonable that we can replace stable by "without the independence property," and then in (A) say "$\mathrm{tp}_*(M_2, M_1)$ is finitely satisfiable in M_0," but this was not checked.

(3) In conclusion 3.6(1) we shall show that when T is universal then (B) implies that some completion of T is unstable.

PROOF. Suppose M_0, M_1, M_2 form a counterexample to (A), and we shall prove (B). Let $\lambda = |T| + \|M_1\| + \|M_2\|$ and choose a model $L = (|L|, <, P, R)$, P a one place predicate, $<$ a (linear) order, R a symmetric and reflexive two-place relation, $L \models (\forall x, y)(P(x) \equiv P(y) \rightarrow xRy)$, which is a λ-homogeneous and λ-universal (i.e. any isomorphism from one submodel of L onto another, both of power $< \lambda$, can be extended to an automorphism of L, and any model L' of power $\leqq \lambda$ satisfying the other conditions can be embedded into L; L may have the power $> \lambda$; see e.g. [1] I 1.8).

We shall now define for every $s \in I \stackrel{\text{def}}{=} \{t : t$ a finite subset of $L\}$ a model $M_s < \mathfrak{C}$ and isomorphism F_h for every $h \in PI$ (see below), Dom $h = s$, such that:

(a) $\langle M_s : s \in I \rangle$ is a stable system.

(b) Let

$$PI = \{h : \text{for some } s, t \in I,\ h \text{ is an isomorphism form } L \restriction s \text{ on } L \restriction t\}.$$

Now for any $h \in PI$ there will be an isomorphism F_h from M_s onto M_t (where $s = \text{Dom}\, h$, $t = \text{Range}(h)$) such that:

(b, α) if $f \subseteq h$ then $F_f \subseteq F_h$,

(b, β) if $h_2 = h_0 h_1$ then $F_{h_2} = F_{h_0} F_{h_1}$,

(b, γ) if h is the identity on $s = \text{Dom}\, h$, then F_h is the identity mapping on M_s.

(c) $M_\varnothing = M_0$, $M_{\{u\}}$ is isomorphic to M_1 over M_0 if $u \in P$ and $M_{\{u\}}$ is isomorphic to M_2 over M_0 if $u \notin P$ (note that $\varnothing \in I$ and we deal exactly with the $u \in L$).

We denote $F_h^n = F_n \restriction \bigcup \{M_s : s \subseteq \text{Dom}\, h, |s| \leqq n\}$. Now the definition is as follows: we define by induction on n, M_s $(s \in I, |s| = n)$ and F_h^n $(h \in PI)$ such that (a), (b), (c) hold in the relevant cases (restricting to F_h^n when appropriate). In the end we shall let $F_h = \bigcup_n F_h^n$.

For $n = 0$ use (c), for $n = 1$ use (c) and the facts on forking (see III 0.1). For $n > 1$ use 3.3: let $\{t_i : i < i(*)\}$ be a list of all $t \in I$, $|t| = n$.

Now we define by induction on $i < i(*)$ the model M_{t_i}. If there is no $j < i$ such that $L \restriction t_i \cong L \restriction t_j$, choose M_{t_i} as any $M < \mathfrak{C}$,

$$A_{t_i} \stackrel{\text{def}}{=} \bigcup_{\substack{s \subseteq t_i \\ s \neq t_i}} M_s \subseteq M, \qquad \|M\| \leqq |T| + |A_{t_i}| \leqq \lambda,$$

and such that $\text{tp}_*(M_{t_i}, \bigcup \{M_s : s \in I, |s| < n\} \cup \bigcup_{j<i} M_{t_j})$ does not fork over A_{t_i} (which is possible by the extension property of non-forking, see III 0.1). If there is $j < i$ such that $L \restriction t_i \cong L \restriction t_j$, choose minimal $j = j(i)$, and let h_i^n be the isomorphism from $L \restriction t_j$ onto $L \restriction t_i$ (it is unique as $<$ is a linear order of L and t_i is finite). Now there is an elementary mapping H_i^n extending $F_{h_i}^{n-1}$ (see above) and whose domain is $M_{t_{j(i)}}$ and $\text{tp}_*(\text{Range}(H_i^n), \bigcup \{M_s : s \in I, |s| < n\} \cup \bigcup_{\alpha < i} M_{t_\alpha})$ does not fork over Range $(F_{h_i}^{n-1})$; but note Range $(F_{h_i}^{n-1}) = A_{t_i}$. Now let $M_{t_i} = \text{Range}(H_i^n)$.

