ISRAEL JOURNAL OF MATHEMATICS **218** (2017), 165–173
DOI: 10.1007/s11856-017-1463-5

CONSISTENTLY $\mathcal{P}(\omega_1)$ IS THE UNION OF LESS THAN $2^{\aleph_1}$ STRONGLY INDEPENDENT FAMILIES

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ABSTRACT

It is consistent that $\mathcal{P}(\omega_1)$ is the union of less than $2^{\aleph_1}$ parts such that if $A_0, \dots, A_{n-1}, B_0, \dots, B_{m-1}$ are distinct elements of the same part, then $|A_0 \cap \dots \cap A_{n-1} \cap (\omega_1 - B_0) \cap \dots \cap (\omega_1 - B_{m-1})| = \aleph_1$.

It is a well known and easily proven statement that $\mathcal{P}(\omega)$ contains a chain of cardinality continuum, i.e., a set $\mathcal{H} \subseteq \mathcal{P}(\omega)$ such that if $A, B \in \mathcal{H}$, then either $A \subseteq B$ or $B \subseteq A$. As it was shown in the seventies, the analogous case for $\mathcal{P}(\omega_1)$ is independent. Baumgartner ([1]) and Shelah ([4]) proved that there is a chain in $\mathcal{P}(\omega_1)$ of size $2^{\aleph_1}$, if either $2^{\aleph_0} = \aleph_1$, or $2^{\aleph_0} = 2^{\aleph_1}$, or $2^{\aleph_0} < \mathrm{cf}(2^{\aleph_1})$ and $2^{\aleph_0} < \aleph_{\omega_1}$. The simplest cases when a chain of size $2^{\aleph_1}$ in $\mathcal{P}(\omega_1)$ may not exist are therefore $2^{\aleph_0} = \aleph_{\omega_1} < 2^{\aleph_1} = \aleph_{\omega_1+1}$, and $2^{\aleph_0} = \aleph_2 < 2^{\aleph_1} = \aleph_{\omega_2}$. In [3], Mitchell proved that either of these are consistent with the nonexistence of a chain as described. In fact, he showed that adding $\aleph_{\omega_1}$ Cohen reals to a model

* Supported by European Research Council Grant 338821. Paper number 1080.
Received November 2, 2015 and in revised form January 31, 2016

of GCH gives a model with $2^{\aleph_0} = \aleph_{\omega_1} < 2^{\aleph_1} = \aleph_{\omega_1+1}$ and with no chain of cardinality $2^{\aleph_1}$ in $\mathcal{P}(\omega_1)$ (see also [1]).

In order to formulate the following results, we introduce the following ad hoc term. A family $\mathcal{H} \subseteq \mathcal{P}(\omega_1)$ of sets is *strongly independent* if the following holds. If $A_0, \dots, A_{k-1}, B_0, \dots, B_{n-1}$ are distinct elements of $\mathcal{H}$, then

$$|(A_0 \cap \cdots \cap A_{k-1}) - (B_0 \cup \cdots \cup B_{n-1})| = \aleph_1.$$

If we only have $(A_0 \cap \cdots \cap A_{k-1}) - (B_0 \cup \cdots \cup B_{n-1}) \neq \emptyset$, we call $\mathcal{H}$ *independent.*

In [6] Shelah extended the above result to showing that in Mitchell's model every family $\mathcal{H} \subseteq \mathcal{P}(\omega_1)$ of cardinality $2^{\aleph_1}$ contains a strongly independent subfamily $\mathcal{H}' \subseteq \mathcal{H}$ of the same cardinality.

Here we give a model in which $\mathcal{P}(\omega_1)$ is the union of less than $2^{\aleph_1}$ strongly independent families (Theorem 2). In particular, $\mathcal{P}(\omega_1)$ is the union of less than $2^{\aleph_1}$ antichains, where $\mathcal{H} \subseteq \mathcal{P}(\omega_1)$ is an antichain, if $A \neq B \in \mathcal{H}$ then $A \not\subseteq B$ and $B \not\subseteq A$. In our model $2^{\aleph_1} = \aleph_{\omega_1+1}$ and $\mathcal{P}(\omega_1)$ is the union of $2^{\aleph_0} = \aleph_{\omega_1}$ strongly independent sets. For a larger gap between the number of sets and $2^{\aleph_1}$ we have the following result: modulo the existence of a supercompact cardinal, for every $\omega_1 \leq \alpha < \omega_2$ it is consistent that $2^{\aleph_1} = \aleph_{\alpha+1}$ and $\mathcal{P}(\omega_1)$ is the union of $2^{\aleph_0} = \aleph_{\omega_1}$ antichains (Theorem 3). An easy argument shows that this does not extend to independent sets: if $\mathcal{P}(\omega_1)$ is the union of μ independent sets, then $\mu^+ \geq 2^{\aleph_1}$ (Theorem 4).

