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More Jonsson Algebras

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Abstract. We prove that on many inaccessible cardinals there is a Jonsson algebra, so e.g. the first regular Jonsson cardinal λ is $\lambda \times \omega$-Mahlo. We give further restrictions on successor of singulars which are Jonsson cardinals. E.g. there is a Jonsson algebra of cardinality $\beth^+_\omega$. Lastly, we give further information on guessing of clubs.

Annotated content

§1 Jonsson algebras on higher Mahlos and $\mathrm{id}^\gamma_{\mathrm{rk}}(\lambda)$.

[We return to the ideal of subsets of $A \subseteq \lambda$ of ranks $< \gamma$ (for self-containment; see [Sh:g, IV],1.1–1.6) for $\gamma < \lambda^+$; we deal again with guessing of clubs (1.11). Then we prove that there are Jonsson algebras on λ for λ inaccessible not $(\lambda \times \omega)$-Mahlo (1.1, 1.25)].

§2 Back to successor of singulars.

[We deal with $\lambda = \mu^+$, μ singular of uncountable cofinality. We give sufficient conditions for $\mu^+ \not\to \left[\mu^+\right]^{<n}_\theta$, (2.6, 2.7), in particular on $\beth^+_\omega$ there is a Jonsson algebra and if $\mathrm{cf}(\mu) < \mu < 2^{<\mu} < 2^\mu$ then on μ^+ there is a Jonsson algebra. Also if $\mathrm{cf}(\mu) \le \kappa$, $2^{\kappa^+} < \mu$, $\mathrm{id}_p(\bar C, \bar I)$ is a proper ideal not weakly κ^+-saturated and each I_δ is κ-based, then λ is close to being "$\mathrm{cf}(\mu)$-supercompact" (note that such $\bar C$ exists if $\lambda \to [\lambda]^2_{\kappa^+}$)].

§3 More on guessing clubs.

[We prove that, e.g. if $\lambda = \aleph_1$, $S \subseteq \{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_1\}$ is stationary, then we can find a strict λ-club system $\bar C = \langle C_\delta : \delta \in S\rangle$ and $h_\delta : C_\delta \to \omega$ such that for every club E of $\aleph_2$ for stationarily many $\delta \in S$, $\mathrm{nacc}(C_\delta) \cap E \cap h^{-1}_\delta\{n\}$ is unbounded in δ for each n. Also we have such $\bar C$ with a property like the one in Fodor's Lemma. Also we have such $\bar C$'s satisfying: for every club E of λ, for stationarily many $\delta \in S \cap \mathrm{acc}(E)$ we have $\{\sup(E \cap C_\delta \cap \alpha) : \alpha \in E \cap \mathrm{nacc}(C_\delta)\}$ is a stationary subset of δ].

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The sections are independent.
This paper is continued in [EiSh 535] getting e.g. $\Pr_1(\lambda, \lambda, \lambda, \aleph_0)$ for e.g. $\lambda = \beth_\omega^+$. It is further continued in [Sh 572] getting e.g. $\Pr_1(\aleph_2, \aleph_2, \aleph_2, \aleph_0)$ and more on guessing of clubs. We thank Todd Eisworth for detecting various mistakes and errors.

1. Jonsson algebras on higher Mahlos and $\mathrm{id}^\gamma_{\mathrm{rk}}(\lambda)$

We continue [Sh:g, III], [Sh:g, IV], see history there, and we use some theorems from there.

Our main result: if λ is inaccessible not $\lambda \times \omega$-Mahlo then on λ there is a Jonsson cardinal. If the reader is willing to lose 1.29 he can ignore also 1.6(1), 1.7, 1.8(2), 1.9, 1.11, 1.12, 1.13, 1.15, 1.16(2), 1.28, 1.29; also, 1.12 is just for "pure club guessing interest". Why "$< \lambda \times \omega$" just as $\gamma \neq \lambda + \gamma \Rightarrow \gamma < \lambda \times \omega$.

1.1 Theorem. *1) Suppose λ is inaccessible and λ is not $(\lambda \times \omega)$-Mahlo.*
Then on λ there is a Jonsson algebra.
2) Instead of "λ not $(\lambda \times \omega)$-Mahlo" it suffices to assume there is a stationary set A of singulars satisfying (on $\mathrm{id}^\gamma_{\mathrm{rk}}(\lambda)$ see below):
$\{\delta < \lambda : \delta \ inaccessible, A \cap \delta \ stationary\} \in \mathrm{id}^\gamma_{\mathrm{rk}}(\lambda)$, $A \notin \mathrm{id}^\gamma_{\mathrm{rk}}(\lambda)$ *and* $\gamma < \lambda \times \omega$.

Proof. 1) If λ is not λ-Mahlo, use [Sh:g, IV,2.14,p.212]. Otherwise this is a particular case of 1.25 as there are $n < \omega$ and $E \subseteq \lambda$, a club of λ such that $\mu \in E \,\&\, \mu$ inaccessible $\Rightarrow \mu$ is not $\mu \times n$-Mahlo. So $S = \{\delta \in E : \mathrm{cf}(\delta) < \delta\}$ is as required in 1.25.
2) Look at 1.25. $\square_{1.1}$

1.2 Definition. *We say $\bar{e}$ is a strict (or strict* or almost strict) λ^+-club system if:*

- (a) $\bar{e} = \langle e_i : i < \lambda^+ \text{ limit}\rangle$,
- (b) *e_i a club of i*
- (c) *$otp(e_i) = \mathrm{cf}(i)$ for the strict case and $\mathrm{otp}(e_i) \leq \lambda$ for the strict* case and $i > \lambda \Rightarrow \mathrm{otp}(e_i) < i$ for the almost strict case (so in the strict* case, $cf(i) < \lambda \Rightarrow \mathrm{otp}(e_i) < \lambda$ and $cf(i) = \lambda \Rightarrow \mathrm{otp}(e_i) = \lambda$).*

1.3 Definition. *1) For λ inaccessible, $\gamma < \lambda^+$, let $S \in \mathrm{id}^\gamma_{\mathrm{rk}}(\lambda)$ iff for every*[1] *strict* λ^+-club system $\bar{e}$, the following sequence $\langle A_i : i \leq \gamma\rangle$ of subsets of λ defined below satisfies "A_γ is not stationary":*

- *(i) $A_0 = S \cup \{\delta < \lambda : S \cap \delta$ stationary in $\delta\}$*
- *(ii) $A_{i+1} = \{\delta < \lambda : A_i \cap \delta$ stationary in δ so $\mathrm{cf}(\delta) > \aleph_0\}$*
- *(iii) if i is a limit ordinal, then for the club e_i of i of order type $\leq \lambda$ we have*[2]*:*

[1] equivalently some — see 1.4
[2] We may consider adding a second clause: (b) if i is inaccessible, $\aleph_0 < i < \lambda$ then $\mathrm{cf}(\delta) > i$; this influences 1.5(6); true, it has only "local" effect that is the two definitions agree for γ except when for some inaccessible i, $\aleph_0 < i \leq \gamma < i + \omega < \lambda$; in [Sh:g, IV] we use the version with clause (b)

$$A_i = \{\delta < \lambda : \text{ if } j \in e_i, \text{ and } [\,\mathrm{cf}(i) = \lambda \Rightarrow \ \mathrm{otp}(j \cap e_i) < \delta] \text{ then } \delta \in A_j\}$$

2) We define $\mathrm{rk}_\lambda(A)$ *as* $\mathrm{Min}\{\gamma : A \in \ \mathrm{id}^{\gamma}_{\mathrm{rk}}(\lambda)\}$ *for* $A \subseteq \lambda$.
3) $\mathrm{id}^{<\gamma}_{\mathrm{rk}}(\lambda) = \bigcup_{\beta<\gamma} \mathrm{id}^{\beta}_{\mathrm{rk}}(\lambda)$.
4) Let $A^{[i,\bar e]}$ *be* A_i *from part (1) for our* $\bar e$ *and* $S =: A$*; if* $i < \lambda \times \omega$ *we may omit* $\bar e$ *meaning* $e_\delta = \{j : \lambda + j \geq \delta\}$ *for limit* $\delta \leq i$.
5) For λ *a cardinal of uncountable cofinality and ordinal* $\gamma < \lambda$ *we define* $id^{\gamma}_{\mathrm{rk}}(\lambda)$, $\mathrm{rk}_\lambda(A)$ *and* $A^{[i]}$ *as above (so* $e_\delta = \delta$ *for limit* $\delta \leq \gamma$*)*

1.4 Claim. Let λ be inaccessible or a limit cardinal of uncountable cofinality.
0) If $\alpha < \beta < \lambda^+, S, \bar e, A^{[i,\bar e]}$ are as in Definition 1.3 <u>then</u> $A^{[\beta,\bar e]}\backslash A^{[\alpha,\bar e]}$ is a non-stationary[3] subset of λ and $\{\zeta < \lambda : \zeta \notin A^{[\alpha,\bar e]}, \mathrm{cf}(\zeta) > \aleph_0$ but $A^{[\alpha,\bar e]}$ is a stationary subset of $\zeta\}$ is not stationary in λ, (in fact, both are empty if $\beta < \alpha + \lambda$).
1) If $\gamma < \lambda^+$, $S \subseteq \lambda$ and for some strict* λ^+-club system $\bar e$, the condition in Definition 1.3 holds, <u>then</u> $S \in \mathrm{id}^{\gamma}_{\mathrm{rk}}(\lambda)$ (i.e. this holds for every such $\bar e$).
2) If $\bar e$, $\langle A_i : i \leq \gamma\rangle$ are as in Definition 1.3 then $i + \mathrm{rk}_\lambda(A_i) = \mathrm{rk}_\lambda(A_0)$.
3) If $\delta \in A^{[\gamma,\bar e]}$ so a limit ordinal and $\lambda > \gamma > 0$, <u>then</u> $\mathrm{cf}(\delta) \geq \aleph_\gamma$ and if $\gamma \geq \lambda$ then λ is inaccessible.
4) Let $\bar e$ be a strict* λ^+-club system. If $\gamma < \mu = cf(\mu) < cf(\lambda)$ and $\langle A_i : i < \mu\rangle$ is an increasing sequence of subsets of λ with union A and $(\forall \delta \in A)(cf(\delta) > \mu)$ or $(\forall\delta < \lambda)(cf(\delta) = \mu \to A \cap \delta$ not stationary in δ), <u>then</u> $A^{[\gamma,\bar e]} = \bigcup_{i<\mu} A_i^{[\gamma,\bar e]}$, note also that $\langle A_i^{[\gamma,\bar e]} : i < \mu\rangle$ is increasing.
5) Let $\bar e$ be a strict* λ^+-club system. If λ is inaccessible, $\langle A_i : i < \lambda\rangle$ is an increasing sequence of subsets of λ and $A = \{\delta < \lambda : \delta \in \bigcup_{i<\delta} A_i\}$ and $\gamma < cf(\lambda)$ <u>then</u> $A^{[\gamma,\bar e]}\backslash(\gamma + 1) \subseteq \cup\{\delta < \lambda : \delta \in \bigcup_{i<\delta} A_i^{[\gamma,\bar e]}$ and $\delta > \gamma\}$.
6) If $\mathrm{cf}(\lambda) \leq \aleph_\gamma < \lambda$, <u>then</u> $\mathrm{id}^{\gamma}_{\mathrm{rk}}(\lambda) = \mathcal{P}(\lambda)$.

Proof. 0) By induction on β.
1) For $\ell = 1, 2$ let $\bar e^\ell$ be a strict* club system and let $\langle A_i^\ell : i \leq \gamma\rangle$ be defined as in Definition 1.3 using $\bar e^\ell$. We can prove by induction on $\beta \leq \gamma$ that

- $(*)_\beta$ there is a club C_β of λ such that for each $\alpha \leq \beta$, the symmetric difference of $A^1_\alpha \cap C_\beta$ and $A^2_\alpha \cap C_\beta$ is bounded (in λ).

2) Check.
3) By induction on γ.
4) We prove this by induction on γ. For $\gamma = 0$ this is trivial. For γ successor, by Definition 1.4(1)(iii) this is easy by the last assumption. For γ limit, by clause (iii) in 1.3(1), if $\delta \in A^{[\gamma,\bar e]}$ then $(\forall j \in e_\gamma)[\delta \in A^{[j,\bar e]}]$, recalling $\gamma < \mu < \lambda$. So for $j \in e_\gamma$ as $\langle A_i^{[j,\bar e]} : i < \mu\rangle$ is increasing with union $A^{[j,\bar e]}$ by the induction hypothesis for some $i(j,\delta) < \mu$ we have $i \in [i(j,\delta),\mu) \Rightarrow \delta \in A_i^{[j,\bar e]}$. As $|e_\gamma| \leq \gamma < \mu = \ \mathrm{cf}(\mu)$ necessarily $i(\delta) = \sup\{i(j,\delta) : j \in e_\delta\} < \mu$, so $\delta \in \bigcap_{j\in e_\delta} A_{i(\delta)}^{[j,\bar e]}$ which means $\delta \in A_{i(\delta)}^{[\gamma,\bar e]}$. As δ was any member of $A^{[\gamma,\bar e]}$ we can conclude that $A^{[\gamma,\bar e]} \subseteq \bigcup_{i<\mu} A_i^{[\gamma,\bar e]}$, but by monotonicity of the function

[3] in fact, bounded

$B \mapsto B^{[\gamma,\bar{e}]}$ we get $A_i^{[\gamma,\bar{e}]} \subseteq A^{[\gamma,\bar{e}]}$, hence we are done.
5) Similar proof.
6) By part (3). $\square_{1.4}$

1.5 Claim. Let λ be inaccessible or a limit cardinal of uncountable cofinality.

0) For $\gamma < \lambda^+$, the family $\mathrm{id}^{\gamma}_{\mathrm{rk}}(\lambda)$ is an ideal on λ including all non-stationary subsets of λ.
1) If $S \subseteq \lambda, \gamma = \mathrm{rk}_\lambda(S), \zeta < \gamma, S' = S^{[\zeta,\bar{e}]}$ ($\bar{e}$ as in Definition 1.3(1)) <u>then</u> $\zeta + \mathrm{rk}_\lambda(S') = \gamma$.
2) In (1) if $\zeta < \gamma = \zeta + \gamma$ (e.g. $\zeta < \lambda \leq \gamma$) <u>then</u> $\mathrm{rk}_\lambda(S') = \gamma$.
3) Assume $S \subseteq \lambda, \zeta < \lambda$ and δ is a limit ordinal $\delta \in S^{[\zeta,\bar{e}]}$ and let $\varepsilon = \zeta + 1$ except that when $\zeta < \omega$ or $\zeta = i + n \,\&\, 0 < i < \lambda \,\&$ [i inaccessible] we let $\varepsilon = \zeta$. <u>Then</u> we have: $\mathrm{cf}(\delta) \geq \aleph_\varepsilon$, moreover $\mathrm{cf}(\delta) \geq \mathrm{Min}\{cf(\alpha)^{+\varepsilon} : \alpha \in S\}$.
4) Assume
   (a) $\mu \leq \lambda$ inaccessible
   (b) $\gamma = \lambda \times n + \beta, n < \omega, \beta < \mu$
   (c) $A \subseteq \lambda$.
   <u>Then</u> $A^{[\gamma]} \cap \mu = (A \cap \mu)^{[\mu \times n + \beta]}$, recalling Definition 1.3(4).
5) Assume $\gamma < \mathrm{cf}(\mu) \leq \mu < \lambda$, $A \subseteq \lambda$ <u>then</u> $A^{[\gamma]} \cap \mu = (A \cap \mu)^{[\gamma]}$.
6) If $\mu = cf(\mu) < cf(\lambda)$ and $\gamma < \mu$ <u>then</u> $\mathrm{id}^{\gamma}_{\mathrm{rk}}(\lambda) + \{\delta < \lambda : cf(\delta) \leq \mu\}$ is μ-indecomposable (see Definition 1.6(2) below and Claim 1.4(4) above).
7) If $\gamma < cf(\lambda)$ <u>then</u> $\mathrm{id}^{\gamma}_{\mathrm{rk}}(\lambda)$ is a weakly normal ideal (see Definition 1.6(1) below, possibly it is $\mathcal{P}(\lambda)$).
8) For λ inaccessible and $\gamma < \lambda^+$ we have: λ is γ-Mahlo iff $\lambda \notin id^{\gamma}_{\mathrm{rk}}(\lambda)$.
9) For λ inaccessible, $n < \omega, \beta < \lambda$ and $A \subseteq \lambda$ we have: $\mathrm{rk}_\lambda(A) \leq \lambda \times n + \beta$ <u>iff</u> for some club E of λ we have $\mu \in E \,\&\, cf(\mu) > \aleph_0 \Rightarrow rk_\mu(A \cap \mu) < \mu \times n + \beta$.

Proof. Straight (parts (6), (7) like the proof of 1.11(6)). $\square_{1.7}$

Recall

1.6 Definition. *1) An ideal I on a cardinal λ of uncountable cofinality is called <u>weakly normal</u> if it contains all bounded subsets of λ and: for every $f : \lambda \to \lambda$ satisfying $f(\alpha) < 1 + \alpha$ and $A \in I^+$, for some $\beta < \lambda$ we have $\{\alpha \in A : f(\alpha) < \beta\} \in I^+$.*

2) An ideal I is μ-indecomposable when: for any sequence $\langle A_i : i < \mu\rangle$ of subsets of λ if $\bigcup_{i<\mu} A_i \in I^+$ then for some $w \subseteq \mu$ of cardinality $< \mu$ we have $\bigcup_{i \in w} A_i \in I^+$; clearly if μ is regular then without loss of generality *$\langle A_i : i < \mu\rangle$ is increasing.*

1.7 Observation. Suppose $\langle I_i : i < \lambda\rangle$ is an increasing sequence of μ-indecomposable ideals on the regular cardinal λ, each including the bounded subsets of λ, $\mu < \lambda$ is regular and

$$I = \Big\{A \subseteq \lambda : \text{ there is a pressing down function } h \text{ on } A \text{ such that for each } \alpha < \lambda, \{\beta \in A : h(\beta) < \alpha\} \in \bigcup_{i<\lambda} I_i\Big\}.$$

Then $I' =: I + \{\delta < \lambda : \mathrm{cf}(\delta) \le \mu\}$ is weakly normal and μ-indecomposable.

Remark. If I is an ideal on λ and I is κ-indecomposable for every regular $\kappa < \mu$, then I is μ-complete.

Proof. I' is weakly normal by its definition (first note that for every club C of λ the set $\lambda \setminus C$ belongs to I: use h_C where $h_C(\alpha) = \sup(\alpha \cap C)$; then we use a pairing function $< -, - >$ such that $\langle \alpha, \beta \rangle < \mathrm{Min}\{\delta : \alpha, \beta < \delta = \omega \times \delta < \lambda\}$).

For μ-indecomposability, assume $\langle A_i : i < \mu \rangle$ is an increasing continuous sequence of members of I', $A_\mu = \bigcup_{i<\mu} A_i$ and we shall prove that $A_\mu \in I'$, this suffices as μ is regular. Without loss of generality A_μ is disjoint to $\{\delta < \lambda : \mathrm{cf}(\delta) \le \mu\}$ hence $i < \mu \Rightarrow A_i \in I$. Let h_i be a pressing down function witnessing $A_i \in I$, so for $\alpha < \lambda$ for some $\zeta(\alpha, i) < \lambda$ we have $\{\beta \in A_i : h_i(\beta) < \alpha\} \in I_{\zeta(\alpha,i)}$.

For each $\alpha < \lambda$ let $\zeta(\alpha) = \bigcup_{i<\mu} \zeta(\alpha, i)$, so as $\mu < \lambda$ clearly $\zeta(\alpha) < \lambda$. Let us define a function h with $\mathrm{Dom}(h) = A_\mu$ by setting $h(\alpha) = \cup\{h_i(\alpha) : \alpha \in A_i \text{ and } i < \mu\}$. Let $\alpha < \lambda$, so for each $i < \mu$ we have $\{\beta \in A_i : h(\beta) < \alpha\} \subseteq \{\beta \in A_i : h_i(\beta) < \alpha\} \in I_{\zeta(\alpha,i)} \subseteq I_{\zeta(\alpha)}$ (remember $\langle I_i : i < \lambda \rangle$ is increasing). For $i \le \mu$ let $B_i^\alpha =: \{\beta \in A_i : h(\beta) < \alpha\}$, so $\langle B_i^\alpha : i \le \mu \rangle$ is increasing continuous, and for $i < \mu$ we have $B_i^\alpha \subseteq \{\beta \in A_i : h_i(\beta) < \alpha\} \in I_{\zeta(\alpha)}$. So as $I_{\zeta(\alpha)}$ is μ-indecomposable $\{\beta \in A_\mu : h(\beta) < \alpha\} \in I_{\zeta(\alpha)}$. So if $\alpha \in A_\mu$, as A_μ is disjoint to $\{\delta < \lambda : \mathrm{cf}(\delta) \le \mu\}$ then $h(\alpha) < \alpha$ hence h witnesses $A_\mu \in I \subseteq I'$. So clearly $I' = I + \{\delta < \lambda : \mathrm{cf}(\delta) \le \mu\}$ is μ-indecomposable. $\square_{1.7}$

1.8 Observation. Let $\langle I_i : i < \delta \rangle$ be an increasing sequence of ideals on λ, each I_i is μ-indecomposable, μ regular.

(1) If $\mathrm{cf}(\delta) \neq \mu$, then $\bigcup_{i<\delta} I_i$ is a μ-indecomposable ideal.
(2) If each I_i is weakly normal, $\delta < \lambda$ then $\bigcup_{i<\delta} I_i$ is a weakly normal ideal on λ.

Proof. Check.

* * *

1.9 Definition. *1) Let λ be a limit cardinal of uncountable cofinality, $\gamma = \lambda \times n + \beta$ (where $[\mathrm{cf}(\lambda) < \lambda \Rightarrow n = 0 \,\&\, \gamma = \beta < \mathrm{cf}(\lambda)]$ and $[\mathrm{cf}(\lambda) = \lambda \Rightarrow \beta < \lambda]$). We define $\mathrm{id}^\gamma(\lambda)$, an ideal on λ (temporarily — a family of subsets of λ, see 1.11); this is defined by induction on λ:*

(a) if $\gamma = 0$ it is the family of non-stationary subsets of λ

(b) if $\gamma < \lambda$ it is the family of $A \subseteq \lambda$ such that:
$\{\mu < \lambda : A \cap \mu \notin \bigcup_{\alpha<\gamma} \mathrm{id}^\alpha(\mu)\}$ *is not a stationary subset of λ.*

(c) If $n > 0$, $\beta = 0$ it is the family of $A \subseteq \lambda$ such that for some pressing down function h on A, for each $i < \lambda$ the set
$$\left\{\mu : \mu < \lambda \text{ inaccessible, } h(\mu) = i \text{ and } A \cap \mu \notin \bigcup_{\alpha<\mu\times n} \mathrm{id}^\alpha(\mu)\right\}$$
is not a stationary subset of λ.

(d) If $n > 0$, $\beta > 0$ it is the family of $A \subseteq \lambda$ such that
$$\left\{\mu : \mu < \lambda \text{ inaccessible and } A \cap \mu \notin \bigcup_{\alpha<\beta} \mathrm{id}^{\mu\times n+\alpha}(\mu)\right\}$$
is not a stationary subset of λ.

2) $rk^*_\lambda(A) = \mathrm{Min}\{\gamma : A \in \mathrm{id}^\gamma(\lambda), \gamma < \lambda \times \omega$ *or* $\gamma = \lambda^+\}$.
3) $id^{<\gamma}(\lambda) = \cup\{\mathrm{id}^\beta(\lambda) : \beta < \lambda\}$*, an ideal too (well for* $\gamma > 0$*)*

1.10 Remark. 1) If in clause (c) we imitate clause (d), we get the ideal from Definition 1.3. We can continue this to all $\gamma < \lambda^+$.

2) Also this definition can be continued for $\gamma \in [\lambda \times \omega, \lambda^+]$ using a strictly* λ^+-club system $\bar{e}$, proving its choice is immaterial, $\mathrm{id}^\gamma_{\mathrm{rk}}(\lambda) \subseteq \mathrm{id}^\gamma(\lambda))$ and other parts of 1.11.

3) We can replace the closure to normal ideal to one for weakly normal ideal.

4) Also we can divide the ordinals $< \lambda \times \omega$ differently between those three operations: reflecting, normality and weak normality. All are O.K. in 1.16, but no need here.

5) Trivially, $\mathrm{id}^\gamma(\lambda)$ increase with γ and is an ideal on λ (possibly equal to $\mathcal{P}(\lambda)$).

1.11 Observation.

0) $\mathrm{id}^\gamma(\lambda)$ is an ideal on λ.
1) For λ of uncountable cofinality, $\gamma < \lambda$, $S \subseteq \lambda$ we have: $S \in \mathrm{id}^\gamma_{\mathrm{rk}}(\lambda) \Leftrightarrow S \in \mathrm{id}^\gamma(\lambda)$, i.e. $\mathrm{id}^\gamma_{\mathrm{rk}}(\lambda) = \mathrm{id}^\gamma(\lambda)$.
2) If λ is inaccessible, $\lambda \leq \gamma < \lambda \times \omega$ and $S \subseteq \lambda$ <u>then</u> $\mathrm{id}^\gamma_{\mathrm{rk}}(\lambda) \subseteq \mathrm{id}^\gamma(\lambda)$.
3) Assume λ is inaccessible ($> \aleph_0$), $\lambda \leq \gamma < \lambda \times \omega$, $\gamma = \mathrm{rk}_\lambda(\lambda)$ and $\theta = \mathrm{cf}(\theta) < \lambda$, $S = \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ <u>then</u> we have $(*)_S$ where

   $(*)_S$ for some $\beta < \lambda \times \omega$ we have $S \notin \bigcup_{i<\lambda} \mathrm{id}^{\beta+i}(\lambda)$,
   but $\{\mu : \mu$ inaccessible, $S \cap \mu$ stationary$\} \in \mathrm{id}^\beta(\lambda)$.

4) For λ inaccessible, $S \subseteq \lambda$ and $\mathrm{rk}_\lambda(S) < \lambda \times \omega$ <u>then</u> $\mathrm{Min}\{\lambda, \mathrm{rk}_\lambda(S)\} \leq \mathrm{rk}^*_\lambda(S)$.
5) Let λ be inaccessible and $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ be stationary
   (a) if $\lambda \leq \gamma = \mathrm{rk}^*_\lambda(S) < \lambda \times \omega$ <u>then</u> $(*)_S$ from part (3) holds
   (b) if $\lambda \leq \mathrm{rk}_\lambda(S) < \lambda \times \omega$ <u>then</u> for some $\gamma, \lambda \leq \gamma = \mathrm{rk}^*_\lambda(S) < \lambda \times \omega$ hence $(*)_S$ of part (3) holds
   (c) if λ is γ-Mahlo not $(\gamma + 1)$-Mahlo and $\lambda \leq \gamma < \lambda \times \omega$ <u>then</u> for some $\gamma, \lambda \leq \gamma \leq \gamma_1 < \lambda \times \omega$ we have $(*)_S$ from part (3) or $\mathrm{rk}^*_\lambda(S) < \lambda$.
6) For λ inaccessible and $\gamma = \lambda \times n + \beta, \beta < \lambda$, the ideal $\mathrm{id}^\gamma(\lambda) + \{\delta < \lambda : \mathrm{cf}(\delta) \leq \sigma\}$ (also $\mathrm{id}^{<\gamma}(\lambda) + \{\delta < \lambda : \mathrm{cf}(\delta) \leq \sigma\}$) is σ-indecomposable for any $\sigma = \mathrm{cf}(\sigma) \in [|\beta|^+, \lambda)$ and is weakly normal.
7) If λ is inaccessible, $S \subseteq \lambda$, $\mathrm{rk}^*_\lambda(S) = \lambda \times n^* + \gamma, \gamma < \lambda$ <u>then</u> we can find a club E of λ such that
   (a) if $\delta \in E, \mathrm{cf}(\delta) > \aleph_0$ then $\mathrm{rk}^*_\delta(S) \leq \delta \times n^* + \gamma$
   (b) if $\gamma > 0, \delta \in E, \mathrm{cf}(\delta) > \aleph_0$ then $\mathrm{rk}^*_\delta(S) < \delta \times n^* + \gamma$.
8) Assume $S \subseteq \lambda$ and $S^+ = \{\delta : \delta$ is inaccessible and $\delta \in S \vee (\delta \cap S$ is stationary$)\}$. Then $\mathrm{rk}^*_\lambda(S) \leq \mathrm{rk}^*_\lambda(S) + \lambda$.
9) If $\mathrm{rk}^*_\lambda(S) = \gamma + 1$ then for some club C of λ, $\{\delta < \lambda$: $\mathrm{rk}^*_\delta(S \cap C) \geq \gamma\}$ is a stationary nonreflecting subset of λ.

Proof. Let $\bar{e}$ be a strict λ^+-club system as in 1.3(4).