So we have defined all the M_{t_i} but still have to define F_h^n for $h \in PI$. Let $\alpha(0) < \cdots < \alpha(k-1)$ be a list of $\{i : t_i \subseteq \text{Dom}\, h\}$, and let $t_{\beta(l)}$ be the range of $h \restriction t_{\alpha(l)}$. Now we can define F_h^n as $F_h^{n-1} \cup \bigcup_{l<k} H_{\beta(l)}^n (H_{\alpha(l)}^n)^{-1}$. It is easy to check that this is a well-defined one-to-one function with the suitable range and

domain. F^n_h is an elementary mapping by 3.3F and if g extends h, both in PI then F^n_g extend F^n_h. So we have defined M_s, F_h $(s \in I, h \in PI)$ as required.

Now let

$$J = \{s \in I : \text{for every } u \neq v \in s, L \models uRv\}$$

and for any $S \subseteq L$, let $B_S = \mathrm{cl} \bigcup \{M_s : s \subseteq S, s \in J\}$. By the hypothesis (that M_0, M_1, M_2 exemplify the failure of 3.4(A)), if $u \in P$, $v \notin P$, $u, v \in L$ but not uRv then $\mathfrak{C} \restriction B_{\{u,v\}}$ is not an elementary submodel of $\mathfrak{C}$. By 3.3(2), the pair $(B_{\{u,v\}}, B_L)$ satisfies the Tarski–Vaught condition (inside $\mathfrak{C}$). Hence $\mathfrak{C} \restriction B_L$ is not an elementary submodel of $\mathfrak{C}$. So there are $\bar{a} \in B_L$, and a first order formula φ, such that $\mathfrak{C} \models \varphi[\bar{a}]$, $B_L \restriction \mathfrak{C} \models \neg \varphi[\bar{a}]$, so for some $n < \omega$ and finite $t \subseteq L$, φ is a Σ_n-formula or Π_n-formula and $\bar{a} \subseteq B_t$. Among all possible $\bar{a}$, φ, n, t choose an example with minimal n, and for the fixed n, a minimal $|t|$, and for the minimal n and $|t|$, maximal $|\{s : s \subseteq t, s \in J\}|$. It is easy to prove that φ cannot be quantifier-free, nor a Π_n-formula, so $n \geqq 1$, $\varphi(\bar{x}) = (\exists \bar{y})\psi(\bar{y}, \bar{x})$, ψ a Π_{n-1}-formula. It is also very easy to see that necessarily for some $u, v \in t$, $\neg uRv$. Hence $B_t = \mathrm{cl}(B_{t-\{u\}} \cup B_{t-\{v\}})$. So w.l.o.g. there are $\bar{b} \in B_{t-\{v\}}$, $\bar{c} \in B_{t-\{u\}}$ so that $\bar{a}$ is algebraic over $\bar{b} {}^\wedge \bar{c}$, in fact for some Σ_1-formula Θ, $\models \Theta[\bar{a}, \bar{b}, \bar{c}]$ and $\Theta(\bar{x}, \bar{b}, \bar{c})$ is algebraic. Let k be the number of $\bar{a}'$ satisfying $\Theta(\bar{x}, \bar{b}, \bar{c}) \wedge \varphi(\bar{x})$ (in $\mathfrak{C}$) and let $\varphi^*(\bar{y}, \bar{z})$ be defined as

$$(\exists^{\geqq k} \bar{x})(\Theta(\bar{x}, \bar{y}, \bar{z}) \wedge \varphi(\bar{x})).$$

Now for every set $Y \subseteq \lambda \times \lambda$ we can define in L elements u_i, v_i $(i < \lambda)$ such that:

(α) In L, u, u_i realize the same quantifier-free type over $t - \{u, v\}$; and similarly v, v_i realize the same quantifier-free type over $t - \{u, v\}$.

(β) $u_i R v_j$ holds iff $\langle i, j \rangle \in Y$.

(γ) If $u < v$ then for every $i, j < \lambda$, $u_i < v_j$ (in L) and if $v < u$ then for every i, $j < \lambda$, $v_j < u_i$ (in L).