Notation and Definitions. We use the notation and definitions of axiomatic set theory. In particular, ordinals are von Neumann ordinals, and each cardinal is identified with the least ordinal of that cardinality. If f is a function, A a set, then $f[A] = \{f(x) : x \in A\}$; $\mathcal{P}(S)$ denotes the power set of S; $\mathrm{otp}(A)$ denotes the order type of set A. If A, B are sets, then ${}^{B}A$ denotes the set of functions from B to A. Notice that $|{}^{B}A| = |A|^{|B|}$. In this usage, ${}^{\omega_2}2$ is the set of functions from ω_2 to 2, while $2^{\aleph_2}$ is a cardinal and so an ordinal. If S is a set, κ a cardinal, then $[S]^{\kappa} = \{x \subseteq S : |x| = \kappa\}$, and similarly $[S]^{<\kappa} = \{x \subseteq S : |x| < \kappa\}$, $[S]^{\leq\kappa} = \{x \subseteq S : |x| \leq \kappa\}$. If κ is a cardinal, then $\exp_0(\kappa) = \kappa$, and then by recursion $\exp_{r+1}(\kappa) = 2^{\exp_r(\kappa)}$ $(r < \omega)$.

If $(P, \leq)$ is a forcing notion, $p, q \in P$, then $p \leq q$ denotes that p is stronger than q.

If $\kappa \geq \omega$ is regular, $\mu \geq \kappa$, then $P = \mathrm{Add}(\kappa, \mu)$ is the Cohen notion of forcing that adds μ subsets to κ: $p \in \mathrm{Add}(\kappa, \mu)$ iff p is a function with $\mathrm{Dom}(p) \in [\mu]^{<\kappa}$; $\mathrm{Ran}(p) \subseteq \{0, 1\}$; $p \leq q$ iff p extends q.

If $A \subseteq \mu$, then $P|A = \{q \in P : \mathrm{Dom}(q) \subseteq A\}$. It is a well known fact that for each nonempty $A \subseteq \mu$, $P|A$ is a regular subordering of P, and so, if G is generic over P, then $G|A$ is generic over $P|A$.

If $\theta > \aleph_0$ is a regular cardinal, then $\mathcal{H}(\theta)$ denotes the set of those sets with hereditary cardinality less than θ. Notice that $|\mathcal{H}(\kappa^+)| = 2^\kappa$.

Lemma 1: *There is a coloring of* $[2^{\aleph_3}]^{\leq \aleph_1}$ *with* $2^{\aleph_2}$ *colors, such that if* A, B *obtain the same color, then* $\mathrm{otp}(A) = \mathrm{otp}(B)$ *and the elements of* $A \cap B$ *occupy the same positions in* A *and* B, *i.e., if* $x \in A \cap B$, *then* $\mathrm{otp}(A \cap x) = \mathrm{otp}(B \cap x)$.

Proof. Enumerate ${}^{\omega_3}2$ as $\{r_\alpha : \alpha < 2^{\aleph_3}\}$. If $A \in [2^{\aleph_3}]^{\leq \aleph_1}$, then let $A = \{x^A_\xi : \xi < \mathrm{otp}(A)\}$ be its increasing enumeration.

Let $\delta(A) < \omega_3$ be minimal such that

$$r_{x^A_\xi}|\delta(A) \neq r_{x^A_\eta}|\delta(A) \quad (\xi \neq \eta).$$

Define the coloring as

$$\Phi(A) = \left\langle r_{x^A_\xi}|\delta(A) : \xi < \mathrm{otp}(A) \right\rangle.$$

One sees immediately that the range of Φ has cardinality $2^{\aleph_2}$.

