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# Localizations of infinite subsets of $\omega$

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## 0 Introduction

### *0.1 Preliminaries*

When we say *reals* we mean one of the nicely defined Polish spaces or their (finite or countable) products like: the real line $\mathbb{R}$, the Cantor space $2^\omega$, the Baire space $\omega^\omega$ or the space of infinite sets of integers $[\omega]^\omega$.
In the present paper we are interested in properties of forcing notions (or, generally, extensions of models of ZFC) which measure in a sense the distance between the ground model reals and the reals in the extension. In particular we look at the ways the "new" reals can be localized (or: aproximated) by "old" reals. There are two extreme cases here: *there are no new reals* and *the old reals are countable*. However, between these two extremes we have a wide spectrum of properties among which the localizations by slaloms seem to be the most popular. A systematic study of slaloms and related localization properties and cardinal invariants was presented in [Bar1].

*A slalom* is a function $S : \omega \longrightarrow [\omega]^{<\omega}$ such that $(\forall n \in \omega)(|S(n)| = n + 1)$. We say that a slalom $S$ *localizes* a function $f \in \omega^\omega$ whenever $(\forall n \in \omega)(f(n) \in S(n))$. In this situation we can think that the slalom $S$ is an approximation of the function $f$. It does not determine the function but it provides some bounds on possible values of $f$. Bartoszyński, Cichoń, Kamburelis et al. studied the localizations by slaloms and those investigations gave the following surprising result.

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**Theorem 0.1 (Bartoszyński, [Bar2])** *Suppose that* $\mathbf{V} \subseteq \mathbf{V}'$ *are models of ZFC. Then the following conditions are equivalent:*

1. *Any function from* $\omega^\omega \cap \mathbf{V}'$ *can be localized by a slalom from* $\mathbf{V}$.
2. *Any Borel (Lebesgue) null set coded in* $\mathbf{V}'$ *can be covered by a Borel null set coded in* $\mathbf{V}$.

On localizations by slaloms see Chapter VI of [Sh:b] too; other localizations of slalom-like type appeared in [GoSh:448].
A stronger localization property was considered in [NeRo]. Fix a natural number $k \geq 2$. By a *$k$-tree* on $\omega$ we mean a tree $T \subseteq \omega^{<\omega}$ such that each finite node in $T$ has at most $k$ immediate successors in $T$. We say that a $k$-tree $T$ *localizes* a function $f \in \omega^\omega$ whenever $f$ is a branch through $T$ (i.e. $(\forall n \in \omega)(f{\restriction}n \in T)$). Clearly, if any function from $\omega^\omega \cap \mathbf{V}'$ can be localized by a $k$-tree from $\mathbf{V}$, $k \geq 2$, $\mathbf{V} \subseteq \mathbf{V}'$ then each function from $\omega^\omega \cap \mathbf{V}'$ can be localized by a slalom from $\mathbf{V}$. Moreover the localization by a $k$-tree implies the localization by a $k+1$-tree (but not conversely).
In the definition below we formulate general localization properties for Polish spaces $X, Y$. In practice, however, these spaces will be various examples of reals only.

**Definition 0.2** Assume that $X, Y$ are Polish spaces and $R \subseteq X \times Y$ is a Borel relation. Suppose that $\mathbf{V} \subseteq \mathbf{V}'$ are models of ZFC and that all parameters we need are in $\mathbf{V}$. We say that the pair $(\mathbf{V}, \mathbf{V}')$ has *the property of the $R$-localization* if

$$(\forall x \in X \cap \mathbf{V}')(\exists y \in Y \cap \mathbf{V})((x, y) \in R)$$

If $x \in X \cap \mathbf{V}'$, $y \in Y \cap \mathbf{V}$ and $(x, y) \in R$ they we say that $y$ *$R$-localizes* $x$.

In the examples we gave earlier $X$ was $\omega^\omega$ and $Y$ was the space of slaloms or the space of all $k$-trees, respectively. The respective relations should be obvious. Those localizations were to approximate functions in an extension by objects from the ground model. They are not useful if we consider infinite subsets of $\omega$. Though each member of $[\omega]^\omega$ can be identified with its increasing enumeration, the localization (either by slaloms or by $k$-trees) of the enumeration does not provide satisfactory information on successive points of the set. The localization gives us "candidates" for the $n$-th point of the set but the same candidates can appear several times for distinct $n$. That led to a suggestion that we should consider disjoint subsets of $\omega$ as sets of "candidates" for successive points of the localized set (the approach was suggested by B. Węglorz). Now we have two possibilities. Either we can demand that each set from the localization contains a limited number of members of the localized set or we can postulate that each intersection of that kind is large. Localizations of this kind are studied in Sect. 1.
In the second section we investigate localizations of infinite subsets of $\omega$ by sets of integers from the ground model. These localizations might be thought as localizations by partitions of $\omega$ into successive intervals. A starting point for our considerations was the following observation.

**Proposition 0.3** *Suppose that* $\mathbf{V} \subseteq \mathbf{V}'$ *are (transitive) models of ZFC. Then:*

1. $\mathbf{V} \cap \omega^\omega$ *is unbounded in* $\mathbf{V}' \cap \omega^\omega$ *if and only if* *for every set* $X \in [\omega]^\omega \cap \mathbf{V}'$ *there exists a set* $Y \in [\omega]^\omega \cap \mathbf{V}$ *such that* **infinitely often** *between two successive points of* $Y$ *there are at least 2 points of* $X$.
2. $\mathbf{V} \cap \omega^\omega$ *is dominating in* $\mathbf{V}' \cap \omega^\omega$ *if and only if* *for every set* $X \in [\omega]^\omega \cap \mathbf{V}'$ *there exists a set* $Y \in [\omega]^\omega \cap \mathbf{V}$ *such that* **for all but finitely many** *pairs of two successive points of* $Y$ *there are at least 2 points of* $X$ *between them.*

Now we try to replace the quantifier *for infinitely many* above by stronger quantifiers (but still weaker than *for all but finitely many*), like *for infinitely many* $n$, *for both* $n$ *and* $n+1$. Finally, in Sect. 3 we formulate several corollaries to the results of previous sections for cardinal invariants related to the notions we study.

### *0.2 Notation*

Our notation is rather standard and essentially compatible with that of [Jec] and [Kun]. In forcing considerations, however, we will use the convention that *a stronger condition is the greater one*.