Let g_i be the function with domain $t - \{v\}$, $g_i(u) = u_i$, $g_i(u') = u'$ for $u' \in t - \{u, v\}$. Let h_i be the function with domain $t - \{u\}$, $h_i(v) = v_i$, $h_i(v') = v'$, for $v' \in t - \{u, v\}$. It is easy to check that $g_i, h_i \in PI$ and $g_i \cup h_j \in PI$ iff $\neg u_i R v_j$ (iff $\langle i, j \rangle \notin Y$). Let $\bar{b}_i = F_{g_i}(\bar{b})$, $\bar{c}_j = F_{h_j}(\bar{c})$.

Fact A. $\mathrm{tp}(\bar{b} {}^\wedge \bar{c}, \emptyset) = \mathrm{tp}(\bar{b}_i {}^\wedge \bar{c}_j, \emptyset)$ (in $\mathfrak{C}$).

This is because $F_{g_i} \cup F_{h_j}$ is an elementary mapping (by 3.3F), and $F_{g_i} \cup F_{h_j}(\bar{b} {}^\wedge \bar{c}) = \bar{b}_i {}^\wedge \bar{c}_j$.

Fact B. $(\mathfrak{C} \restriction B_L) \models \varphi^*[\bar{b}_i, \bar{c}_j]$ if $u_i R v_j$.

We have chosen k such that $\mathfrak{C} \models (\exists^{\geq k}\bar{x})[\Theta(\bar{x}, \bar{b}, \bar{c}) \times \varphi(\bar{x})]$, i.e. $\mathfrak{C} \models \varphi^*[\bar{b}, \bar{c}]$, hence by Fact A, $\mathfrak{C} \models \varphi^*[\bar{b}_i, \bar{c}_j]$.

Now if $\mathfrak{C} \restriction B_L \models \neg \varphi^*[\bar{b}_i, \bar{c}_j]$, then $\bar{b}_i {}^\wedge \bar{c}_j$, φ^*, n, $t^* = t \cup \{u_i, v_j\} - \{u, v\}$ contradict the choice of $\bar{a}$, φ, n, t, i.e. φ^* is a Σ_n-formula as Θ is a Σ_1-formula, φ a Σ_n-formula and $n > 0$, also $|t^*| = |t|$, however the maximality of $|\{s : s \subseteq t, s \in J\}|$ is contradicted.

FACT C. $(\mathfrak{C} \restriction B_L) \models \neg \varphi^*[\bar{b}_i, \bar{c}_j]$ if $\neg u_i R v_j$.

If $\bar{a}' \in B_L$, $\mathfrak{C} \restriction B_L \models \Theta[\bar{a}', \bar{b}, \bar{c}] \wedge \varphi[\bar{a}']$ then $\mathfrak{C} \models \Theta[\bar{a}', \bar{b}, \bar{c}]$, $\mathfrak{C} \models \varphi[\bar{a}']$ (by the minimality of n, as φ is a Σ_n-formula; for Θ — trivially). Hence the set of $\bar{a}' \in B_L$ for which $(\mathfrak{C} \restriction B_L) \models \Theta[\bar{a}', \bar{b}, \bar{c}] \wedge \varphi[\bar{a}']$ is a subset of the set of $\bar{a}' \in B_L$ for which $\mathfrak{C} \models \Theta[\bar{a}', \bar{b}, \bar{c}] \wedge \varphi[\bar{a}']$ which is a proper subset of the set of $\bar{a}' \in \mathfrak{C}$ for which $\mathfrak{C} \models \Theta[\bar{a}', \bar{b}, \bar{c}] \wedge \varphi[\bar{a}']$ (as witnessed by $\bar{a}$). So we have just proved $(\mathfrak{C} \restriction B_L) \models \neg \varphi^*[\bar{b}, \bar{c}]$. So it is enough to find an automorphism of $\mathfrak{C} \restriction B_L$ taking $\bar{b} {}^\wedge \bar{c}$ to $b_i {}^\wedge \bar{c}_j$. Now, as we have noted before, $g_i \cup h_j \in PI$, and $F_{g_i \cup h_j}(\bar{b} {}^\wedge \bar{c}) = b_i {}^\wedge c_j$. By choice of L, $g_i \cup h_j$ can be extended to an automorphism f of L. Now $\{F_{f \restriction t} : t \in PI\}$ is a directed family of elementary mapping, hence its union, F, is an elementary mapping. We shall prove that $F^* = F \restriction B_L$ is as required.

(i) $F^*(\bar{b} {}^\wedge \bar{c}) = \bar{b}_i {}^\wedge c_j$: as $F_{g_i \cup h_j}$ belong to the family.