In order to show that Φ is as required, assume that $\Phi(A) = \Phi(B)$. Clearly, $\mathrm{otp}(A) = \mathrm{otp}(B)$ and $\delta(A) = \delta(B) = \delta$. It suffices to show that if $\xi \neq \eta$ then $x^A_\xi \neq x^B_\eta$. Assume indirectly that $x^A_\xi = x^B_\eta$ with $\xi \neq \eta$. Then

$$r_{x^A_\xi}|\delta = r_{x^B_\eta}|\delta = r_{x^A_\eta}|\delta \neq r_{x^A_\xi}|\delta$$

a contradiction (the first equality holds by $x^A_\xi = x^B_\eta$, the second by $\Phi(A) = \Phi(B)$, the last inequality follows from the definition of $\delta(A)$). ∎

Theorem 2: *It is consistent that* $\mathcal{P}(\omega_1)$ *is the union of less than* $2^{\aleph_1}$ *strongly independent systems.*

Proof. Set $\mu = \aleph_{\omega_1}$, $\lambda = \mu^+$. Start with a model V which satisfies $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$, $2^{\aleph_2} = \aleph_3$, $2^{\aleph_3} = \lambda$, and $\mu^{\aleph_0} = \mu$. This can be arranged by forcing over a model of GCH, with the poset $\mathrm{Add}(\omega_3, \mu)$.

Fix, for each $\alpha < \lambda$, $\mathrm{cf}(\alpha) = \omega$ a sequence $\{\gamma^\alpha_k : k < \omega\}$ of successor ordinals converging to α.

Define $\chi_0 = \exp_7^+(\lambda)$, $\chi_1 = \exp_9^+(\lambda)$.

We next force with $Q = \mathrm{Add}(\omega, \mu)$. After forcing with Q, we have $2^{\aleph_0} = \mu$ and $2^{\aleph_1} = \lambda$. Let G be a $V - Q$-generic filter.

In $V[G]$, define the equivalence relation $\sim$ on $\mathcal{P}(\omega_1)$ by $A \sim B$ iff $|A \triangle B| \leq \aleph_0$, i.e., the symmetric difference of A and B is countable. Let $\mathcal{A}$ be a system of representatives from the equivalence classes.

As the truth of $|A_0 \cap \cdots \cap A_{n-1} \cap (\omega_1 - B_0) \cap \cdots \cap (\omega_1 - B_{m-1})| = \aleph_1$ will stay unchanged if one (or all) of the sets $A_0, \ldots, A_{n-1}, B_0, \ldots, B_{m-1}$ is replaced by an equivalent one, it suffices to give an appropriate coloring $F : \mathcal{A} \to \mu$.

Let $\underline{\mathcal{A}}$ be a name for $\mathcal{A}$ such that 1 forces it to be a system of representatives modulo $\sim$.

Let $\mathcal{F}$ be a canonical set of names of elements of $\underline{\mathcal{A}}$. That is, $\mathcal{F}$ is a set of names forced to be elements of $\underline{\mathcal{A}}$, if $\tau \neq \tau'$ are in $\mathcal{F}$, then $1 \Vdash \tau \neq \tau'$, and 1 forces that $\{\tau^G : \tau \in \mathcal{F}\} = \mathcal{A}$.

In V, fix $\langle M_\alpha : 1 \leq \alpha < \lambda \rangle$, a continuous, increasing sequence of elementary submodels of $(\mathcal{H}(\chi_1), \in, <_w)$ such that if $\alpha > 0$, then $(\mu+1) \cup \mathcal{P}(\omega_1) \cup \{\mathcal{F}\} \subseteq M_\alpha$, $|M_\alpha| = \mu$, $\{M_\alpha\} \cup [M_{\alpha+1}]^\omega \subseteq M_{\alpha+1}$. Here and later $<_w$ denotes a well ordering of $\mathcal{H}(\chi_1)$ or $\mathcal{H}(\chi_0)$. For technical reasons we put $M_0 = \emptyset$. Notice that if $u' \subseteq u \in M_\alpha$, $|u| = \aleph_1$, then $u' \in M_\alpha$.

Also, if τ is a name for a subset of ω_1, $\tau \in M_\alpha$, $\tau' \in \mathcal{F}$, $p \Vdash \tau \sim \tau'$ for some $p \in Q$, then $\tau' \in M_\alpha$.