0) Should be clear.
1) Clearly also $\mathrm{id}^\gamma(\lambda)$ is an ideal which includes all bounded subsets of λ. We prove the equality by induction on λ and then by induction on γ.

So if $\gamma < \lambda$, $A \subseteq \lambda$; let for any B, $B^{[i]}$ be defined as in Definition 1.3 (for $\bar{e}$), we can discard the case $\gamma = 0$; and without loss of generality $\lambda = \sup(A)\ \&\ A \cap (\gamma+1) = \emptyset$; now (ignoring the case γ is inaccessible for simplicity)

$$A \in \mathrm{id}^\gamma(\lambda) \Leftrightarrow$$

$$\Big\{\mu < \lambda : \mu > \gamma \text{ and } \mu \cap A \notin \bigcup_{\alpha<\gamma} \mathrm{id}^\alpha(\mu)\Big\} \text{ is not stationary } \Leftrightarrow$$

$$\Big\{\mu < \lambda : \mu > \gamma \text{ and } \bigwedge_{\alpha<\gamma}[\mu \cap A \notin \mathrm{id}^\alpha(\mu)]\Big\} \text{ is not stationary } \Leftrightarrow$$

$$\Big\{\mu < \lambda : \mu > \gamma \text{ and } \bigwedge_{\alpha<\gamma}[\mu \cap A \notin \mathrm{id}^\alpha_{\mathrm{rk}}(\mu)]\Big\} \text{ is not stationary } \Leftrightarrow$$

$$\Big\{\mu < \lambda : \bigwedge_{\alpha<\gamma}[(\mu \cap A)^{[\alpha]} \text{ is stationary in } \mu]\Big\} \text{ is not stationary } \Leftrightarrow$$

$$\Big\{\mu < \lambda : \bigwedge_{\alpha<\gamma}[(\mu \cap A) \cap A^{[\alpha]} \text{ is stationary in } \mu]\Big\} \text{ is not stationary } \Leftrightarrow$$

$$\Big\{\mu < \lambda : \bigwedge_{\alpha<\gamma}[\mu \cap A^{[\alpha]} \text{ is stationary in } \mu]\Big\} \text{ is not stationary } \Leftrightarrow$$

$$\Big\{\mu < \lambda : \mu \in \bigcap_{\alpha<\gamma} A^{[\alpha+1]}\Big\} \text{ is not stationary } \Leftrightarrow$$

$$A^{[\gamma]} \text{ not stationary } \Leftrightarrow$$

$$A \in \mathrm{id}^\gamma_{\mathrm{rk}}(\lambda).$$

2) We prove this by induction on λ, and for each λ by induction on γ. For $\gamma < \lambda$ use part (1). For $\gamma \geq \lambda$ successor ordinal, read the definitions (and 1.10(3)). So assume $\gamma \in [\lambda, \lambda \times \omega)$ is a limit ordinal. For every $A \in \mathrm{id}^\gamma_{\mathrm{rk}}(\lambda)$, we know $A^{[\gamma,\bar{e}]}$ is not stationary, so for some club E of λ, $A^{[\gamma,\bar{e}]} \cap E = \emptyset$. So if we define $h : E \to \lambda$ by $h(\delta) = \mathrm{Min}\{\mathrm{otp}(j \cap e_\gamma) : j \in e_\gamma, \delta \notin A^{[j,\bar{e}]}, \mathrm{otp}(j \cap e_\gamma) < \delta\}$, by the definition of $A^{[\gamma,\bar{e}]}$ it is well defined, and $h(\delta) < \delta\ \&\ h(\delta) < \mathrm{otp}(e_\gamma)$. Let $\gamma = \lambda \times n + \beta$, $\beta < \lambda$, so $n \geq 1$.

Clearly, possibly replacing E by a thinner club of λ

⊠ for every $\delta \in E$

(α) $\delta > \beta$ is a limit cardinal and $\delta = \sup(A)$

(β) if $\mathrm{cf}(\delta) > \aleph_0 \,\&\, \gamma = \lambda$ then $A \cap \delta \in \mathrm{id}^{h(\delta)}_{\mathrm{rk}}(\delta)$

(γ) if δ is inaccessible, $\gamma = \lambda \times n, n > 1$ (so $\beta = 0$) then $A \cap \delta \in \mathrm{id}^{\delta \times (n-1)+h(\delta)}_{\mathrm{rk}}(\delta)$ and $h(\delta) < \delta$

(ε) if δ is inaccessible, $\gamma = \lambda \times n + \beta > \lambda \times n, n \geq 1$ then $A \cap \delta \in \mathrm{id}^{\delta \times n + h(\delta)}_{\mathrm{rk}}(\delta)$ and $h(\delta) < \beta$.

Now we can case by case prove that $A \in \mathrm{id}^{\gamma}(\lambda)$, using the induction hypothesis on λ and on γ (or part (1)) and the definition of $\mathrm{id}^{\gamma}(-)$.
3), 4) Check.
5) For the second statement note that by parts (1) + (2) we have $\lambda \leq \mathrm{rk}^*_\lambda(S) \leq \mathrm{rk}_\lambda(S) < \lambda \times \omega$ so $\gamma =: \mathrm{rk}_\lambda(S)$ is as required.
6) We prove this by induction on λ and for a fix λ by induction on γ.

Case 1: $\gamma < \lambda$.

By part (1) we know that $\mathrm{id}^{\gamma}(\lambda) = \mathrm{id}^{\gamma}_{\mathrm{rk}}(\lambda)$ and the latter $+\{\delta < \lambda : \mathrm{cf}(\delta) \leq \sigma\}$ is weakly normal by 1.5(7) and is σ-indecomposable for any regular $\sigma \in (|\gamma|^+, \lambda)$ by 1.5(6). Alternatively, the proofs are similar to those of case (3).

Case 2: $\gamma = \lambda \times n, 1 \leq n < \omega$.

By Definition 1.9 clause (c) obviously $\mathrm{id}^{\gamma}(\lambda)$ contains the family of bounded subsets of λ and is even normal hence λ-complete hence σ-indecomposable for any $\sigma < \lambda$.

Case 3: $\gamma = \lambda \times n + \beta, 1 \leq n < \omega, 1 \leq \beta < \lambda$.

First we prove the indecomposability part, so let $\sigma = \mathrm{cf}(\sigma) \in [|\beta|^+, \lambda)$ and assume $\langle A_i : i \leq \sigma \rangle$ is an increasing continuous sequence of subsets of λ and assume $A_\sigma \notin \mathrm{id}^{\gamma}(\lambda)$ and we should prove that for some $i < \sigma$ we have $A_i \notin \mathrm{id}^{\gamma}(\lambda)$.

Let us define for $i \leq \sigma$:

$$B_i =: \{\mu < \lambda : \mu \text{ inaccessible and } A_\sigma \cap \mu \notin \bigcup_{\alpha < \beta} \mathrm{id}^{\mu \times n + \alpha}(\mu)\}.$$

For each inaccessible $\mu < \lambda$ which is $> \sigma$ and $\alpha < \beta$ we apply the induction hypothesis with $\lambda' = \mu, \gamma' = \mu \times n + \alpha$ and $\langle A'_i : i \leq \sigma \rangle = \langle A_i \cap \mu : i \leq \sigma \rangle$ and get: for every $\mu \in B_\sigma$ for some $i(\mu, \alpha) < \sigma$ we have $A_{i(\mu,\alpha)} \cap \mu \notin \mathrm{id}^{\mu \times n + \alpha}(\mu)$, but $\gamma < \sigma$ hence $i(\mu) =: \sup\{i(\mu, \alpha) : \alpha < \gamma\} < \sigma$, and clearly $\mu \in B_{i(\mu)}$, as the A_j's are increasing. As $\sigma < \lambda$ and B_σ stationary (by assumptions) we have: B_σ is a stationary subset of λ and $B_\sigma \subseteq \bigcup_{i<\sigma} B_i \cup \sigma^+$, hence for some $i(*) < \sigma$ the set $B_{i(*)}$ is stationary, hence $A_{i(*)} \notin \mathrm{id}^{\lambda \times n + \gamma}(\lambda)$ is as required.

Second we prove the weak normality part. So let $A \subseteq \lambda$, $A \notin \mathrm{id}^{\gamma}(\lambda)$ and h a function with domain A, $h(i) < 1 + i$, and let $A_j = \{\alpha \in A : h(\alpha) < j\}$. We define $B_i =: \{\mu < \lambda : \mu \text{ inaccessible } > i, \text{ and } A \notin \bigcup_{\alpha<\beta} \mathrm{id}^{\mu \times n + \alpha}(\mu)\}$, $B =: \{\mu < \lambda : \mu \text{ inaccessible and } A_i \cap \mu \notin \bigcup_{\alpha<\beta} \mathrm{id}^{\mu \times n+\alpha}(\mu)\}$.

Again we assume that B is stationary and has to prove that some B_j is stationary. For every inaccessible $\mu \in B$ and $\alpha < \beta$ applying the induction hypothesis to $\mu, A \cap \mu, h \restriction (A \cap \mu)$ for some $i(\mu,\alpha) < \mu$ the set $\{\mu' < \mu : \mu'$ inaccessible, $A^{\mu}_{i(\mu,\alpha)} \cap \mu' \notin \mathrm{id}^{\mu' \times n+\alpha}(\mu')\}$ is stationary where $A^{\mu}_{i(\mu,\alpha)} = \{\zeta \in A \cap \mu : (h \restriction (A \cap \mu))(\zeta) < i(\mu,\alpha)\}$. Let $i(\mu) = \sup\{i(\mu,\alpha) : \alpha < \beta\}$ so it is $< \mu$, and clearly $A_{i(\mu) \cap \mu} \notin \bigcup_{\alpha<\beta} \mathrm{id}^{\mu \times n+\beta}(\lambda)$. So $B \subseteq \bigcup_{j<\lambda} B_j$, and we easily finish.

7) By induction on the rank.
8) By induction on λ.
9) Easy. $\square_{1.11}$

* * *

1.12 Claim. Suppose λ is inaccessible, $S \subseteq \lambda$ a stationary set of inaccessibles $> \sigma$, $S_1 \subseteq \{\delta < \lambda : \delta$ a limit cardinal $> \sigma$ of cofinality $> \aleph_0$ and $\neq \sigma\}$ is stationary, $\lambda > \sigma = \mathrm{cf}(\sigma)$ and for $\delta \in S$ the ideal I_δ is a weakly normal σ-indecomposable ideal on $\delta \cap S_1$ and J is a weakly normal σ-indecomposable ideal on S, (and of course all are proper ideals which contains the bounded subsets of their domain; of course we demand $\delta \in S \Rightarrow \delta = \sup(S_1 \cap \delta)$ so $\delta \in S \Rightarrow \delta > \sigma$). Further let $\bar{C}^1 = \langle C^1_\alpha : \alpha \in S_1 \rangle$ be a strict S_1-club system satisfying:

(∗) for every club E of λ

$$\left\{\delta \in S : \{\alpha \in S_1 \cap \delta : E \cap \delta \backslash C^1_\alpha \text{ unbounded in } \alpha\} \in I^+_\delta\right\} \in J^+.$$

<u>Then</u>: (1) We can find an S_1-club system $\bar{C}^2 = \langle C^2_\alpha : \alpha \in S_1 \rangle$ such that for every club E of λ the set of $\delta \in S$ satisfying the following is not in J:

$$\left\{\alpha < \delta : \alpha \in S_1 \cap E \text{ and } \{\mathrm{cf}(\beta) : \beta \in \mathrm{nacc}(C^2_\alpha) \text{ and } \beta \in E\} \text{ is unbounded in } \alpha\right\} \in I^+_\delta.$$

(2) Suppose in addition $\cup\{\mathrm{cf}(\alpha) : \alpha \in S_1\} < \lambda$. <u>Then</u> we can demand that for some $\theta < \lambda$, $\alpha \in S_1 \Rightarrow |C^2_\alpha| < \theta$. Also if $\bar{C}^1$ is almost strict <u>then</u> we can demand that $\bar{C}^2$ is almost strict.
(3) Suppose $\cup\{\mathrm{cf}(\alpha) : \alpha \in S_1\} < \lambda$ and for arbitrarily large regular $\kappa < \lambda$ we have $\{\delta \in S : I_\delta$ not κ-*indecomposable*$\} \in J$.

<u>Then</u> we can strengthen the conclusion to: $\bar{C}^2$ is a nice strict S_1-club system such that for every club E of λ the set of $\delta \in S$ satisfying the following is not in J:

$$\left\{\alpha < \delta : \alpha \in S_1 \cap E \text{ and } C^2_\alpha \backslash E \text{ is bounded in } \alpha\right\} \neq \emptyset \bmod I_\delta.$$

(4) In part (1) (and (2), (3)) instead of "I_δ weakly normal σ-indecomposable" it suffices to assume: if δ belongs to S and $h_1 : \delta \cap S_1 \to \delta$ is pressing down and $h_2 : \delta \cap S_1 \to \sigma$ <u>then</u> for some $j_1 < \delta$, $\zeta < \sigma$ we have $\{\alpha \in \delta \cap S_1 : h_1(\alpha) < j$ and $h_2(\alpha) < \zeta\} \in I^+_\delta$.
5) We can replace $\langle\{\delta : \delta < \lambda, \mathrm{cf}(\delta) \geq \theta\} : \theta < \lambda\rangle$ by $\langle S_\theta : \theta < \lambda\rangle$ such that

(i) $\bigcap_{\theta<\lambda} S_\theta = \emptyset$,
(ii) S_θ decreasing in θ and
(iii) for no $\delta \in \lambda \setminus S_\theta$ do we have $\mathrm{cf}(\delta) > \aleph_0$ and $S_\theta \cap \delta$ stationary subset of δ; and
(iv) $\mathrm{Min}(S_\theta) > \theta$.

6) Assume $A \subseteq \lambda$ is stationary such that $A^{[0,\bar{e}]} = A$ (any $\bar{e}$ will do).
Then in part (1) we can add $\mathrm{nacc}(C^2_\alpha) \subseteq A$ and waive $\delta \in S \Rightarrow \mathrm{cf}(\delta) > \aleph_0$.

1.13 Remark. 1) This is similar to [Sh:g, IV, 1.7, p.188]. We can replace "S is a set of inaccessibles $> \sigma$" by "S is a set of cardinals of cofinality $\neq \sigma$" and get a generalization of [Sh:g, IV,1.7,p.188].
2) Note that $(*)$ of 1.12 holds if S_1 is a set of singulars and $\mathrm{otp}(C^1_\alpha) < \alpha$ for every $\alpha \in S_1$.
Concerning $(*)$ see [Sh 276, 3.7,p.370] or [Sh:g, III,2.12,p.134], it is a very weak condition, a strong version of not being weakly compact.
3) This claim is not presently used here (but its relative 1.14 will be used) but still has interest.

Proof. 1) Let $\bar{e}$ be a strict λ-club system.

It suffices to show that for some regular $\theta < \lambda$ and club E^2 of λ the sequence $\bar{C}^{2,E^2,\theta} = \langle C^{2,E^2,\theta}_\alpha = g\ell^1_\theta(C^1_\alpha, E^2, \bar{e}) : \theta < \alpha \in S_1 \rangle$ satisfies the conclusion (on $g\ell^1_\theta$ see [Sh 365], Definition 2.1(2) and uses in §2 there). So we shall assume that this fails. This means that for every club E^2 of λ and regular cardinal $\theta < \lambda$ some club $E = E(E^2, \theta)$ exemplifies the "failure" of $\bar{C}^{2,E^2,\theta}$. This means that for some $Y = Y(E^2, \theta) \in J$ for every $\delta \in S \setminus Y$ we have

$$\Big\{\alpha < \delta : \alpha \in S_1 \cap E \text{ and } \{\mathrm{cf}(\beta) : \beta \in \mathrm{nacc}(C^{2,E^2,\theta}_\alpha) \text{ and } \beta \in E\} \text{ is unbounded in } \alpha\Big\} \in I_\delta.$$

We now define by induction on $\zeta \leq \sigma$ a club E_ζ of λ:
<u>for $\zeta = 0$</u>: $E_\zeta =: \lambda$
<u>for ζ limit</u>: $E_\zeta =: \bigcap_{\xi<\zeta} E_\xi$
<u>for $\zeta = \xi + 1$</u>:

$$E_\zeta =: \Big\{\delta : \delta \text{ a limit cardinal } < \lambda, \delta \in E_\xi, \delta > \sigma \text{ and : } \theta = \mathrm{cf}(\theta) < \delta \Rightarrow \delta \in E(E_\xi, \theta)\Big\}.$$

Let $E^+ = \Big\{i < \lambda : i \text{ a cardinal }, i \in E_\sigma, \text{ moreover } i = \mathrm{otp}(E_\sigma \cap i)\Big\}$.

By $(*)$ (in the assumption)

$$B =: \{\delta \in S : A_\delta \in I^+_\delta\} \in J^+$$

and let

$$A = \bigcup_{\delta \in S} A_\delta$$

where for $\delta \in S$

$$A_\delta =: \{\alpha \in S_1 \cap \delta : E^+ \cap \alpha \backslash C^1_\alpha \text{ unbounded in } \alpha\}.$$

Note that if $\delta \in B$ or $\delta \in A$ then $\delta = \sup(\delta \cap E^+) \in E^+$; note also that $A \subseteq S_1$ and $B \subseteq S$. Now as $\alpha \in S_1 \Rightarrow \mathrm{cf}(\alpha) \neq \sigma$, for each $\alpha \in A$ there are $\zeta(\alpha) < \sigma$ and $\theta(\alpha) = \mathrm{cf}[\theta(\alpha)] < \alpha$ such that:

$$(*)_0 \qquad \theta(\alpha) \leq \theta = \mathrm{cf}(\theta) < \alpha \;\&\; \zeta(\alpha) \leq \zeta < \sigma \Rightarrow$$
$$\alpha = \sup\left\{\mathrm{cf}(\beta) : \beta \in \mathrm{nacc}(C^{2,E_\zeta,\theta}_\alpha) \cap E_{\zeta+1}\right\}.$$

[Why? We can find an increasing sequence $\langle \alpha_i, \beta_i : i < \mathrm{cf}(\alpha)\rangle, \alpha_i$ increasing with i with limit $\alpha, \alpha_i \in C^1_\alpha, \beta_i \in E_\sigma, \alpha_i < \mathrm{cf}(\beta_i) \leq \beta_i < \mathrm{Min}\left(C^1_\alpha \backslash (\alpha_i + 1)\right)$ (possible by the definition of the set A_δ and of the club E^+). For each $i < \mathrm{cf}(\alpha)$ we can find $\zeta_i < \sigma, \theta_i < \bigcup_{j<i} \alpha_j$ and γ_i such that $\zeta_i \leq \zeta < \sigma \;\&\; \theta_i \leq \theta < \bigcup_{j<i} \alpha_j \;\&\; \theta = \mathrm{cf}(\theta) \Rightarrow \mathrm{Min}(C^{2,E_\zeta,\theta}_\alpha \backslash \beta_i) = \gamma_i$
(check definition of $g\ell^1_\theta$!). So by the definition of $g\ell^1_\theta$ we have $\alpha_i \leq \gamma_i \leq \beta_i$ and $\mathrm{cf}(\gamma_i) \geq \bigcup_{j<i} \alpha_j$ and $\zeta_i \leq \zeta < \sigma \;\&\; \theta_i \leq \theta = \mathrm{cf}(\theta) < \bigcup_{j<i} \alpha_j \Rightarrow \gamma_i \in \mathrm{nacc}\left(C^{2,E_\zeta,\theta}_\alpha\right)$, this implies the statement $(*)_0$].

Now if $\delta \in B$, we have: $A_\delta \in I^+_\delta$ and A_δ is the union of $\langle\{\alpha \in A_\delta : \zeta(\alpha) \leq \zeta\} : \zeta < \sigma\rangle$ which is increasing.
As I_δ is σ-indecomposable, and $A_\delta \in I^+_\delta$ for some $\xi = \xi(\delta) < \sigma$,

$$A_{\delta,\xi} =: \{\alpha \in A_\delta : \zeta(\alpha) \leq \xi\} \in I^+_\delta.$$

Similarly, as I_δ is weakly normal, for some regular cardinal $\tau = \tau(\delta) < \delta$, we have

$$A^\tau_{\delta,\xi} = \{\alpha \in A_\delta : \zeta(\alpha) \leq \xi \text{ and } \theta(\alpha) \leq \tau\} \in I^+_\delta.$$

Similarly, as the ideal J is σ-indecomposable weakly normal ideal on $S \subseteq \lambda$, for some $\epsilon < \sigma$ and $\tau^* < \lambda$ we have:

$$B^+ =: \{\delta \in B : A^{\tau^*}_{\delta,\varepsilon} \in I^+_\delta\} \in J^+.$$

In particular B^+ cannot be a subset of $Y(E_\epsilon, \tau^*)$ (as the latter is a member of J, it was chosen in the first paragraph of the proof). Choose $\delta \in B^+ \backslash Y(E_\epsilon, \tau^*)$, which is $> \tau^*$.
By the definition of $Y(E_\varepsilon, \tau^*)$,

$$\big\{\alpha < \delta : \alpha \in S_1 \cap E(E_\varepsilon, \tau^*) \text{ and}$$
$$\alpha = \sup\{\mathrm{cf}(\beta) : \beta \in \mathrm{nacc}(C^{2,E_\varepsilon,\tau^*}_\alpha) \cap E(E_\varepsilon, \tau^*)\}\big\} \in I_\delta.$$

If $\alpha \in A^{\tau^*}_{\delta,\varepsilon} \backslash \tau^* + 1$ then $\alpha \in S_1 \cap E(E_\varepsilon, \tau^*)$ and since $\zeta(\alpha) \leq \varepsilon$ and $\theta(\alpha) \leq \tau^*$, we have by $(*)_0$

$$\alpha = \sup\{\mathrm{cf}(\beta) : \beta \in \ \mathrm{nacc}(C^{2,E_\varepsilon,\tau^*}_\alpha) \cap E_{\varepsilon+1}\}$$

hence

$$\alpha = \sup\{\mathrm{cf}(\beta) : \beta \in \ \mathrm{nacc}(C^{2,E_\varepsilon,\tau^*}_\alpha) \cap E(E_\varepsilon, \tau^*)\}.$$

Since $A^{\tau^*}_{\delta,\varepsilon} \backslash \tau^* + 1 \notin I_\delta$, we have a contradiction.

2) By the proof of part (1) for some regular $\theta < \lambda$ and club E^2 of λ, $\bar{C}^2 = \bar{C}^{2,E^2,\theta}$ is as required. So $|C^2_\alpha| < \theta + |C^1_\alpha|^+$ as we repeat the proof of part (1) for such $\bar{C}^1$, so the second phrase (in 1.12(2)) follows. For the first phrase $\theta + \sup_{\alpha \in S_1} |C^1_\alpha|^+ < \lambda$ is as required (remember $\bar{C}^1$ is a strict S_1-club system).

3) Let $\bar{C}^2, \theta$ be as in part (2). Let κ be regular be such that $\theta < \kappa < \lambda, \alpha \in S_1 \Rightarrow |C^2_\alpha| < \kappa$ and $\{\delta \in S : I_\delta$ not κ-indecomposable$\} \in J$.
For any club E of λ we define $\bar{C}^{3,E} = \langle \bar{C}^{3,E}_\alpha : \alpha \in S_1 \rangle$ as follows: <u>if</u> $C^2_\alpha \cap E$ is a club of α and $\alpha = \cup\{\mathrm{cf}(\beta) : \beta \in \ \mathrm{nacc}(C^2_\alpha \cap E)\}$ <u>then</u> $C^{3,E}_\alpha = C^2_\alpha \cap E$, otherwise $C^{3,E}_\alpha$ is a club of α of order type $\mathrm{cf}(\alpha)$ with $\mathrm{nacc}(C^{3,E}_\alpha)$ consisting of successor cardinals (remember each $\alpha \in S_1$ is a limit cardinal).

If for some club E of λ, $\bar{C}^{3,E}$ satisfies: for every club E^1 of λ the set $\big\{\delta \in S : \{\beta \in S_1 \cap \alpha : C^{3,E}_\beta \backslash E^1$ bounded in $\beta\} \in I^+_\delta\big\} \in J^+$ <u>then</u> we essentially finish, as we can choose $C^3_\alpha \subseteq C^{3,E}_\alpha$ which is closed of order type $\mathrm{cf}(\alpha)$ and $[\beta \in \mathrm{nacc}|C^3_\alpha| \Rightarrow \ \mathrm{cf}(\beta) > \sup(C^3_\alpha \cap \beta)]$, and $\langle C^3_\beta : \beta \in S_1 \rangle$ is as required. So assume that for every club E of λ for some club $E' = E'(E)$ this fails. We choose by induction on $\zeta < \kappa$, a club E_ζ of λ, as follows:

$$E_0 = \lambda$$

$$E_{\zeta+1} = E'(E_\zeta)$$

$$E_\zeta = \bigcap_{\xi<\zeta} E_\xi \text{ for } \zeta \text{ limit}$$

and recalling the choice of κ we easily get a contradiction.
4), 5) Same proof.
6) In the proof of part (1) choose $\bar{e}$ such that:

$$\text{for limit } \alpha < \lambda, \alpha \notin A \Rightarrow e_\alpha \cap A = \emptyset.$$

Then we replace the definition of $C^{2,E^2,\theta}_\alpha$ by $C^{2,E^2,A}_\alpha = g\ell^1_A(C^1_\alpha, E^2, \bar{e})$. $\square_{1.12}$

1.14 Claim. Assume

(*a*) λ inaccessible

(*b*) $A \subseteq \lambda$ is a stationary set of limit ordinals and $\delta < \lambda$ & ($A \cap \delta$ stationary in $\delta) \Rightarrow \delta \in A$

(*c*) J is a σ-indecomposable ideal on λ containing the nonstationary ideal

(*d*) $S \in J^+$ and $S \cap A = \emptyset$

(*e*) $\sigma = \mathrm{cf}(\sigma) < \lambda$ and $\delta \in S \Rightarrow \mathrm{cf}(\delta) \neq \sigma$.

Then for some S-club system $\bar{C} = \langle C_\delta : \delta \in S \rangle$ we have

$\boxtimes$ for every club E of λ
$\{\delta \in S : \delta = \sup(E \cap \mathrm{nacc}(C_\delta) \cap A)\} \in J^+$.

Proof. As usual let $\bar{e} = \langle e_\alpha : \alpha < \lambda \rangle$ be a strict λ-club system but such that for every limit $\delta \in \lambda \backslash A$ we have $e_\delta \cap A = \emptyset$. For any set $C \subseteq \lambda$ and club E of λ we define $g\ell_n^2(C, E, \bar{e}, A)$ by induction on $n < \omega$ as follows: for $n = 0$, $g\ell_n^2(C, E, \bar{e}, A) = \{\sup(\alpha \cap E) : \alpha \in C\}$ and

$$\begin{aligned} g\ell_{n+1}^2(C, E, \bar{e}, A) = g\ell_n^2(C, E, \bar{e}, A) \cup \{&\sup(\alpha \cap E) : \text{for some} \\ &\beta \in \mathrm{nacc}(g\ell_n^2(C, E, \bar{e}, A)) \text{ we have } \beta \notin A, \text{ and} \\ &\sup(\alpha \cap E) > \sup(\beta \cap g\ell_n^2(C, E, \bar{e}, A)) \text{ and} \\ &\sup(\alpha \cap E) \geq \sup(\alpha \cap e_\beta) \text{ and } \alpha \in e_\beta\} \end{aligned}$$

and

$$g\ell^2(C, E, \bar{e}, A) = \bigcup_{n<\omega} g\ell_n^2(C, E, \bar{e}, A).$$

If C is a club of some $\delta \in \mathrm{acc}(E)$, clearly $g\ell_n^2(C, E, \bar{e}, A)$, $g\ell^2(C, E, \bar{e}, A)$ are clubs of δ.
If for some club E of λ, letting $C_{\delta,E}$ be $g\ell^2(e_\delta, E, \bar{e}, A)$ when $\delta \in \mathrm{acc}(E)$, and letting $C_{\delta,E}$ be e_δ otherwise, the sequence $\bar{C}_E =: \langle C_{\delta,E} : \delta \in S \rangle$ is as required, then fine, we are done. Assume not, so for any club E of λ for some club $\mathbf{E}(E)$ of λ the set $Y_E =: \{\delta \in S : \delta = \sup(\mathbf{E}(E) \cap A \cap \mathrm{nacc}(C_{\delta,E}))\}$ belongs to J.

As we can replace $\mathbf{E}(E)$ by any club $E' \subseteq \mathbf{E}(E)$ of λ, without loss of generality $\mathbf{E}(E) \subseteq E$.
We choose E_ε by induction on $\varepsilon < \sigma$ such that:

(*i*) E_ε is a club of λ

(*ii*) $\zeta < \varepsilon \Rightarrow E_\varepsilon \subseteq E_\zeta$

(*iii*) if $\varepsilon = \zeta + 1$ then $E_\varepsilon \subseteq \mathbf{E}(E_\zeta)$.