(ii) F maps $\bigcup_{s \in J} M_s$ onto itself by the properties of the F_f's, and as $\mathrm{cl}(\bigcup_{s \in J} M_s) \subseteq \mathrm{Dom}\, F$, clearly by the properties of cl, F^* has to map $\mathrm{cl}(\bigcup_{s \in J} M_s)$ onto itself, so it is an automorphism of $\mathfrak{C} \restriction B_L$.

So we have proved that

$$\mathfrak{C} \restriction B_L \models \varphi^*[\bar{b}_i, \bar{c}_j] \qquad \text{iff } \langle i, j \rangle \in Y$$

where $Y \subseteq \lambda \times \lambda$ was arbitrary. So for some Y, we get that $\mathfrak{C} \restriction B_L$ has a theory with the independence properties.

We can get some more relevant facts.

3.5. CLAIM. (1) *In* 3.4 *we can replace* "cl *is a* Σ_1-*closure operation*" *by* "cl *is a* Φ-*closure operation,*" *provided that*:

(∗) *if* $A = \mathrm{cl}(\bigcup_{i<\alpha} M_i)$, $\bar{a} \in A$, $\varphi(\bar{x}) \in \Phi$ *and* $\mathfrak{C} \restriction A \models \varphi[\bar{a}]$ *then* $\mathfrak{C} \models \varphi[\bar{a}]$.

(2) *If* cl *is local, then* "(A, B) *satisfies the Tarski–Vaught condition*" *implies* "$(\mathrm{cl}\, A, \mathrm{cl}\, B)$ *satisfies the Tarski–Vaught condition.*"

(3) *If* 3.4(A) *holds then for every such* $M_0 < M_1, M_2$, $\mathrm{cl}(M_1 \cup M_2) = \mathrm{acl}(M_1 \cup M_2)$.

PROOF. (1) We replace in the proof Σ_n, Π_n by Σ'_n, Π'_n defined below by induction on n, and then the proof of 3.4 proves the assertion.

Σ'_n: if $n = 0$,, Σ'_n is the set of quantifier-free formulas;
if $n = 1$, Σ'_n is the set Φ of formulas satisfying $(*)$ of 3.5(1);
if $n > 1$, Σ'_n is the closure of Π_{n-1} by conjunctions, disjunctions and existential quantifiers.

Π'_n is the set of negations of Σ'_n formulas.

(2) Trivial.

(3) Trivial.

3.6. CONCLUSION. (1) If T is a stable universal theory (i.e., every completion of T is stable), then for every model $\mathfrak{C}$ of T, 3.4(A) holds.

(2) We can replace "universal" by Σ_2 (i.e. every sentence of T is a Σ_2, or even Σ_2^{al} (defined below).

3.7. DEFINITION. (1) Let Π_1^{al} be the closure of the family of quantifier-free formulas by: universal quantification, conjunctions, disjunctions, and $(\exists^{[1,k]}x)\ldots$ which mean: there are at least one but no more than k $\bar{x}$'s satisfying....

(2) Let $\Sigma_2^{\text{al}} = \{(\exists \bar{y})\varphi(\bar{y},\bar{x}) : \varphi(\bar{y},\bar{x}) \in \Pi_1^{\text{al}}\}$, $\Sigma_1^{\text{al}} = \{\neg\varphi : \varphi \in \Pi_1^{\text{al}}\}$.

PROOF. So $\text{Th}(\mathfrak{C} \restriction B_L)$ is unstable, so what? Here comes the main use of the assumption that the theory T is universal (this is the only use of that fact). Remember the definition of B_L: it was obtained by applying cl on a subset of $\mathfrak{C}$. By part (1) (i) of Definition 3.1 and part (1) (ii) the set B_L is closed under the functions of the model $\mathfrak{C}$ hence it is a submodel of $\mathfrak{C}$. But since T has Π_1 axiomatization, really $\mathfrak{C} \restriction B_L \models T$. So we found a model of T which is unstable (even has the independence property), namely a completion of T $(= \text{Th}(\mathfrak{C} \restriction B_L))$ which is not stable.

(2) By adding constants, translating the Σ_2 to Π_1 axioms, i.e., when applying 3.5, the set B_L such that $\text{Th}(\mathfrak{C} \restriction B_L)$ has the independence property, satisfies: $B_L = \text{cl}\, B_L$ and B_L extend some elementary submodel of $\mathfrak{C}$. We can conclude that $\mathfrak{C} \restriction B_L$ is a model of T.