**Basic Definitions 0.4**

1. A tree on $\omega$ is a set $T \subseteq \omega^{<\omega}$ closed under initial segments. For the tree $T$ the body $[T]$ of $T$ is the set
$$\{x \in \omega^\omega : (\forall l \in \omega)(x \restriction l \in T)\}.$$
If $t \in T$ then $\mathrm{succ}_T(t) = \{s \in T : t \subseteq s \ \&\ \mathrm{lh}(t) + 1 = \mathrm{lh}(s)\}$.
2. By a model of ZFC we will mean *a transitive model* of (enough of) ZFC. Models of ZFC will be denoted by $\mathbf{V}$, $\mathbf{V}'$ etc.
3. We will be interested in extensions of models, i.e. in pairs $(\mathbf{V}, \mathbf{V}')$ of models such that $\mathbf{V} \subseteq \mathbf{V}'$. If a property of an extension is defined then we extend this definition to notions of forcing. We say that a notion of forcing $\mathbb{P}$ has the property whenever for any generic filter $G \subseteq \mathbb{P}$ over $\mathbf{V}$ the extension $\mathbf{V} \subseteq \mathbf{V}[G]$ has the considered property.
4. We will use the quantifiers $(\forall^\infty n)$ and $(\exists^\infty n)$ as abbreviations for
$$(\exists m \in \omega)(\forall n > m) \quad \text{and} \quad (\forall m \in \omega)(\exists n > m),$$
respectively.
5. The Baire space $\omega^\omega$ of all functions from $\omega$ to $\omega$ is endowed with the partial order $\leq^*$:
$$f \leq^* g \iff (\forall^\infty n)(f(n) \leq g(n)).$$
A family $F \subseteq \omega^\omega$ is unbounded in $(\omega^\omega, \leq^*)$ if
$$\neg(\exists g \in \omega^\omega)(\forall f \in F)(f \leq^* g)$$
and it is dominating in $(\omega^\omega, \leq^*)$ if
$$(\forall g \in \omega^\omega)(\exists f \in F)(g \leq^* f).$$

6. The unbounded number $\mathfrak{b}$ is the minimal size of an unbounded family in $(\omega^\omega, \leq^*)$; the dominating number $\mathfrak{d}$ is the minimal size of a dominating family in that order.
7. The size of the continuum is denoted by $\mathfrak{c}$, $[\omega]^*$ stands for the family of infinite co-infinite subsets of $\omega$.

*0.3 Acknowledgement*

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## 1 $R_k^\forall$, $R_k^\exists$ - localizations

In this section we show that a localization of infinite subsets of $\omega$ suggested by B. Węglorz implies that the considered extension adds no new real.

**Definition 1.1** 1. A partition of $\omega$ into finite sets is a sequence $\langle K_n : n \in \omega\rangle$ of disjoint finite sets such that $\bigcup_{n\in\omega} K_n = \omega$.

2. $\mathscr{P}_k$ is the set of all partitions $\langle K_n : n \in \omega\rangle$ of $\omega$ into finite sets such that $(\forall n\in\omega)(|K_n| > k)$.
   *[Note that $\mathscr{P}_k$ is a $\Pi^0_2$-subset of $([\omega]^{<\omega})^\omega$ so it is a Polish space.]*
3. We define relations $R_k^\forall \subseteq [\omega]^* \times \mathscr{P}_k$ and $R_k^\exists \subseteq [\omega]^* \times \mathscr{P}_{k+1}$ by

$$(X, \langle K_n : n\in\omega\rangle) \in R_k^\forall \ \equiv\ (\forall^\infty n\in\omega)(|X\cap K_n| \leq k)$$

$$(X, \langle K_n : n\in\omega\rangle) \in R_k^\exists \ \equiv\ (\exists^\infty n\in\omega)(|X\cap K_n| \leq k).$$

   Their complements (in $[\omega]^* \times \mathscr{P}_k$, $[\omega]^* \times \mathscr{P}_{k+1}$) are denoted by $cR_k^\forall$, $cR_k^\exists$, respectively.

If we want to approximate an infinite co-infinite subset of $\omega$ by an object in a given model we can look for a separation of distinct members of the set by a sequence of sets from the model. Thus we could ask if it is possible to find a partition of $\omega$ (in $\mathbf{V}$) such that the localized set is *a partial selector* of the partition. More generally we may ask for $R_k^\forall$–localization; recall definition 0.2. Thus the $R_k^\forall$-localization property means that for every infinite set of integers $X$ from the extension there exists a partition $\langle K_n : n \in \omega\rangle \in \mathscr{P}_k$ from the ground model such that for almost all $n\in\omega$ the intersection $X\cap K_n$ is of size at most $k$. The following result shows that the $R_k^\forall$-localization fails if we add new reals.

**Theorem 1.2** *Suppose that $\mathbf{V} \subseteq \mathbf{V}'$ are models of ZFC such that $\mathbf{V}\cap 2^\omega \neq \mathbf{V}'\cap 2^\omega$. Then there is a set $X \in [\omega]^* \cap \mathbf{V}'$ such that for* **no** *$k\in\omega$ there is a partition $\langle K_n : n\in\omega\rangle \in \mathbf{V}$ of $\omega$ such that*

$$(\forall^\infty n)(|X\cap K_n| \leq k \ \&\ |K_n| > k).$$

*Consequently the extension $\mathbf{V} \subseteq \mathbf{V}'$ does not have the $R_k^\forall$-localization property (for any $k$).*

*Proof.* Let $x \in 2^\omega$ be a new real (i.e. $x \in \mathbf{V}' \setminus \mathbf{V}$) and let

$$X = 2^{<\omega} \setminus \{x{\restriction}i : i \in \omega\}.$$

As we can identify $\omega$ with $2^{<\omega}$ we may think that $X \in [\omega]^*$. Now the following claim works.

**Claim 1.2.1** *Suppose that $x \in 2^\omega$, $x \notin \mathbf{V}$. Let $\langle K_n : n \in \omega\rangle \in \mathbf{V}$ be a sequence of disjoint subsets of $2^{<\omega}$ such that $(\forall n \in \omega)(|K_n| > k)$, $k \in \omega$. Then for some $n \in \omega$ the set $K_n \setminus \{x{\restriction}i : i \in \omega\}$ has at least $k+1$ points.*

*Proof of Claim.* Suppose not. Thus for each $n \in \omega$ we have

$$|K_n \setminus \{x{\restriction}i : i \in \omega\}| \le k \quad \text{and thus} \quad K_n \cap \{x{\restriction}i : i \in \omega\} \ne \emptyset.$$

First note that each $K_n$ is finite. If not then the tree

$$\{s \in 2^{<\omega} : (\exists t \in K_n)(s \subseteq t)\}$$

has exactly one infinite branch - the branch is $x$. As the tree is in $\mathbf{V}$ we would get $x \in \mathbf{V}$.
Let

$$u(n) = \max\{\mathrm{lh}(s) : s \in K_n\} \quad \text{and} \quad d(n) = \min\{\mathrm{lh}(s) : s \in K_n\}$$