For $\varepsilon = 0$ let $E_\varepsilon = \lambda$, for ε limit let $E_\varepsilon = \bigcap_{\zeta<\varepsilon} E_\zeta$, for $\varepsilon = \zeta + 1$ let $E_\varepsilon = \mathbf{E}(E_\zeta) \cap E_\zeta$.

This is straightforward and let $E = \bigcap_{\varepsilon<\sigma} E_\varepsilon$, it is a club of λ hence $E \cap A$ is stationary hence $E' = \{\delta \in E : \delta = \sup(E \cap A \cap \delta)\}$ is a club of λ hence $\lambda \backslash E' \in J$. Now for each $\delta \in E' \cap S$, choose an increasing sequence $\langle \beta_{\delta,i} : i < \mathrm{cf}(\delta) \rangle$ of members of $A \cap E \cap \delta$ with limit δ; as $\delta \in S$ clearly $\delta \notin A$ hence $e_\delta \cap A = \emptyset$ hence $\{\beta_{\delta,i} : i < \mathrm{cf}(\delta)\} \cap e_\delta = \emptyset$. Now for each $i < \mathrm{cf}(\delta)$ and $\varepsilon < \sigma$, we can prove by induction on n that $g\ell_n^2(e_\delta, E_\varepsilon, \bar{e}, A) \cap \beta_{\delta,i}$ is bounded in $\beta_{\delta,i}$ and $\langle \min(g\ell_n^2(e_\delta, E_\varepsilon, \bar{e}, A) \backslash \beta_{\delta,i}) : n < \omega \rangle$ is decreasing hence eventually constant say for $n \geq n(\delta, \varepsilon, i)$ hence $\min(g\ell_n^2(e_\delta, E_\varepsilon, \bar{e}, A) \backslash \beta_{\delta,i})$ is a member of $C_{\delta,E_\varepsilon} = \bigcup_n g\ell_n^2(e_\delta, E_\varepsilon, \bar{e}, A)$ moreover of $\mathrm{nacc}(C_{\delta,E_\varepsilon})$ and so necessarily $\in A$

as only the demand "$\beta \notin A$" prevent $g\ell^2_{n+1}$ having unboundedly many members below $\min(g\ell^2_n(e_\delta, E_\varepsilon, \bar{e}, A)\backslash\beta_{\delta,i})$.

Also as usual for each $i < \mathrm{cf}(\delta)$ for some $\varepsilon_{i,\delta} < \sigma$ we have $\varepsilon_{i,\delta} \le \zeta < \sigma \Rightarrow \mathrm{Min}(C_{\delta,E_\zeta}\backslash\beta_{\delta,i}) = \mathrm{Min}(C_{\delta,E_{\varepsilon_{i,\delta}}}\backslash\beta_{\delta,i})$ as for each n, the sequence $\langle \mathrm{Min}(g\ell^2_n(e_\delta, E_\varepsilon, \bar{e}, A)\backslash\beta_{\delta,i}) : \varepsilon < \sigma\rangle$ is nonincreasing hence eventually constant. But $\mathrm{cf}(\delta) \in \{\mathrm{cf}(\delta') : \delta' \in S\}$ hence $\mathrm{cf}(\delta) \ne \sigma$, so for some ε_δ we have $\mathrm{cf}(\delta) = \sup\{i : \varepsilon_{i,\delta} \le \varepsilon_\delta\}$. So easily $\varepsilon_\delta \le \varepsilon < \sigma \Rightarrow \delta \in Y_{E_\varepsilon}$, see definition below.

Let $Y_\varepsilon = \cap\{Y_{E_\zeta} : \zeta \ge \varepsilon \text{ and } \zeta < \sigma\}$. Clearly $Y_\varepsilon \subseteq Y_{E_\varepsilon} \in J$ so $Y_\varepsilon \in J$ and $\varepsilon_1 < \varepsilon_2 \Rightarrow Y_{\varepsilon_1} \subseteq Y_{\varepsilon_2}$. As J is σ-indecomposable, necessarily $\bigcup_{\varepsilon<\sigma} Y_\varepsilon \in J$, but by the previous paragraph $\delta \in E' \cap S \;\&\; \bigwedge_{\varepsilon \ge \varepsilon_\delta} \delta \in Y_{E_\varepsilon} \Rightarrow \delta \in Y_{\varepsilon_\delta} \Rightarrow \delta \in \bigcup_{\varepsilon<\sigma} Y_\varepsilon$, so $E' \cap S \subseteq \bigcup_{\varepsilon<\sigma} Y_\varepsilon \in J$ but $S \in J^+$, $\lambda\backslash E' \in J$, a contradiction. $\square_{1.14}$

1.15 Claim. 1) Suppose $\lambda > \theta+\sigma$, λ inaccessible, θ regular uncountable, σ regular, $\sigma \ne \theta$, $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ stationary, J a weakly normal σ-indecomposable ideal on S (proper, of course).

Then for some S-club system $\langle C_\delta : \delta \in S\rangle$:

(*a*) $\delta \in S \;\&\; \alpha \in \mathrm{nacc}(C_\delta) \Rightarrow \mathrm{cf}(\alpha) > \sup(\alpha \cap C_\delta)$

(*b*) for every club E of λ, $\{\delta \in S : \delta = \sup(E \cap \mathrm{nacc}(C_\delta))\} \in J^+$

(*c*) $\sup_{\delta\in S} |C_\delta| < \lambda$.

2) If in addition $\{\kappa < \lambda : \mathrm{cf}(\kappa) = \kappa$, J is κ-indecomposable$\}$ is unbounded in λ we can demand $\bar{C}$ is nice and strict.

Proof. Like 1.12 or 1.14 but easier (and see [Sh:g, III,2.7,p.128]). More specifically part (1) is proved like 1.12(1) (but simpler) and part (2) like 1.12(3). $\square_{1.15}$

1.16 Claim. 1) Assume λ is an inaccessible Jonsson cardinal, $n^* < \omega$, $\theta = \aleph_{\gamma(*)} < \lambda$, $S \subseteq \lambda$, and $S^+ = \{\delta < \lambda : S \cap \delta$ is stationary and δ is inaccessible$\}$, satisfy $\delta \in S \Rightarrow \theta \le \mathrm{cf}(\delta) < \delta$ and

$(*)(\alpha)$ $\lambda \times n^* \le \mathrm{rk}_\lambda(S) < \lambda \times (n^*+1)$ and

(β) $\mathrm{rk}_\lambda(S^+) < \mathrm{rk}_\lambda(S)$

(γ) if $\theta > \aleph_0$ then $n^* > 0$ or at least $\gamma(*) \times \omega < \mathrm{rk}_\lambda(S)$, (note: if $\theta = \aleph_0$ this holds trivially; similarly for clause (δ))

(δ) if $\theta > \aleph_0$, then for some $\alpha(*)$ we have $\gamma(*) + \mathrm{rk}_\lambda(S^+) \le \alpha(*) < \mathrm{rk}_\lambda(S)$ (recall $\theta = \aleph_{\gamma(*)}$), and $\mathrm{id}^{\alpha(*)}_{\mathrm{rk}}(\lambda) \restriction S$ is θ-complete (of course, $\theta = \aleph_{\gamma(*)}$).

$(**)(\alpha)$ $\bar{C}$ is an S-club system,

(β) $\lambda \notin \mathrm{id}_p(\bar{C}, \bar{I})$, see definition below, where $\bar{I} = \langle I_\delta : \delta \in S\rangle$, $I_\delta =: \{A \subseteq C_\delta :$ *for some* $\sigma < \delta$ *and* $\alpha < \delta$, $(\forall\beta \in A)(\beta < \alpha \vee \mathrm{cf}(\beta) < \sigma \vee \beta \in \mathrm{acc}(C_\delta)\}$, moreover

(γ) for every club E of λ we have $\alpha(*) < \mathrm{rk}_\lambda(\{\delta \in S :$ *for every* $\sigma < \delta$ *we have* $\delta = \sup(E \cap \mathrm{nacc}(C_\delta) \cap \{\alpha < \delta : \mathrm{cf}(\alpha) > \sigma\}))$.

Then $\mathrm{id}^j_\theta(\bar{C})$ is a proper ideal (see 1.18 below).
2) Like part (1) using id^γ, rk^*_λ instead of $\mathrm{id}^\gamma_{\mathrm{rk}}$, rk_λ respectively.

1.17 Remark. The ideals $\mathrm{id}_j(\bar{C})$, $\mathrm{id}^j_\theta(\bar{C})$ are defined below; they are from [Sh:g, IV,Definition 1.8(2),(3),p.190] [4] but $\mathrm{id}_j(\lambda) = \mathrm{id}^j_{\aleph_0}(\lambda)$ and the definition of $\mathrm{rk}^j_\theta(\lambda)$ is repeated in the proof below, and the ideal $\mathrm{id}_p(\bar{C}, \bar{I})$ in [Sh:g, III,3.1,p.139] is:

1.18 Definition. *For λ regular $> \aleph_0$, $\bar{C} = \langle C_\delta : \delta \in S\rangle$, $C_\delta \subseteq \delta = \sup(C_\delta)$, $S \subseteq \lambda = \sup(S)$, $\bar{I} = \langle I_\delta : \delta \in S\rangle$, I_δ an ideal on C_δ let $id_p(\bar{C}, \bar{I})$ be the family $\{A \subseteq \lambda :$ for some club E of λ for no $\delta \in \mathrm{Dom}(\bar{C}) \cap \mathrm{acc}(E)$ do we have $A \cap E \cap C_\delta \notin I_\delta\}$.*

1.19 Definition. *1) For λ an inaccessible Jonsson cardinal, $\bar{C} = \langle C_\delta : \delta \in S\rangle$, $C_\delta \subseteq \delta$, $S \subseteq \lambda = \sup(S)$ and $\theta = \mathrm{cf}(\theta) < \lambda$ let $id^j_\theta(\bar{C})$ be the family of $A \subseteq \lambda$ such that: for every $\chi > \lambda$ and $x \in \mathcal{H}(\chi)$ there is a sequence $\bar{M}$ exemplifying $A \in \mathrm{id}^j_\theta(\lambda)$ for x (and $\bar{C}, \chi$) where:*
2) $\bar{M}$ exemplify $A \in \mathrm{id}^j_\theta(\lambda)$ for $x \in \mathcal{H}(\chi)$ (and $\chi > \lambda$ and λ) if:

- $\boxtimes_0$ *$\bar{M} = \langle M_\zeta : \zeta < \xi\rangle$, $\xi < \theta$,*
- $\boxtimes_1$ *$\xi < \theta, \theta + 1 \subseteq M_\zeta \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ and $|M_\zeta \cap \lambda| = \lambda$ and $x \in M_\zeta$ and $\lambda \in M_\zeta$, $\bar{C} \in M_\zeta$, $S \in M_\zeta$ and $\lambda \not\subseteq M_\zeta$*
- $\boxtimes_2$ *for some $\alpha^* < \lambda$ for no $\delta \in S \backslash \alpha^*$ do we have:*
  - *(a) $\delta = \sup(M_\zeta \cap \delta)$ for $\zeta < \xi$*
  - *(b) for every $\beta < \delta$ for some α we have: $\alpha \in \mathrm{nacc}(C_\delta)\backslash\beta$, $cf(\alpha) \geq \beta$ and*

    $\circledast$ *for every $\zeta < \xi$ we have: $\alpha \in M_\zeta$ or $Min(M_\zeta \backslash \alpha)$ is singular.*

Proof of 1.16. 1) Recall $\theta = \aleph_{\gamma(*)}$, note that $\gamma(*) + \mathrm{rk}_\lambda(S^+) < \mathrm{rk}_\lambda(S)$, if $\theta > \aleph_0$ by clause $(*)(\delta)$, if $\theta = \aleph_0$ trivially.
Without loss of generality $\delta < \lambda \Rightarrow \mathrm{rk}_\delta(S \cap \delta) < \delta \times \omega$ and even $\mathrm{rk}_\delta(S \cap \delta) < \delta \times n^* + (\mathrm{rk}_\lambda(S) - \lambda \times n^*) < \delta \times n^* + \delta$ (in part (2) the first inequality is $\leq$).

Toward contradiction assume $\lambda \in \mathrm{id}^j_\theta(\bar{C})$ let $x = \langle \lambda, \bar{C}, S\rangle$ and let $\langle M_\zeta : \zeta < \xi\rangle$ exemplify $\lambda \in \mathrm{id}^j_\theta(\bar{C})$ for x which means that $\boxtimes_0, \boxtimes_1, \boxtimes_2$ of Definition 1.19(2) hold and let α^* be as in $\boxtimes_2$.

Let: $E = \{\delta < \lambda : \delta \not\subseteq M_\zeta$ and $\delta = \sup(M_\zeta \cap \delta)$ for every $\zeta < \xi$ and $\delta > \alpha^*$ for the α^* from $\boxtimes_2$ of 1.19(2)$\}$ and let

$$S^* = \{\delta \in S : \text{ for every } \sigma < \delta, \{\alpha \in E \cap \mathrm{nacc}(C_\delta) : \mathrm{cf}(\alpha) > \sigma\} \text{ is unbounded in } \delta\}.$$

So E is a club of λ with every member a limit cardinal, $S^* \subseteq S$ is stationary (as $\lambda \notin \mathrm{id}_p(\bar{C}, \bar{I})$) and even $S^* \notin \mathrm{id}^{\alpha(*)}_{\mathrm{rk}}(\lambda)$ (see clause $(**)(\gamma)$ in the assumption) and using $\boxtimes_2$ of Definition 1.19(2) we shall look only at $\delta \in S^*$.

For each $i < \lambda$ and $\zeta < \xi$ let $\beta^i_\zeta =: \mathrm{Min}(M_\zeta \backslash i)$. As $\langle M_\zeta : \zeta < \xi\rangle$ exemplifies $\lambda \in \mathrm{id}^j_\theta(\bar{C})$, we have

[4] but the "same x" in line 4 should be "every x"

$\boxtimes_3$ for each $\delta \in S^*$ for some $\zeta < \xi$, $\beta^\delta_\zeta = \mathrm{cf}(\beta^\delta_\zeta) > \delta$ hence β^δ_ζ is inaccessible.

Proving this will take some steps. First for some $\beta^* < \delta$ we have:

$\boxtimes_4$ $\alpha \in \mathrm{nacc}(C_\delta)\backslash\beta^* \,\&\, \mathrm{cf}(\alpha) \geq \beta^* \rightarrow (\exists\zeta < \xi)[\mathrm{Min}(M_\zeta\backslash\alpha)$ is an inaccessible $> \alpha]$.

[Why? In the definition of id^j_θ, i.e. clause (b) of $\boxtimes_2$ of Definition 1.19(2) we do not speak on β^δ_ζ for $\delta \in S$, we speak on β^α_ζ, for $\alpha \in \mathrm{nacc}(C_\delta) \cap E$. As $\delta \in S^*$ we have $\delta \in E$ so $\delta > \alpha^*$ hence δ cannot satisfy (a) + (b) of $\boxtimes_2$, but as $\delta \in E$ it satisfies (a) hence for some $\beta^* < \delta$, we have $\boxtimes_4$.]

Next note

$\boxtimes_5$ $\beta^\delta_\zeta = \delta \,\&\, \alpha \in E \cap \mathrm{nacc}(C_\delta) \Rightarrow \beta^\alpha_\zeta = \alpha$.

[Why? So we have $\delta = \beta^\delta_\zeta \in M_\zeta$ hence $C_\delta \in M_\zeta$ so $(\forall\gamma \in \delta \cap M_\zeta)[\mathrm{Min}(C_\delta\backslash\gamma) \in M_\zeta]$, and now for every $\alpha \in E \cap \mathrm{nacc}(C_\delta)$ we can find $\gamma \in M_\zeta \cap \alpha$ satisfying $\gamma > \sup(C_\delta \cap \alpha)$ so $\alpha = \mathrm{Min}(C_\delta\backslash\gamma) \in M_\zeta$ as required in $\boxtimes_5$.]

$\boxtimes_6$ β^δ_ζ singular $\&\, \alpha \in E \cap \mathrm{nacc}(C_\delta) \,\&\, \mathrm{cf}(\alpha) > \mathrm{cf}(\beta^\delta_\zeta) \Rightarrow \beta^\alpha_\zeta = \alpha$.

[Why? Fix such α. There is a club e of β^δ_ζ of order type $\mathrm{cf}(\beta^\delta_\zeta)$ which belongs to M_ζ; also $\mathrm{cf}(\beta^\delta_\zeta) \in M_\zeta \cap \delta$ so $\mathrm{cf}(\beta^\delta_\zeta) < \delta$. Also for every $\delta' \in e_0 = \{\delta' \in e \cap S : \alpha \notin \mathrm{acc}(C_{\delta'})\}$ there is $\gamma_{\delta'}$ such that $\sup(C_{\delta'} \cap \alpha) < \gamma_{\delta'} < \alpha$, hence $\gamma^* = \sup\{\gamma_{\delta'} : \delta' \in e_0\} < \alpha$ (as $\mathrm{cf}(\alpha) > \mathrm{cf}(\beta^\delta_\zeta)$ by assumption). As $\alpha \in \mathrm{acc}(E)$ there is $\gamma^1 \in M_\zeta \cap \alpha$, $\gamma^1 > \gamma^*$. So α is the minimal ordinal α' satisfying $\gamma^1 < \alpha' \,\&\, (\exists\delta' \in e \cap S)[\alpha' \in \mathrm{nacc}(C_{\delta'})] \,\&\, (\forall\delta' \in e \cap S)[\delta' \in \mathrm{nacc}(C_{\delta'}) \rightarrow \sup(\alpha' \cap C_{\delta'}) < \gamma^1]$ hence $\alpha \in M_\zeta$ hence $\beta^\alpha_\zeta = \alpha$ as required.]

Of course, $[\beta^\delta_\zeta$ singular $\Rightarrow \mathrm{cf}(\beta^\delta_\zeta) < \delta]$ as $\mathrm{cf}(\beta^\delta_\zeta) \in M_\zeta \cap \beta^\delta_\zeta = M_\zeta \cap \delta$; so together $\boxtimes_3$ actually holds.

Letting $S^*_\zeta =: \{\delta \in S^* : \beta^\delta_\zeta = \mathrm{cf}(\beta^\delta_\zeta) > \delta\}$, we have $S^* = \bigcup_{\zeta<\xi} S^*_\zeta$, hence for some $\zeta(*) < \xi$ the set $S^*_{\zeta(*)}$ is stationary. Moreover, if $\theta > \aleph_0$ by clause (δ) of $(*)$ in our assumption and if $\theta = \aleph_0$ by 1.5(0) (for the $\mathrm{id}^\gamma_{\mathrm{rk}}$ case) or 1.11(0) (for the id^γ case) we can choose $\zeta(*)$ such that $\mathrm{rk}_\lambda(S^*_{\zeta(*)}) > \alpha(*)$.

So to get the contradiction it suffices to prove $\mathrm{rk}_\lambda\left(S^*_{\zeta(*)}\right) \leq \alpha(*)$. Stipulate $\beta^\lambda_{\zeta(*)} = \lambda$.

Let $\alpha^\delta_{\zeta(*)} =: \mathrm{rk}_{\beta^\delta_{\zeta(*)}}\left(S^+ \cap \beta^\delta_{\zeta(*)}\right)$ for $\delta \leq \lambda$.

Let $\alpha^\delta_{\zeta(*)} = \beta^\delta_{\zeta(*)} \times n^\delta_{\zeta(*)} + \gamma^\delta_{\zeta(*)}$ where $\gamma^\delta_{\zeta(*)} < \beta^\delta_{\zeta(*)}$ (see the assumption in the beginning of the proof). For $\delta < \lambda$, as $\{\lambda, S\} \subseteq M_{\zeta(*)}$ and $\beta^\delta_{\zeta(*)} \in M_{\zeta(*)}$ clearly $\alpha^\delta_{\zeta(*)} \in M_{\zeta(*)}$ hence $\gamma^\delta_{\zeta(*)} \in M_{\zeta(*)} \cap \delta$ hence $\gamma^\delta_{\zeta(*)} < \delta$.

We now prove by induction on $i \in E \cup \{\lambda\}$ that

$$\otimes \qquad \mathrm{rk}_i\left(S^*_{\zeta(*)} \cap i \cap E\right) \leq i \times n^i_{\zeta(*)} + \gamma^i_{\zeta(*)}.$$

This suffices as for $i = \lambda$ (as $\alpha^i_{\zeta(*)} \leq \alpha(*)$) it gives: $\mathrm{rk}_\lambda\left(S^*_{\zeta(*)}\right) = \mathrm{rk}_\lambda(S^*_{\zeta(*)} \cap E) = \mathrm{rk}_\lambda(S^*_{\zeta(*)} \cap \lambda \cap E) \leq \alpha^\lambda_{\zeta(*)} \leq \mathrm{rk}_\lambda(S^+) \leq \alpha(*)$, contradicting the choice of $\zeta(*)$ (and $\alpha(*)$).

Proof of $\otimes$. The case $\mathrm{cf}(i) \leq \aleph_0 \vee i \in \mathrm{nacc}(E) \vee i \in \mathrm{nacc}(\mathrm{acc}(E))$ is trivial; so we assume

$\circledast_1$ $i \in \mathrm{acc}(\mathrm{acc}(E))$ & $\mathrm{cf}(i) > \aleph_0$ hence $\mathrm{rk}_i\left(S^*_{\zeta(*)} \cap i \cap E\right) = \mathrm{rk}_i\left(S^*_{\zeta(*)} \cap i\right)$.

For a given i, clearly for every club e of $\beta^i_{\zeta(*)}$ which belongs to $M_{\zeta(*)}$ we have $i = \sup(e \cap i)$ (as M_ζ "think" e is an unbounded subset of $\beta^i_{\zeta(*)}$ and $i = \sup(i \cap M_\zeta)$ as $i \in E$) and for a given i, by the definition of rk there is a club e of $\beta^i_{\zeta(*)}$ satisfying $\mathrm{Min}(e) > \gamma^i_{\zeta(*)}$ such that one of the following occurs:

(a) $\alpha^i_{\zeta(*)} = 0$ and $\varepsilon \in e \Rightarrow \mathrm{rk}_\varepsilon(S^+ \cap \varepsilon) = 0$ & $S^+ \cap e = \emptyset$

(b) $\alpha^i_{\zeta(*)} > 0$ and $\varepsilon \in e \Rightarrow \mathrm{rk}_\varepsilon(S^+ \cap \varepsilon) < \varepsilon \times n^i_{\zeta(*)} + \gamma^i_{\zeta(*)}$.

As $S^+, \beta^i_{\zeta(*)} \in M_{\zeta(*)}$ without loss of generality $e \in M_{\zeta(*)}$ hence $i \in \mathrm{acc}(e)$. Necessarily

$\circledast_2$ if $\varepsilon \in i \cap \mathrm{acc}(e) \cap \mathrm{acc}(E)$, then $\beta^\varepsilon_{\zeta(*)} \in e$.

[Why? Otherwise $\sup(\beta^\varepsilon_{\zeta(*)} \cap e)$ is a member of e (as e is closed, $\beta^\varepsilon_{\zeta(*)} \geq \varepsilon \in \mathrm{acc}(e)$ so $\beta^\varepsilon_{\zeta(*)} > \mathrm{Min}(e)$), is $\geq \varepsilon$ (as $\varepsilon \in \mathrm{acc}(e)$) and is $< \beta^\varepsilon_{\zeta(*)}$ and it belongs to $M_{\zeta(*)}$ (as $e, \beta^\varepsilon_{\zeta(*)} \in M_{\zeta(*)}$), contradicting the choice of $\beta^\varepsilon_{\zeta(*)}$.]

Hence one of the following occurs:

(A) $\alpha^i_{\zeta(*)} = 0$ and e is disjoint to S^+

(B) $\alpha^i_{\zeta(*)} > 0$ and $\mathrm{rk}_{\beta^\varepsilon_{\zeta(*)}}\left(S^+ \cap \beta^\varepsilon_{\zeta(*)}\right) < \beta^\varepsilon_{\zeta(*)} \times n^i_{\zeta(*)} + \gamma^i_{\zeta(*)}$ for every $\varepsilon \in \mathrm{acc}(e) \cap \mathrm{acc}(E)$.

First assume (A). Now for any $\delta \in \mathrm{acc}(E) \cap S^*_{\zeta(*)}$ we have $\beta^\delta_{\zeta(*)}$ is inaccessible (as $\delta \in S^*_{\zeta(*)}$ and the definition of $S^*_{\zeta(*)}$) and $\beta^\delta_{\zeta(*)} \cap S$ is stationary in $\beta^\delta_{\zeta(*)}$ (otherwise there is a club $e' \in M_{\zeta(*)}$ of $\beta^\delta_{\zeta(*)}$ disjoint to S, but necessarily $\delta \in e'$ but our present assumption is $\delta \in S^*_{\zeta(*)} \subseteq S$, contradiction); together $\beta^\delta_{\zeta(*)} \in S^+$ hence $\beta^\delta_{\zeta(*)} \notin e$ (e from above, after $\circledast_1$), so necessarily $\delta \neq \beta^i_{\zeta(*)} \Rightarrow \delta \notin \mathrm{acc}(e)$. So $\mathrm{acc}(e) \cap \mathrm{acc}(E) \cap i$ is a club of i disjoint to $S^*_{\zeta(*)}$ hence $\mathrm{rk}_i\left(S^*_{\zeta(*)} \cap i\right) = 0$ which suffices for $\otimes$.

If (B) above occurs, then for $\varepsilon \in \mathrm{acc}(e) \cap \mathrm{acc}(E)$ we have $\beta^\varepsilon_{\zeta(*)} \times n^\varepsilon_{\zeta(*)} + \gamma^\varepsilon_{\zeta(*)} < \beta^\varepsilon_{\zeta(*)} \times n^i_{\zeta(*)} + \gamma^i_{\zeta(*)}$.

Since $\gamma^i_{\zeta(*)} < \mathrm{Min}(e)$, we have $(n^\varepsilon_{\zeta(*)}, \gamma^\varepsilon_{\zeta(*)}) <_{\mathrm{lex}} (n^i_{\zeta(*)}, \gamma^i_{\zeta(*)})$, hence $\varepsilon \times n^\varepsilon_{\zeta(*)} + \gamma^\varepsilon_{\zeta(*)} < \varepsilon \times n^i_{\zeta(*)} + \gamma^i_{\zeta(*)}$ for all $\varepsilon \in \mathrm{acc}(e) \cap \mathrm{acc}(E)$. Using the induction hypothesis, we see for $\varepsilon \in e \cap \mathrm{acc}(E) \setminus \mathrm{Min}(e)$ that

$$\mathrm{rk}_\varepsilon(S^*_{\zeta(*)} \cap \varepsilon \cap E) \leq \varepsilon \times n^\varepsilon_{\zeta(*)} + \gamma^\varepsilon_{\zeta(*)} < \varepsilon \times n^i_{\zeta(*)} + \gamma^i_{\zeta(*)}$$

hence by the definition of rk_i the statement $\otimes$ holds for i; which as said above is enough.

2) We repeat the proof of part (1), replacing rk_i by rk^*_i up to and including the phrasing of $\otimes$ and the explanation of why it suffices. For any ordinal $i < \lambda$ and $\zeta < \xi$ let $M_{\zeta,i}$ be the Skolem Hull in $(\mathcal{H}(\chi), \in, <^*_\chi)$ of $M_\zeta \cup \{j : j \leq \beta^i_\zeta\}$. But $\delta \in S^*_{\zeta(*)} \Rightarrow \mathrm{cf}(\beta^\delta_{\zeta(*)}) = \beta^\delta_{\zeta(*)} > \delta$ hence clearly

$\boxtimes_7$ $M_{\zeta,i}$ increases with i, $M_{\zeta,i} \prec (\mathcal{H}(\chi), \in, <^*_\chi)$, and

$\boxtimes_8$ $\delta \in M_\zeta$ & $\mathrm{cf}(\delta) > \beta^i_\zeta \Rightarrow \sup(M_{\zeta,i} \cap \delta) = \sup(M_\zeta \cap \delta)$.