The proof for Σ_2^{al} is similar.

§4. Examples

4.1. *Unstable*

Let the language have one two-place function, the theory: empty. This T is a variety, not stable and $I(\aleph_\alpha, T) = 2^{\aleph_\alpha}$.

4.2. *Stable not Superstable*

Let the language have ω one-place function F_n, and T be

$$\{(\forall x)[F_n(F_m(x)) = F_m(F_n(x)) = F_n(x)] : n \leqq m < \omega\}.$$

Clearly T is a variety stable not superstable, $I(\aleph_\alpha, T) = 2^{\aleph_\alpha}$. A "natural" model is $({}^{\omega\geqq}\lambda, F_0, \dots, F_n, \dots)$, $F_n(\eta) = \eta \restriction n$.

4.3. *Superstable with the dop*

Let us define a model M: let G be an abelian group of order 2, $|M| = G \times \omega$, with the functions:

c	$\langle 0,0 \rangle$	(c an individual constant),
D	$D(\langle a,n\rangle, \langle b,m\rangle, \langle c,k\rangle) = \langle a-b+c, 0\rangle$	(D a three-place function),
F_k	$F_k(\langle a,n\rangle) = \langle a,k\rangle$	(F_k a one-place function for each $k<\omega$).

Let T be the set of sentences $(\forall x_1 \dots)(\tau = \sigma)$, $(\forall x_1 \dots)(\tau \neq \sigma)$ for terms τ, σ, which M satisfies.

T is universal, superstable with the dop.

4.4. *Superstable without the dop, but Deep*

If L contains just one-place function F, T is empty, then T is a variety and as required.

4.5. Definition. Call T suitable if $I(\aleph_\alpha, T) \geqq |\alpha| + \aleph_0$, $I(\aleph_\alpha, T) \leqq I(\aleph_\beta, T)$ for $\alpha \leqq \beta$.

4.6. Claim. (1) *For universal T_i $(i \leqq \alpha)$ there is a universal T, $|T| = \Sigma_{i<\alpha} |T_i|$ (the power of T_i is $|L(T_i)| + \aleph_0$) such that: if each T_i is suitable then T is too and*

$$I(\lambda, T) = \sum_{i<\alpha} I(\lambda, T_i).$$

(2) *Similarly with $I(\lambda, T) = \Pi_{i<\alpha} I(\lambda, T_i)$.*

Remark. (1) The suitability hypothesis is just to simplify the computation, and anyhow we here encounter only such T's.

(2) E.g. in (2) we should of course write $\mathrm{Min}\{2^\lambda, \Pi_{i<\alpha} I(\lambda, T_i)\}$, but we shall ignore this.

Proof. (1) W.l.o.g. $L(T_i)$ $(i<\alpha)$ are pairwise disjoint. Let c_i $(i<\alpha)$ be new individual constants, P a monadic predicate and let T consist of the following sentences:

$c_i \neq c_\alpha \rightarrow c_j = c_\alpha$	for $i \neq j < \alpha$.
$\neg P(c_i)$	
$c_i = c_\alpha \rightarrow (\forall \bar{x}) R(\bar{x})$	$R \in L(T_i)$ a predicate.
$c_i = c_\alpha \rightarrow (\forall x_1, \dots)[F(x_1, \dots) = x_1]$	$F \in L(T_i)$ a function symbol.
$c_i \neq c_\alpha \rightarrow \psi^P$	$\psi \in T_i$.
$R(x_1, \dots, x_n) \rightarrow \bigwedge_{i=1}^{n} P(x_l)$	$R \in L(T_i)$ a predicate.
$F(x_1, x_2, \dots) \neq x_1 \rightarrow \bigwedge_{l=1}^{n} P(x_l)$	$F \in L(T_i)$ a function symbol.

(2) Let $\{P_i : i < \alpha\}$ be new monadic predicates. T will say: the P_i's are pairwise disjoint, P_i is a model of T_i, and the predicates and function symbols of T_i are trivial when applied to elements not all of which are in P_i.