(remember that each $K_n$ is finite). Choose an increasing sequence $\langle n_\ell : \ell<\omega\rangle$ (in $\mathbf{V}$) such that

$$(\forall \ell \in \omega)(u(n_\ell) < d(n_{\ell+1}))$$

(possible as the $K_n$'s are disjoint). Let

$$F_\ell = \{s{\restriction}d(n_\ell) : s \in K_{n_\ell}\}.$$

Note that $x{\restriction}d(n_\ell) \in F_\ell$ as an initial segment of $x$ belongs to $K_{n_\ell}$. Moreover $|F_\ell| \le k+1$ as only one element of $F_\ell$ may be an initial segment of $x$ and above each member of $F_\ell$ there is an element of $K_{n_\ell}$ (remember $|K_{n_\ell} \setminus \{x{\restriction}i : i \in \omega\}| \le k$). Clearly $\langle F_\ell : \ell \in \omega\rangle \in \mathbf{V}$. Consider the set

$$A = \{y \in 2^\omega : (\forall \ell \in \omega)(y{\restriction}d(n_\ell) \in F_\ell)\}.$$

It is a finite set from $\mathbf{V}$. But $x \in A$ – a contradiction. $\square\square$

Thus the $R^\forall_k$-localization is the trivial one. The complementary $cR^\forall_k$-localization is not of special interest either. Every extension $\mathbf{V} \subseteq \mathbf{V}'$ has the $cR^\forall_0$-localization. The description of the $cR^\forall_k$-localization for $k > 0$ is given by the following observation.

**Proposition 1.3** *Let $\mathbf{V} \subseteq \mathbf{V}'$ be an extension of models of ZFC. Then the following conditions are equivalent:*

1. *For each $k > 0$ the pair $(\mathbf{V}, \mathbf{V}')$ has the $cR^\forall_k$-localization.*
2. *For some $k > 0$ the pair $(\mathbf{V}, \mathbf{V}')$ has the $cR^\forall_k$-localization.*

3. $\mathbf{V}\cap\omega^\omega$ *is unbounded in* $\mathbf{V}'\cap\omega^\omega$.

*Proof.* $2 \Rightarrow 3$ Let an increasing function $f \in \omega^\omega \cap \mathbf{V}'$ be given. Take an increasing function $f' \in \mathbf{V}'\cap\omega^\omega$ such that

$$(\forall n \in \omega)(f(f'(n)) + 1 < f'(n+1)).$$

Consider the set $\mathrm{rng}(f') \in [\omega]^*$. Let $\langle K_n : n \in \omega\rangle \in \mathbf{V}$ be the partition of $\omega$ given for this set by the $cR_k^\forall$-localization. Let $k_n = \min K_n$ and put $g(k_n) = 1 + \max K_n$. Extend $g$ to $\omega$ putting $g(m) = 0$ if $m \notin \{k_n : n \in \omega\}$. Clearly $g \in \mathbf{V}$.
Note that $|K_n \cap \mathrm{rng}(f')| > k > 0$ implies that $f(k_n) < g(k_n)$. Consequently

$$(\exists^\infty m \in \omega)(f(m) < g(m)).$$

$3 \Rightarrow 1$ Given $k > 0$. Let $X \in [\omega]^\omega \cap \mathbf{V}'$. Define

$$f(n) = \min\{m > n : |X \cap [n,m)| > 2k\} \quad \text{for } n \in \omega.$$

Since $\mathbf{V}\cap\omega^\omega$ is unbounded in $\mathbf{V}'\cap\omega^\omega$ we find an increasing function $g \in \mathbf{V}\cap\omega^\omega$ such that $(\exists^\infty n \in \omega)(f(n) < g(n))$. Let $k_n \in \omega$ be defined by:

$$k_0 = 0, \quad k_{n+1} = k + 1 + k_n + g(k_n).$$

Put $K_n = [k_n, k_{n+1})$. Clearly $\langle K_n : n \in \omega\rangle \in \mathscr{R}_k \cap \mathbf{V}$. Now suppose that $m \in K_n$ is such that $f(m) < g(m)$. As $g$ is increasing we have $g(m) < g(k_{n+1}) < k_{n+2}$. Consequently $|[m, k_{n+2}) \cap X| > 2k$ and hence either $|K_n \cap X| > k$ or $|K_{n+1} \cap X| > k$. Hence $(\exists^\infty n \in \omega)(|K_n \cap X| > k)$. □

In a similar way one can prove the analogous result for the $cR_k^\exists$-localization.

**Proposition 1.4** *Let* $\mathbf{V} \subseteq \mathbf{V}'$ *be models of ZFC. Then the following conditions are equivalent:*

1. *For each* $k \in \omega$ *the pair* $(\mathbf{V}, \mathbf{V}')$ *has the* $cR_k^\exists$*-localization.*
2. *For some* $k \in \omega$ *the pair* $(\mathbf{V}, \mathbf{V}')$ *has the* $cR_k^\exists$*-localization.*
3. $\mathbf{V}\cap\omega^\omega$ *is a dominating family in* $\mathbf{V}'\cap\omega^\omega$. □

For the $R_k^\exists$-localization we did not find a full description. First note that the requirement that members of the partition have to have at least $k+2$ elements (i.e. $R_k^\exists \subseteq [\omega]^* \times \mathscr{R}_{k+1}$) is to avoid a trivial localization. If we divide $\omega$ into $k+1$-element intervals then for each set $X \in [\omega]^*$ infinitely many intervals contain at most $k$ members of $X$.

**Definition 1.5** Let $\mathbf{V} \subseteq \mathbf{V}'$ be models of ZFC and let $k < l < \omega$. A set $X \in [\omega]^* \cap \mathbf{V}'$ is called $(l,k)$*-large (over* $\mathbf{V}$*)* if for every sequence $\langle K_n : n \in \omega\rangle \in \mathbf{V}$ of disjoint $l$-element subsets of $\omega$ we have: $(\forall^\infty n \in \omega)(|K_n \cap X| > k)$.
[Note that we do not require that $\langle K_n : n \in \omega\rangle$ is a partition of $\omega$.]

**Theorem 1.6** *All* $R_k^\exists$*-localizations (for* $k \in \omega$*) are equivalent, i.e. if an extension* $\mathbf{V} \subseteq \mathbf{V}'$ *has the* $R_k^\exists$*-localization property for some* $k \in \omega$ *then it has the* $R_k^\exists$*-localization for each* $k \in \omega$.

*Proof.* Let $\mathbf{V} \subseteq \mathbf{V}'$ be models of ZFC. Let $k < l < \omega$.