We define

$$\mathcal{F}_\alpha = \{\tau \in M_{\alpha+1} : 1 \Vdash \tau \in \underline{\mathcal{A}}, \tau' \in M_\alpha \longrightarrow 1 \Vdash \tau \neq \tau'\}.$$

Claim 1: *If G is Q-generic, then*

$$\{\tau^G \in \mathcal{A} : \tau \in M_{\alpha+1}\} - \{\tau^G \in \mathcal{A} : \tau \in M_\alpha\} = \{\tau^G : \tau \in \mathcal{F}_\alpha\}.$$

Proof. It suffices to notice that if $p \Vdash \tau \in \underline{\mathcal{A}} - M_\alpha$, then we can build a maximal antichain $\{p_i : i < \omega\}$ and names $\{\tau_i : i < \omega\}$ such that $1 \Vdash \tau_i \in \underline{\mathcal{A}} - M_\alpha$, $p_0 = p$, $\tau_0 = \tau$ by borrowing later elements (possible, as $|M_\alpha| < \lambda$). ∎

Enumerate $\mathcal{F}_\alpha$ as $\mathcal{F}_\alpha = \{\tau_{\alpha,\beta} : \beta < \mu\}$.

For every $\alpha < \lambda$, $\beta < \mu$, choose an elementary submodel

$$\omega_1 \cup \{\langle \alpha, \beta \rangle, \tau_{\alpha,\beta}\} \subseteq N_{\alpha,\beta} \prec (\mathcal{H}(\chi_0), \in, <_w)$$

with $|N_{\alpha,\beta}| = \aleph_1$, $N_{\alpha,\beta} \in M_{\alpha+1}$. The latter condition, i.e., that $N_{\alpha,\beta}$ be an element of $M_{\alpha+1}$, can be satisfied as $\mathcal{H}(\chi_0)$ is an element of $\mathcal{H}(\chi_1)$ and so $x \prec (\mathcal{H}(\chi_0), \in, <_w)$ is first order in $(\mathcal{H}(\chi_1), \in, <_w)$.

Set $A_{\alpha,\beta} = N_{\alpha,\beta} \cap \mu$, $Q_{\alpha,\beta} = Q|A_{\alpha,\beta}$.

Claim 2: *If $p \in Q_{\alpha,\beta}$, $\eta < \omega_1$, $u \in M_\alpha$, $u \subseteq A_{\alpha,\beta}$, then there are $\xi \geq \eta$, $q_0, q_1 \in Q_{\alpha,\beta}$, $q_0, q_1 \leq p$, $q_0|u = q_1|u$, such that $q_0 \Vdash \xi \notin \tau_{\alpha,\beta}$ and $q_1 \Vdash \xi \in \tau_{\alpha,\beta}$.*

Proof. If the statement fails, then the following holds. If G is generic with $p \in G$, then $\tau_{\alpha,\beta}|(\omega_1 - \eta)$ can be calculated from $G|u$, that is, there is a canonical name τ over u with $p \Vdash \tau_{\alpha,\beta}|(\omega_1 - \eta) = \tau$. As $u \in M_\alpha$ and $|u| \leq \aleph_1$, we have $\tau \in M_\alpha$. By the above closure properties of M_α, there is $\tau' \in M_\alpha$, $1 \Vdash \tau' \in \underline{\mathcal{A}}$ such that $1 \Vdash \tau \sim \tau'$. Then, $p \Vdash \tau' \sim \tau_{\alpha,\beta}$, which is impossible as $\tau' \in M_\alpha$, $\tau_{\alpha,\beta} \notin M_\alpha$. ∎

If $\alpha < \lambda$, $\beta < \mu$, $\gamma \leq \alpha$, define the following set $D_{\gamma,\alpha\,\beta} \subseteq Q_{\alpha,\beta}$.