4.7. Claim. *For a universal T_0 there is a universal T_1, $|T_1| = |T_0|$, such that if T_0 is suitable, T_1 is suitable too; and*

$$I(\lambda, T_1) = 2^{I(\lambda T_0)}.$$

Proof. W.l.o.g. T_0 has no individual constants (replace c by $F_c(x)$, adding $(\forall x, y)\ [F_c(x) = F_c(y)]$ to T_0). T_1 "says": E is an equivalence relation, each equivalence class is a model of T_0, and

$$P(x_1, \dots, x_n) \rightarrow \bigwedge_{l=1}^{n} x_l E x_1,$$

$$F(x_1, \dots, x_n) \neq x_1 \rightarrow \bigwedge_{l=1}^{n} x_l E x_1$$

for every predicate P and function symbol F of T_0.

4.8. Claim. *Claims 4.6 and 4.7 hold for "quasi-varieties" and for "totally transcendental universal" and for "total transcendental quasi-varieties" instead of "universal" provided that for the quasi-variety cases the language has no function symbols.*

Proof. The problem is taking care of the "quasi-variety".

Case I. 4.6(1) We assume w.l.o.g. $L(T_i)$ are pairwise disjoint and with no individual constant. Let P_i $(i < \alpha)$ be new monadic predicates. T will say:

(a) $P_i(x) \wedge P_j(y) \rightarrow \varphi$ for $i \neq j$, φ an atomic formula,

(b) $P_i(x) \to P_i(y)$ for $i < \alpha$,

(c) $R(x_1, \ldots, x_n) \to P_i(y)$ for R a predicate of $L(T_i)$,

(d) ψ $\psi \in T_i$ for some $i < \alpha$.

Of course $L(T) = \bigcup_{i<\alpha} L(T_i) \cup \{P_i : i < \alpha\}$. Let M be a model of T. If $P_i^M = \emptyset$ for $i < \alpha$ by (c), (d) all the relations of M are empty so M is determined up to isomorphism by its cardinality.

If $P_{i(0)}^M \neq \emptyset$, $P_{i(1)}^M \neq \emptyset$ and $i(0) \neq i(1)$ the model M is again determined up to isomorphism by its cardinality: by (a).

Lastly suppose $P_{i(0)}^M \neq \emptyset$, $P_j^M = \emptyset$ for $j \neq i(0)$. By (b) $P_{i(0)}^M \models |M|$, by (c) $R^M = \emptyset$ for $R \in L(T_j)$, $j \neq i(0)$, and by (d) $M \upharpoonright L(T_{i(0)})$ is a model of $T_{i(0)}$. As in the cases with inclusion we can conclude:

$$I(\lambda, T) = 2 + \sum_{i<\alpha} I(\lambda T_i).$$

Case II. 4.6(2) Let T say:

(a) $P_i(x) \times P_j(x) \to \varphi$, $i < j < \alpha$, φ any atomic formula,

(b) $R(x_1, \ldots, x_n) \to P_i(x_l)$, R an n-place predicate is $L(T_i)$, $l \in \{1, \ldots, n\}$,

(c) $n \bigwedge_m P_i(x_m) \wedge \bigwedge_{l=1}^{m} \psi_l \to \psi$, $\bigwedge_{l=1}^{n} \psi_l \to \psi$ an axiom of T_i.

Case III. 4.7 Similar.

REMARK. We can add to the definition of *suitable*

(iii) $I(\lambda, T \cup \{c = c\}) = I(\lambda, T)$ for c a new individual constant.

Then also for quasi-varieties we can allow the languages to have individual constants and get similar results.

4.9. CLAIM. (1) *There is a variety T_κ ($\kappa \geq \aleph_0$ a cardinal) $|T_\kappa| = \kappa$, $I(\aleph_\alpha, T) = (|\alpha| + \aleph_0)^{2^\kappa}$.*

(2) *There is a variety T, $I(\aleph_\alpha, T)$ is 1 for $\alpha > 0$ and $\aleph_0$ for $\alpha = 0$, $|T| = \aleph_0$.*

(3) *There is a variety T, $I(\aleph_\alpha, T) = 1$, $|T| = \aleph_0$.*

(4) *For any $n < \omega$, there is a variety T, $I(\aleph_\alpha, T) = \beth_n(|\alpha| + \aleph_0)$.*

PROOF. (1) Let the language have κ one-place functions F_i ($i < \kappa$), and

$$T = \{(\forall x) F_i(F_i(x)) = x : i < \kappa\}.$$

(2) The axioms of T are those of a vector space over the rationals (i.e., for each rational number there is a one-place function symbol for multiplication by it and of course we have the addition function).

(3) T is empty.