**Claim 1.6.1** *If $X \in [\omega]^* \cap \mathbf{V}'$ is $(l,k)$-large, $k+1 < l < \omega$ and $m \geq 1$ then $X$ is $(lm, lm-(l-k))$-large.*

*Proof of Claim.* Let $\langle K_n : n \in \omega \rangle \in \mathbf{V}$ be a sequence of disjoint subsets of $\omega$ of the size $lm$. Let $K_n = \{a_{n,i} : i < lm\}$ be the increasing enumeration and let $K_n^A = \{a_{n,i} : i \in A\}$ for $A \subseteq lm$. Fix a set $A \in [lm]^l$ and consider the sequence $\langle K_n^A : n \in \omega \rangle \subseteq [\omega]^l$. This sequence is in $\mathbf{V}$ and its members are disjoint. Thus we find $N(A)$ such that $(\forall n \geq N(A))(\mid K_n^A \cap X \mid > k)$. Let $N = \max\{N(A) : A \in [lm]^l\}$. Then for each $n \geq N$ and $A \in [lm]^l$ we have $\mid K_n^A \cap X \mid > k$ and hence $\mid K_n^A \setminus X \mid < l-k$. Hence we conclude that $\mid K_n \setminus X \mid < l-k$ for each $n \geq N$ (just take a suitable $A$). Thus $\mid K_n \cap X \mid > lm-(l-k)$ for each $n \geq N$ and the claim is proved. □

**Claim 1.6.2** *If $X \in [\omega]^* \cap \mathbf{V}'$ is $(l,k)$-large, $k+1 < l < \omega$ and $m \geq l-k$ then $X$ is $(m, m-(l-k))$-large.*

*Proof of Claim.* Let $\langle K_n : n \in \omega \rangle \in \mathbf{V}$ be a sequence of disjoint $m$-element sets. Put $K_n^* = K_{ln} \cup K_{ln+1} \cup \ldots \cup K_{ln+(l-1)}$. Clearly $\langle K_n^* : n \in \omega \rangle \in \mathbf{V}$, $|K_n^*| = lm$ and the sets $K_n^*$'s are disjoint. It follows from 1.6.1 that the set $X$ is $(lm, lm-(l-k))$-large and hence there is $N$ such that

$$(\forall n \geq N)(\mid K_n^* \cap X \mid > lm-(l-k)).$$

So $\mid K_n^* \setminus X \mid < l-k$ for $n \geq N$ and hence $\mid K_n \setminus X \mid < l-k$ for $n \geq lN$. This implies $(\forall n \geq lN)(\mid K_n \cap X \mid > m-(l-k))$ and the claim is proved. □

**Claim 1.6.3** *Assume that the extension $\mathbf{V} \subseteq \mathbf{V}'$ has the $R_0^\exists$-localization property. Then it has the $R_k^\exists$-localization for each $k \in \omega$.*

*Proof of Claim:* Let $k > 0$ and assume that the $R_k^\exists$-localization fails. Then we have a set $X \in [\omega]^*$ witnessing it, i.e. such that

$$(\forall \langle K_n : n \in \omega \rangle \in \mathscr{P}_{k+1} \cap \mathbf{V})(\forall^\infty n \in \omega)(\mid K_n \cap X \mid > k).$$

Then, in particular, the set $X$ is $(l,k)$-large for each $l > k+1$. By claim 1.6.2 it is $(l,0)$-large for each $l \geq 2$. By the $R_0^\exists$-localization we find a partition $\langle K_n : n \in \omega \rangle \in \mathscr{P}_1 \cap \mathbf{V}$ such that $(\exists^\infty n \in \omega)(K_n \cap X = \emptyset)$. Each set $K_n$ we partition into 2- and 3-element sets:

$$K_n = \bigcup \{K_{n,i}^2 : i \in w_n^2\} \cup \bigcup \{K_{n,i}^3 : i \in w_n^3\}$$

(everything should be done in $\mathbf{V}$, of course). Next look at

$$\langle K_{n,i}^2 : n \in \omega, i \in w_n^2 \rangle, \qquad \langle K_{n,i}^3 : n \in \omega, i \in w_n^3 \rangle \in \mathbf{V}.$$

These are sequences of disjoint 2- (or 3-, respectively) element subsets of $\omega$. At least one of them is infinite; for simplicity we assume that both are infinite. As $X$ is both $(2,0)$- and $(3,0)$-large we find $N$ such that for each $n \geq N$, $j \in \{2,3\}$

and $i \in w_n^j$ we have $K_{n,i}^j \cap X \neq \emptyset$. But this implies that for each $n \geq N$ the intersection $K_n \cap X$ is not empty - a contradiction to the choice of $\langle K_n : n \in \omega \rangle$. $\square$

**Claim 1.6.4** *Assume that $k \in \omega$, $\mathbf{V} \subseteq \mathbf{V}'$ has the $R_k^\exists$-localization property. Then the extension $\mathbf{V} \subseteq \mathbf{V}'$ has the $R_0^\exists$-localization.*

*Proof of Claim.* Suppose that the $R_0^\exists$-localization fails and this is witnessed by a set $X \in [\omega]^* \cap \mathbf{V}'$. As earlier we conclude from this that the set $X$ is $(l, 0)$-large and hence, by 1.6.2, it is $(l+k, k)$-large for each $l \geq 2$. By the $R_k^\exists$-localization we find $\langle K_n : n \in \omega \rangle \in \mathscr{P}_{k+1} \cap \mathbf{V}$ such that $(\exists^\infty n \in \omega)(| K_n \cap X | \leq k)$. For $i \in \omega$ let $A_i = \{n \in \omega : |K_n| = i\}$ (some of these sets can be finite or even empty). For $n \notin \bigcup_{i \leq 2k+2} A_i$ partition $K_n$ into 2- and 3-element sets to have more than $k$ pieces:

$$K_n = \bigcup \{K_{n,i}^2 : i \in w_n^2\} \cup \bigcup \{K_{n,i}^3 : i \in w_n^3\}, \qquad |w_n^2| + |w_n^3| > k$$

(everything is done in $\mathbf{V}$, of course). Consider the sequences

$$\begin{aligned} &\langle K_{n,i}^2 : i \in w_n^2, n \notin \textstyle\bigcup_{j \leq 2k+2} A_j \rangle, \\ &\langle K_{n,i}^3 : i \in w_n^3, n \notin \textstyle\bigcup_{j \leq 2k+2} A_j \rangle, \\ &\langle K_n : n \in A_{k+2} \rangle, \ldots, \langle K_n : n \in A_{2k+2} \rangle \end{aligned}$$

(note that $A_i = \emptyset$ for $i < k+2$). These are sequences of disjoint sets of the sizes $2, 3, k+2, \ldots, 2k+2$, respectively, and all sequences are in $\mathbf{V}$. Since $X$ is $(l+k, k)$-large for each $l \geq 2$ and it is (2,0)- and (3,0)-large we find $N$ such that

(a) if $n > N$, $n \in A_i$, $k+2 \leq i \leq 2k+2$ then $|K_n \cap X| > k$
(b) if $n > N$, $n \notin \bigcup_{j \leq 2k+2} A_j$, $i \in w_n^x$, $x \in \{2, 3\}$ then $K_{n,i}^x \cap X \neq \emptyset$.

The condition (b) implies that if $n > N$, $n \notin \bigcup_{j \leq 2k+2} A_j$ then $| K_n \cap X | > k$ (recall that we have more than $k$ sets $K_{n,i}^x$). Consequently $| K_n \cap X | > k$ for all $n > N$ which contradicts the choice of $\langle K_n : n \in \omega \rangle$. $\square\square$

The above proof suggests to consider $(m, 0)$-large sets (over $\mathbf{V}$) and ask if the existence of such sets depends on $m \geq 2$. The answer is given by the next result.