But $\delta \in S^*_{\zeta(*)} \Rightarrow \mathrm{cf}(\beta^\delta_{\zeta(*)}) = \beta^\delta_{\zeta(*)} > \delta$ hence clearly $j < \delta \in S^*_{\zeta(*)} \Rightarrow j < \delta \,\&\, \delta = \sup(M_{\zeta(*)} \cap \beta^\delta_{\zeta(*)}) \Rightarrow j < \delta \,\&\, \delta = \sup(M_{\zeta(*),j} \cap \beta^\delta_{\zeta(*)}) \Rightarrow \beta^\delta_{\zeta(*)} = \mathrm{Min}(M_{\zeta(*),j} \cap \lambda \backslash \delta)$. Now for $j < \lambda$ let $\mathcal{W}_j = \{w : w$ belongs to $M_{\zeta(*),j}$ and $w \subseteq S\}$ and for $w \in \mathcal{W}_j$ we let $w^+ = \{\delta < \lambda : \delta$ inaccessible and $w \cap \delta$ is a stationary subset of $\delta\}$, let $\beta^i_{\zeta(*),j,w} = \beta^i_{\zeta(*),j} = \mathrm{Min}(M_{\zeta(*),j} \cap \lambda \backslash i)$. Also for $j < \lambda$, $w \in \mathcal{W}_j$ and $i > \beta^j_{\zeta(*),j,w}$ let $\alpha^i_{\zeta(*),j,w} = \mathrm{rk}^*_{\beta^i_{\zeta(*),j,w}}(w^+ \cap \beta^i_{\zeta(*),j,w})$, so as $w^+ \subseteq S^+$ necessarily $\alpha^i_{\zeta(*),j,w} = \beta^i_{\zeta(*),j,w} \times n^i_{\zeta(*),j,w} + \gamma^i_{\zeta(*),j,w}$ with $n^i_{\zeta(*),j,w} < \omega$ and $\gamma^i_{\zeta(*),j,w} < \beta^i_{\zeta(*),j}$. By the definition of $M_{\zeta,j}$ and $\beta^i_{\zeta(*),j,w}$ clearly $\beta^i_{\zeta(*),j,w}$ decrease with j and by $\boxtimes_8$ we have $\beta^j_{\zeta(*)} < i \in E \,\&\, \mathrm{cf}(i) > \beta^j_{\zeta(*)} \Rightarrow \beta^i_{\zeta(*),j,w} = \beta^i_{\zeta(*)}$. Now we prove by induction on $i \in E \cup \{\lambda\}$ that

$\otimes^+$ if $j < \lambda$, $\beta^j_{\zeta(*)} < i \in E$, $w \in \mathcal{W}_j$ then
$\mathrm{rk}_i(S^*_{\zeta(*)} \cap w \cap i \cap E) \le i \times n^i_{\zeta(*),j,w} + \gamma^i_{\zeta(*),j,w}$.

This clearly suffices (for $w = S$ we shall get $\otimes$ for each $M_{\zeta(*),j}$ which is more than enough).

Proof of $\otimes^+$. The case $\mathrm{cf}(i) \le \aleph_0 \vee i \in \mathrm{nacc}(E) \vee i \in \mathrm{nacc}(\mathrm{acc}(E))$ is trivial; so we assume

$\circledast_3$ $i \in \mathrm{acc}(\mathrm{acc}(E)) \,\&\, \mathrm{cf}(i) > \aleph_0$ hence $\mathrm{rk}^*_i\left(S^*_{\zeta(*)} \cap w \cap i \cap E\right)$
$= \mathrm{rk}^*_i\left(S^*_{\zeta(*)} \cap w \cap i\right)$.

For a given $w \in \mathcal{W}_j$ and $i \in E \backslash \beta^j_{\zeta(*),j,w}$ clearly for every club e of $\beta^i_{\zeta(*),j,w}$ which belongs to $M_{\zeta(*),j}$ we have $i = \sup(i \cap e)$; (this because "M_ζ thinks" e is an unbounded subset of $\beta^i_{\zeta(*)}$ and $i \in E$ implies $i = \sup(i \cap M_\zeta)$ is a limit ordinal); so $i \in \mathrm{acc}(e)$ even $i \in \mathrm{acc}(\mathrm{acc}(e))$, etc. By the definition of $\mathrm{rk}^*_{\beta^i_{\zeta(*),j,w}}$, for our i, there is a club e of $\beta^i_{\zeta(*),j,w}$ with $\mathrm{Min}(e) > \gamma^i_{\zeta(*),j,w}$ and h (for case (c)) such that one of the following cases occurs:

(a) $\gamma^i_{\zeta(*),j,w} = 0 \,\&\, n^i_{\zeta(*),j,w} = 0$ that is $\alpha^i_{\zeta(*),j,w} = 0$ and $w^+ \cap e = \emptyset$ so $\varepsilon \in e \Rightarrow \mathrm{rk}^*_\epsilon(w^+ \cap \varepsilon) = 0 \,\&$

(b) $\gamma^i_{\zeta(*),j,w} > 0$ and $\varepsilon \in e \Rightarrow \mathrm{rk}^*_\varepsilon(w^+ \cap \varepsilon) < \varepsilon \times n^i_{\zeta(*),j,w} + \gamma^i_{\zeta(*),j,w}$

(c) $\gamma^i_{\zeta(*),j,w} = 0 \,\&\, n^i_{\zeta(*),j,w} > 0$, h a pressing down function on $w^+ \cap i$ such that for each $j < i$ we have $j < \varepsilon \in e \,\&\, h(\varepsilon) = j \Rightarrow \mathrm{rk}^*_\varepsilon(w^+ \cap \varepsilon) < \varepsilon \times n^i_{\zeta(*),j,w} + \gamma^i_{\zeta(*),j,w}$.

For $j < \lambda$, $w \in \mathcal{W}_j$ and $i < \lambda$, clearly $\beta^i_{\zeta(*),j,w}$ and w belongs to $M_{\zeta(*),j}$ hence also $\alpha^i_{\zeta(*),j,w} \in M_{\zeta(*),j}$ and so also ($n^i_{\zeta(*),j,w}$ and) $\gamma^i_{\zeta(*),j,w}$ belongs to $M_{\zeta(*),j}$. So without loss of generality to clauses (a), (b), (c) we can add:

$\circledast_4$ $e \in M_{\zeta(*),j}$ and $h \in M_{\zeta(*),j}$ when defined (and $i = \sup(i \cap e)$).

Necessarily

$\circledast_5$ if $\varepsilon \in i \cap \operatorname{acc}(e) \cap \operatorname{acc}(E)$ then $\beta^{\varepsilon}_{\zeta(*),j,w} \in e$.

[Why? Otherwise:

(i) $\beta^{\varepsilon}_{\zeta(*),j,w} < i$ (as $\varepsilon < i \,\&\, i \in \operatorname{acc}(E)$ and the definition of $\beta^{\varepsilon}_{\zeta(*),j,w}$ and the choice of E)

(ii) $\sup(\beta^{\varepsilon}_{\zeta(*),j,w} \cap e)$ is a member of e (as e is a closed unbounded subset of $\beta^{i}_{\zeta(*),j,w}$ and $\operatorname{Min}(e) < \beta^{\varepsilon}_{\zeta(*),j,w} < i \leq \beta^{i}_{\zeta(*),j,w}$)

(iii) $\sup(\beta^{\varepsilon}_{\zeta(*),j,w} \cap e) \geq \varepsilon$ (as $\varepsilon \in \operatorname{acc}(e) \,\&\, \varepsilon \leq \beta^{\varepsilon}_{\zeta(*),j,w}$)

(iv) $\beta^{\varepsilon}_{\zeta(*),j,w} \in M_{\zeta(*),j}$ (by its definition)

(v) $\sup(\beta^{\varepsilon}_{\zeta(*),j,w} \cap e) \in M_{\zeta(*),j}$ (as $e, \beta^{\varepsilon}_{\zeta(*)} \in M_{\zeta(*),j}$).

So $\sup(\beta^{\varepsilon}_{\zeta(*),j,w} \cap e) \in \lambda \cap M_{\zeta(*),j} \backslash \varepsilon$ hence is $\geq \operatorname{Min}(\lambda \cap M_{\zeta(*),j} \backslash \varepsilon) = \beta^{\varepsilon}_{\zeta(*),j,w}$, but trivially $\sup(\beta^{\varepsilon}_{\zeta(*),j,w} \cap e) \leq \beta^{\varepsilon}_{\zeta(*),j,w}$ so we get the $\beta^{\varepsilon}_{\zeta(*),j,w} = \sup(\beta^{\varepsilon}_{\zeta(*),j,w} \cap e)$ and it belongs to e by (ii) so we have proved $\circledast_5$.]
So by the choice of e, one of the following cases occurs:

(A) $\alpha^{i}_{\zeta(*),j,w} = 0$ and e is disjoint to w^{+}

(B) $\gamma^{i}_{\zeta(*),j,w} > 0$ and $\operatorname{rk}^{*}_{\beta^{\epsilon}_{\zeta(*),j,w}}\left(w^{+} \cap \beta^{\varepsilon}_{\zeta(*),j,w}\right) < \beta^{\varepsilon}_{\zeta(*),j,w} \times n^{i}_{\zeta(*),j,w} + \gamma^{i}_{\zeta(*),j,w}$ for every $\epsilon \in \operatorname{acc}(e) \cap \operatorname{acc}(E)$

(C) $\gamma^{i}_{\zeta(*),j,w} = 0, n^{i}_{\zeta(*),j,w} > 0, h \in M_{\zeta(*),j}$ a pressing down funtion on e such that: $\varepsilon < \mu \in e \,\&\, (\mu$ inaccessible$) \Rightarrow \operatorname{rk}^{*}_{\mu}(\{\gamma < \mu : \gamma \in w^{+} \cap e$ and $h(\gamma) = \varepsilon\}) < \mu \times n^{i}_{\zeta(*),j,w}$ (read Definition 1.9(1) clause (c) and use diagonal intersection; remember that for singular μ, $\operatorname{rk}^{*}_{\mu}(\mu) = \operatorname{rk}_{\mu}(\mu) < \mu$).

First assume (A). Now for any $\delta \in \operatorname{acc}(E) \cap S^{*}_{\zeta(*)} \cap w$ necessarily $\beta^{\delta}_{\zeta(*),j,w}$ is inaccessible (as $\delta \in S^{*}_{\zeta(*)}$ and the definition of $S^{*}_{\zeta(*)}$) and $\beta^{\delta}_{\zeta(*),j,w} \cap w$ is stationary in $\beta^{\delta}_{\zeta(*),j,w}$ (otherwise there is a club $e' \in M_{\zeta(*),j}$ of $\beta^{\delta}_{\zeta(*),j,w}$ disjoint to w, but necessarily $\delta \in e'$ and $\delta \in w$, contradiction); together $\beta^{\delta}_{\zeta(*),j,w} \in w^{+}$ hence $\beta^{\delta}_{\zeta(*),j,w} \notin e$ (e from above), so as $e \in M_{\zeta(*),j}$ necessarily $\delta \neq \beta^{i}_{\zeta(*),j,w} \Rightarrow \delta \notin \operatorname{acc}(e)$. So $\operatorname{acc}(e) \cap \operatorname{acc}(E) \cap i$ is a club of i disjoint to $S^{*}_{\zeta(*)} \cap w$ hence $\operatorname{rk}^{*}_{i}\left(S^{*}_{\zeta(*)} \cap w \cap i\right) = 0$ which suffices for $\otimes^{+}$.

Secondly, assume clause (B) occurs; then for every $\varepsilon \in \operatorname{acc}(e) \cap \operatorname{acc}(E)$ we have $\beta^{\varepsilon}_{\zeta(*),j,w} \times n^{\varepsilon}_{\zeta(*),j,w} + \gamma^{\varepsilon}_{\zeta(*),j,w} < \beta^{\varepsilon}_{\zeta(*),j,w} \times n^{i}_{\zeta(*),j,w} + \gamma^{i}_{\zeta(*),j,w}$. Since $\gamma^{i}_{\zeta(*),j,w} \leq \operatorname{Min}(e)$ we have $(n^{\varepsilon}_{\zeta(*),j,w}, \gamma^{\varepsilon}_{\zeta(*),j,w}) <_{\ell ex} (n^{i}_{\zeta(*),j,w}, \gamma^{i}_{\zeta(*),j,w})$ hence $\varepsilon \times n^{\varepsilon}_{\zeta(*),j,w} + \gamma^{\varepsilon}_{\zeta(*),j,w} < \varepsilon \times n^{i}_{\zeta(*),j,w} + \gamma^{i}_{\zeta(*),j,w}$ for every $\varepsilon \in \operatorname{acc}(e) \cap \operatorname{acc}(E)$. Using the induction hypothesis we get for every $\varepsilon \in \operatorname{acc}(e) \cap \operatorname{acc}(E)$ that

$$\operatorname{rk}^{*}_{\varepsilon}(S^{*}_{\zeta(*),j,w} \cap i \cap E) \leq \varepsilon \times n^{\varepsilon}_{\zeta(*),j,w} + \gamma^{\varepsilon}_{\zeta(*),j,w} < \varepsilon \times n^{i}_{\zeta(*),j,w} + \gamma^{i}_{\zeta(*),j,w}.$$

Lastly, assume that clause (C) holds and let $e, h \in M_{\zeta(*),j}$ be as there, without loss of generality i is inaccessible (otherwise the conclusion is trivial), so $e \cap i$, $E \cap i$ are clubs of i, and let $j^* =: h(i)$, $j_1 = \text{Max}\{j, j^*\}$ so $j \le j_1 < i$ and $M_{\zeta(*), j_1}$ is well defined (and $j^*, j_1 \in M_{\zeta(*), j_1}$). Clearly $\beta^i_{\zeta(*), j^*, w} = \beta^i_{\zeta(*), j, w}$ [because $\beta^i_{\zeta(*), j, w}$ is inaccessible (as otherwise $\alpha^i_{\zeta(*), j, w} < \beta^i_{\zeta(*), j, w}$ contradicting our case) hence $j \le j' < i \Rightarrow \beta^i_{\zeta(*), j^*, w} = \beta^i_{\zeta(*), j, w}$ as in previous cases.]

Let $u_{j_1} = \{\alpha \in w \cap e : h(\alpha) = j^*\} \in M_{\zeta(*), j_1}$ and as $j_1 < i \le \beta^i_{\zeta(*), j, w}$ clearly $\delta \in e \Rightarrow \text{rk}^*_\delta(S^*_{\zeta(*), j} \cap u_{j_1} \cap \delta) < n^i_{\zeta(*), j, w} \times \delta$ hence by the induction hypothesis $\delta \in i \cap \text{acc}(e) \cap \text{acc}(E) \Rightarrow \text{rk}^*_\delta(S^*_{\zeta(*), j_1} \cap u_j \cap \delta) < n^i_{\zeta(*), j, w} \times \delta$, hence $\text{rk}_i(S^*_{\zeta(*), j_1} \cap w \cap i) \le n^i_{\zeta(*), j, w} \times i$ as required. $\square_{1.16}$

1.20 Claim. Assume

(a) (i) $\text{cf}(\lambda) > \mu$

(ii) $S \subseteq \{\delta < \lambda : \mu < \text{cf}(\delta) < \delta\}$

(iii) $\text{rk}_\lambda(S) = \gamma^* = \lambda \times n^* + \zeta^*$ where $\zeta^* < \lambda, n^* < \omega$

(b) (i) J an $\aleph_1$-complete ideal on μ containing the singletons

(ii) if $A \in J^+$, (i.e. $A \subseteq \mu$, $A \notin J$) and $f \in {}^A\lambda$ <u>then</u> $\|f\|_{J \restriction A} < \lambda$ (if e.g. $J = J^{\text{bd}}_\mu$, μ regular, then $A = \mu$ suffices as $J \restriction A \cong J$)

(iii) if $A \in J^+$ and $f \in {}^A(\zeta^*)$ <u>then</u> $\|f\|_{J \restriction A} < \zeta^*$.

<u>Then</u> $\text{id}^{<\gamma^*}_{\text{rk}}(\lambda) \restriction S$ is J-indecomposable (see Definition 1.21 below).

1.21 Definition. *An ideal I on λ is J-indecomposable where J is an ideal on μ, if: for any $S_\mu \subseteq \lambda$, $S_\mu \notin I$, and $f : S_\mu \to J$ there is $i < \mu$ such that $S_i =: \{\alpha \in S_\mu : i \notin f(\alpha)\} \notin I$; note that given S_μ, f can be defined from $\langle S_i : i < \mu\rangle$ and vice versa.*

Clearly

1.22 Claim. 1) If $J = J^{\text{bd}}_\mu$, μ regular then "I is J^{bd}-indecomposable" is equivalent to "I is μ-indecomposable".

2) If J is a $|\zeta^*|^+$-complete ideal on μ, then the assumption (b) (iii) of 1.20 holds automatically.

Proof of Claim 1.20. We prove this by induction on γ^*. Assume toward contradiction that the conclusion fails as exemplified by S_μ, f, S_i (for $i < \mu$), so $f : S_\mu \to J$ we have $S_i = \{\alpha \in S_\mu : i \notin f(\alpha)\}$ and without loss of generality $S_\mu \subseteq S$ such that $S_\mu \notin \text{id}^{<\gamma^*}_{\text{rk}}(\lambda)$, but $S_i \in \text{id}^{<\gamma^*}_{\text{rk}}(\lambda)$ for each $i < \mu$. Now let $\text{rk}_\lambda(S_i) = \lambda \times n_i + \zeta_i$ with $\zeta_i < \lambda$; clearly $\delta \in S_\mu \Rightarrow \{i < \mu : \delta \notin S_i\} = f(\delta) \in J$. Without loss of generality $S = S_\mu$ and clearly $S_i \subseteq S_\mu = \bigcup_{j<\mu} S_j$. By our assumption toward contradiction clearly $n_i < n^* \vee (n_i = n^* \,\&\, \zeta_i < \zeta^*)$ for each $i < \mu$.

As we can replace S by $S \cap E$ for any club E of λ, without loss of generality

$(*)_0$ if $\delta < \lambda$ then $\text{rk}_\delta(S \cap \delta) < \delta \times n^* + (\text{rk}_\lambda(S) - \lambda \times n^*) = \delta \times n^* + \zeta^*$ and $\text{rk}_\delta(S_i \cap \delta) < \delta \times n_i + \zeta_i$ and $\text{Min}(S) > \zeta^*, \zeta_i$ for $i < \mu$.

Recalling 1.3(1), (4), for $\delta \in S^{[0]}_\mu \cup \{\lambda\}$ and $n \leq n^*$ let: $A^\delta_n = \{i < \mu : \delta \times n \leq \mathrm{rk}_\delta(S_i \cap \delta) < \delta \times (n+1)\}$ and let $f^\delta_n : A^\delta_n \to \delta$ be defined by $f^\delta_n(i) =: \mathrm{rk}_\delta(S_i \cap \delta) - \delta \times n$ and let $n(\delta) = \mathrm{Min}\{n : A^\delta_n \notin J\}$ so by $(*)_0$ clearly $n(\delta)$ is well defined and $\leq n^*$.

For $i < \mu$ and $\delta < \lambda$ let $\mathrm{rk}_\delta(S_i \cap \delta) = \delta \times m_{\delta,i} + \varepsilon_{\delta,i}$, where $m_{\delta,i} \leq n^*$ and $\varepsilon_{\delta,i} < \delta$; so for some E_0

$(*)_1$ E_0 is a club of λ, and if $\delta < \lambda$, $A^\delta_n \notin J$ and $n \leq n^*$, then $\|f^\delta_n\|_{J \restriction A^\delta_n} < \mathrm{Min}(E_0 \backslash (\delta+1))$

(possible as $f^\delta_n : A^\delta_n \to \delta \subseteq \lambda$ and hypothesis (b)(ii)).

Now we shall prove for $\delta \in S^{[0]} \cup \{\lambda\}$ that, recalling $S^{[0]} = \{\delta : \delta \in S$ or $S \cap \delta$ is stationary in $\delta\}$:

$$\bigotimes_\delta \quad \mathrm{rk}_\delta(S_\mu \cap E_0 \cap \delta) \leq \delta \times n(\delta) + \|f^\delta_{n(\delta)}\|_{J \restriction A^\delta_{n(\delta)}} < \delta \times n(\delta) + \delta.$$

Why does this suffice? For $\delta = \lambda$, first note: if $n(\lambda) < n^*$ then $\mathrm{rk}_\lambda(S_\mu) \leq \lambda \times n(\lambda) + \|f^\delta_{n(\delta)}\|_{J \restriction A^\delta_{n(\delta)}} \leq \lambda \times (n^*-1) + \|f^\delta_{n(\delta)}\|_{J \restriction A^\delta_{n(\delta)}} < \lambda \times (n^*-1) + \lambda \leq \lambda \times n^* \leq \mathrm{rk}_\lambda(S) = \mathrm{rk}_\lambda(S_\mu)$ [why? first inequality by $\otimes_\lambda$, second inequality by $n(\lambda) < n^*$ (see above), third inequality by assumption (b)(ii), as for $i \in A_{n(\lambda)}$, $f^\delta_{n(\delta)}(i)$, that is $f^\lambda_{n(\lambda)}(i)$ is $\zeta_i < \lambda$ by our assumption toward contradition; the fourth inequality is an ordinal addition and the fifth we have assumed] and this is a contradiction.

So we can assume $n(\lambda) = n^*$, but then by $\otimes_\lambda$, we know $\mathrm{rk}_\lambda(S_\mu) \leq \lambda \times n(\lambda) + \|f^\delta_{n(\delta)}\|_{J \restriction A^\delta_{n(\delta)}}$.

But for $i \in A^\delta_{n(\delta)} = A^\lambda_{n^*}$, by the definition of the A^δ_n's we know that $n_i = n(\delta) = n(\lambda) = n^*$, and so we know $\lambda \times n_i + \zeta_i = \mathrm{rk}_\lambda(S_i) < \mathrm{rk}(S_\mu) = \gamma = \lambda \times n^* + \zeta^*$ so we know $f^\delta_{n(\delta)}(i) = \mathrm{rk}_\delta(S_i \cap \delta) - \delta \times n(\delta) = \zeta_i < \zeta^*$ so by assumption (b) (iii), $\|f^\delta_{n(\delta)}\|_{J \restriction A^\delta_{n(\delta)}} < \zeta^*$, so by $\otimes_\lambda$, $\mathrm{rk}_\lambda(S_\mu) < \lambda \times n^* + \zeta^*$, contradiction.

So it actually suffices to prove $\otimes_\delta$. We prove it by induction on δ.

If $\mathrm{cf}(\delta) = \aleph_0$, or $\delta \notin \mathrm{acc}(E_0)$ or more generally $S_\mu \cap \delta$ is not a stationary subset δ, then $\mathrm{rk}_\delta(S_\mu \cap \delta) = 0$, and $\mathrm{rk}_\delta(S_i \cap \delta) = 0$ hence $\|f^\delta_{n(\delta)}\| = 0$ so the inequality $\otimes_\delta$ holds trivially.

So assume otherwise; for each $i < \mu$, for some club e_i of δ we have:

$$(*)_2 \quad \delta(1) \in e_i \Rightarrow (m_{\delta(1),i} < m_{\delta,i}) \vee (m_{\delta(1),i} = m_{\delta,i} \;\&\; \varepsilon_{\delta(1),i} < \varepsilon_{\delta,i}).$$

Without loss of generality $e_i \subseteq E_0$. As $S_\mu \cap \delta$ is a stationary in δ (as we are assuming "otherwise") by hypothesis (a) (ii) of the claim, $\mathrm{cf}(\delta) \geq \mathrm{Min}\{\mathrm{cf}(\alpha) : \alpha \in S\} > \mu$, so $e =: \bigcap_{i \in A^\delta_{n(\delta)}} e_i$ is a club of δ.

As $\varepsilon_{\delta,i} < \delta$ (see its choice) and $\mathrm{cf}(\delta) > \mu$ (by hypothesis (a) (ii)) clearly $\varepsilon = \sup_{i<\mu} \varepsilon_{\delta,i} < \delta$, hence $\sup(\mathrm{Rang}(f^\delta_{n(\delta)})) < \delta$ hence $\|f^\delta_{n(\delta)}\|_{J \restriction A^\delta_{n(\delta)}} < \delta$ (see $(*)_1$, as $\delta \in E_0$), so the second inequality in $\otimes_\delta$ holds; so without loss of generality $\varepsilon_{\delta,i} < \min(e)$ and $\|f^\delta_{n(\delta)}\|_{J \restriction A^\delta_{n(\delta)}} < \min(e)$.

Suppose the first inequality in $\otimes_\delta$ fails, so $\mathrm{rk}_\delta(S_\mu \cap E_0 \cap \delta) > \delta \times n(\delta) + \|f^\delta_{n(\delta)}\|_{J \restriction A^\delta_{n(\delta)}}$, hence

$$B = \left\{\delta(1) \in e : \mathrm{rk}_{\delta(1)}(S_\mu \cap E_0 \cap \delta(1)) \geq \delta(1) \times n(\delta) + \|f^\delta_{n(\delta)}\|_{J \restriction A^\delta_{n(\delta)}}\right\}$$

is a stationary subset of δ; note that

$\delta(1) \in B \Rightarrow \delta(1) \in e \Rightarrow \|f^\delta_{n(\delta)}\|_{J\restriction A^\delta_{n(\delta)}} < \min(e) \Rightarrow \|f^\delta_{n(\delta)}\|_{J\restriction A^\delta_{n(\delta)}} < \delta(1)$.

But by the induction hypothesis

$$\begin{aligned}\delta(1) \in B \Rightarrow \ & \mathrm{rk}_{\delta(1)}(S_\mu \cap E_0 \cap \delta(1)) \leq \delta(1) \times n(\delta(1)) + \|f^{\delta(1)}_{n(\delta(1))}\|_{J\restriction A^{\delta(1)}_{n((\delta)(1))}} \\ & < \delta(1) \times n(\delta(1)) + \delta(1).\end{aligned}$$

Let $\delta(1) \in B$; putting this together with the definition of "$\delta(1) \in B$" we get

$$(*)_3 \ \ \delta(1) \times n(\delta) + \|f^\delta_{n(\delta)}\|_{J\restriction A^\delta_{n(\delta)}} \leq \delta(1) \times n(\delta(1)) + \|f^{\delta(1)}_{n(\delta(1))}\|_{J\restriction A^{\delta(1)}_{n(\delta(1))}}.$$

Now by $(*)_2$ necessarily $n(\delta(1)) \leq n(\delta)$ so by $(*)_3$ we have $n(\delta(1)) = n(\delta)$ (remember $\|f^{\delta(1)}_{n(\delta(1))}\|_{J\restriction A^{\delta(1)}_{n(\delta(1))}} < \delta(1)$ by the induction hypothesis). So

$$(*)_4 \ \ \|f^\delta_{n(\delta)}\|_{J\restriction A^\delta_{n(\delta)}} \leq \|f^{\delta(1)}_{n(\delta(1))}\|_{J\restriction A^{\delta(1)}_{n(\delta(1))}}.$$

Now by $(*)_2$ (as we have $n(\delta) = n(\delta(1))$)

$$\left\{i \in A^\delta_{n(\delta)} : i \notin A^{\delta(1)}_{n(\delta(1))}\right\} \subseteq \bigcup_{n<n(\delta(1))} A^{\delta(1)}_n$$

now as $n(\delta(1)) = \mathrm{Min}\{n : A^{\delta(1)}_n \notin J\}$ and J an ideal, clearly $\bigcup_{n<n(\delta(1))} A^{\delta(1)}_n \in J$. So we have shown $A^\delta_{n(\delta)} \setminus A^{\delta(1)}_{n(\delta(1))} \in J$. Also for $i \in A^\delta_{n(\delta)} \cap A^{\delta(1)}_{n(\delta(1))}$, we have $f^\delta_{n(\delta)}(i) = \varepsilon^\delta_{\delta,i} > \varepsilon_{\delta(1),i} = f^{\delta(1)}_{n(\delta(1))}(i)$. Together (and by the properties of $\| - \|_-$)

$$\begin{aligned}\|f^\delta_{n(\delta)}\|_{J\restriction A^\delta_{n(\delta)}} &= \|f^\delta_{n(\delta)} \restriction (A^\delta_{n(\delta)} \cap A^{\delta(1)}_{n(\delta(1))})\|_{J\restriction(A^\delta_{n(\delta)} \cap A^{\delta(1)}_{n(\delta(1))})} \\ &> \|f^{\delta(1)}_{n(\delta(1))} \restriction (A^\delta_{n(\delta)} \cap A^{\delta(1)}_{n(\delta(1))})\|_{J\restriction(A^\delta_{n(\delta)} \cap A^{\delta(1)}_{n(\delta(1))})} \\ &\geq \|f^{\delta(1)}_{n(\delta(1))} \restriction A^{\delta(1)}_{n(\delta(1))}\|_{J\restriction A^{\delta(1)}_{n(\delta(1))}}\end{aligned}$$

contradicting $(*)_4$. $\square_{1.20}$

1.23 Claim. If J is an ideal on μ, $\mu < \lambda$, γ a limit ordinal, J is μ-complete, $\gamma < \mu$, <u>then</u> $I = \mathrm{id}^{<\gamma}_{\mathrm{rk}}(\lambda) \restriction S$ is J-indecomposable.

Proof. Assume $S_\mu \in I^+$ and $f : S_\mu \to J$ and $S_i =: \{\alpha \in S_\mu : i \notin f(\alpha)\}$.