(4) When $n > 0$ let T be $(\forall x)F^{n+1}(x) = c$. For $n = 0$ let $T = \{(\forall x)F(F(x)) = x\}$. A model of T, $M = \langle |M|, F^M\rangle$ is determined up to isomorphism by the following two cardinals: $|\{a \in M : F(a) = a\}|$ and $|\{a \in M : F(a) \neq a\}|$.

4.10. Claim. (1) *There is a countable variety* T, $I(\aleph_\alpha, T) = \beth_m((|\alpha| + \aleph_0)^{2^{\aleph_0}})$.
(2) *There is a countable variety* T, $I(\aleph_\alpha, T) = \beth_m((|\alpha| + \aleph_0)^{\aleph_0})$.

Proof. (1) For $m = 0$, let T consist of

$$(\forall x)F_n(F_n(x)) = x \quad (\text{for } n < \omega).$$

For $m > 0$ let T consist of (G, F_n are unary functions):

$$G^{m+2}(x) = G^{m+1}(x),$$

$$F_n(F_n(x)) = x,$$

$$F_n(G(x)) = G(x),$$

$$G(F_n(x)) = G(x) \text{ for } n < \omega.$$

(2) First let $m = 0$, and T_0 will consist of $(\forall x)[F(F^{-1}(x)) = x = F^{-1}(F(x))]$ (i.e., essentially one unary one-to-one function). So a model M is characterized by the cardinals

$$\lambda_n(M) = |\{x \in M : n \geqq 1 \text{ is minimal such that } F^n(x) = x\}/E\,|$$

where

$$xEy \stackrel{\text{def}}{=} (\exists n)\Big(x = F^n(y) \vee y = F^n(x)\Big).$$

(E is an equivalence relation by our axioms, F^n is the nth power of F.)
For $m > 0$, $T = T_m$ will consist of (F, G, G^{-1} are unary functions)

$$F^{m+1}(x) = F^{m+2}(x),$$

$$G(F(x)) = F(G(x)),$$

$$F(G(x)) = F(x),$$

$$G(G^{-1}(x)) = x = G^{-1}(G(x)).$$

Note that if $M \models T$, $A_l = \text{Range}(F^l)$, then G is the identity on A_1: for $x \in A_1$, there is y, $x = F(y)$ so $x = F(y) = F(G(y)) = G(F(y)) = G(x)$.

4.11. Conjecture. For every variety T, either $I(\lambda, T) = 2^\lambda$ for $\lambda > |T|$ or $I(\aleph_\alpha, T) < \beth_\omega(|\alpha| + |T|)$ for every α.

§5. Counterexample

5.1. Example. For each $n<\omega$ there is a universal theory T_n, which has an unstable completion, but if $\varphi(\bar{x},\bar{y})$ is an unstable formula in such a completion, then φ is not Σ_n.

We first prove:

5.2. Claim. *We can define by induction on n a theory T_n such that:*

(A) *T_n is universal, countable $\aleph_0$-stable, and in fact every completion of it has a finite depth* (see [2, §4]), *in fact depth n, and $I(\aleph_\alpha,T_n)=\beth_n(|\alpha|+\aleph_0)$.*

(B) *T_n is complete for $\bigcup_{m<n}(\Sigma_m\cup\Pi_m)$ sentences.*

(C) *T_n is not complete for Σ_n-sentences, but only sentences from $\Sigma_n\cup\Pi_m$ are needed for completion, i.e. if T is a completion of T_n then*

$$T_n\cup\{\varphi:\varphi\in T,\ \varphi \text{ a } \Sigma_n \text{ or } \Pi_n \text{ sentence}\}$$

is complete.

(D) *For every completion T of T_n, there is a complete universal countable $\aleph_0$-stable theory $S(T)$ of depth n (maybe in a larger language), $T\subseteq S(T)$, $S(T)$ as in* (A), *such that $I(\aleph_\alpha,S(T))\leqq\beth_n(|\alpha|+\aleph_0)$.*

(E) *$T_n^*=T_n\cup\{\Theta:\Theta$ a Σ_n-sentence consistent with $T_n\}$ is a consistent theory.*

(F) *L_n, the language of T_n, as well as the languages of $S(T)$ from* (D), *consists of predicates and one-place functions only.*

(G) *The language of $S(T)$ (for T a completion of T_n) is L_n^+ (the language does not depend on the theory T, only on n), and for every sentence $\psi\in L(T_n)$ there is a sentence $\Theta_\psi^n\in L_n$ such that for every completion T of T_n $[\psi\in T\leftrightarrow\Theta_\psi^n\in S(T)]$.*

Proof. *For* $n=1$. We let L_0 contain the equality sign only, and $T_0=\{(\forall x)x=x\}$.