$D_{\gamma,\alpha,\beta}$ is the set of all $q \in Q_{\alpha,\beta}$ such that *if*

(a) $n < \omega$, $\langle \varepsilon_i : i < \omega \rangle \in {}^n2$,

(b) $u_{i,j} \subseteq A_{\alpha,\beta}$, $u_{i,j} \in M_\gamma$ $(i \neq j < n)$ such that $u_{i,j} = u_{j,i}$ and $u_{i,j} \cap u_{j,k} \subseteq u_{i,k}$,

(c) $q_i \leq q$, $q_i \in Q_{\alpha,\beta}$ $(i < n)$,

(d) $q_i|u_{i,j} = q_j|u_{i,j}$,

(e) $\eta < \omega_1$,

then there exist $\eta \leq \xi < \omega_1$ and $r_i \in Q_{\alpha,\beta}$ $(i < n)$ $r_i \leq q_i$ such that

(f) $r_i|u_{i,j} = r_j|u_{i,j}$,

(g) $r_i \Vdash \xi \in \tau_{\alpha,\beta} \longleftrightarrow \varepsilon_i = 1$.

Claim 3: *If $\mathrm{cf}(\gamma) \neq \omega$, i.e., γ is either successor or $\mathrm{cf}(\gamma) > \omega$, then $D_{\gamma,\alpha,\beta}$ is dense in $Q_{\alpha,\beta}$.*

Proof. Otherwise, there exists a $p \in Q_{\alpha,\beta}$ such that for each $q \leq p$, $q \in Q_{\alpha,\beta}$, there exist $n < \omega$, $\langle \varepsilon_i : i < n \rangle \in {}^n2$, $\langle u_{i,j} : i \neq j < n \rangle$, $\eta < \omega_1$, $q_i \leq q$ as in (a–e) of the definition above such that there are no $\eta \leq \xi < \omega_1$, $r_i \leq q_i$ satisfying (f–g).

Pick a countable elementary submodel M^* of $\langle \mathcal{H}(\chi_1^+), \in, <_w \rangle$ such that $p \in M^*$. Set $u^* = \bigcup (M^* \cap M_\gamma)$, $A^* = M^* \cap A_{\alpha,\beta}$. Notice that $A^* \subseteq u^*$, $|A^*| = \omega$ but u^* may be uncountable. As $\mathrm{cf}(\gamma) \neq \omega$, $u^*, A^* \in M_\gamma$.

Apply Claim 2 to p, u^*, $\eta = \omega_1 \cap M^* < \omega_1$ and obtain $\xi \geq \eta$, $q^+, q^- \leq p$ such that $q^+|u^* = q^-|u^*$, $q^- \Vdash \xi \notin \tau_{\alpha,\beta}$, $q^+ \Vdash \xi \in \tau_{\alpha,\beta}$. Set $q = q^+|A^*$. As q is finite, $q \in M^*$. By the construction of M^*, there are $n < \omega$, $\langle \varepsilon_i : i < n\rangle \in {}^n2$, $\langle u_{i,j} : i \neq j\rangle$, $q_i \leq q$ as described in the definition of $D_{\gamma,\alpha,\beta}$. Notice that $q_i \in M^*$ and so $\mathrm{Dom}(q_i) \subseteq A^*$.

Define

$$r_i = q_i \cup \begin{cases} q^- & (\varepsilon_i = 0), \\ q^+ & (\varepsilon_i = 1). \end{cases}$$

Observe that r_i is a condition as $q_i \leq q = q^\pm | A^*$ and $\mathrm{Dom}(q_i) \subseteq A^*$. Further, if $i \neq j$, then $r_i | u_{i,j} = r_j | u_{i,j}$ as r_i, r_j only differ outside u^*.

As $r_i \leq q^- \Vdash \xi \notin \tau_{\alpha,\beta}$ $(\varepsilon_i = 0)$ and $r_i \leq q^+ \Vdash \xi \in \tau_{\alpha,\beta}$ $(\varepsilon_i = 1)$ we have reached a contradiction. ∎

Fix, in V, an isomorphism $\pi_{\alpha,\beta} : N_{\alpha,\beta} \to \overline{N}_{\alpha,\beta}$ to some structure $\overline{N}_{\alpha,\beta}$ for $\alpha < \lambda$, $\beta < \mu$, such that if $N_{\alpha,\beta}$, $N_{\alpha',\beta'}$ are isomorphic, then $\overline{N}_{\alpha,\beta} = \overline{N}_{\alpha',\beta'}$.

We define, in $V[G]$, the following coloring of $\mathcal{A}$.