Now we prove by induction on $\beta < \gamma$ that: if $\delta < \lambda$, $\mathrm{rk}_\delta(S_\mu \cap \delta) \geq 2\beta$ and $\mathrm{cf}(\delta) \neq \mu$, <u>then</u> $A_\beta =: \{i : \mathrm{rk}_\delta(S_i \cap \delta) \geq \beta\} = \mu \bmod J$. Note that we have "$\geq 2\beta$" in the assumption but $\geq \beta$ in the conclusion; we can "get away" with this as γ is a limit ordinal. As J is μ-complete, $\mu > |\gamma|$ this implies that $\{i : \mathrm{rk}_\delta(S_i \cap \delta) \geq \gamma\} = \mu \bmod J$. So let us carry the induction; if $\beta = 0$ this is trivial and for β limit use $\beta < \gamma < \mu$ and the induction hypothesis (and J being μ-complete). So assume $\beta = \alpha + 1$, $\delta < \lambda$, $\mathrm{cf}(\delta) \neq \mu$, $\mathrm{rk}_\delta(S_\mu \cap \delta) \geq 2\beta = 2\alpha + 2$, hence $S' =: \{\delta' < \delta \colon \mathrm{rk}_{\delta'}(S_\mu \cap \delta') \geq 2\alpha + 1\}$ is a stationary subset of δ.

So $\delta' \in S'$ & $\mathrm{cf}(\delta') \neq \mu \Rightarrow \delta' \in A_\alpha$ by the induction hypothesis so if $\{\delta' \in S' : \mathrm{cf}(\delta') \neq \mu\}$ is a stationary subset of δ we are done. Otherwise, still $[\delta' \in S' \Rightarrow \{\delta'' < \delta' : \delta'' \in A_{2\alpha}\}$ is a stationary subset of $\delta']$ hence $S'' = \{\delta'' < \delta : \mathrm{cf}(\delta'') < \mu$ and $\delta'' \in A_{2\alpha}\}$ is a stationary subset of δ, and we can finish as before. $\square_{1.23}$

1.24 Remark. 1) It is more natural to demand only J is κ-complete and $\kappa > \gamma$; and allow γ to be a successor, but this is not needed and will make the statement more cumbersome because of the "problematic" cofinalities in $[\kappa, \mu]$.
2) We can prove more in 1.23:

$\otimes$ if $\beta < \mu$, $\mathrm{rk}_\lambda(S_\mu) > \beta$ then $\{i < \mu : \mathrm{rk}_\lambda(S_i) \geq \beta\} = \mu \bmod J$.

1.25 Theorem. *Assume λ is inaccessible and there is $S \subseteq \lambda$ stationary such that* $\mathrm{rk}_\lambda(\{\kappa < \lambda : \kappa$ *is inaccessible and* $S \cap \kappa$ *is stationary in* $\kappa\}) < \mathrm{rk}_\lambda(S)$.

<u>Then</u> on λ there is a Jonsson algebra.

Proof. Assume toward contradiction that there is no Jonsson algebra on λ. Let $S^+ =: \{\delta < \lambda : \delta$ inaccessible and $S \cap \delta$ is stationary in $\delta\}$.
Note that without loss of generality

$\circledast$ S is a set of singulars and $\mathrm{rk}_\lambda(S)$ is a limit ordinal.

[Why? Let $S' = \{\delta \in S : \delta$ a singular ordinal $\}$, $S'' = \{\delta \in S : \delta$ is a regular cardinal$\}$, so $\mathrm{rk}_\lambda(S) = \mathrm{rk}_\lambda(S' \cup S'') = \mathrm{Max}\{\mathrm{rk}(S'), \mathrm{rk}(S'')\}$ by 1.5(0). Now if $\mathrm{rk}_\lambda(S'') < \mathrm{rk}_\lambda(S)$, then necessarily $\mathrm{rk}_\lambda(S') = \mathrm{rk}_\lambda(S)$ so we can replace S by S'. If $\mathrm{rk}_\lambda(S'') = \mathrm{rk}(S)$ then $\mathrm{rk}_\lambda(S'') > \mathrm{rk}_\lambda(S^+)$ and clearly $S'' \cap \delta$ stationary $\Rightarrow \delta \in S^+$, so necessarily $\mathrm{rk}_\lambda(S'')$ is finite hence λ has a stationary set which does not reflect and we are done; see [Sh:g]. If $\mathrm{rk}_\lambda(S)$ is a successor ordinal we are done similarly.]

By the definition of rk_λ, $\gamma^* =: \mathrm{rk}_\lambda(S) < \lambda + \mathrm{rk}_\lambda(S^+)$, but we have assumed $\mathrm{rk}_\lambda(S^+) < \mathrm{rk}_\lambda(S)$ so $\mathrm{rk}_\lambda(S) < \lambda + \mathrm{rk}_\lambda(S)$, which implies $\mathrm{rk}_\lambda(S) < \lambda \times \omega$. So for some $n^* < \omega$ we have $\lambda \times n^* \leq \mathrm{rk}_\lambda(S) < \lambda \times n^* + \lambda$.

Let $\mathrm{rk}_\lambda(S^+) = \beta^* = \lambda \times m^* + \varepsilon^*$ with $\varepsilon^* < \lambda$. We shall now prove 1.25 by induction on λ. By [Sh:g, Ch.III], without loss of generality $\beta^* > 0$. By 1.5(9) we can find a club E of λ such that:

(A) $\delta \in E \Rightarrow \mathrm{rk}_\delta(S \cap \delta) < \delta \times n^* + (\mathrm{rk}_\lambda(S) - \lambda \times n^*)$

(B) $\delta \in E \Rightarrow \mathrm{rk}_\delta(S^+ \cap \delta) < \delta \times m^* + \varepsilon^*$.

Note that $\delta \times m^* + \varepsilon^* > 0$ for $\delta \in E$ (or just $\delta > 0$) as $\beta^* > 0$. Let $A =: \{\delta \in E : \delta$ inaccessible, $\varepsilon^* < \delta$ and $\mathrm{rk}_\delta(S \cap \delta) \geq \delta \times m^* + \varepsilon^*\}$.

Clearly $\delta \in A$ implies $S \cap \delta$ is a stationary subset of δ. By the induction hypothesis and the choice of A and clause (B) every member of A has a Jonsson algebra on it and by the definition of A (and 1.5(9)) we have $[\alpha < \lambda$ & $A \cap \alpha$ is stationary in $\alpha \Rightarrow \alpha \in A]$; note that as A is a set of inaccessibles, any ordinal in which it reflects is inaccessible. If A is not a stationary subset of λ, then without loss of generality $A = \emptyset$, and we get $\mathrm{rk}_\lambda(S) \leq \lambda \times m^* + \varepsilon^* = \beta^* < \mathrm{rk}_\lambda(S)$, a contradiction. So without loss of generality (using the induction hypothesis on λ):

$\bigoplus$ A is stationary, $A^{[0]} \subseteq A$, i.e. $(\forall \delta < \lambda)(A \cap \delta$ is stationary in $\delta \Rightarrow \delta \in A)$, each $\delta \in A$ is an inaccessible with a Jonsson algebra on it.

So by [Sh:g, IV, 2.12, p.209] without loss of generality for arbitrarily large $\kappa < \lambda$ (even κ inaccessible):

$\bigotimes_\kappa$ $\kappa = \mathrm{cf}(\kappa) > \aleph_0, \kappa < \lambda$ and for every $f \in {}^\kappa\lambda$ we have $\|f\|_{J^{\mathrm{bd}}_\kappa} < \lambda$.

So choose such $\kappa < \lambda$ satisfying $\kappa > \mathrm{rk}_\lambda(S) - \lambda \times n^*$. We shall show that

$(*)$ $\mathrm{id}^{<\gamma^*}_{\mathrm{rk}}(\lambda) \restriction S$ is J^{bd}_κ-indecomposable

hence it follows by 1.22(1)

$(*)'$ $\mathrm{id}^{<\gamma^*}_{\mathrm{rk}}(\lambda) \restriction S$ is κ-indecomposable.

Why $(*)$ holds? If $\gamma^* \geq \lambda$ by 1.5(1),(3) we know that $\mathrm{rk}_\lambda(\{\delta \in S^{[0]} : \mathrm{cf}(\delta) > \kappa\}) = \mathrm{rk}_\lambda(S)$, so without loss of generality $\mathrm{Min}\{\mathrm{cf}(\delta) : \delta \in S\} > \kappa$ and we can use 1.20 and the statement $\bigotimes$ above to get $(*)$. If $\gamma^* < \lambda$ use 1.23. So $(*)$ and $(*)'$ holds.

Note that S^+ satisfies the assumptions on A in 1.14, i.e. clause (b) there and letting $\sigma = \kappa$, the ideal $\mathrm{id}^{<\gamma^*}_{\mathrm{rk}}(\lambda)$ is κ-indecomposable by $(*)'$ above. Hence by 1.14 applied to $J = \mathrm{id}^{<\gamma^*}_{\mathrm{rk}}(\lambda), \sigma = \kappa, S, A$, we get that for some S-club system $\bar{C}$ we have:

(a) $\delta \in S \Rightarrow \mathrm{nacc}(C_\delta) \subseteq A$

(b) for every club E of λ,
$\mathrm{rk}_\lambda(\{\delta \in S : \delta = \sup(E \cap \mathrm{nacc}(C_\delta))\}) \geq \gamma^*$.

We now apply 1.16(1) for our S, S^+, n^*, λ and $\theta = \aleph_0$. Why its assumptions hold? Now λ is a Jonsson cardinal by our assumption toward contradiction. Clauses $(*)(\alpha) + (*)(\beta)$ hold by our choice of S, S^+, clauses $(*)(\gamma) + (*)(\delta)$ holds as $\theta = \aleph_0$, clause $(**)(\alpha)$ holds by the choice of $\bar{C}$, clause $(**)(\beta)$ holds by $(**)(\gamma)$. Last and the only problematic assumption of 1.16 is clause (γ) of $(**)$ there, which holds by clause (b) above because $\mathrm{nacc}(C_\delta) \subseteq A$, each $\alpha \in A$ is inaccessible. So the conclusion of 1.16 holds, i.e. $\lambda \notin \mathrm{id}^j_{\aleph_0}(\bar{C})$. Now if $\delta \in S, \alpha \in \mathrm{nacc}(C_\delta)$ then α is from A but by the choice of A (and the induction hypothesis on λ) this implies that on α there is a Jonsson algebra, so we finish by 1.26(1) below. $\square_{1.25}$

1.26 Claim. 1) Assume

(a) λ is inaccessible

(b) $\bar{C} = \langle C_\delta : \delta \in S\rangle$, S a stationary subset of λ

(c) $\mathrm{id}^j_{\aleph_0}(\bar{C})$ is a proper ideal

(d) if $\alpha \in \bigcup_{\delta \in S} \mathrm{nacc}(C_\delta)$ then on α there is a Jonsson algebra and α is inaccessible.

Then on λ there is a Jonsson algebra (so we get a contradiction to (c)).
2) We can replace $(c) + (d)$ by

$(c)^+$ $\mathrm{id}_k(\bar{C}, \bar{I})$ is a proper ideal[5] and $\sigma < \delta \;\&\; \delta \in S \Rightarrow \{\alpha \in C_\delta : \alpha \in \mathrm{acc}(C_\delta) \vee \mathrm{cf}(\alpha) < \sigma\} \in I_\delta$

[5] see [Sh:g, IV, Def.1.8(1), p.190], only in line 4 replace "some" by "every"; but not used

$(d)'$ if $\alpha \in \bigcup_{\delta \in S} \mathrm{nacc}(C_\delta)$ then on $\mathrm{cf}(\alpha)$ there is a Jonsson algebra.

3) In clause (d) of part (1) we can omit "α is inaccessible".

Proof. 1) Very similar to the proof of [Sh:g, IV, p.192].

Let χ be large enough, M an elementary submodel of $(\mathcal{H}(\chi), \in, <^*_\chi)$ such that $\lambda \in M$, $|M \cap \lambda| = \lambda$, and it suffices to prove $\lambda \subseteq M$; assume toward contradiction that this fails. Without loss of generality $\bar{C} \in M$ and let $E = \{\delta < \lambda : \delta$ a limit ordinal, $\delta \not\subseteq M$ and $\delta = \sup(M \cap \delta)\}$. Clearly E is a club of λ, so by the choice of $\bar{C}$, i.e. "$\mathrm{id}^j_{\aleph_0}(\bar{C})$ a proper ideal" there is $\delta \in S \cap \mathrm{acc}(E)$ such that $\delta = \sup(B_\delta)$ where $B_\delta = \{\alpha \in \mathrm{nacc}(C_\delta) \cap E : \beta_\alpha = \alpha \vee \mathrm{cf}(\beta_\alpha) < \beta_\alpha\}$ where $\beta_\alpha =: \mathrm{Min}(M \cap \lambda \backslash \alpha)$, it exists as $|M \cap \lambda| = \lambda$ and clearly $\mathrm{cf}(\beta_\delta) < \delta \equiv \mathrm{cf}(\beta_\delta) < \beta_\delta$. But for $\alpha \in B_\delta$ we know that α is inaccessible so β_α cannot be singular so $\beta_\alpha = \alpha$, that is $\alpha \in M$. But for $\alpha \in B_\delta$, $\alpha \in \mathrm{acc}(E)$ by the definition of B_δ hence: $\alpha \in M$, $\sup(\alpha \cap M) = \alpha$, α is inaccessible on which there is a Jonsson algebra hence $\alpha \subseteq M$. But $\delta = \sup(B_\delta)$ so $\delta \subseteq M$, contradicting $\delta \in E$.

2) Similar.

3) In the proof of part (1) we use $E = \{\mu : \mu$ a limit cardinal, $\mu = \aleph_\mu = |M \cap \mu|, \mu \not\subseteq M\}$. Now if β_α is singular (hence α is singular) we consider M', the Skolem Hull of $M \cup \{i : i \leq \mathrm{cf}(\beta_\alpha)\}$ as in the proof of 1.16(2). $\square_{1.26}$

Minimal cases we do not know are

1.27 Question.

1) Can the first λ which is $\lambda \times \omega$-Mahlo be a Jonsson cardinal?
2) Let λ be the first ω-Mahlo cardinal; is $\lambda \to [\lambda]^2_\lambda$ consistent?
3) Is it enough to assume that for some set S of inaccessibles $0 < \mathrm{rk}_\lambda(S) < \lambda^+$ to deduce that there is a Jonsson algebra on λ (or even have $\mathrm{Pr}_1(\lambda, \lambda, \aleph_0)$)?

1.28 Remark. 1) Instead of J^{bd}_μ we could have used $[\mu]^{<\kappa}$, $\kappa \leq \mu$, but there was no actual need.

2) We can replace in 1.25, rk_λ by rk^*_λ. We can also axiomatize our demand on the rank for the proof to work.

1.29 Theorem. *Assume*

(a) λ *is inaccessible,*

(b) $S \subseteq \lambda$ *is stationary, and let* $S^+ = \{\mu < \lambda : S \cap \mu$ *is stationary and* μ *is inaccessible*$\}$

(c) *if* $\mathrm{rk}^*_\lambda(S^+) < \mathrm{rk}^*_\lambda(S)$.

Then on λ *there is a Jonsson algebra.*

Proof. In essence, we repeat the proof of 1.25, replacing rk_λ by rk^*_λ, and 1.16(2) instead of 1.16(1) only the proof is shorter.

As in the proof of 1.25 without loss of generality $\delta \in S \Rightarrow \mathrm{cf}(\delta) < \delta$ and we prove this by induction on λ.

If $\mathrm{rk}^*_\lambda(S) < \lambda$, <u>then</u> also $\mathrm{rk}^*_\lambda(S^+) < \lambda$, by 1.11 (1) $\mathrm{rk}_\lambda(S) = \mathrm{rk}^*_\lambda(S), \mathrm{rk}_\lambda(S^+) = \mathrm{rk}^*_\lambda(S^+)$ and so 1.25 apply so we are done, so we can assume $\mathrm{rk}^*_\lambda(S) \geq \lambda$. Let $\gamma^* = \mathrm{rk}^*_\lambda(S)$ be $\lambda \times n^* + \zeta^*, \zeta^* < \lambda$ and let $\sigma \in (\aleph_0 + |\zeta^*|^+, \lambda)$ be regular. Now $\mathrm{rk}^*_\lambda(S^{[\sigma+1]}) \geq \gamma^*$ as $\gamma^* \geq \lambda$, so without loss of generality we have $(\forall \delta \in S)(\mathrm{cf}(\delta) > \sigma)$. By 1.11(6), the ideal $\mathrm{id}^{<\gamma^*}(\lambda)$ is σ-indecomposable. Let $A = S^+ = \{\mu < \lambda : \mu$ inaccessible and $S \cap \mu$ is stationary$\}$, without loss of generality A is a stationary subset of λ (otherwise we are done by [Sh:g, Ch.III]), as in the proof of 1.25, without loss of generality $\mu \in A \Rightarrow$ on μ there is a Jonsson algebra. Now we can apply claim 1.14 to λ, A, S, $\mathrm{id}^{<\gamma^*}(\lambda)$, σ; its assumption holds as $\delta \in S \Rightarrow \mathrm{cf}(\delta) < \delta$, while $\delta \in A \Rightarrow \delta$ inaccessible). Now we can repeat the last paragraph of the proof of 1.25, using 1.16(2) + 1.26(1). $\square_{1.29}$

2. Back to successor of singulars

Earlier we have that if $\lambda = \mu^+$, $\mu > \mathrm{cf}(\mu)$ and μ is "small" in the alephs sequence, <u>then</u> on λ there is a Jonsson algebra. Here we show that we can replace "small in the aleph sequence" by other notions of smallness, like "small in the beth sequence". This shows that on $\beth^+_\omega$ there is a Jonsson algebra. Of course, we feel that being a Jonsson cardinal is a "large cardinal property" and for successor of singulars it is very large, both in consistency strength and in relation to actual large cardinals. We have some results materializing this intuition. If $\lambda = \mu^+$ is Jonsson $\mu > \mathrm{cf}(\mu)$, then μ is a limit of cardinals close to being measurable (expressed by games). If in addition $\mathrm{cf}(\mu) > \aleph_0$, $2^{(\mathrm{cf}(\mu))^+} < \mu$, <u>then</u> λ is close to being $\mathrm{cf}(\mu)$-compact, i.e. there is a uniform $\mathrm{cf}(\mu)$-complete ideal I on λ that is close to being an ultrafilter (the quotient is small).

2.1 Definition. *We define the game $Gm_n(\lambda, \mu, \gamma)$ for $\lambda \geq \mu$ cardinals, γ an ordinal and $n \leq \omega$. A play last γ moves; in the α-th move the first player chooses a function F_α from $[\lambda]^{<n} = \{w \subseteq \lambda : |w| < n\}$ into μ, and the second player has to choose a subset A_α of λ such that $A_\alpha \subseteq \bigcap_{\beta<\alpha} A_\beta$, $|A_\alpha| = \lambda$ and $Rang(F_\alpha \restriction [A_\alpha]^{<n})$ is a proper subset of μ. Second player loses if he has no legal move for some $\alpha < \gamma$; wins otherwise.*

2.2 Claim. We can change the rules slightly without changing the existence of winning strategies:

(a) instead of $\mathrm{Rang}(F_\alpha)$ being $\subseteq \mu$, just $|\mathrm{Rang}(F_\alpha)| = \mu$ and the demand on A_α is changed to: $\mathrm{Rang}(F_\alpha \restriction [A_\alpha]^{<n})$ is a proper subset of $\mathrm{Rang}(F_\alpha)$. and/or

(b) the second player can decide in the $\alpha -$th move to make it void, but defining the outcome of a play, if $\mathrm{otp}(\{\alpha < \gamma : \alpha\text{-th move non-void}\}) < \gamma$ he loses and/or

(c) in (a) instead of $|\mathrm{Rang}(F_\alpha)| = \mu$, we can require just $|\mathrm{Rang}(F_\alpha)| \geq \mu$.

Proof. Easy.

2.3 Claim. 1) If $\theta \not\to [\theta]^{<n}_{\kappa,<\kappa}$ (where $\theta \geq \kappa \geq \aleph_0 \geq n$) <u>then</u> first player wins $\mathrm{Gm}_n(\theta,\kappa,\kappa^+)$ (where "$\theta \not\to [\theta]^{<n}_{\kappa,<\kappa}$" means: there is $F : [\theta]^{<n} \to \kappa$ such that if $A \subseteq \theta$, $|A| = \theta$ then $|\mathrm{Rang}(F \restriction A)| = \kappa$).

2) If $\theta \not\to [\theta]^{<n}_{\kappa,<\sigma}$ (where $\theta \geq \kappa > \sigma \geq \aleph_0 \geq n$) and $\kappa > \sigma$ <u>then</u> for some $\tau \in [\sigma,\kappa]$ first player wins $\mathrm{Gm}_n(\theta,\tau,\tau^+)$ (where $\theta \not\to [\theta]^{<n}_{\kappa,<\sigma}$ means: there is $F : [\theta]^{<n} \to \kappa$ such that if $A \subseteq \theta$, $|A| = \theta$ then $|\mathrm{Rang}(F \restriction [A]^{<n})| \geq \sigma$.

Proof. 1) Let F exemplify $\theta \not\to [\theta]^{<n}_{\kappa,<\kappa}$. For any subset A of κ of cardinality κ let $h_A : \kappa \to \kappa$ be $h_A(\alpha) =$ otp$(\alpha \cap A)$ so $h_A \restriction A$ is one to one from A onto κ. Now a first player strategy is to choose $F_\alpha = h_{B_\alpha} \circ F$ where $B_\alpha =:$ $\mathrm{Rang}(F \restriction [\bigcap_{\beta<\alpha} A_\beta]^{<n})$ so $F_\alpha(x) = h_{B_\alpha}(F_\alpha(x))$ (note: we can instead use (a) of 2.2). Note that $|\mathrm{Rang}(F_\alpha)| = \kappa$ by the choice of F. So if $\langle F_\alpha, A_\alpha : \alpha < \kappa^+\rangle$ is a play in which this strategy is used then $\langle \mathrm{Rang}(F \restriction [A_\alpha]^{<n}) : \alpha < \kappa^+\rangle$ is a strictly decreasing sequence of subsets of κ, contradiction; i.e. for some α the second player has no legal move hence he loses.

2) Let $F : [\theta]^{<n} \to \kappa$ exemplify $\theta \not\to [\theta]^{<n}_{\kappa,<\sigma}$, and let $B \subseteq \theta$, $|B| = \theta$ be with $|\mathrm{Rang}(F \restriction [B]^{<n})|$ minimal, so let $\tau =: |\mathrm{Rang}(F \restriction [B]^{<n})|$, so B, F exemplify $\theta \not\to [\theta]^{<n}_{\tau,<\tau}$, and use part (1). $\square_{2.3}$

2.4 Claim.

1) *If* $\theta \leq 2^\kappa$ *but* $(\forall \mu < \kappa) 2^\mu < \theta$ *<u>then</u>* $\theta \not\to [\theta]^2_{\kappa,<\kappa}$.
2) *If* $\mathrm{cf}(\kappa) \leq \sigma < \kappa < \theta, \mathrm{pp}^+_\sigma(\kappa) > \theta = \mathrm{cf}(\theta)$ *<u>then</u>* $\theta \not\to [\theta]^2_{\kappa_1,<\kappa_1}$ *for some* $\kappa_1 \in [\kappa,\theta)$.
3) *If* $\theta = \mu^+$ *and* $\mu \not\to [\mu]^n_{\kappa,<\kappa}$, *<u>then</u>* $\theta \not\to [\theta]^{n+1}_{\kappa,<\kappa}$. *If* $\beth_n(\kappa) < \lambda \leq \beth_{n+1}(\kappa)$ *and* $\theta < \kappa \Rightarrow \beth_{n+1}(\theta) < \lambda$ *<u>then</u>* $\lambda \not\to [\lambda]^{n+2}_{\kappa,<\kappa}$.
4) *If* $\kappa + |T| < \theta$, T *is a tree with* κ *levels and* $\geq \theta$ κ*-branches and for any set* Y *of* κ*-branches* $|Y| \geq \theta \Rightarrow |\{\eta \cap \nu : \eta \neq \nu \in Y\}| \geq \kappa_0$, *<u>then</u>* $\theta \not\to [\theta]^2_{\kappa_1,<\kappa_1}$ *for some* $\kappa_1 \in [\kappa_0, |T|] \subseteq [\kappa_0,\theta)$ *hence the first player has a winning strategy in* $\mathrm{Gm}_2(\theta,\kappa_1,\kappa_1^+)$.
5) *Assume:* $f_\alpha : \kappa \to \sigma$, $f_\alpha(i) < \sigma_i < \sigma$ *for* $\alpha < \theta, i < \kappa$ *and* $\theta \geq \kappa, \tau \leq \sigma_i$ *and for no* $Y \subseteq \theta$, $|Y| = \theta$ *do we have* $i < \kappa \Rightarrow \sigma_i > |\{f_\alpha(i) : \alpha \in Y\}|$. *<u>Then</u> the first player wins in* $\mathrm{Gm}_2(\theta,\tau,\sigma+1)$. *Hence if cf*$(\kappa) \leq \sigma \leq \tau < \kappa < \theta =$ $\mathrm{cf}(\theta) < pp^+_\sigma(\theta)$ *<u>then</u> first player wins in* $\mathrm{Gm}_2(\theta,\tau,\sigma+1)$.
6) *If the first player does not win* $\mathrm{Gm}_n(\lambda,\kappa,\gamma)$, $\kappa \leq \theta$ *and* $[\beta < \gamma \Rightarrow \beta + \theta^+ \leq \gamma]$, *(equivalently, there is a limit ordinal* β *such that* $\theta^+ \times \beta = \gamma$*) <u>then</u> the first player does not win in the following variant of* $\mathrm{Gm}_n(\lambda,\theta,\gamma)$*: the second player has to satisfy* $|\mathrm{Rang}(F_\alpha \restriction [A_\alpha]^{<n})| < \kappa$.
7) $\kappa_1 \leq \kappa_2$ & $\gamma_1 \geq \gamma_2$ & $n_1 \geq n_2$ & *second player wins* $\mathrm{Gm}_{n_1}(\theta,\kappa_1,\gamma_1) \Rightarrow$ *second player wins* $\mathrm{Gm}_{n_2}(\theta,\kappa_2,\gamma_2)$.
8) *If* $\kappa_1 \leq \kappa_2, \gamma_1 \geq \gamma_2, n_1 \geq n_2$ *and first player wins* $\mathrm{Gm}_{n_2}(\theta,\kappa_2,\gamma_2)$ *<u>then</u> it wins* $\mathrm{Gm}_{n_1}(\theta,\kappa_1,\gamma_1)$.

Remark. On 2.4, 2.6, 2.7 see more in [EiSh 535], particularly on colouring theorems (instead of, e.g., no Jonsson algebras).

Proof. 1) Let $\langle A_\alpha : \alpha < \theta\rangle$ be a list of distinct subsets of κ, and define $F(\alpha,\beta) =:$ $\mathrm{Min}\{\gamma : \gamma \in A_\alpha \equiv \gamma \notin A_\beta\}$.

2) Easy, too, but let us elaborate.

First case. There is a set $\mathfrak{a}$ of $\leq \sigma$ regular cardinals $< \theta$, with no last element, $\sigma <$ min$(\mathfrak{a})$ and sup$(\mathfrak{a}) \in [\kappa, \theta)$ such that $\kappa_1 \in \mathfrak{a} \Rightarrow$ max pcf$(\mathfrak{a} \cap \kappa_1) < \kappa_1$ and max pcf$(\mathfrak{a}) = \theta$. Clearly it suffices to prove $\theta \not\to [\theta]^2_{\sup \mathfrak{a}, <\sup \mathfrak{a}}$.

Let J be an ideal on $\mathfrak{a}$ extending $J^{\rm bd}_{\mathfrak{a}}$ such that $\theta = \mathrm{tcf}(\Pi\mathfrak{a}, <_J)$ and let $\langle f_\alpha : \alpha < \theta\rangle$ be a $<_J$-increasing cofinal sequence in $\Pi\mathfrak{a}$ such that for $\mu \in \mathfrak{a}$, $|\{f_\alpha \restriction \mu : \alpha < \theta\}| < \mu$ (exists by [Sh:g, II,3.5,p.65]). Let $F(\alpha, \beta) = f_\beta(i(\alpha, \beta))$ where $i(\alpha, \beta) =$ Min$\{i : f_\alpha(i) \neq f_\beta(i)\}$.

The rest should be clear after reading the proof of $\mathrm{Pr}_1(\mu^+, \mu^+, \mathrm{cf}(\mu), \mathrm{cf}(\mu))$ in [Sh:g, II, 4.1].

Second case. For some ordinal[6] $\delta < \kappa$ we have $\mathrm{pp}^+_{J^{\rm bd}_\delta}(\kappa) > \theta$.

Hence (by [Sh:g, II, 2.3(1)]) for some strictly increasing sequence $\langle \sigma_i : i < \delta\rangle$ of regulars with limit κ such that tcf $\prod_{i<\delta} \sigma_i / J^{\rm bd}_\delta$ is equal to θ and let $f_\alpha(\alpha < \theta)$ exemplify this. Let $F(\alpha, \beta) = f_\beta(i(\alpha, \beta))$ where $i = i(\alpha, \beta)$ is maximal such that $\alpha < \beta \equiv f_\alpha(i) > f_\beta(i)$ if there is such i and zero otherwise (or probably more transparent $i = \sup\{j+1 : j < \delta$ and $\alpha < \beta \equiv f_\alpha(i) \geq f_\beta(i)\}$). The proof should be clear after reading [Sh:g, II, 4.1].