**Proposition 1.7** *Suppose $\mathbf{V} \subseteq \mathbf{V}'$ are models of ZFC, $m \geq 2$. Then there exists an $(m, 0)$-large set over $\mathbf{V}$ if and only if there exists an $(m+1, 0)$-large set over $\mathbf{V}$.*

*Proof.* Clearly each $(m, 0)$-large set is $(m+1, 0)$-large. So suppose now that $X \in [\omega]^* \cap \mathbf{V}'$ is $(m+1, 0)$-large over $\mathbf{V}$. If it is $(m, 0)$-large then we are done. So assume that $X$ is not $(m, 0)$-large and this is witnessed by $\langle K_n : n \in \omega \rangle \in \mathbf{V}$ (so $|K_n| = m$, $K_n$'s are disjoint and $(\exists^\infty n \in \omega)(K_n \cap X = \emptyset)$). Let

$$Y = \{n \in \omega : K_n \cap X \neq \emptyset\}.$$

Clearly $Y$ is infinite co-infinite. We are going to show that $Y$ is $(2, 0)$-large (and hence $(m, 0)$-large) over $\mathbf{V}$.

Suppose that $\langle L_n : n \in \omega\rangle \in \mathbf{V}$ is a sequence of disjoint 2-element sets. Let $K_n^* = \bigcup_{l\in L_n} K_l$. Thus $|K_n^*| = 2m$ and $K_n^*$'s are disjoint. Obviously the sequence $\langle K_n^* : n \in \omega\rangle$ is in $\mathbf{V}$. Since the set $X$ is $(m+1,0)$-large (and hence, by 1.6.2, $(2m, m-1)$-large) we have

$$(\forall^\infty n \in \omega)(|\ K_n^* \cap X\ | > m-1).$$

But $K_n^* \cap X \neq \emptyset$, $L_n = \{l^0, l^1\}$ imply that either $K_{l^0} \cap X \neq \emptyset$ or $K_{l^1} \cap X \neq \emptyset$ and hence $L_n \cap Y \neq \emptyset$. Consequently $Y$ is (2,0)-large. $\square$

**Corollary 1.8** *Let $\mathbf{V} \subseteq \mathbf{V}'$ be models of ZFC, $m \geq 2$, $k \in \omega$. Then the following conditions are equivalent:*

1. *there is no $(m,0)$-large set in $[\omega]^* \cap \mathbf{V}'$ over $\mathbf{V}$*
2. *there is no $(2,0)$-large set in $[\omega]^* \cap \mathbf{V}'$ over $\mathbf{V}$*
3. *$\mathbf{V} \subseteq \mathbf{V}'$ has the $R_0^\exists$-localization property*
4. *$\mathbf{V} \subseteq \mathbf{V}'$ has the $R_k^\exists$-localization property.* $\square$

*Remark.* One can consider a modification of the notion of $(m,0)$-largeness giving (probably) more freedom. For an increasing function $f \in \omega^\omega \cap \mathbf{V}$ we say that a set $X \in [\omega]^*$ is $f$-large over $\mathbf{V}$ if for every sequence $\langle K_n : n \in \omega\rangle \in \mathbf{V}$ of disjoint finite subsets of $\omega$ we have

$$\text{either } (\exists n \in \omega)(|K_n| < f(n)+2) \quad \text{ or } (\forall^\infty n \in \omega)(|K_n \cap X| > f(n)).$$

**Proposition 1.9** *Let $\mathbf{V} \subseteq \mathbf{V}'$ be models of ZFC.*
**a)** *If $\mathbf{V} \cap 2^\omega$ is not meager in $\mathbf{V}'$ then the pair $(\mathbf{V}, \mathbf{V}')$ has the $R_0^\exists$-localization property.*
**b)** *If the pair $(\mathbf{V}, \mathbf{V}')$ has the $R_0^\exists$-localization property then $\mathbf{V} \cap \omega^\omega$ is unbounded in $\mathbf{V}' \cap \omega^\omega$.*

*Proof.* **a)** If $\mathbf{V} \cap 2^\omega$ is not meager in $\mathbf{V}'$ then

$$(*) \qquad (\forall f \in \mathbf{V}' \cap \omega^\omega)(\forall Y \in \mathbf{V}' \cap [\omega]^\omega)(\exists g \in \mathbf{V} \cap \omega^\omega)(\exists^\infty n \in Y)(f(n) = g(n))$$

and $\mathbf{V} \cap \omega^\omega$ is unbounded in $\mathbf{V}' \cap \omega^\omega$ (see [Bar1]). By proposition 1.3, the pair $(\mathbf{V}, \mathbf{V}')$ has the $cR_1^\forall$-localization property. Suppose that $X \in [\omega]^* \cap \mathbf{V}'$. By the $cR_1^\forall$-localization we find a partition $\langle K_n : n \in \omega\rangle \in \mathbf{V} \cap \mathscr{P}_1$ such that $(\exists^\infty n \in \omega)(|\ K_n \setminus X\ | \geq 2)$. In $\mathbf{V}'$ we define

$$f(n) = K_n \setminus X \in [\omega]^{<\omega} \quad \text{(for } n \in \omega\text{)},$$
$$Y = \{n \in \omega : |f(n)| \geq 2\}.$$

By $(*)$ we find $g \in \mathbf{V}$, $g : \omega \longrightarrow [\omega]^{<\omega}$ such that $g(n) \in [K_n]^{\geq 2}$ and

$$(\exists^\infty n \in Y)(f(n) = g(n)).$$

Since $f(n) = g(n)$ implies $g(n) = K_n \setminus X$ we get $(\exists^\infty n \in \omega)(g(n) \cap X = \emptyset)$. Hence we easily get that $X$ can be $R_0^\exists$-localized by a partition from $\mathbf{V}$.

**b)** Since $|K| > 1$ & $K \cap (\omega \setminus X) = \emptyset$ implies $|K \cap X| \geq 2$ we get that $R_0^\exists$-localization implies the $cR_1^\forall$-localization. Now Proposition 1.3 works. $\square$

The next result gives some bounds on possible improvements of the previous one.

**Proposition 1.10** *1. The Cohen forcing notion has the $R_0^{\exists}$-localization property. Consequently, the $R_0^{\exists}$-localization does not imply that the old reals are a dominating family.*

*2. The Random real forcing does not have the $R_0^{\exists}$-localization property. Consequently, the localization is not implied by the fact that there is no unbounded real in the extension.*

*Proof. 1.* As in the extensions via the Cohen forcing the ground model reals are not meager, we may apply 1.9.