Clearly T_n^* is (equivalent to) $\{(\exists x_1\cdots x_k)\bigwedge_{i<j}x_i\neq x_j:k<\omega\}$ (hence T_n^* is consistent, i.e. (E) holds). Also any other completion of T_0 is the theory of a finite model, i.e. it is

$$T_0^l=\left\{\{(\exists x_1\cdots x_l)\}\bigwedge_{i<j}x_i\neq x_j\right\}\cup\left\{(\forall x_1\cdots x_{l+1})\bigvee_{i<j}x_i=x_j\right\}\qquad\text{for some } l.$$

It is now easy to prove (D): e.g., for $T=T_0^k$

$$S(T)=\{(\forall x)(\forall y)F_n(x)=F_n(y):n<k\}$$
$$\cup\left\{(\forall x_1,\ldots,x_{l+1}),\ \bigvee_{i<j}x_i=x_j\right\}$$
$$\cup\{(\forall x)F_n(x)\neq F_m(x):n<m<k\}.$$

So $L_n^+ = L_n \cup \{F_n : n < \omega\}$. (We use one place functions instead of individual constants just for the convenience of the induction, formally — so that (F) holds.) The other parts are very easy, too.

For $n+1$. So T_n, L_n, T_n^*, $S(T)$, L_n^+ are defined and satisfy (A)–(F).

Let S^l $(l < \omega)$ be each a copy of some theories $S(T)$ such that every sentence in L_n consistent with T_n appears in infinitely many S^l, their languages are disjoint outside L_n, and F_n^m, $E_n \in S^l$ $(l, m < \omega)$. Now L_{n+1} consists of $L_n \cup \{E_n\} \cup \{F_l^n : l < \omega\} \cup \bigcup_{l<\omega} L(S^l)$, E_n a two place relation, F_n^m a one place function symbol (so it is still countable). T_{n+1} will consist of sentences saying the following:

(i) E is an equivalence relation, and for any function symbol F of L_n (necessarily one place by (F)), and the choice of F_l^n

$$(\forall x)[xE_nF(x)].$$

(ii) Each E_n-equivalence class is a model of T_n, i.e. if $\forall x_1 \cdots x_{k-1}\psi \in T_n$ (ψ quantifier-free) then

$$(\forall x_0 \cdots x_{k-1})\left(\bigwedge_{l<k} x_0E_nx_l \rightarrow \psi\right).$$

(iii) F_n^l is really a hidden individual constant, i.e.

$$\forall x \forall y (F_n^l(x) = F_n^l(y))$$

and those "constants" are not E_n-equivalent, i.e.

$$(\forall x)[\neg F_n^l(x)E_nF_n^k(x)] \qquad \text{for } l < k < \omega.$$

(iv) The E_n-equivalence class of $F_n^l(x)$ is a model of S^l. It is easy to check that T_{n+1} is as required.

(v) Every function symbol of S^l not in L_n is the identity outside the equivalence class which is a model of S^l (and similarly for other non-logical symbols).

REMARK. In 5.2(A) we could add that T_n has only countably many completions.

PROOF OF 5.1. Let P be a place predicate, F a two-place function, E a two-place predicate, all not in T_n from 5.2. Now T will say:

(a) F is a pairing function from P into $\neg P$:

$$(\forall xy)\Big[\neg P(x) \vee \neg P(y) \rightarrow F(x, y) = x\Big],$$

$$(\forall xy)\left[P(x) \wedge P(y) \rightarrow \neg P(F(x,y))\right],$$
$$(\forall x_1 x_2 y_1 y_2)\Big[P(x_1) \wedge P(y_1) \wedge P(x_2) \wedge P(y_2) \rightarrow$$
$$\left[F(x_1, y_1) = F(x_2, y_2) \equiv \left(x_1 = x_2 \wedge y_1 = y_2\right)\right]\Big].$$

(b) E is an equivalence relation on $\neg P$, and each equivalence class contains at most one $F(x, y)$.

(c) Each equivalence class is a model of T_n and the unary functions of $L(T_n)$ are the identity on $\neg P$.

(d) If $\neg P(F(x, y))$, then $F(x, y)$ is $F^0_{n-1}(F(x, y))$, i.e. it is one of individual constants of T_n which we have hidden for technical reasons as one place functions.

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