We finish by

2.5 Observation. At least one case holds.

Proof. As $\mathrm{pp}^+_\sigma(\kappa) > \theta$, by [Sh:g, II, 2.3] there is $\mathfrak{a}' \subseteq \kappa = \sup(\mathfrak{a}')$, $|\mathfrak{a}'| \leq \sigma$ such that $\mathfrak{a}'$ is a set of regular cardinals $> \sigma$ and there is an ideal J extending $J^{\rm bd}_{\mathfrak{a}'}$ such that $\mathrm{tcf}(\Pi\mathfrak{a}'/J) = \theta$; without loss of generality max pcf$(\mathfrak{a}') = \theta$ and $\theta \cap$ pcf$(\mathfrak{a}')$ has no last element. If $J_{<\theta}[\mathfrak{a}'] \subseteq J^{\rm bd}_{\mathfrak{a}'}$ we use the second case. If not, we try to choose inductively on $i < \sigma^+$, $\tau_i \in$ pcf$(\mathfrak{a}')\backslash\{\theta\}\backslash\kappa$, such that $\theta, \tau_i >$ max pcf$\{\tau_j : j < i\}$. As $J_{<\theta}[\mathfrak{a}'] \nsubseteq J^{\rm bd}_{\mathfrak{a}'}$ we can choose for $i = 0$, for i successor pcf$\{\tau_j : j < i\}$ has a last element but pcf$(\mathfrak{a}')\backslash\{\theta\}\backslash\kappa$ does not, so we can choose τ_i recalling that pcf$(\{\tau_j : j < i\}) \subseteq$ pcf$(\mathfrak{a}')$ by [Sh:g, I]. By localization (i.e. [Sh:g, VIII,3.4]) we cannot arrive to $i = |\mathfrak{a}'|^+ \leq \sigma^+$, so for some limit $\delta < |\mathfrak{a}'|^+ \leq \sigma^+$ we have: τ_i is defined iff $i < \delta$. So $\{\tau_i : i < \delta\}$ is as required in the first case. So we can apply the first case.

Continuation of the proof of 2.4.

3) — 6) Left to the reader.

7) Let $h : \kappa_2 \to \kappa_1$ be $\quad h(\alpha) = \begin{cases} \alpha \text{ if } \alpha < \kappa_1 \\ 0 \text{ if } \kappa_1 \leq \alpha < \kappa_2. \end{cases}$
During a play $\langle F_\alpha, A_\alpha : \alpha < \gamma_2\rangle$ of $\mathrm{Gm}_{n_2}(\theta, \kappa_2, \gamma_2)$, the second player simulates (an initial segment of) a play of $\mathrm{Gm}_{n_1}(\theta, \kappa_1, \gamma_1)$, where for $t \subseteq \theta, n_1 \leq |t| < n_2$ we let $h \circ F_\alpha(t) = 0$ and in the simulated play $\langle h \circ F_\alpha, A_\alpha : \alpha < \gamma_2\rangle$ the second player uses a winning strategy.

[6] of course, without loss of generality, δ is a regular cardinal

8) During a play of $\mathrm{Gm}_{n_1}(\theta, \kappa_1, \gamma_1)$, the first player simulates a play of the game $\mathrm{Gm}_{n_2}(\theta, \kappa_2, \gamma_2)$. The simulated play is $\langle F_\alpha, A_\alpha : \alpha < \gamma_1 \rangle$, the actual one $\langle h \circ F_\alpha, A_\alpha : \alpha < \gamma_1 \rangle$ (so first player wins before he must, if $\gamma_1 \neq \gamma_2$). $\square_{2.4}$

2.6 Theorem. *1) If* $\lambda = \mu^+, \mathrm{cf}(\mu) < \mu, \gamma^* < \mu, \kappa < \mu$ *and for every large enough regular* $\theta \in \mathrm{Reg} \cap \mu$ *the first player wins* $\mathrm{Gm}_\omega(\theta, \kappa, \gamma^*)$ *<u>then</u>* $\lambda \not\to [\lambda]^{<\omega}_\kappa$.

2) Instead of $\mathrm{Gm}_\omega(\theta, \kappa, \gamma)$ *we can use* $\mathrm{Gm}_\omega(\theta, \kappa(\theta), \gamma^*)$ *with* $\kappa = \lim_{\theta \in \mathrm{Reg} \cap \mu} \kappa(\theta) \leq \mu$*; e.g.* $\langle \kappa(\theta) : \theta \in \mathrm{Reg} \cap \mu \rangle$ *is non-decreasing with limit* $\kappa \leq \mu$ *(so possibly* $\kappa = \mu$*; and then we can get* $\lambda \not\to [\lambda]^{<\omega}_\lambda$*).*

Proof of 2.6. (1) Compare with [Sh:g, III, §2, §3]. If $\kappa \leq \mathrm{cf}(\mu)$ we know this (see [Sh:g, II, 4.1(1), p.67]) so let $\kappa > \mathrm{cf}(\mu)$. So let $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \mathrm{cf}(\mu)\}$ be stationary. If $\mathrm{cf}(\mu) > \aleph_0$ let $\bar{C}^1$ be a nice strict S-club system with $\lambda \notin \mathrm{id}_p(\bar{C}^1)$, (exists by [Sh:g, III, 2.6]) and let $\bar{J} = \langle J_\delta : \delta \in S \rangle$, $J_\delta = J^{\mathrm{bd}}_{C^1_\delta}$. If $\mathrm{cf}(\mu) = \aleph_0$, without loss of generality S is such that $[\delta \in S \Rightarrow \mu$ divides $\delta]$, let $\bar{C}^1 = \langle C^1_\delta : \delta \in S \rangle$ be such that: $C^1_\delta \subseteq \delta = \sup(C^1_\delta)$, $\mathrm{otp}(C^1_\delta) = \mu$, C^1_δ closed and $\lambda \notin \mathrm{id}_p(\bar{C}^1, \bar{J})$ where $\bar{J} = \langle J_\delta : \delta \in S \rangle$, $J_\delta = \{A \subseteq C^1_\delta$: for some $\beta < \delta$ and $\theta < \mu$, we have $(\forall \alpha)[\alpha \in A \,\&\, \alpha \geq \beta, \alpha \in \mathrm{nacc}(C^1_\delta) \rightarrow \mathrm{cf}(\alpha) < \theta]\}$, (exists by [Sh:g, III,p.131]).

Let $\bar{C}^2 = \langle C^2_\delta : \delta < \lambda \rangle$ be a strict λ-club system such that for every club E of λ, we have:

$$\left\{ \delta < \lambda : (\forall \beta < \delta)(\exists \alpha \in E)[\alpha \in \mathrm{nacc}(C^2_\delta) \,\&\, \alpha > \beta] \right\} \notin \mathrm{id}_p(\bar{C}^1, \bar{J}).$$

[We can build together $\bar{C}^1, \bar{C}^2$ like this as in the proof of 1.12 or use [Sh:g, III, 2.6] as each J_δ is $\mathrm{cf}(\mu)$-based.]

Let $\mu = \sum_{i < \mathrm{cf}(\mu)} \mu_i$ where $\mu_i < \mu$. Let $\sigma^+ < \mu, \gamma^* < \sigma^+, \sigma$ regular $\geq \mathrm{cf}(\mu)$. Let $\mu^* < \mu$ be such that first player has a winning strategy in $\mathrm{Gm}_\omega(\theta, \kappa, \gamma^*)$ if $\mu^* \leq \theta = \mathrm{cf}(\theta) < \mu$. For each $\delta < \lambda$, if the first player has a winning strategy in $\mathrm{Gm}_\omega(\mathrm{cf}(\delta), \kappa, \gamma^*)$, let St_δ be a winning strategy for him in the variant of the play where we use $\mathrm{nacc}(C^2_\delta)$ instead of $\mathrm{cf}(\delta)$ as domain, and allow the second player to pass (see 2.2(b)); we let the play last σ^+ moves (this is even easier for first player to win). So St_δ is well defined if $\mathrm{cf}(\delta) \geq \mu^*$.

We try successively σ^+ times to build an algebra on λ witnessing the conclusion, while at the same time for each $\delta < \lambda$ of cofinality $\geq \mu^*$ playing on C^2_δ a play of $\mathrm{Gm}_\omega(\mathrm{cf}(\delta), \kappa, \sigma^+)$ in which the first player uses the strategy St_δ. In stage $\zeta < \sigma^+$ (i.e. the ζ-th try), initial segments of length ζ of all those plays have already been defined; now for $\delta < \lambda, \mathrm{cf}(\delta) \geq \mu^*$, first player chooses $F_{\delta,\zeta} : [\mathrm{nacc}(C^2_\delta)]^{<\omega} \rightarrow \kappa$. Let F_ζ code all those functions $F_\zeta : [\lambda]^{<\omega} \rightarrow \lambda$ (so δ is viewed as a variable) and enough set theory; specifically we demand:

$\circledast_1$ if $t \in [\lambda]^{<\omega}$ and then

- (i) $F_\zeta(t)$ belongs to $A_{\zeta,t}$, the Skolem Hull of $t \cup \{F_{\delta,\zeta}(s) : \delta \in t, s \subseteq t \cap C^2_\delta\}$ in $(\mathcal{H}(\lambda^+), \in, <^*_{\lambda^+}, \bar{C}^1, \bar{C}^2, \kappa)$
- (ii) if $x \in A_{\zeta,t}$, <u>then</u> for infinitely many $k < \omega$ we have:

$$t \lhd t^+ \in [\lambda]^k \Rightarrow F_\zeta(t^+) = x.$$

Now let F'_ζ be

$$F'_\zeta(t) = \begin{cases} F_\zeta(t) & \text{if } F_\zeta(t) \in \kappa \\ 0 & \text{otherwise .} \end{cases}$$

Let $B_\zeta \in [\lambda]^\lambda$ exemplify that F'_ζ is not as required in 2.6, that is $\kappa \not\subseteq \{F'(t) : t \in [B_\zeta]^{<\aleph_0}\}$. Without loss of generality B_ζ is closed under F_ζ (possible by the choice of F_ζ).

Let $E_\zeta = \left\{\delta : \delta \not\subseteq B_\zeta \text{ and } \delta = \sup(\delta \cap B_\zeta)\right\} \cap \bigcap_{j<\zeta} E_j$.

It is a club of λ. For each $\delta \in E_\zeta$ such that $\mathrm{cf}(\delta) \geq \mu^*$, in the game $\mathrm{Gm}_\omega(C^2_\delta, \kappa, \sigma^+)$, second player has to make a move. The move is $\{\alpha \in \mathrm{nacc}(C^2_\delta) : \alpha \in E_\zeta\}$ if this is a legal move and $\delta \in B_\zeta$; otherwise the second player makes it void; i.e. pass (see 2.2(b)).

Having our σ^+ moves we shall get a contradiction. Let E be $\bigcap_{\zeta<\sigma^+} \mathrm{acc}(E_\zeta)$, this is a club of λ, hence by the choice of $\bar{C}^1, \bar{C}^2$ for some $\delta(*) \in S$ we have $\delta(*) = \sup(A_1)$ moreover $A_1 \in J^+_{\delta(*)}$ where

$$A_1 =: \left\{\delta : \delta \in \mathrm{nacc}(C^1_{\delta(*)}) \text{ and } (\forall \beta < \delta)(\exists \alpha \in E)[\alpha \in \mathrm{nacc}(C^2_\delta) \,\&\, \alpha > \beta]\right\}.$$

For $\zeta < \sigma^+$ define

$$i(\zeta) = \mathrm{Min}\{i : \mu_i \geq \mathrm{cf}\,[\mathrm{Min}(B_\zeta \setminus \delta(*))]\}.$$

Since B_ζ is closed under F_ζ and F_ζ codes enough set theory, the proof of [Sh:g, III,1.9], (similar things are in §1 here) shows that

(*) if $\delta \in A_1, \mathrm{cf}(\delta) > \mu_{i(\zeta)}$ then $\delta \in B_\zeta$ and $(\forall\alpha)[\alpha \in \mathrm{nacc}(C^2_\delta) \cap E_\zeta \Rightarrow \alpha \in B_\zeta]$.

Now as $\sigma \geq \mathrm{cf}(\mu)$ (whereas there are $\mathrm{cf}(\mu)$ cardinals μ_i) for some $i(*) < \mathrm{cf}(\mu)$ we have

$$\sigma^+ = \sup(U) \text{ where } U =: \{\zeta < \sigma^+ : i(\zeta) \leq i(*)\}.$$

Choose $\delta \in A_1$ with $\mathrm{cf}(\delta) > \mu_{i(*)}$ (why is this possible? if $\mathrm{cf}(\mu) = \aleph_0$ as $\delta(*) = \sup(A_1)$ and $\bar{C}^1$ is nice; if not as $A_1 \in J^+_{\delta(*)}$ see [Sh:g, III,1.1]). By (*) we have $\zeta \in U \Rightarrow \delta \in B_\zeta$ and by the choice of E and $\delta(*)$, δ clearly $E_\zeta \cap \mathrm{nacc}(C^2_\delta)$ has cardinality $\mathrm{cf}(\delta)$; so for every $\zeta \in U$ the second player (in the play of $\mathrm{Gm}_\omega(C^2_\delta, \kappa, \sigma^+)$) make a non-void move. As $|U| = \sigma^+$, this contradicts "St_δ is a winning strategy for the first player in $\mathrm{Gm}_\omega(C^2_\delta, \kappa, \sigma^+)$".

(2) Similar proof (for $\kappa = \mu$ see [Sh:g, II, 355].) $\square_{2.6}$

An example of an application is

2.7 Conclusion.

1) On $\beth^+_\omega$ there is a Jonsson algebra.
2) If $\beth_{n+1}(\kappa) < \lambda \leq \beth_{n+2}(\kappa)$ <u>then</u> the first player wins in $\mathrm{Gm}_{n+2}\left(\lambda, \kappa^+, (2^\kappa)^+\right)$.

3) If μ is singular not strong limit, $\sigma < \kappa^{<\sigma} < \mu \leq \kappa^\sigma$ and $\lambda = \mu^+$ but $\bigwedge_{\theta<\kappa} \theta^\sigma < \mu$ then $\lambda \not\to [\lambda]^{<\omega}_\kappa$.
4) If μ singular not strong limit, $\lambda = \mu^+$, $\mu^* + \kappa < \mu \leq \kappa^\sigma$, $\sigma \leq \kappa$ and there is a tree T $\kappa = |T| < \mu$, T has $\geq \mu$ σ-branches, and $T' \subseteq T \,\&\, |T'| < \kappa \Rightarrow T'$ has $\leq \mu^*$ σ-branches then $\lambda \not\to [\lambda]^2_\kappa$.
5) Assume $\lambda = \mu^+$, $\mathrm{cf}(\mu) < \mu$, and for every $\mu_0 < \mu$ there is a singular $\chi \in (\mu_0, \mu)$ satisfying $\mathrm{pp}(\chi) \geq \mu$. Then on λ there is a Jonsson algebra.
6) Assume $\lambda = \mu^+$, $\mu >$ $\mathrm{cf}(\mu)$, $\mathrm{cf}(\chi) \leq \kappa < \chi < \chi^+ < \lambda$, $\mathrm{pp}^+_\kappa(\chi) > \lambda$. Then $\lambda \not\to [\lambda]^{<\omega}_\chi$.
7) If μ singular not strong limit, $2^{<\kappa} \leq \mu \leq 2^\kappa$, $\kappa =$ $\mathrm{Min}\{\sigma : 2^\sigma \geq \mu\} < \mu$ then $\mu^+ \not\to [\mu^+]^{<\omega}_\kappa$.
8) There is on μ^+ a Jonsson algebra if $\mathrm{cf}(\mu) < \mu < 2^{<\mu} < 2^\mu$ (i.e. μ singular not strong limit and $\langle 2^\lambda : \lambda < \mu\rangle$ is not eventually constant).

Proof.

1) It is enough to prove for each $n < \omega$ that $\beth^+_\omega \not\to [\beth^+_\omega]^{<\omega}_{\beth_n}$. By part 2) (and monotonicity in n – see 2.4(8)) for every regular $\theta < \beth_\omega$ large enough, first player wins in $\mathrm{Gm}_\omega(\theta, \beth^+_n, \beth^+_{n+1})$. So by 2.6 we get $\beth^+_\omega \not\to [\beth^+_\omega]^{<\omega}_{\beth_n}$, and as said above, this suffices.
2) Let κ_1 be $\mathrm{Min}\{\sigma : \beth_{n+1}(\sigma) \geq \lambda\}$, so $\kappa_1 > \kappa$ (as $\beth_{n+1}(\kappa) < \lambda$) and $2^\kappa \geq \kappa_1$ (as $\beth_{n+1}(2^\kappa) = \beth_{n+2}(\kappa) \geq \lambda$), also $\lambda \leq \beth_{n+1}(\kappa_1)$ (by the definition of κ_1) and $\beth_n(\kappa_1) < \lambda$ (as $\kappa_1 \leq 2^\kappa$ and $\beth_{n+1}(\kappa) < \lambda$), moreover $\mu < \kappa_1 \Rightarrow \beth_{n+1}(\mu) < \lambda$ by the choice of κ_1. By 2.4(3) the second phrase we have $\lambda \not\to [\lambda]^{n+2}_{\kappa_1, <\kappa_1}$. By 2.3(1) the first player wins $Gm_{n+2}(\lambda, \kappa_1, \kappa_1^+)$. By monotonicity properties (2.4(8)) the first player wins $\mathrm{Gm}_{n+2}\left(\lambda, \kappa^+, (2^\kappa)^+\right)$.
3) By 2.4(4) for every regular $\theta \in (\kappa^{<\sigma}, \kappa^\sigma)$, first player wins in $\mathrm{Gm}_2(\theta, \kappa, (\kappa^{<\sigma})^+)$. Now apply 2.6.
4) Similar to (3).
5) If $\mathrm{cf}(\chi) < \chi$, $\mathrm{pp}^+(\chi) > \theta =$ $\mathrm{cf}(\theta) > \chi$ and $\tau < \chi$ then the first player wins the game $\mathrm{Gm}_2(\theta, \tau, \chi + 1)$ (by 2.4(5)). So by 2.6 if $\mathrm{cf}(\chi) < \chi < \mu \leq pp^+(\chi)$ we have $\tau < \chi \Rightarrow \lambda \not\to [\lambda]^{<\omega}_\tau$ hence easily we are done.
6) Similar to (5).
7) If $2^{<\kappa} < \mu$ we apply 2.4(1) and then 2.3 + 2.6. So assume $2^{<\kappa} = \mu$, so necessarily κ is a limit cardinal $< \mu$ and $\mathrm{cf}(\mu) =$ $\mathrm{cf}(\kappa) \leq \kappa < \mu$. Now for every regular $\theta \in (\kappa, \mu)$ letting $\kappa(\theta) =$ $\mathrm{Min}\{\sigma : 2^\sigma \geq \theta\}$ we get $\kappa(\theta) < \kappa$ hence by the regularity of θ, $2^{<\kappa(\theta)} < \theta$, so by 2.4(1) + 2.3 player I wins $\mathrm{Gm}_2(\theta, \kappa(\theta), \kappa(\theta)^+)$ hence he wins $\mathrm{Gm}_2(\theta, \kappa(\theta), \kappa)$. Use 2.6(2) to derive the conclusion.
8) By part (4) and [Sh 430, 3.4]. $\square_{2.7}$

2.8 Remark. In 2.9 below, remember, an ideal I is θ-based if for every $A \subseteq \mathrm{Dom}(I)$, $A \notin I$ there is $B \subseteq A$, $|B| < \theta$ such that $B \notin I$; also I is weakly κ-saturated if $\mathrm{Dom}(I)$ cannot be partitioned to κ sets not in I. The case we think of in 2.9 is $\lambda = \mu^+$, μ singular of uncountable cofinality.

2.9 Claim. Suppose

(a) $\lambda = \mathrm{cf}(\lambda) > (2^{\kappa^+})^+$ and $\theta = \kappa$

(b) $\bar{C}$ is an S-club system, $S \subseteq \lambda$ stationary and $\bar{I} = \langle I_\delta : \delta \in S\rangle$, I_δ an ideal on C_δ containing $J^{\mathrm{bd}}_{C_\delta}$ and $\mathrm{id}_p(\bar{C}, \bar{I})$ is (see 1.17, a proper ideal and) weakly κ^+-saturated and

(c) $(*)^{2^\kappa,\theta}_{I_\delta}$ if $A \subseteq \mathrm{Dom}(I_\delta)$, $A \notin I_\delta$ then for some $Y \subseteq A$, $|Y| \le \theta$, $Y \notin I_\delta$

hence $|\mathcal{P}(Y)/I_\delta| \le 2^\theta$.

<u>Then</u>:

(i) $\mathcal{P}(\lambda)/\mathrm{id}_p(\bar{C}, \bar{I})$ has cardinality $\le 2^\kappa$

(ii) for every $A \in \mathcal{P}(\lambda)\backslash\mathrm{id}_p(\bar{C}, \bar{I})$, there is $B \subseteq A$, $B \in \mathcal{P}(\lambda)\backslash\mathrm{id}_p(\bar{C}, \bar{I})$ and an embedding of $\mathcal{P}(\lambda)/\left[\mathrm{id}_p(\bar{C}, \bar{I}) + (\lambda\backslash B)\right]$ into some $\mathcal{P}(Y)/I_\delta$ for some $\delta \in S$, $Y \subseteq C_\delta$ such that $Y \notin I_\delta$,

(iii) moreover, in (ii) we can find $h : B \to \theta$ such that for every $B' \subseteq B$ for some $A' \subseteq \theta$ we have $B' \equiv h^{-1}(A') \bmod \mathrm{id}_p(\bar{C}, \bar{I})$. (In fact for some $g : Y \to \theta$ and ideal J^* on θ for every $B' \subseteq B$ we have: $B' \in \mathrm{id}_p(\bar{C}, \bar{I}) \Leftrightarrow g^{-1}(h(B')) \in J^*$.)

2.10 Remark. 1) The use of θ and κ though $\theta = \kappa$ is to help considering the case they are not equal.

2) The point of 2.9 is that e.g. if $\lambda = \mu^+, \mu > \mathrm{cf}(\mu)$, $S \subseteq \lambda$, <u>then</u> we can find $\bar{C} = \langle C_\delta : \delta \in S\rangle$ and $\bar{I} = \langle I_\delta : \delta \in C\rangle$ such that $\lambda \notin \mathrm{id}_p(\bar{C}, \bar{I})$ and I_δ is $(\mathrm{cf}(\mu))$-based and $\delta \in S, \beta < \delta, \theta < \mu \Rightarrow \{\alpha \in C_\delta : \alpha \in \mathrm{acc}(C_\delta)$ or $\alpha < \beta$ or $\mathrm{cf}(\alpha) < \theta\} \in I_\delta$. Now if $\mathrm{id}_p(\bar{C}, \bar{I})$ is not weakly χ-saturated then $\lambda \not\to [\lambda]^{<\omega}_\chi$ and more; see [Sh:g, III].

Proof. There is a sequence $\langle A_i : i < i^*\rangle$ such that: $A_0 = \emptyset$, $A_i \subseteq \lambda$, $[i \ne j \Rightarrow A_i \ne A_j \bmod \mathrm{id}_p(\bar{C}, \bar{I})]$ and: $i^* = (2^\kappa)^+$ or: $i^* < (2^\kappa)^+$ and for every $B \subseteq \lambda$ for some $i < i^*$ we have $B \equiv A_i \bmod \mathrm{id}_p(\bar{C}, \bar{I})$. Let $\mathcal{P}$ be the closure of $\{A_i : i < i^*\}$ under finitary Boolean operations and the union of $\le \kappa^+$ members. So in particular $\mathcal{P}$ includes the family of sets of the form $(A_i\backslash A_j)\backslash \bigcup_{\zeta<\kappa^+}\left(A_{i_\zeta}\backslash A_{j_\zeta}\right)$ (where $i, j, i_\zeta, j_\zeta < i^*$), clearly $|\mathcal{P}| \le 2^{\kappa^+} + (2^\kappa)^+ < \mu$ and if $|i^*| \le 2^\kappa$ then $|\mathcal{P}| \le 2^{\kappa^+}$.

For each $A \in \mathcal{P}$ which is in $\mathrm{id}_p(\bar{C}, \bar{I})$, choose a club E_A of λ witnessing it (and if $A \in \mathcal{P}\backslash\mathrm{id}_p(\bar{C}, \bar{I})$ let $E_A = \lambda$).

As $(2^{\kappa^+})^+ < \lambda$ clearly $|\mathcal{P}| < \lambda$ hence $E =: \bigcap_{A\in\mathcal{P}} E_A$ is a club of λ.

So $S^* = \{\delta \in S : E \cap C_\delta \notin I_\delta\}$ is a stationary subset of λ. For proving (i) suppose $i^* = (2^\kappa)^+$ and eventually we shall get a contradiction. We now choose by induction on $\zeta < \kappa^+$ ordinals $i_1(\zeta), i_2(\zeta) < i^*$ and $\delta_\zeta \in S^*$ and sets $Y_\zeta \subseteq A_{i_2(\zeta)}\backslash A_{i_1(\zeta)} \cap E \cap C_{\delta_\zeta}$ such that $Y_\zeta \notin I_{\delta_\zeta}$, $|\mathcal{P}(Y_\zeta)/I_{\delta_\zeta}| \le 2^\kappa$, $|Y_\zeta| \le \theta$, $A_{i_2(\zeta)}\backslash A_{i_1(\zeta)} \notin \mathrm{id}_p(\bar{C}, \bar{I})$ and $\xi < \zeta \Rightarrow \left(A_{i_2(\zeta)}\backslash A_{i_1(\zeta)}\right) \cap Y_\xi = \emptyset$.

Why can we choose $i_1(\zeta), i_2(\zeta)$ and Y_ζ? There is a natural equivalence relation $\approx_\zeta$ on i^*:

$$i \approx_\zeta j \text{ \underline{iff} for every } \xi < \zeta,\ A_i \cap Y_\xi = A_j \cap Y_\xi$$

and it has $\le (2^\theta)^\kappa = 2^\kappa$ equivalence classes. So for some $j_1 \ne j_2$ we have $j_1 \approx_\zeta j_2$.

By assumption $A_{j_1} \neq A_{j_2} \bmod \mathrm{id}_p(\bar{C}, \bar{I})$, so without loss of generality $A_{j_2} \nsubseteq A_{j_1} \bmod \mathrm{id}_p(\bar{C}, \bar{I})$, hence $A_{j_2} \backslash A_{j_1} \notin \mathrm{id}_p(\bar{C}, \bar{I})$. By this for some $\delta_\zeta \in S^* \cap \mathrm{acc}(E)$ we have $\left(A_{j_2} \backslash A_{j_1}\right) \cap C_{\delta_\zeta} \cap E \notin I_{\delta_\zeta}$, so there is $Y_\zeta \subseteq \left(A_{j_2} \backslash A_{j_1}\right) \cap C_{\delta_\zeta}$ satisfying $|Y_\zeta| \leq \theta$ and $|\mathcal{P}(Y_\zeta)/I_{\delta_\zeta}| \leq 2^\kappa$ and $Y_\zeta \notin I_{\delta_\zeta}$.

Let $i_2(\zeta) = j_2, i_1(\zeta) = j_1$.

So $\langle A_{i_1(\zeta)}, A_{i_2(\zeta)}, \delta_\zeta, Y_\zeta : \zeta < \kappa^+ \rangle$ is well defined. Let $B^1_\zeta =: A_{i_2(\zeta)} \backslash A_{i_1(\zeta)}$, $B_\zeta =: B^1_\zeta \backslash \bigcup_{\xi \in (\zeta, \kappa^+)} B^1_\xi$ (for $\zeta < \kappa^+$). So each B_ζ is in $\mathcal{P}$, and they are pairwise disjoint. Also $Y_\zeta \subseteq B^1_\zeta$ (by the choice of Y_ζ) and $\zeta < \xi < \kappa^+ \Rightarrow Y_\zeta \cap B^1_\xi = \emptyset$ (see the inductive choice of $A_{i_2(\zeta)}, A_{i_1(\zeta)}$) hence $Y_\zeta \subseteq B_\zeta$. Next we prove that $B_\zeta \notin \mathrm{id}_p(\bar{C}, \bar{I})$, but otherwise $E \subseteq E_{B_\zeta}$, and $\delta_\zeta, Y_\zeta \subseteq E$ contradict the choice of E_{B_ζ}. Now $\langle B_\zeta : \zeta < \kappa^+ \rangle$ contradicts "$\mathrm{id}_p(\bar{C}, \bar{I})$ is weakly κ^+-saturated". So $i^* < (2^\kappa)^+$, i.e. (i) holds.
Let $\mathfrak{B}$ be the Boolean Algebra of subsets of λ generated by $\{A_i : i < i^*\}$. Now we prove clause (ii), so let $A \subseteq \lambda$, $A \notin \mathrm{id}_p(\bar{C}, \bar{I})$.