*2.* The Random algebra $\mathbb{B}$ is the quotient algebra of Borel subsets of $2^{\omega}$ modulo the ideal of Lebesgue null sets. We define a $\mathbb{B}$-name for an element of $[\omega]^*$ that cannot be localized:

Let $l_0 = 0$, $l_{k+1} = l_k + 2^{k^2}$ (for $k \in \omega$).
For each $k \in \omega$ fix disjoint Borel sets $A_m \subseteq 2^{\omega}$ for $l_k \leq m < l_{k+1}$ such that $\mu(A_m) = 2^{-k^2}$, where $\mu$ is the Lebesgue measure on $2^{\omega}$.
$\dot{X}$ is a $\mathbb{B}$-name for a subset of $\omega$ such that if $m \in [l_k, l_{k+1})$, $k \in \omega$ then $[\![m \notin \dot{X}]\!]_{\mathbb{B}} = [A_m]_{\mu}$.

It should be clear that $\Vdash_{\mathbb{B}} (\forall k \in \omega)(|\, [l_k, l_{k+1}) \setminus \dot{X}\ | = 1)$.
Suppose now that $\langle K_n : n \in \omega\rangle \in \mathscr{P}_1 \cap \mathbf{V}$. Let $k_n^0 = \min K_n$ and $k_n^1 = \max K_n$. Suppose that $l_m \leq k_n^0 < l_{m+1}$. Note that

$$k_n^1 < l_{m+1} \quad \Rightarrow \quad [\![k_n^0 \notin \dot{X} \ \&\ k_n^1 \notin \dot{X}]\!]_{\mathbb{B}} = \mathbf{0}$$

and

$$k_n^1 \geq l_{m+1} \quad \Rightarrow \quad \mu([\![k_n^1 \notin \dot{X}]\!]_{\mathbb{B}}) \leq 2^{-(m+1)^2}.$$

Hence

$$\mu([\![\dot{X} \cap K_n = \emptyset]\!]_{\mathbb{B}}) \leq \mu([\![k_n^0 \notin \dot{X} \ \&\ k_n^1 \notin \dot{X}]\!]_{\mathbb{B}}) \leq 2^{-(m+1)^2}.$$

Consequently, for each $m \geq 0$:

$$\mu([\![(\exists n \in \omega)(l_m \leq k_n^0 \ \&\ K_n \cap \dot{X} = \emptyset)]\!]_{\mathbb{B}}) \leq \sum_{n\in\omega} \mu([\![l_m \leq k_n^0 \ \&\ K_n \cap \dot{X} = \emptyset]\!]_{\mathbb{B}})$$

$$= \sum_{r\geq m} \Big( \sum_{k_n^0 \in [l_r, l_{r+1})} \mu([\![\dot{X} \cap K_n = \emptyset]\!]_{\mathbb{B}})\Big) \leq \sum_{r \geq m} 2^{r^2} 2^{-(r+1)^2} = \frac{1}{3} 2^{1-2m}.$$

Hence we can conclude that $\mu([\![(\exists^\infty n \in \omega)(K_n \cap \dot{X} = \emptyset)]\!]_{\mathbb{B}}) = 0$ which means $\Vdash_{\mathbb{B}}$"$\langle K_n : n \in \omega\rangle$ does not $R_0^{\exists}$-localize $\dot{X}$". $\square$

Though the random real forcing is an example of a forcing notion adding a (2,0)-large set over $\mathbf{V}$ (without adding an unbounded real!) it does not seem to be the minimal one. A canonical example of a forcing notion without the $R_0^{\exists}$-localization property is given below. (Recall that a forcing notion $\mathbb{Q}$ is $\sigma$-centered if it can be presented as a countable union of sets which all finite subsets have upper bounds in $\mathbb{Q}$.)

*Example 1.11* There is a $\sigma$-centered (Borel) forcing notion $\mathbb{Q}$ adding no dominating real and without the $R_0^{\exists}$-localization property.

*Proof.* The forcing notion $\mathbb{Q}$ consists of pairs $(u,\mathscr{K})$ such that $u \in [\omega]^{<\omega}$ and $\mathscr{K}$ is a finite set of families of disjoint 2-element subsets of $\omega$ (so $F \in \mathscr{K} \Rightarrow F \subseteq [\omega]^2$). The order of $\mathbb{Q}$ is given by

$(u_0,\mathscr{K}_0) \le (u_1,\mathscr{K}_1)$ if and only if
$u_1 \cap (1+\max u_0) = u_0$, $\mathscr{K}_0 \subseteq \mathscr{K}_1$ and if $K \in F \in \mathscr{K}_0$ and $K \subseteq u_1$ then $K \subseteq u_0$.

For $u \in [\omega]^{<\omega}$, $m \in \omega$ let

$$Q_u = \{(u,\mathscr{K}) : (u,\mathscr{K}) \in \mathbb{Q}\} \quad \text{and} \quad Q_u^m = \{(u,\mathscr{K}) \in Q_u : |\mathscr{K}| = m\}.$$

Since each $Q_u$ is obviously centered we get that $\mathbb{Q}$ is $\sigma$-centered. Let $\dot{w}$ be a $\mathbb{Q}$-name such that $\dot{w}^G = \bigcup\{u : (\exists \mathscr{K})((u,\mathscr{K}) \in G)\}$ for each generic filter $G \subseteq \mathbb{Q}$ over $\mathbf{V}$. Note that $\Vdash_{\mathbb{Q}} \dot{w} \in [\omega]^*$. If $K \in F \in \mathscr{K}$, $(u,\mathscr{K}) \in \mathbb{Q}$ and $\max u < \min K$ then $(u,\mathscr{K}) \Vdash_{\mathbb{Q}} K \setminus \dot{w} \neq \emptyset$. Hence we conclude that

$$\Vdash_{\mathbb{Q}} \text{“}\omega \setminus \dot{w} \text{ is } (2,0)\text{-large over } \mathbf{V}\text{”}.$$

So $\mathbb{Q}$ does not have the $R_0^{\exists}$-localization property. Suppose now that $\tau$ is a $\mathbb{Q}$-name for a member of $\omega^{\omega}$.