$F(\tau^G_{\alpha,\beta}) = \langle F_i(\tau^G_{\alpha,\beta}) : i < 5\rangle$ where

$F_0(\tau^G_{\alpha,\beta}) = \beta$,

$F_1(\tau^G_{\alpha,\beta}) = \mathrm{tp}(N_{\alpha,\beta}, \in, A_{\alpha,\beta}, \langle \alpha, \beta\rangle, \tau_{\alpha,\beta})$,

$F_2(\tau^G_{\alpha,\beta}) = \Phi(A_{\alpha,\beta})$,

if $\mathrm{cf}(\alpha) \neq \aleph_0$, then $F_3(\tau^G_{\alpha,\beta}) = \{\pi_{\alpha,\beta}[X] : X \subseteq A_{\alpha,\beta}, X \in M_\alpha\}$,

if $\mathrm{cf}(\alpha) = \aleph_0$, then $F_3(\tau^G_{\alpha,\beta}) = \big\langle \{\pi_{\alpha,\beta}[X] : X \subseteq A_{\alpha,\beta}, X \in M_{\gamma^\alpha_k}\} : k < \omega\big\rangle$,

if $\mathrm{cf}(\alpha) \neq \aleph_0$, then $F_4(\tau^G_{\alpha,\beta})$ is $\pi_{\alpha,\beta}(p)$ for some $p \in G \cap D_{\alpha,\alpha,\beta}$,

if $\mathrm{cf}(\alpha) = \aleph_0$, then $F_4(\tau^G_{\alpha,\beta})$ is $\langle \pi_{\alpha,\beta}(p_k) : k < \omega\rangle$ where $p_k \in G \cap D_{\gamma^\alpha_k,\alpha,\beta}$.

Here $\mathrm{tp}(M)$ denotes the isomorphism type of structure M.

Notice that it is possible that a subset gets more than one color, in which case an arbitrary choice is made for the value of F.

The range of F_0 is μ. The ranges of F_1 and F_2 have cardinality $2^{\aleph_1} = \aleph_2$ each. The range of F_3 has cardinality $2^{2^{\aleph_1}} = \aleph_3$ and the range of F_4 has cardinality $\mu^{\aleph_0}$ as calculated in $V[G]$, which is μ. Consequently, F is a coloring with μ colors, that is, it gives a decomposition of $\mathcal{A}$ into μ parts.

In order to conclude the proof of the Theorem, assume that $2 \leq n < \omega$, $\langle \varepsilon_0, \dots \varepsilon_{n-1}\rangle \in {}^n2$, $\eta < \omega_1$, $\langle \alpha_0, \beta_0\rangle, \dots, \langle \alpha_{n-1}, \beta_{n-1}\rangle$ are distinct, $p \in G$ forces that $F(\tau_{\alpha_0,\beta_0}) = \cdots = F(\tau_{\alpha_{n-1},\beta_{n-1}})$, and $\bigcap_{i<n} \tau^{\varepsilon_i}_{\alpha_i,\beta_i} \subseteq \eta$. By the definition of F_0 above, $\beta_0 = \cdots = \beta_{n-1}$; let the common value be denoted by β. Necessarily $\alpha_0, \dots, \alpha_{n-1}$ are different and so we may assume without loss of generality that $\alpha_0 < \cdots < \alpha_{n-1}$. Let $\pi_i : N_{\alpha_i,\beta} \to N_{\alpha_{n-1},\beta}$ be the isomorphism guaranteed by the common value of F_1. By the definition of F_4, either $\mathrm{cf}(\alpha_i) \neq \omega$ holds for all $i < n$ or $\mathrm{cf}(\alpha_i) = \omega$ holds for all $i < n$. In the former case let $\gamma_i = \alpha_i$, in the latter case pick $k < \omega$ so large that $\alpha_{i-1} < \gamma_k^{\alpha_i}$ holds for $1 \le i < n$ and set $\gamma_i = \gamma_k^{\alpha_i}$.

Define $u_{i,j} = u_{j,i} = \pi_i[A_{\alpha_i,\beta} \cap A_{\alpha_j,\beta}]$ $(i < j)$.

Claim 4: $u_{i,j} \in M_{\gamma_{n-1}}$ $(i < j < n)$.