Let $i_2 < i^*$ be such that $A \equiv A_{i_2} \bmod \mathrm{id}_p(\bar{C}, \bar{I})$, choose $\delta \in S \cap \mathrm{acc}(E)$ such that $A \cap A_{i_2} \cap C_\delta \cap E \notin I_\delta$, and choose $Y \subseteq A \cap A_{i_2} \cap C_\delta$ such that $|Y| \leq \theta, Y \notin I_\delta, |\mathcal{P}(Y)/I_\delta| \leq 2^\kappa$. Now we try to choose by induction on $\zeta < \kappa^+$, $\langle i_1(\zeta), i_2(\zeta), \delta_\zeta, Y_\zeta \rangle$ as before, except that we demand in addition that $Y \cap \left(A_{i_2(\zeta)} \backslash A_{i_1(\zeta)}\right) = \emptyset$. Necessarily for some $\zeta(*) < \kappa^+$ we are stuck. Let $B = A_{i_2} \backslash \bigcup_{\zeta < \zeta(*)} \left(A_{i_2(\zeta)} \backslash A_{i_1(\zeta)}\right)$, it belongs to $\mathcal{P}$ (as $A_{i_2} = A_{i_2} \backslash A_0$, remember $A_0 = \emptyset$), also $Y \subseteq B$, but $E \subseteq E_B$ hence $B \notin \mathrm{id}_p(\bar{C}, \bar{I})$. The mapping $H : \mathcal{P}(B) \to \mathcal{P}(Y)$ defined by $H(X) = X \cap Y$ induce a homomorphism $H_1 = H \restriction \mathfrak{B}$ from $\mathfrak{B}$ into $\mathcal{P}(Y)$. Now if $X \in \mathfrak{B} \cap \mathrm{id}_p(\bar{C}, \bar{I})$ then $X \in \mathcal{P}$ (as $\mathfrak{B} \subseteq \mathcal{P}$ because $A_i = A_i \backslash A_0 \in \mathcal{P}$ and $\mathcal{P}$ closed under the (finitary) Boolean operations). Hence $X \in \mathfrak{B} \cap \mathrm{id}_p(\bar{C}, \bar{I}) \Rightarrow X \cap Y \in I_\delta$. Hence H_1 induces a homomorphism H_2 from $\mathfrak{B}/\mathrm{id}_p(\bar{C}, \bar{I})$ into $\mathcal{P}(Y)/I_\delta$. By the choice of B, this homomorphism is one to one on $(\mathcal{P}(B) \cap \mathfrak{B})/\mathrm{id}_p(\bar{C}, \bar{I})$ and as $\mathcal{P}(\lambda)/\left[\mathrm{id}_p(\bar{C}, \bar{I}) + (\lambda \backslash B)\right]$ is essentially equal to $(\mathcal{P}(B) \cap \mathfrak{B})/\mathrm{id}_p(\bar{C}, \bar{I})$, we have finished proving clause (ii).
We are left with clause (iii).

Let $\mathfrak{B}^*$ be the closure of $\{A_i : i < i^*\}$ under finitary Boolean operations and unions of $\leq \theta$ sets. So $|\mathfrak{B}^*| \leq 2^\theta$. For each $A \in \mathfrak{B}^* \cap \mathrm{id}_p(\bar{C}, \bar{I})$ let E_A witness this, and let $E^* =: \cap\{E_A : A \in \mathfrak{B}^* \cap \mathrm{id}_p(\bar{C}, \bar{I})\}$. Without loss of generality $E^* = E$. For any $A \in \mathcal{P}(\lambda) \backslash \mathrm{id}_p(\bar{C}, \bar{I})$ choose δ, Y, B as in the proof of (ii), fix them.

Let $B^* = \left\{\alpha \in B : \text{for no } \gamma \in Y \text{ do we have } \bigwedge_{i < i^*} \alpha \in A_i \equiv \gamma \in A_i\right\}$.
Now

(∗) $B^* \in \mathrm{id}_p(\bar{C}, \bar{I})$
[why? if not, there is $\delta(1) \in S$ such that $B^* \cap E^* \cap C_{\delta(1)} \notin I_{\delta(1)}$ hence there is $Y_1 \subseteq B^* \cap E^* \cap C_{\delta(1)}$ such that $Y_1 \notin I_{\delta(1)}, |Y_1| \leq \theta$. By the definition of B^* for every $\alpha \in Y_1, \beta \in Y$ (as necessarily $\alpha \in B^*$) there is $A_{\alpha,\beta} \in \{A_i : i < i^*\} \subseteq \mathfrak{B}^*$, such that $\alpha \in A_{\alpha,\beta} \,\&\, \beta \notin A_{\alpha,\beta}$. Hence $A^*_1 = B \cap \bigcup_{\alpha \in Y_1} \bigcap_{\beta \in Y} A_{\alpha,\beta}$ belongs to $\mathfrak{B}^*$ and $Y_1 \subseteq A^*_1$, (as $\alpha \in Y_1 \,\&\, \beta \in Y \Rightarrow \alpha \in A_{\alpha,\beta}$) and $Y \cap A^*_1 = \emptyset$ (because for each $\beta \in Y$ we have $\alpha \in Y_1 \,\&\, \beta \in Y \Rightarrow \beta \notin A_{\alpha,\beta}$). As $A^*_1 \subseteq B, Y \cap A^*_1 = \emptyset$

by the choice of B we have $A_1^* \in \mathrm{id}_p(\bar{C}, \bar{I})$. But Y_1 (and E^*) witness $A_1^* \notin \mathrm{id}_p(\bar{C}, \bar{I})$, contradiction.]

Define $h_0 : (B\backslash B^*) \to Y/\approx$ by $h(\alpha)$ is $\{\gamma \in Y : \bigwedge_{i<i^*} \alpha \in A_i \equiv \gamma \in A_i\}$ where for $\gamma_1, \gamma_2 \in Y$ we let $\gamma_1 \approx \gamma_2 \Leftrightarrow \bigwedge_{i<i^*} \gamma_1 \in A_i \equiv \gamma_2 \in A_i$. The rest should be clear. $\square_{2.9}$

2.11 Remark. 1) In 2.9 we can replace κ^+ by κ, then instead of $2^\kappa < \lambda$ we have $2^{<\kappa} < \lambda$ and in (i) we get $\leq 2^\theta$ for some $\theta < \kappa$.

2) If $I_\delta = J^{\mathrm{bd}}_{\mathrm{nacc}(C_\delta)}$, $\theta = \kappa$, and $[\delta \in S \Rightarrow \mathrm{cf}(\delta) \leq \kappa]$ then the demand "θ based ideal on C_δ containing $J^{\mathrm{bd}}_{C_\delta}$" on $\bar{I}$ holds.

3. More on guessing Clubs

Here we continue the investigation of guessing clubs in a successor of regulars.

3.1 Claim. Assume e.g.
$S \subseteq \{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_1$ and δ is divisible by $(\omega_1)^2\}$ is stationary.
There is $\bar{C} = \langle C_\delta : \delta \in S\rangle$, a strict club system such that $\aleph_2 \notin \mathrm{id}_p(\bar{C})$ and $[\alpha \in \mathrm{nacc}(C_\delta) \Rightarrow \mathrm{cf}(\alpha) = \aleph_1]$; moreover, there are $h_\delta : C_\delta \to \omega$ for $\delta \in S$ such that for every club E of $\aleph_2$, for some δ,

$$\bigwedge_{n<\omega} \delta = \sup \left[h_\delta^{-1}(\{n\}) \cap E \cap \mathrm{nacc}(C_\delta)\right].$$

Proof. Let $\bar{C} = \langle C_\delta : \delta \in S\rangle$ be a strict S-club system such that $\lambda \notin \mathrm{id}_p(\bar{C})$ and $[\alpha \in \mathrm{nacc}(C_\delta) \Rightarrow \mathrm{cf}(\delta) = \aleph_1]$ (exist by [Sh:g, III, 2.4, p.126]). For each $\delta \in S$ let $\langle \eta_\delta^\alpha : \alpha \in C_\delta\rangle$ be a sequence of pairwise distinct members of ${}^\omega 2$. We try to define by induction on $\zeta < \omega_1$, E_ζ, $\langle T_\alpha^\zeta : \alpha \in E_\zeta\rangle$ such that:

$$E_\zeta \text{ is a club of } \aleph_2, \text{ decreasing with } \zeta,$$

$$T_\delta^\zeta = \left\{\nu \in {}^{\omega>}2 : \delta = \sup\{\alpha : \alpha \in E_\zeta \cap \mathrm{nacc}(C_\delta) \text{ and } \nu \trianglelefteq \eta_\delta^\alpha\}\right\}$$

$$E_{\zeta+1} \text{ is such that } \left\{\delta \in S : T_\delta^\zeta = T_\delta^{\zeta+1} \text{ and } \delta \in \mathrm{acc}(E_{\zeta+1})\right\} \text{ is not stationary}.$$

We necessarily will be stuck say for $\zeta < \omega_1$. Then for each $\delta \in S \cap \mathrm{acc}(E_\zeta)$ let $\{\nu_n^\delta : n < \omega\} \subseteq T_\delta^\zeta$ be a maximal set of pairwise incomparable (exist as T_δ^ζ has $\geq \aleph_1$ branches), and let $h_\delta(\alpha) =$ the n such that $\nu_n^\delta \triangleleft \eta_\delta^\alpha$ if there is one, zero otherwise. $\square_{3.1}$

3.2 Remark. 0) Where is "δ divisible by $(\omega_1)^2$ used? If not, then there is no club C of δ such that $\alpha \in \mathrm{nacc}(C_\delta) \Rightarrow \mathrm{cf}(\alpha) = \aleph_1$.

1) We can replace $\aleph_0, \aleph_1, \aleph_2$ by $\sigma, \lambda, \lambda^+$ when $\lambda = \mathrm{cf}(\lambda) > \kappa \geq \sigma$ and for some tree $T, |T| = \kappa, T$ has $\geq \lambda$ branches, such that: if $T' \subseteq T$ has $\geq \lambda$ branches then T' has an antichain of cardinality $\geq \sigma$. We can replace "branches" by "θ-branches" for some fixed θ. More in [Sh 572].

2) In the end of the proof no harm is done if h_δ is a partial function. Still we could have chosen ν^δ_n so that it always exists: e.g. if without loss of generality $\{\eta^\delta_\alpha : \alpha \in C_\delta\}$ contains no perfect subset of ${}^\omega 2$, we can choose $\nu^\delta \in {}^\omega 2 \setminus \{\eta^\delta_\alpha : \alpha \in C_\delta\}$ such that $n < \omega \Rightarrow \nu^\delta \restriction n \in T^{\zeta(*)}_\delta \,\&\, (\exists \rho)[\nu^\delta \restriction n \triangleleft \rho \in T^{\zeta(*)}_\delta \,\&\, \neg(\rho \triangleleft \nu^\delta)]$, and then we can choose $\{\eta^\alpha_\delta : \alpha \in C_\delta\}$ be $\eta^\alpha_\delta = (\nu^\delta \restriction k_n)\char`^\langle 1 - \nu^\delta(k_n)\rangle$ where $k_n < k_{n+1} < k$ and $(\nu^\delta \restriction k)\char`^\langle 1 - \nu^\delta(k)\rangle \in T^{\zeta(*)}_\delta$ iff $(\exists n)(k = k_n)$.

3.3 Claim. Suppose λ is regular uncountable and $S, S_0 \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda\}$ are stationary. Then:

1) We can find $\bar{C} = \langle C_\delta : \delta \in S\rangle$ such that:
   (A) C_δ is a club of δ
   (B) for every club E of λ^+ and function f from λ^+ to λ^+ satisfying $f(\alpha) < 1+\alpha$ <u>there are</u> stationarily many $\delta \in S \cap \mathrm{acc}(E)$ such that for some $\zeta < \lambda^+$ we have $\delta = \sup\{\alpha \in \mathrm{nacc}(C_\delta) : \alpha \in E \cap S_0 \text{ and } \zeta = f(\alpha)\}$
   (C) for each $\alpha < \lambda^+$ the set $\{C_\delta \cap \alpha : \delta \in S\}$ has cardinality $\le \lambda^{<\lambda}$; moreover, for any chosen strict λ^+-club system $\bar{e}$ we can demand:

$$(\alpha) \quad \left[\bigwedge_{\alpha<\lambda^+} \left|\{e_\delta \cap \alpha : \delta < \lambda^+\}\right| \le \lambda \Rightarrow \bigwedge_{\alpha<\lambda^+} \left|\{C_\delta \cap \alpha : \delta < \lambda^+\}\right| \le \lambda\right] \textit{and}$$

$$(\beta) \quad \Big[\bigwedge_{\alpha<\lambda^+} |\{e_\delta \cap \alpha \ : \ \alpha \in \mathrm{nacc}(e_\delta), \delta < \lambda^+\}| \le \lambda$$
$$\Rightarrow \bigwedge_{\alpha<\lambda^+} \left|\{C_\delta \cap \alpha : \alpha \in \mathrm{nacc}(C_\delta), \delta < \lambda^+\}\right| \le \lambda\Big].$$

2) Assume $\lambda = \lambda^{<\lambda}$. We can find $\bar{C} = \langle C_\delta : \delta \in S\rangle$ such that:
   (A),(B),(C) as above and
   (D) For some partition $\langle S^\xi : \xi < \lambda\rangle$ of S_0, for every club E of λ^+, there are stationarily many $\delta \in S \cap \mathrm{acc}(E)$ such that for every $\xi < \lambda$, we have $\delta = \sup\{\alpha \in \mathrm{nacc}(C_\delta) : \alpha \in E \cap S^\xi\}$.

3.4 Remark.

1) The main point is (B) and note that $\mathrm{otp}(C_\delta)$ may be $> \lambda$.
2) In clause (B) we can make ζ not depend on δ.
3) In clause (D) we can have $\mathrm{nacc}(C_\delta) \cap E \cap S^\xi$ has order type divisible say by λ^n for any fixed n.

Proof. 1) Let $\bar{e}$ be a strict λ^+-club system (as assumed for clause (C)); note

(∗) $\delta < \lambda^+ \,\&\, \alpha \in \mathrm{acc}(e_\delta) \Rightarrow \mathrm{cf}(\alpha) < \lambda$
$\alpha = \beta + 1 < \lambda^+ \Rightarrow e_\alpha = \{0, \beta\}$.

For each $\beta < \lambda^+$ and $n < \omega$ we define C^n_β, by induction on n : $C^0_\beta = e_\beta$, $C^{n+1}_\beta = C^n_\beta \cup \{\alpha : \alpha \in e_{\mathrm{Min}(C^n_\beta \setminus \alpha)}\}$. Clearly $\beta = \bigcup_n C^n_\beta$ (as for $\alpha \in \beta \setminus \bigcup_n C^n_\beta$, the sequence $\langle \mathrm{Min}(C^n_\beta \setminus \alpha) : n < \omega \text{ and } \alpha \notin C^n_\beta \rangle$ is a strictly decreasing sequence of ordinals hence is finite), [also this is a case of the well known paradoxical decomposition as $\mathrm{otp}(C^{n+1}_\beta) \le \lambda^n$ (ordinal exponentiation)]. Also clearly C^n_β is a closed subset of β and if β is a limit ordinal then it is unbounded in β.

Note:

$(*)'$ $\beta < \lambda^+ \,\&\, \alpha < \beta \,\&\, \mathrm{cf}(\alpha) = \lambda \Rightarrow (\exists n)\Big[\alpha \in C^n_\beta \setminus \bigcup_{\ell < n} C^\ell_\beta$ $\&\, \alpha \in \mathrm{nacc}(C^n_\beta)\Big]$.

Now for some $n < \omega$, $\langle C^n_\delta : \delta \in S \rangle$ is as required; why? we can prove by induction on $n < \omega$ that for every $\alpha < \lambda^+$ we have $|\{C^n_\delta \cap \alpha : \delta \in S\}| \le \lambda^{<\lambda}$, moreover also the second phrase of clause (C) is easy to check; we have noted above that clause (A) holds. So clause (C) holds for every n; also clause (A) holds for every n. So if the sequence fails we can choose E_n, f_n such that E_n, f_n exemplify $\langle C^n_\delta : \delta \in S \rangle$ is not as required in clause (B).

Now $E =: \bigcap_{n<\omega} E_n$ is a club of λ^+, and $f(\delta) =: \sup\{f_n(\delta) + 1 : n < \omega\}$ satisfies:

$(*)''$ if $\delta < \lambda^+$, $\mathrm{cf}(\delta) > \aleph_0$ then $f(\delta) < \delta$:

hence by Fodor's Lemma for some $\alpha^* < \lambda^+$ we have $S_1 =: \{\alpha \in S_0 : f(\alpha) = \alpha^*\}$ is stationary (remember: $\delta \in S_0 \Rightarrow \mathrm{cf}(\delta) = \lambda > \aleph_0$). Let $\alpha^* = \bigcup_{\zeta < \lambda} A_\zeta$, $|A_\zeta| < \lambda$, A_ζ increasing in ζ, so easily for some ζ we have $S_2 =: \{\delta \in S_1 : n < \omega \Rightarrow f_n(\delta) \in A_\zeta\}$ is a stationary subset of λ^+ (remember $\lambda = \mathrm{cf}(\lambda) > \aleph_0$). Note that if $(\forall \alpha)[\alpha < \lambda \to |\alpha|^{\aleph_0} < \lambda]$ we can shorten the proof a little.

So also $E \cap S_2$ is stationary, hence for some $\delta \in S$ we have: $\delta = \sup(E \cap S_2)$. Hence (remembering $(*)'$) for some $n, \delta = \sup(E \cap S_2 \cap \mathrm{nacc}(C^n_\delta))$. Now as $\mathrm{cf}(\delta) = \lambda > |A_\zeta|$ there is $B \subseteq E \cap S_1 \cap \mathrm{nacc}(C^n_\delta)$ unbounded in δ such that $f_n \restriction B$ is constant, contradicting the choice of E_n.

2) For simplicity we ignore here clause (B). Let $\bar{e}$, $\langle < C^n_\alpha : n < \omega >: \alpha < \lambda^+ \rangle$ be as in the proof of part (1). We prove a preliminary fact. Let $\kappa < \lambda$, let κ^* be κ if $\mathrm{cf}(\kappa) > \aleph_0$, κ^+ if $\mathrm{cf}(\kappa) = \aleph_0$ and $\langle S_{0,\epsilon} : \epsilon < \kappa^* \rangle$ be a sequence of pairwise disjoint stationary subsets of S_0. For every club E of λ^+, let $E' = \{\delta < \lambda : \text{ for every } \epsilon < \kappa^*, \delta = \sup(E \cap S_{0,\epsilon})\}$, it too is a club of λ^+. Now for every $\delta \in E' \cap S$ and $\epsilon < \kappa^*$ for some $n_E(\delta, \epsilon) < \omega$ we have $\delta = \sup(S_{0,\varepsilon} \cap E \cap \mathrm{nacc}(C^{n_E(\delta,\varepsilon)}_\delta))$ hence (as $\mathrm{cf}(\kappa^*) > \aleph_0$, see its choice) for some $n_E(\delta) < \omega$, $u^\delta_E =: \{\epsilon < \kappa^* : n_E(\delta, \epsilon) = n_E(\delta)\}$ has cardinality κ^*. Without loss of generality, $n_E(\delta, \varepsilon)$, $n_E(\delta)$ are minimal. So for some n^* for every club E of λ^+, for stationarily many $\delta \in E \cap S$, we have $\delta \in E'$ and $n_E(\delta) = n^*$. Now if $\mathrm{cf}(\kappa) = \aleph_0$, for some $\epsilon(*) < \kappa^*$ for every club E of λ^+ for stationarily many $\delta \in E \cap S$ we have $n_E(\delta) = n^*$ and $|u^\delta_E \cap \epsilon(*)| = \kappa$. If $\mathrm{cf}(\kappa) > \aleph_0$ let $\epsilon(*) = \kappa$. Now there is a club E of λ^+ such that: if $E_0 \subseteq E$ is a club then for stationarily many $\delta \in S \cap E$, $n_E(\delta) = n_{E_0}(\delta) = n^*$, $u^\delta_E \cap \epsilon(*) = u^\delta_{E_0} \cap \epsilon(*)$ and it has cardinality

κ (just remember $\varepsilon(*) < \lambda$ in all cases so after $\le \lambda$ tries of E_0 we succeed). As $\kappa < \lambda = \lambda^{<\lambda}$, we conclude:

(∗) for some $w \subseteq \kappa^*$, $|w| = \kappa$ (in fact $w \subseteq \varepsilon(*)$), for every club E of λ^+ for stationarily many $\delta \in S \cap E$, for every $\epsilon \in w$ we have $\delta = \sup\{\alpha \in \operatorname{nacc}(C_\delta^{n^*}) : \alpha \in S_{0,\varepsilon} \cap E\}$.

Let $\langle S_{1,\xi} : \xi < \lambda \rangle$ be a sequence of pairwise disjoint stationary subsets of S_0. For each ξ we can partition $S_{1,\xi}$ into $|\xi + \omega|^+$ pairwise disjoint stationary subsets $\langle S_{1,\xi,\varepsilon} : \varepsilon < |\xi + \omega|^+ \rangle$, and apply the previous discussion (i.e. $S_{1,\xi}$, $|\xi + \omega|$, $S_{1,\xi,\varepsilon}$ here stand for S_0, κ, $S_{0,\varepsilon}$ there) hence for some n^*_ξ, $\langle S_{1,\xi,\epsilon} : \epsilon < \xi \rangle$

$(*)_\xi$ $n^*_\xi < \omega$, $\langle S_{1,\xi,\epsilon} : \epsilon < \xi \rangle$ is a sequence of pairwise disjoint stationary subsets of $S_{1,\xi}$ such that for every club E of λ^+ for stationarily many $\delta \in S \cap E$, for every $\epsilon < \xi$ we have
$$\delta = \sup\left\{\alpha \in \operatorname{nacc}(C_\delta^{n^*_\xi}) : \alpha \in S_{1,\xi,\epsilon} \cap E\right\}.$$

This is not what we really want but it will help. We shall next prove that

$(*)'$ for some n, for every club E of λ^+, for stationarily many $\delta \in S \cap E$ we have; letting $S_{2,\epsilon} = \cup\{S_{1,\xi,\epsilon} : \xi \in (\epsilon, \lambda)\}$: for every $\epsilon < \lambda$,
$$\delta = \sup\left\{\alpha : \alpha \in E \cap \operatorname{nacc}(C_\delta^n) \cap S_{2,\varepsilon}\right\}.$$

If not for every n, there is a club E_n of λ^+ such that for some club E'_n of λ no $\delta \in S \cap E'_n$ is as required in $(*)'$ for δ.

Let $E =: \bigcap_{n<\omega} E_n \cap \bigcap_{n<\omega} E'_n$, it is a club of λ^+. Now for each $\xi < \lambda$, by the choice of $\langle S_{1,\xi,\epsilon} : \epsilon < \xi \rangle$ we have

$$S^\xi =: \left\{\delta \in S : \text{ for every } \epsilon < \xi \text{ we have } \delta = \sup\{\alpha \in \operatorname{nacc}(C_\delta^{n^*_\xi}) : \alpha \in S_{1,\xi,\epsilon} \cap E\}\right\}$$

is a stationary subset of λ^+, so

$$E^+ = \{\delta < \lambda^+ : \delta \in \operatorname{acc}(E) \text{ is divisible by } \lambda^2 \text{ and } \delta \cap S^\xi \cap E \text{ has order type } \delta \text{ for every } \xi < \lambda\}$$

is a club of λ^+.

Let us choose $\delta^* \in S \cap E^+$, and let $e_{\delta^*} = \{\alpha^*_i : i < \lambda\}$ (α^*_i increasing continuous). We shall show that for some n, δ^* is in E'_n and is as required in $(*)'$ for E_n, thus deriving a contradiction. Let for $\xi < \lambda$

$$A_\xi = \{i < \lambda : (\alpha^*_i, \alpha^*_{i+1}) \cap S^\xi \neq \emptyset\}.$$

As $\delta^* = \operatorname{otp}(\delta^* \cap S^\xi \cap E)$ clearly A_ξ is an unbounded subset of λ; hence we can choose by induction on $\xi < \lambda$, a member $i(\xi) \in A_\xi$ such that $i(\xi) > \xi$ & $i(\xi) > \bigcup_{\zeta<\xi} i(\zeta)$. Now for each ξ we have $\left(\alpha_{i(\xi)}, \alpha_{i(\xi)+1}\right) \subseteq \bigcup_{n<\omega} C^n_{\alpha_{i(\xi)+1}}$ hence for some $m(\xi) < \omega$ we have $\left(\alpha_{i(\xi)}, \alpha_{i(\xi)+1}\right) \cap S^\xi \cap \left(C^{m(\xi)}_{\alpha_{i(\xi)+1}} \setminus \bigcup_{\ell<m(\xi)} C^\ell_{\alpha_{i(\xi)+1}}\right)$

$\neq \emptyset$ so choose δ_ξ in this intersection; as $\delta_\xi \in S^\xi \subseteq S$ clearly $\mathrm{cf}(\delta_\xi) = \lambda$. Looking at the inductive definition of the C^n_δ's, it is easy to check that $\big(\alpha_{i(\xi)}, \alpha_{i(\xi)+1}\big) \cap C^{m(\xi)+n^*_\xi+1}_{\delta*} \cap \delta_\xi$ contains an end-segment of $C^{n^*_\xi}_{\delta_\xi}$ hence for every $\epsilon < \xi$, $\big(\alpha_{i(\xi)}, \alpha_{i(\xi)+1}\big) \cap E \cap \mathrm{nacc}(C^{m(\xi)+n^*_\xi+1}_{\delta*}) \cap S_{1,\xi,\varepsilon} \neq \emptyset$ hence by the definition of $S_{2,\varepsilon}$ we have $(\alpha_{i(\xi)}, \alpha_{i(\xi)+1}) \cap E \cap \mathrm{nacc}(C^{m(\xi)+n^*_\xi+1}_{\delta*}) \cap S_{2,\varepsilon} \neq \emptyset$. Now for some $k < \omega$ we have $B = \{\xi < \lambda : m(\xi) + n^*_\xi + 1 = k\}$ is unbounded in λ, hence for each $\epsilon < \lambda$, $S_{2,\epsilon} \cap E \cap \mathrm{nacc}(C^k_{\delta*})$ is unbounded in δ^*, contradicting $\delta^* \in E \subseteq E'_k$. $\square_{3.3}$

3.5 Claim. If $\lambda = \mu^+$, $\mu = \kappa^+$ and $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \mu\}$ stationary then for some strict S-club system $\bar{C}$ with $C_\delta = \{\alpha_{\delta,\zeta} : \zeta < \mu\}$, (where $\alpha_{\delta,\zeta}$ is strictly increasing continuous in ζ) we have: for every club $E \subseteq \lambda$ for stationarily many $\delta \in S$,

$$\{\zeta < \mu : \alpha_{\delta,\zeta+1} \in E\} \text{ is stationary (as subset of } \mu).$$

Remark. So this is stronger than previous statements saying that this set is unbounded in μ. A price is the demand that μ is not just regular but is a successor cardinal (for inaccessible we can get by the proof a less neat result, see more [Sh 572]).

Proof. We know that for some strict S-club system $\bar{C}^0 = \langle C^0_\delta : \delta \in S\rangle$ we have $\lambda \notin \mathrm{id}_p(\bar{C}^0)$ (exists, e.g. as in 3.1). Let $C^0_\delta = \{\alpha^\delta_\zeta : \zeta < \mu\}$ (increasing continuously in ζ). We claim that for some sequence of functions $\bar{h} = \langle h_\delta : \delta \in S\rangle$ with $h_\delta : \mu \to \kappa$ we have:

$(*)_{\bar{h}}$ for every club E of λ for stationarily many $\delta \in S \cap \mathrm{acc}(E)$, for some $\epsilon < \kappa$ the following subset of μ is stationary

$$A^{\delta,\varepsilon}_E = \Big\{\zeta < \mu : \alpha^\delta_\zeta \in E \text{ and the ordinal } \mathrm{Min}\{\alpha^\delta_\xi : \xi > \zeta, h_\delta(\xi) = \epsilon\} \text{ belongs to } E\Big\}.$$

This suffices: for each $\epsilon < \kappa$ let $C_{\epsilon,\delta}$ be the closure in C^0_δ of $\{\alpha^\delta_\xi \in E : \xi < \mu, h_\delta(\alpha^\delta_\xi) = \epsilon\}$, so for each club E of λ for stationarily many $\delta \in S \cap \mathrm{acc}(E)$ for some ordinal ε the set $A^{\delta,\varepsilon}_E$ is stationary hence for one ε_E this holds for stationarily many $\delta \in E$; but $E_1 \subseteq E_2$ implies ε_{E_1} is O.K. for E_2 hence for some ϵ the sequence $\langle C_{\epsilon,\delta} : \delta \in S\rangle$ is as required.