**Claim 1.11.1** *Let $u \in [\omega]^{<\omega}$, $m,n \in \omega$. Then there is $f(u,m,n) < \omega$ such that:*

*for every $p \in Q_u^m$ there is $q = (u^q,\mathscr{K}^q) \ge p$ such that $q$ decides the value of $\tau(n)$ and $\max u^q < f(u,m,n)$.*

*Proof of Claim.* The space $\mathscr{X}$ of all families $F \subseteq [\omega]^2$ of two-element disjoint subsets of $\omega$ can be equipped with a natural topology. For $F \in \mathscr{X}$, $N \in \omega$ the $N$-th basic open neighbourhood of $F$ is

$$\{F' \in \mathscr{X} : \{K \cap N : K \in F\} = \{K \cap N : K \in F'\}\}.$$

This topology is compact. It introduces a (product) topology on $Q_u^m$ such that if $q \in Q_{u'}$, $q \ge p$, $p \in Q_u^m$ then for some open neighbourhood $V$ of $p$ (in $Q_u^m$) each member of $V$ has an extension in $Q_{u'}$. Applying this fact and the compactness of $Q_u^m$ we get the claim. $\square$

Now we define a function $g \in \omega^{\omega}$ putting

$$g(k) = 1 + \max\{l : \ (\exists u \subseteq k)(\exists m,n \le k)(\exists v) \ (u \subseteq v \subseteq f(u,m,n) \ \text{ and } \ (\exists q \in Q_v)(q \Vdash \tau(n) = l))\}.$$

Given $p \in Q_u^m$, $l \in \omega$. Take $k > \max\{l,m,\max u\}$. By the definition of $f(u,m,k)$ we find $v$ such that $u \subseteq v \subseteq f(u,m,k)$ and some $q \in Q_v$, $q \ge p$ decides the value of $\tau(k)$. By the definition of the function $g$, the condition $q$ forces “$\tau(k) < g(k)$”. $\square$

*Remark.* An example of a forcing notion $\mathbb{P}$ with the $R_0^{\exists}$-localization property and such that

$$\Vdash_{\mathbb{P}} \text{“}\mathbf{V} \cap 2^{\omega} \text{ is meager”}$$

is an application of a general framework of [RoSh:470] and will be presented there.

## 2 Between dominating and unbounded reals

In this section we are interested in some localizations which are between the $cR_k^{\forall}$-localization and $cR_k^{\exists}$-localization (so between not adding a dominating real and not adding an unbounded real). The localizations are similar to that considered in the previous section. The difference is that we will consider partitions of $\omega$ into intervals and we will introduce quantifiers stronger than $\exists^{\infty} n$ but weaker than $\forall^{\infty} n$.

For an infinite subset $X$ of $\omega$ let $\mu_X : \omega \longrightarrow X$ be the increasing enumeration of $X$. A set $X \in [\omega]^{\omega}$ can be identified with the partition

$$\langle [\mu_X(n), \mu_X(n+1)) : n \in \omega \rangle$$

of $\omega \setminus \mu_X(0)$, so essentially $[\omega]^{\omega} \subseteq \mathscr{P}_0$. Now, for $k > 0$ and an increasing function $\phi \in \omega^{\omega}$, we define relations $S_k, S_+, S_{+\epsilon}, S_+^{\phi} \subseteq [\omega]^{\omega} \times [\omega]^{\omega}$:

$$(X, Y) \in S_k \equiv (\exists^{\infty} n \in \omega)(\forall i < k)(|\, [\mu_Y(n+i), \mu_Y(n+i+1)) \cap X \,| \geq 2)$$

$$(X, Y) \in S_+ \equiv (\forall m \in \omega)(\exists n \in \omega)(\forall i < m)(|\, [\mu_Y(n+i), \mu_Y(n+i+1)) \cap X \,| \geq 2)$$

$$(X, Y) \in S_{+\epsilon} \equiv (\exists^{\infty} n \in \omega)(\forall i < 2^n)(|\, [\mu_Y(2^n+i), \mu_Y(2^n+i+1)) \cap X \,| \geq 2)$$

$$(X, Y) \in S_+^{\phi} \equiv (\exists^{\infty} n \in \omega)(\forall i < \phi(n))(|\, [\mu_Y(n+i), \mu_Y(n+i+1)) \cap X \,| \geq 2).$$

[The relation $S_{+\epsilon}$ appears here for historical reasons only: it determines a cardinal invariant which was of serious use in [RoSh:475].]
Note that $S_1 \supseteq S_2 \supseteq S_3 \supseteq \ldots \supseteq S_+ \supseteq S_{+\epsilon} \cup S_+^{\phi}$ (remember that $\phi$ is increasing). If the function $\phi$ is increasing fast enough (e.g. $\phi(n) > 2^{2n}$) then $S_{+\epsilon} \supseteq S_+^{\phi}$.

It should be clear that if we consider $S_*$-localizations we could put any integer greater than 2 in place of 2 in the definitions above. However we do not know if replacing 2 by 1 provides the same notions of localizations.

**Proposition 2.1** *Let $\mathbf{V} \subseteq \mathbf{V}'$ be models of ZFC.*

*(a) The pair $(\mathbf{V}, \mathbf{V}')$ has the $S_1$-localization property if and only if $\mathbf{V} \cap \omega^{\omega}$ is unbounded in $\mathbf{V}' \cap \omega^{\omega}$.*
*(b) If the pair $(\mathbf{V}, \mathbf{V}')$ has the $S_{k+1}$-localization property then it has the $S_k$-localization. The $S_+$-localization property implies the $S_k$-localization for each $k > 0$ and is implied by both the $S_+^{\phi}$ and the $S_{+\epsilon}$-localization properties ($\phi$ - an increasing function).*
*(c) If $\mathbf{V} \cap \omega^{\omega}$ is dominating in $\mathbf{V}' \cap \omega^{\omega}$ then the extension $\mathbf{V} \subseteq \mathbf{V}'$ has the $S_+^{\phi}$-localization property for every increasing function $\phi \in \mathbf{V} \cap \omega^{\omega}$.*

**Proposition 2.2** *If $\mathbf{V} \subseteq \mathbf{V}'$ are models of ZFC such that $\mathbf{V} \cap 2^{\omega}$ is not meager in $\mathbf{V}'$, $\phi \in \mathbf{V} \cap \omega^{\omega}$ is increasing then the pair $(\mathbf{V}, \mathbf{V}')$ has the $S_+^{\phi}$-localization property.*

*Proof.* The proof is almost the same as that of 1.9(a). Suppose that $X \in \mathbf{V}' \cap [\omega]^{\omega}$. Let $X_0 \in [X]^{\omega}$ be such that for each $n \in \omega$

$$|\ [\mu_{X_0}(n), \mu_{X_0}(n+1)) \cap X\ | > 3\phi(\mu_{X_0}(n) + 4) + 6.$$

As $\omega^{\omega} \cap \mathbf{V}$ is unbounded in $\omega^{\omega} \cap \mathbf{V}'$ we find a set $Y_0 \in \mathbf{V} \cap \omega^{\omega}$ such that the set

$$X_1 = \{\mu_{Y_0}(n) : n \in \omega \ \&\ 2 \leq |\ [\mu_{Y_0}(n), \mu_{Y_0}(n+1)) \cap X_0\ |\}$$

is infinite. Let $f : \omega \longrightarrow [\omega]^{<\omega}$ be such that for every $\mu_{Y_0}(n) \in X_1$ we have

$$f(\mu_{Y_0}(n)) = [\mu_{Y_0}(n), \mu_{Y_0}(n+1)) \cap X$$

(so, for $\mu_{Y_0}(n) \in X_1$, $|\ f(\mu_{Y_0}(n))\ | > 3\phi(\mu_{X_0}(k) + 4) + 6$ where $k$ is the first such that $\mu_{Y_0}(n) \leq \mu_{X_0}(k) < \mu_{Y_0}(n+1)$). By $(*)$ from the proof of 1.9 we find a function $g \in \mathbf{V}$, $g : \omega \longrightarrow [\omega]^{<\omega}$ such that

$$(\exists^{\infty} n \in \omega)(\mu_{Y_0}(n) \in X_1 \quad \& \quad g(\mu_{Y_0}(n)) = f(\mu_{Y_0}(n))).$$