Proof. Set $A = A_{\alpha_i,\beta} \cap A_{\alpha_j,\beta}$. As $A \subseteq A_{\alpha_i,\beta} \in M_{\alpha_i+1}$, we have $A \in M_{\alpha_i+1} \subseteq M_{\gamma_j}$ as M_{α_i+1} is closed under subsets of elements of size $\aleph_1$ and by the choice of γ_j.

By $F_2(\tau^G_{\alpha_i,\beta}) = F_2(\tau^G_{\alpha_j,\beta})$ we have that the common elements of $A_{\alpha_i,\beta}$ and $A_{\alpha_j,\beta}$ occupy the same positions and so $\pi_i[A] = \pi_j[A]$.

As $A \in M_{\gamma_j}$, by $F_3(\tau^G_{\alpha_i,\beta}) = F_3(\tau^G_{\alpha_j,\beta})$ we have $u_{i,j} = \pi_j[A] \in M_{\gamma_{n-1}}$. ∎

By the definition of F_4 there are $p_i \in G \cap D_{\gamma_i,\alpha_i,\beta}$ such that $\pi_i(p_i) = p_{n-1}$, i.e., they are mapped onto each other by the isomorphisms between the $N_{\alpha_i,\beta}$'s.

As $p, p_i \in G$, p and p_i are compatible.

Set

$$q_i = \pi_i\left((p|A_{\alpha_i,\beta}) \cup p_i\right).$$

Claim 5: $q_i|u_{i,j} = q_j|u_{i,j}$ $(i < j < n)$.

Proof. Define $A = A_{\alpha_i,\beta} \cap A_{\alpha_j,\beta}$. First,

$$\pi_i\left(p|A_{\alpha_i,\beta}\right)|u_{i,j} = \pi_i\left(p|A\right) = \pi_j\left(p|A\right) = \pi_j\left(p|A_{\alpha_j,\beta}\right)|u_{i,j}$$

as π_i, π_j agree on A, by F_2.

Next, $\pi_i(p_i) = \pi_j(p_j)$, and so $\pi_i(p_i|A) = \pi_j(p_j|A)$ by F_2. ∎

We can therefore use the definition of $D_{\gamma_{n-1},\alpha_{n-1},\beta}$ and obtain $\eta \le \xi < \omega_1$ and $r_i \in Q_{\alpha_{n-1},\beta}$, $r_i \le q_i$ $(i < n)$, $r_i|u_{i,j} = r_j|u_{i,j}$, such that $r_i \Vdash \xi \in \tau_{\alpha_n,\beta} \leftrightarrow \varepsilon_i = 1$. As $\pi_i^{-1}(r_i) \le p|A_{\alpha_i,\beta}$ and $\pi_i^{-1}(r_i)$, $\pi_j^{-1}(r_j)$ agree on $A_{\alpha_i,\beta} \cap A_{\alpha_j,\beta}$, $r = p \cup \bigcup \pi_i^{-1}(r_i)$ is a condition and

$$r \Vdash \xi \in \bigcap_{i<n} \tau^{\varepsilon_i}_{\alpha_i,\beta}$$

contradicting $r \le p$. ∎

Theorem 3: *If the existence of a supercompact cardinal is consistent and $\omega_1 \le \nu < \omega_2$, then it is consistent that $2^{\aleph_1} = \aleph_{\nu+1}$ and $\mathcal{P}(\omega_1)$ is the union of $2^{\aleph_0} = \aleph_{\omega_1}$ antichains.*

Proof. Set $\mu = \aleph_{\omega_1}$. Start with a model V in which $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$, $2^{\aleph_2} = \aleph_3$, $2^{\aleph_3} = \mu$, $\mu^{\aleph_0} = \mu$, $\mu^{\aleph_1} = \aleph_{\nu+1}$. This is possible, from a model in which μ is strong limit and $\mu^{\aleph_1} = \aleph_{\nu+1}$ (obtained from a supercompact in Shelah [5]).

We force with $Q = \mathrm{Add}(\omega, \mu)$. Let G be a $V - Q$-generic filter. We prove that in $V[G]$, if $\mathcal{A} \subseteq \mathcal{P}(\omega_1)$, in which $A \neq B$ implies $|A \triangle B| = \aleph_1$, then $\mathcal{A}$ is the union of μ antichains. This suffices, as each family is the union of μ such subfamilies.