So assume for no $\bar{h}$ does $(*)_{\bar{h}}$ holds, and we define by induction on $n < \omega$, $E_n, \bar{h}^n = \langle h^n_\delta : \delta \in S\rangle, \bar{e}^n = \langle e^n_\delta : \delta \in S\rangle$ with E_n a club of λ, e^n_δ club of μ and $h^n_\delta : \mu \to \kappa$ as follows:
let $E_0 = \lambda$, $h^0_\delta(\zeta) = 0$, $e^n_\delta = \mu$.
If $E_0, \ldots, E_n, \bar{h}^0, \ldots, \bar{h}^n, \bar{e}^0, \ldots, \bar{e}^n$ are defined, necessarily $(*)_{\bar{h}^n}$ fails, so for some club $E_{n+1} \subseteq \mathrm{acc}(E_n)$ of λ for every $\delta \in S \cap \mathrm{acc}(E_{n+1})$ and $\epsilon < \kappa$ there is a club $e_{\delta,\epsilon,n} \subseteq e^n_\delta$ of μ, such that:

$$\zeta \in e_{\delta,\epsilon,n} \Rightarrow \mathrm{Min}\{\alpha^\delta_\xi : \xi > \zeta \text{ and } h_\delta(\xi) = \epsilon\} \notin E_{n+1}.$$

Choose $h^{n+1}_\delta : \mu \to \kappa$ such that $\left[h^{n+1}_\delta(\zeta) = h^{n+1}_\delta(\xi) \Rightarrow h^n_\delta(\zeta) = h^n_\delta(\xi)\right]$ and

$$\Big[\Big[\zeta \neq \xi \;\&\; \zeta < \kappa \;\&\; \xi < \kappa \;\&\; \bigvee_{\epsilon<\kappa} \mathrm{Min}\{\gamma \in e_{\delta,n,\epsilon} : \gamma > \zeta\} = \mathrm{Min}\{\gamma \in e_{\delta,n,\epsilon} : \gamma > \xi\}\Big] \Rightarrow h^{n+1}_\delta(\zeta) \neq h^{n+1}_\delta(\xi)\Big].$$

Note that we can do this as $\mu = \kappa^+$.

Lastly let $e^{n+1}_\delta = \bigcap_{\epsilon<\kappa} e_{\delta,\epsilon,n} \cap \mathrm{acc}(e^n_\delta)$.

There is no problem to carry out the definition. By the choice of $\bar{C}^0$ for some $\delta \in \mathrm{acc}(\bigcap_{n<\omega} E_n)$ we have $\delta = \sup(A')$ where $A' = \mathrm{acc}(\bigcap_{n<\omega} E_n) \cap \mathrm{nacc}(C^0_\delta)$. Let $A \subseteq \mu$ be such that $A' = \{\alpha^\delta_\zeta : \zeta \in A\}$ with α^δ_ζ increasing with ζ and let

$$\xi =: \sup\Big\{\sup\{\beta \in A : h^n_\delta(\beta) = \epsilon\} : n < \omega, \epsilon < \kappa \text{ and } \{\beta \in A : h^n_\delta(\beta) = \epsilon\} \text{ is bounded in } A\Big\}.$$

(so we get rid of the uninteresting ε's).

As A' is unbounded in δ, clearly A is unbounded in μ and $\mu = \mathrm{cf}(\mu) = \kappa^+ > \kappa$, whereas the sup is on a set of cardinality $\leq \aleph_0 \times \kappa < \mu$, clearly $\xi < \sup(A) = \mu$, so choose $\zeta \in A$, $\zeta > \xi$ and $\zeta > \mathrm{Min}(e^n_\delta)$ for each n. Now $\langle \sup(e^n_\delta \cap \zeta) : n < \omega\rangle$ is non-increasing (as e^n_δ decreases with n) hence for some $n(*) < \omega : n > n(*) \Rightarrow \sup(e^n_\delta \cap \zeta) = \sup(e^{n(*)}_\delta \cap \zeta)$; and for $n(*) + 1$ we get a contradiction. $\square_{3.5}$

3.6 Remark. If we omit "$\mu = \kappa^+$" in 3.5, we can prove similarly a weaker statement (from it we can then derive 3.5):

(*) if $\lambda = \mu^+$, $\mu = \mathrm{cf}(\mu) > \aleph_0$, $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \mu\}$ is stationary, $\bar{C}^0$ is a strict S-club system, $C^0_\delta = \{\alpha_{\delta,\zeta} : \zeta < \mu\}$ (with $\alpha_{\delta,\zeta}$ strictly increasing with ζ), and $\lambda \notin \mathrm{id}_p(\bar{C}^0)$ <u>then</u> we can find $\bar{e} = \langle e_\delta : \delta \in S\rangle$ such that:
 (a) e_δ is a club of δ with order type μ
 (b) for every club E of λ for stationarily many $\delta \in S$ we have $\delta \in \mathrm{acc}(E)$ and for stationarily many $\zeta < \mu$ we have:
 $\zeta \in e_\delta$ and $(\exists \xi)[\zeta < \xi + 1 < \mathrm{Min}(e_\delta \backslash (\zeta + 1)) \;\&\; \alpha_{\delta,\xi+1} \in E]$

3.7 Remark. In 3.5 we can for each $\delta \in S$ have $h_\delta : \mu \to \kappa$ such that for every club E of λ, for stationarily many $\delta \in S$, for every $\epsilon < \kappa$, for stationarily many $\zeta \in h^{-1}_\delta(\{\epsilon\})$ we have $\alpha_{\delta,\zeta+1} \in E$.

Use Ulam's proof.

3.8 Claim. Suppose $\lambda = \mu^+$, $S \subseteq \lambda$ stationary, $\bar{C} = \langle C_\delta : \delta \in S\rangle$ an S-club system, $\lambda \notin \mathrm{id}^p(\bar{C})$, $\mu > \kappa =: \sup\{\mathrm{cf}(\alpha)^+ : \alpha \in \mathrm{nacc}(C_\delta), \delta \in S\}$.
<u>Then</u> there is $\bar{e}$, a strict λ-club system such that:

(*) for every club E of λ, for stationarily many $\delta \in S$,
 $\delta = \sup\{\alpha \in \mathrm{nacc}(C_\delta) : \alpha \in E$, moreover $e_\alpha \subseteq E$ and $\min(e_\alpha) \to \sup(\alpha \cap C_\delta)\}$.

Proof. Let $\bar{e}$ be a strict λ-club system.

Clearly for some $\theta < \kappa$ for every club E of λ, for stationarily many $\delta \in S$, $\delta = \sup\{\alpha : \alpha \in E, \alpha \in \operatorname{nacc}(C_\delta) \text{ and } \operatorname{cf}(\alpha) = \theta\}$. For any club E of λ and $\varepsilon < \theta$ we let $\bar{e}^\varepsilon_E = \langle e^\varepsilon_{E,\alpha} : \alpha < \lambda\rangle$ be: $e^\varepsilon_{E,\alpha} = \{\sup(\gamma \cap E) : \gamma \in e_\alpha \text{ and } \operatorname{otp}(\gamma \cap e_\alpha) > \varepsilon\}$ if $\alpha \in \operatorname{acc}(E)$ & $\operatorname{cf}(\alpha) = \theta$ and $e^\varepsilon_{E,\alpha} = e_\alpha$ otherwise. It is enough to show that for some club E of λ and $\varepsilon < \theta$ the sequence $\bar{e}^\varepsilon_E$ is as required. If this fails, we choose by induction on $\zeta < \kappa$ a club E_ζ of λ such that $\zeta_1 < \zeta_2 \Rightarrow E_{\zeta_2} \subseteq \operatorname{acc}(E_{\zeta_1})$.

For $\zeta + 1$, for each $\zeta < \kappa$, $\varepsilon < \theta$, let $E_{\zeta,\varepsilon}$ be a club of λ such that $\bar{e}^\varepsilon_{E_\zeta}$ is not as required. Let $E'_{\zeta,\varepsilon}$ a club of λ disjoint to $\{\delta \in S : \delta = \sup\{\alpha \in \operatorname{nacc}(C_\delta)\text{: } \operatorname{cf}(\alpha) = \theta$ and $e^\varepsilon_{E_\zeta,\alpha} \subseteq E \backslash (\sup(C_\delta \cap \alpha)\}$ and lastly $E_{\zeta+1} = \bigcap_{\varepsilon<\theta} E_{\varepsilon,\zeta} \cap \bigcap_{\theta<\theta} E'_{\varepsilon,\zeta} \cap \operatorname{acc}(E_\zeta)$. By the choice of θ we can find $\delta^* \in S \cap \bigcap_{\zeta<\kappa} E_\zeta$ such that the set $A = \{\alpha \in \operatorname{nacc}(C_{\delta^*}) : \operatorname{cf}(\alpha) = \theta, \alpha \in \bigcap_{\varepsilon<\kappa} E_\varepsilon\}$ is unbounded in δ^*. We can easily find $\varepsilon < \theta$, $\zeta < \kappa$ giving contradiction. $\square_{3.8}$

3.9 Claim. Let $\lambda = \mu^+$, $\mu > \operatorname{cf}(\mu) = \kappa$, $\theta = \operatorname{cf}(\theta) < \mu$, $\theta \neq \kappa$ and $S \subseteq \{\delta < \lambda : \operatorname{cf}(\delta) = \theta$ and δ divisible by $\mu\}$ be stationary.

1) For any limit ordinal $\gamma(*) < \mu$ of cofinality θ there is an S-club system $\bar{C}^{\gamma(*)} = \langle C^{\gamma(*)}_\delta : \delta \in S\rangle$ satisfying $\lambda \notin \mathrm{id}^a\left(\bar{C}^{\gamma(*)}\right)$ with $\operatorname{otp}\left(\bar{C}^{\gamma(*)}\right) = \gamma(*)$. Let $C^{\gamma(*)}_\delta = \{\alpha^{\gamma(*),\delta}_i : i < \gamma(*)\}$, $\alpha^{\gamma(*),\delta}_i$ increasing continuous with i.
2) Assume further $\kappa > \aleph_0$, and $\gamma(*)$ is divisible by κ and let $\bar{e}$ be a strict λ-club system.
Then for some σ regular $\sigma < \mu$, and club E^0 of λ, $\bar{C} = \bar{C}^{\gamma(*),\sigma,\bar{e},E^0} = \langle g\ell^1_\sigma(C^{\gamma(*)}_\delta, E^0, \bar{e}) : \delta \in S\rangle$ satisfies:
$(*)^a$ for every club $E \subseteq E^0$ of λ for stationarily many $\delta \in S$, for arbitrarily large $i < \gamma(*)$ we have $\mu = \sup\left\{\operatorname{cf}(\gamma) : \gamma \in \operatorname{nacc}(C_\delta) \cap [\alpha^{\gamma(*),\delta}_i, \alpha^{\gamma(*),\delta}_{i+\kappa}) \cap E\right\}$.
3) We can add in (2): for some club $E^1 \subseteq E^0$ of λ,
$(*)^b$ for every club $E \subseteq E^1$ of λ for some $\delta \in S$ we have $E \cap C_\delta = E^1 \cap C_\delta$ and for arbitrarily large $i < \gamma(*)$,
$\mu = \sup\left\{\operatorname{cf}(\gamma) : \gamma \in C_\delta \cap [\alpha^{\gamma(*),\delta}_i, \alpha^{\gamma(*),\delta}_{i+\kappa}) \cap E\right\}$.
4) In part (1), if $S \in I[\lambda]$ then without loss of generality $|\{C^{\gamma(*)}_\delta \cap \alpha : \delta \in S \text{ and } \alpha \in \operatorname{nacc}(C^{\gamma(*)}_\delta)\}| < \lambda$ for every $\alpha < \lambda$.

Proof. 1) Let $\mu = \sum_{\varepsilon<\kappa} \lambda_\varepsilon$ with $\langle \lambda_\varepsilon : \varepsilon < \kappa\rangle$ increasing continuous, $\lambda_\varepsilon < \mu$. Let for each $\alpha \in [\mu, \lambda)$, $\langle a^\alpha_\varepsilon : \varepsilon < \kappa\rangle$ be an increasing sequence of subsets of α, $|a^\alpha_\varepsilon| = \lambda_\varepsilon$, $\alpha = \bigcup_{\varepsilon<\kappa} a^\alpha_\varepsilon$. Now

$(*)_1$ there is an $\varepsilon < \kappa$ such that

$(*)_{1,\varepsilon}$ for every club E of λ we have

$$S^1_\varepsilon[E] =: \{\delta \in S : a^\delta_\varepsilon \cap E \text{ is unbounded in } \delta \text{ and } \operatorname{otp}(a^\delta_\varepsilon \cap E) \text{ is divisible by } \gamma(*)\}$$

is stationary in λ

[Why? If not, for every $\varepsilon < \kappa$ there is a club E^1_ε of λ such that $S^1_\varepsilon[E^1_\varepsilon]$ is not stationary, so let it be disjoint to the club E^2_ε of λ. Let $E = \bigcap_{\varepsilon<\kappa}(E^1_\varepsilon \cap E^2_\varepsilon)$, clearly it is a club of λ, hence $E^1 = \{\delta < \lambda : \mathrm{otp}(\delta \cap E) = \delta$ and is divisible by μ hence by $\gamma(*)\}$ is a club of λ and choose $\delta^* \in E^1 \cap S$. Now for every $\varepsilon < \kappa$, as $\delta^* \in E^1 \subseteq E \subseteq E^2_\varepsilon$, clearly $\sup(a^{\delta^*}_\varepsilon \cap E^1_\varepsilon) < \delta$ or $\mathrm{otp}(a^{\delta^*}_\varepsilon \cap E^1_\varepsilon)$ is not divisible by $\gamma(*)$ hence $\sup(a^{\delta^*}_\varepsilon \cap E) < \delta \vee [\mathrm{otp}(a^{\delta^*}_\varepsilon \cap E)$ not divisible by $\gamma(*)]$. Choose $\gamma_\varepsilon < \delta^*$ such that $a^{\delta^*}_\varepsilon \cap E \subseteq \beta_\varepsilon$ or $\mathrm{otp}(a^{\delta^*}_\varepsilon \cap E \backslash \beta_\varepsilon) < \gamma(*)$, so always the second holds.

As $\theta \neq \kappa$ are regular cardinals, and $\mathrm{cf}(\delta) = \theta$ necessarily for some $\beta^* < \delta^*$ we have: $b^* = \{\varepsilon < \kappa : \beta_\varepsilon \leq \beta^*\}$ is unbounded in κ. So

$$E \cap \delta^* \backslash \beta^* \subseteq \bigcup_{\varepsilon \in b^*} (E \cap a^{\delta^*}_\varepsilon \backslash \beta^*)$$

hence

$$|E \cap \delta^* \backslash \beta^*| \leq \sum_{\varepsilon \in b^*} |E \cap a^{\delta^*}_\varepsilon \backslash \beta^*| \leq |b^*| \times |\gamma(*)| < \mu.$$

But $\delta^* \in E^1$ hence $\mathrm{otp}(E \cap \delta^*) = \delta^*$ and is divisible by μ, so now $E \cap \delta^* \backslash \beta^*$ has order type $\geq \mu$, a contradiction.]

Let ε from $(*)_1$ be $\varepsilon(*)$.

$(*)_2$ There is a club E^* of λ^+ such that for every club E of λ the set $\{\delta \in S_{\varepsilon(*)}[E^*] : a^\delta_{\varepsilon(*)} \cap E^* \subseteq E\}$ is stationary recalling

$$S_\varepsilon[E^*] = \{\delta \in S : a^\delta_\varepsilon \cap E^* \text{ is unbounded in } \delta \text{ and } \mathrm{otp}(a^\delta_\varepsilon \cap E^*) \text{ is divisible by } \gamma(*)\}$$

[Why? If not, we choose by induction on $\zeta < \lambda^+_{\varepsilon(*)}$ a club E_ζ of λ^+ as follows:

- (a) $E_0 = \lambda$
- (b) if ζ is limit, $E_\zeta = \bigcap_{\xi<\zeta} E_\zeta$
- (c) if $\zeta = \xi + 1$ as we are assuming $(*)_2$ fails, E_ξ cannot serve as E^* so there is a club E^1_ξ of λ such that the set $\{\delta \in S_\varepsilon[E_\xi] : a^\delta_\varepsilon \cap E_\xi \subseteq E^1_\xi\}$ is not stationary, say disjoint to the club E^2_ξ of λ, ($S_\varepsilon[E_\xi]$ is defined above).
  Let $E_\zeta = E_{\xi+1} =: E_\xi \cap E^1_\xi \cap E^2_\xi$.
  So $E = \bigcap_{\zeta<\lambda^+_{\varepsilon(*)}} E_\zeta$ is a club of λ. By the choice of $\varepsilon(*)$ for some $\delta \in E$ we have $\delta = \sup(a^\delta_{\varepsilon(*)} \cap E)$ and $\mathrm{otp}(a^\delta_{\varepsilon(*)} \cap E)$ is divisible by $\gamma(*)$. Now $\langle (a^\delta_{\varepsilon(*)} \cap E_\zeta) : \zeta < \lambda^+_{\varepsilon(*)}\rangle$ is necessarily strictly decreasing sequence of subsets of $a^\delta_{\varepsilon(*)}$, but $|a^\delta_{\varepsilon(*)}| \leq \lambda_{\varepsilon(*)}$, a contradiction.]

Let E^* be as in $(*)_2$.

Let $S' = S_{\varepsilon(*)}[E^*]$ and for $\delta \in S'$ let $C^{\gamma(*)}_\delta$ be a closed unbounded subset of $a^\delta_{\varepsilon(*)} \cap E^*$ of order type $\gamma(*)$ (possible as $\mathrm{otp}(a^\delta_{\varepsilon(*)} \cap E^*)$ is divisible by $\gamma(*)$, has cofinality θ (as $\sup(a^{\delta}_{\varepsilon(*)} \cap E^*) = \delta$ has cofinality θ) and $\mathrm{cf}(\gamma(*)) = \theta$ (by an assumption). For $\delta \in S \setminus S_{\varepsilon(*)}[E^*]$ choose any appropriate $C^{\gamma(*)}_\delta$, so we are done.

2) Assume not, so easily for every regular $\sigma < \mu$ and club E^0 of λ there is a club $E = E(E^0, \sigma)$ of λ such that:

$(*)_1$ the set $S_{E,E^0,\sigma} = \big\{\delta \in S :$ for arbitrarily large $i < \gamma(*)$, $\mu = \sup\{\mathrm{cf}(\gamma) : \gamma \in \mathrm{nacc}(C^{\gamma(*),\sigma,\bar{e},E^0}_\delta) \cap [\alpha^{\gamma(*),\delta}_i, \alpha^{\gamma(*),\delta}_{i+1}) \cap E\}\big\}$ is not a stationary subset of λ so shrinking E further without loss of generality

$(*)^+_1$ the set $S_{E,E^0,\sigma}$ is empty.

Choose a regular cardinal $\chi < \mu$, $\chi > \kappa + \theta + |\gamma(*)|$. We choose by induction on $\zeta < \chi$ a club E_ζ of λ as follows:

for $\zeta = 0$, $E_0 = \lambda$

for ζ limit, $E_\zeta = \bigcap_{\xi<\zeta} E_\zeta$

for $\zeta = \xi + 1$ let $E_\zeta = \cap\{E(E_\varepsilon, \sigma) : \sigma < \mu \text{ regular}\}$.

Let $E = \bigcap_{\zeta<\chi} E_\zeta$, $E' = \{\delta \in E : \mathrm{otp}(E \cap \delta) = \delta\}$ both are clubs of λ and by the choice of $\bar{C}^{\gamma(*)}$ for some $\delta(*) \in S$ we have $C^{\gamma(*)}_{\delta(*)} \subseteq E'$ and $\mu^2 \times \mu$ divides $\delta(*)$. For each $i < \gamma(*)$, the set $b_{\delta^*,i} = \{\beta \in e_{\alpha^{\delta(*)}_{i+1}} : \mathrm{otp}(E \cap \mathrm{Min}(e^{\alpha^{\delta(*)}_{i+1}} \setminus (\beta+1) \setminus \beta)$.

Let $j < \gamma(*)$ be divisible by κ (e.g. $j = 0$). For each $\varepsilon < \kappa$ and $\sigma < \lambda_\varepsilon$, $\zeta < \chi$ we look at

$$\gamma_{j,\varepsilon,\zeta,\sigma} = \mathrm{Min}\big(g\ell^1_\sigma[C^{\gamma(*)}_{\delta(*)}, E_\zeta, \bar{e}] \setminus (\alpha^{\delta(*)}_{j+\varepsilon} + 1)\big).$$

If we change only $\zeta < \chi$, for $\zeta < \chi$ large enough it becomes constant (as in old proofs). Choose $\zeta^* < \chi$ such that $\gamma_{j,\varepsilon,\zeta,\sigma}$ is the same for every $\zeta \in [\zeta^*, \chi)$, for any choice of $j < \gamma(*)$ divisible by $\kappa, \varepsilon < \kappa, \sigma \in \{\lambda_\xi : \xi < \varepsilon\}$. Also $\mathrm{cf}(\gamma_{j,\varepsilon,\zeta,\sigma}) \geq \sigma$ and $\langle \gamma_{j,\varepsilon,\zeta,\lambda_\xi} : \xi < \varepsilon\rangle$ is nonincreasing with ξ so for ε limit it is eventually constant say $\gamma_{j,\varepsilon,\zeta,\lambda_\xi} = \gamma^*_{j,\varepsilon,\zeta,\lambda_\xi}$ for $\xi \in [\xi^*(j,\varepsilon,\zeta), \varepsilon)$. By Fodor for some $\xi^{**} = \xi^{**}(j,\zeta) < \kappa$, $\{\varepsilon : \xi^*(j,\varepsilon,\zeta) = \xi^{**}(j,\zeta)\}$ is a stationary subset of κ; and for some $\xi^{***} = \xi^{**}(\zeta) < \kappa$

$$\gamma(*) = \sup\{j < \gamma(*) : j \text{ divisible by } \kappa, \xi^{**}(j,\zeta) = \xi^{***}\}$$

(recall $\mathrm{cf}(\gamma(*)) = \theta \neq \kappa$). Now choosing $\sigma = \xi^{***}(\zeta^*)$ we are finished.

3) Based on (2) like the proof of (1).

4) Assume $S \in I[\lambda]$, so let $E^1, \bar{b}^1 = \langle b^1_\alpha : \alpha < \lambda\rangle$ witness it, i.e. $b^1_\alpha \subseteq \alpha$ closed in α, $\mathrm{otp}(b^1_\alpha) \leq \theta, \alpha \in \mathrm{nacc}(b^1_\beta) \Rightarrow b^1_\alpha = b^1_\beta \cap \alpha$ and E^1 a club of λ such that $\delta \in S \cap E^1 \Rightarrow \delta = \sup(b_\delta)$. Let $\kappa + \theta + \gamma(*) < \chi = \mathrm{cf}(\chi) < \mu$; by [Sh 420, §1] there is a stationary $S^* \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \chi\}$, $S^* \in I[\lambda]$ and let $E^2, \bar{b}^2 = \langle b^2_\alpha : \alpha < \lambda\rangle$ witness it. There is a club E^3 of λ such that for every club E of λ the set $\{\delta \in S^* : \delta \in \mathrm{acc}(E^3), g\ell(b^2_\alpha, E^3) \subseteq E\}$ is stationary. Let $S^{**} = S^* \cap \mathrm{acc}(E^3)$, $C^2_\alpha = g\ell(b^2_\alpha, E^3)$ for $\alpha \in S^{**}$; clearly C^2_α is a club of α of order type χ and

$$(*) \quad |\{C^2_\alpha \cap \gamma : \gamma \in \mathrm{nacc}(C^2_\alpha)\}| \leq |\{C^2_\beta : \beta \leq \mathrm{Min}(E^3 \setminus \gamma)\}| \leq \mu.$$

Let $b^1_\alpha = \{\beta_{\alpha,\varepsilon} : \varepsilon < \theta\}$, $\beta_{\alpha,\varepsilon}$ increasing continuous with ε. Fix $f_\beta : \beta \to \mu$ be one to one for $\beta < \lambda$. For each $\alpha \in S$ and club E of λ let $b^0_\alpha = b^0_\alpha[E] = b^1_\alpha \cup \{C^2_\beta \backslash (\beta_{\delta,\varepsilon} + 1) : \varepsilon < \theta, \beta \in [\beta_{\delta,\varepsilon}, \beta_{\delta,\varepsilon+1})$ and $C^2_\beta \subseteq E$ and for no such β' is $f_{\beta_{\delta,\varepsilon+2}}(\beta') < \beta\}$. We shall prove that for some club E of λ, $\langle b^0_\alpha[E] : \alpha \in S\rangle$ satisfies: for every club E' of λ for stationarily many $\delta \in S$, $E' \cap b^0[E]$ is an unbounded subset of δ of order type $\chi \times \theta$; this clearly suffices.

First note

(∗) for some $\varepsilon < \kappa$ for every club E of λ for some $\delta \in S \cap \operatorname{acc}(E)$ we have:

$$\theta = \sup\{\varepsilon < \theta : \text{for some } \beta \in [\beta_{\delta,\varepsilon} + 1, \beta_{\delta,\varepsilon+1}) \text{ we have } C^2_\beta \subseteq E \text{ and } f_{\beta_{\delta+\varepsilon+2}}(\beta) < \lambda_\varepsilon\}.$$

[Why? If not, then for every $\varepsilon < \kappa$ there is a club E_ε of λ for which the above fails, let $E = \bigcap_{\varepsilon<\kappa} E_\varepsilon$, it is a club of λ. So $E' = \{\delta < \lambda : \delta$ a limit ordinal and for arbitrarily large $\alpha \in \delta \cap S^{**}$ we have $C^2_\alpha \subseteq E\}$.

Now E' is a club of λ and so for some $\delta^* \in S$ divisible by μ^2 we have $\operatorname{otp}(E' \cap \delta^*) = \delta^*$ and we easily get a contradiction.]

Fix $\varepsilon(*)$, now:

(∗) for some club E^0 of λ for every club $E^1 \subseteq E^0$ of λ for some $\delta \in S \cap \operatorname{acc}[E]$ we have

(a) $\theta = \sup\{\varepsilon < \kappa :$ for some $\beta \in [\beta_{\delta,\varepsilon} + 1, \beta_{\delta,\varepsilon+1}]$ we have $C^2_\beta \subseteq E^0 \cap E^1$ and $f_{\beta_{\delta,\varepsilon+2}}(\beta) < \lambda_{\varepsilon(*)}\}$

(b) if ε is as in (a) then

$$b^0_\alpha[E^1] = b^0_\alpha[E^0].$$

[Why? We try $\lambda^+_{\varepsilon(*)}$ times.]

Now it is easy to check that $\langle b^0_\alpha[E^0] : \alpha \in S\rangle$ is as required. $\square_{3.9}$

3.10 Conclusion. Assume $\lambda = \mu^+$, $\mu > \operatorname{cf}(\mu) = \kappa > \aleph_0$, $\kappa \neq \theta = \operatorname{cf}(\theta) < \lambda$, $\gamma^* < \lambda$, $\operatorname{cf}(\gamma^*) = \theta$, $S \subseteq \{\delta < \lambda : \operatorname{cf}(\delta) = \theta\}$. Then we can find an S-club system $\bar{C}$ such that:

(a) $\lambda \notin \operatorname{id}^a(\bar{C})$

(b) $C_\delta = \{\alpha^\delta_i : i < \kappa \times \gamma^*\}$ increasing, and for each i, $\langle \operatorname{cf}(\alpha^\delta_{i+j+1}) : j < \kappa\rangle$ is increasing with limit μ

(c) if $S \in I[\lambda]$ then $|\{C_\delta \cap \alpha : \delta \in S$ and $\alpha \in \operatorname{nacc}(C'_\delta)\}| < \lambda$.

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References

[EiSh 535] Eisworth, T., Shelah, S.: Further on colouring. *Archive for Mathematical Logic*, **submitted**. math.LO/9808138[7]

[Sh 276] Shelah, S.: Was Sierpiński right? I. *Israel Journal of Mathematics*, **62**, 355–380 (1988)

[Sh 420] Shelah, S.: Advances in Cardinal Arithmetic. In *Finite and Infinite Combinatorics in Sets and Logic*, pages 355–383. Kluwer Academic Publishers, 1993. N.W. Sauer et al (eds.).

[Sh:g] Shelah, S.: *Cardinal Arithmetic*, volume 29 of *Oxford Logic Guides*. Oxford University Press, 1994

[Sh 365] Shelah, S.: There are Jonsson algebras in many inaccessible cardinals. In *Cardinal Arithmetic*, volume 29 of *Oxford Logic Guides*, chapter III. Oxford University Press, 1994

[Sh 430] Shelah, S.: Further cardinal arithmetic. *Israel Journal of Mathematics*, **95**, 61–114 (1996) math.LO/9610226

[Sh 572] Shelah, S.: Colouring and non-productivity of $\aleph_2$-cc. *Annals of Pure and Applied Logic*, **84**, 153–174 (1997) math.LO/9609218

[7] References of the form math.XX/$\cdots$ refer to the xxx.lanl.gov archieve