We prove this by induction on $\tau = |\mathcal{A}|$. Assume that $|\mathcal{A}| = \tau$ and we know the statement for systems of size less than τ. Fix a name $\underline{\mathcal{A}}$ for $\mathcal{A}$. We construct, in V, the sequence $\langle M_\alpha : \alpha < \tau \rangle$ of models, as in the proof of Theorem 2. Define $\mathcal{F}_\alpha$ similarly, and set $\mathcal{F}_\alpha = \{\tau_{\alpha,\beta} : \beta < |M_{\alpha+1}|\}$. Define $\mathcal{B}_\alpha = \{\tau^G_{\alpha,\beta} : \beta < |M_{\alpha+1}|\}$. Then $\mathcal{A} = \bigcup\{\mathcal{B}_\alpha : \alpha < \tau\}$ is a disjoint decomposition. Define $F^0(\tau^G_{\alpha,\beta}) = \langle F_i(\tau^G_{\alpha,\beta}) : 1 \leq i \leq 4 \rangle$ (we cannot use F_0 as its range is not bounded by μ). As in Theorem 2, we see that the range of F^0 has cardinality μ. Let $F^1 : \mathcal{A} \to \mu$ be such that if $A \neq B$ are in the same $\mathcal{B}_\alpha$, $A \subseteq B$ or $B \subseteq A$, then $F^1(A) \neq F^1(B)$. This is possible by the inductive hypothesis. Then $F(A) = \langle F^0(A), F^1(A) \rangle$ is as required. ∎

Theorem 4: *If $\mathcal{P}(\omega_1)$ is the union of μ independent sets, then $\mu^+ \geq 2^{\aleph_1}$.*

Proof. Assume that $F : \mathcal{P}(\omega_1) \to \mu$ and $2^{\aleph_1} \geq \mu^{++}$. By the latter condition we can choose distinct nonempty elements of $\mathcal{P}(\omega_1)$, $\{a_\alpha : \alpha < \mu^+\}$ and $\{b_\beta : \beta < \mu^{++}\}$ such that $a_\alpha \cap b_\beta = \emptyset$ ($\alpha < \mu^+$, $\beta < \mu^{++}$). Set $f(\alpha, \beta) = F(a_\alpha \cup b_\beta)$. Then $f : \mu^+ \times \mu^{++} \to \mu$, and thus by a lemma of Elekes, Erdős, and Hajnal ([2]), there exist $\alpha_0 < \alpha_1$, $\beta_0 < \beta_1$, such that $f(\alpha_0, \beta_0) = f(\alpha_0, \beta_1) = f(\alpha_1, \beta_0) = f(\alpha_1, \beta_1)$. But then the sets $A_{\alpha_0} \cup B_{\beta_0}$, $A_{\alpha_0} \cup B_{\beta_1}$, $A_{\alpha_1} \cup B_{\beta_0}$, $A_{\alpha_1} \cup B_{\beta_1}$ get the same color, and

$$(A_{\alpha_0} \cup B_{\beta_0}) \cup (A_{\alpha_1} \cup B_{\beta_1}) = (A_{\alpha_0} \cup B_{\beta_1}) \cup (A_{\alpha_1} \cup B_{\beta_0}). \qquad \blacksquare$$

The argument in the proof of Theorem 2 gives the following general form.

Theorem 5: *If κ is regular, $\mu = \kappa^{+\kappa^+}$, $\lambda = \mu^+$, then it is consistent that $\mathcal{P}(\kappa^+)$ is the union of $\mu < 2^{\kappa^+} = \lambda$ classes with each class independent in the sense that if the distinct sets $A_0, \ldots, A_{n-1}$ ($2 \leq n < \omega$) are in the same class, $\varepsilon_0, \ldots, \varepsilon_{n-1} \in \{0, 1\}$, then $|A_0^{\varepsilon_0} \cap \cdots \cap A_{n-1}^{\varepsilon_{n-1}}| = \kappa$ where $A_i^1 = A_i$, $A_i^0 = \kappa - A_i$. Specifically, the sets are antichains.* ∎

Acknowledgment. The authors are thankful for the referee's suggestions which greatly improved the exposition.

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