Next, using $g$ and $Y_0$ (both are in $\mathbf{V}$) we define sets $Y_1, Y \in \mathbf{V} \cap [\omega]^{\omega}$:

$$Y_1 = \bigcup_{n \in \omega} g(\mu_{Y_0}(n)) \cap [\mu_{Y_0}(n), \mu_{Y_0}(n+1))$$

and $\mu_Y(n) = \mu_{Y_1}(3n)$ for each $n$. Note that if $n$ is such that $\mu_{Y_0}(n) \in X_1$ and $g(\mu_{Y_0}(n)) = f(\mu_{Y_0}(n))$ then

$$Y_1 \cap [\mu_{Y_0}(n), \mu_{Y_0}(n+1)) = X \cap [\mu_{Y_0}(n), \mu_{Y_0}(n+1))$$

is of the size $> 3\phi(\mu_{Y_0}(n) + 4) + 6$ and consequently $Y$ $S_+^{\phi}$-localizes $X$. $\square$

**Proposition 2.3** *1. The Cohen forcing notion has the $S_+^{\phi}$-localization property for each increasing function $\phi \in \omega^{\omega}$.*

*2. If $\mathbf{V} \subseteq \mathbf{V}' \subseteq \mathbf{V}''$ are models of ZFC, the pair $(\mathbf{V}, \mathbf{V}')$ has the $S_+^{\phi}$-localization property and $\mathbf{V}' \cap \omega^{\omega}$ is dominating in $\mathbf{V}'' \cap \omega^{\omega}$ then the extension $\mathbf{V} \subseteq \mathbf{V}''$ has the $S_+^{\phi}$-localization property.*

*3. The iteration of the Cohen forcing notion and the random real forcing has the $S_+^{\phi}$-localization.*

*Consequently, the $S_+^{\phi}$-localization property implies neither that the ground model reals are dominating in the extension nor that the old reals are not meager.*

**Theorem 2.4** *For each $k > 0$, the $S_k$-localization property does not imply the $S_{k+1}$-localization property.*

*Proof.* To prove the theorem we will define a forcing notion $\mathbb{Q}_k$ which will possess the $S_k$-localization property but not $S_{k+1}$. The forcing notion is similar to that used in [Sh:207], [BsSh:242] and is a special case of the forcing notions of [RoSh:470].

We start with a series of definitions.

1. A function $\mathbf{n}$ is a *nice norm on $A$* if:
   – $\mathbf{n} : \mathscr{P}(A) \longrightarrow \omega$ and it is monotonic
   (i.e. $B \subseteq C \subseteq A \quad \Rightarrow \quad \mathbf{n}(B) \leq \mathbf{n}(C)$),
   – if $B \subseteq C \subseteq A$, $\mathbf{n}(C) > 0$
   then either $\mathbf{n}(B) \geq \mathbf{n}(C) - 1$ or $\mathbf{n}(C \setminus B) \geq \mathbf{n}(C) - 1$,
   – $\mathbf{n}(A) > 0$, if $a \in A$ then $\mathbf{n}(\{a\}) \leq 1$.
2. *A creature* is a tuple $\mathbb{T} = \langle T, nor, L, R\rangle$ such that
   $\alpha$) $T \subseteq \omega^{<\omega}$ is a finite nonempty tree,
   $\beta$) for each $t \in T$, either $\mathrm{succ}_T(t) = \emptyset$ or $|\mathrm{succ}_T(t)| = k$ or $|\mathrm{succ}_T(t)| > k$
   [so we have three kinds of nodes in the tree $T$],
   $\gamma$) $nor$ is a function with the domain
   $$\mathrm{dom}(nor) = \{t \in T : |\mathrm{succ}_T(t)| > k\}$$
   and such that for $t \in \mathrm{dom}(nor)$, $nor(t)$ is a nice norm on $\mathrm{succ}_T(t)$
   [$nor$ stands for "norm"],
   $\delta$) if $s \in \mathrm{succ}_T(t)$ then either $|\mathrm{succ}_T(t)| \neq k$ or $|\mathrm{succ}_T(s)| \neq k$
   [i.e. we do not have two successive $k$-ramifications in $T$],
   $\varepsilon$) $L, R : T \longrightarrow \omega$ are functions such that for each $t \in T$:
   – $L(t) \leq R(t)$
   – if $s \in \mathrm{succ}_T(t)$ then $[L(s), R(s)] \subseteq [L(t), R(t)]$
   – if $s_1, s_2 \in \mathrm{succ}_T(t)$ are distinct
   then $[L(s_1), R(s_1)] \cap [L(s_2), R(s_2)] = \emptyset$
   – if $\mathrm{succ}_T(t) = \emptyset$ then $L(t) = R(t)$
   [$L$ stands for "left" and $R$ is for "right"].
3. We will use the convention that if $a, b$ are indexes and $\mathbb{T}^a_b$ is a creature then its components are denoted by $T^a_b$, $nor^a_b$, $L^a_b$ and $R^a_b$, respectively.
4. Let $\mathbb{T} = \langle T, nor, L, R\rangle$ be a creature. We define its weight $\|\mathbb{T}\|$ and its contribution $\mathrm{cont}(\mathbb{T})$:
   $$\|\mathbb{T}\| = \min\{nor(t)(\mathrm{succ}_T(t)) : t \in \mathrm{dom}(nor)$$
   [if $\mathrm{dom}(nor) = \emptyset$ then we put $\|\mathbb{T}\| = 0$],
   $$\mathrm{cont}(\mathbb{T}) = \{L(t) : t \in T \ \&\ \mathrm{succ}_T(t) = \emptyset\}$$
   [recall that if $t$ is a leaf in $T$ then $L(t) = R(t)$].
5. Let $\mathbb{T}_i$ ($i = 0, 1$) be creatures. We say that the creature $\mathbb{T}_1$ *refines* $\mathbb{T}_0$ (we write: $\mathbb{T}_0 \leq \mathbb{T}_1$) if:
   **a)** $T_1 \subseteq T_0$, $L_1 = L_0{\restriction}T_1$, $R_1 = R_0{\restriction}T_1$,
   **b)** if $t \in T_1$ then
   $|\mathrm{succ}_{T_1}(t)| > k$ iff $|\mathrm{succ}_{T_0}(t)| > k$,
   $|\mathrm{succ}_{T_1}(t)| = k$ iff $|\mathrm{succ}_{T_0}(t)| = k$, and
   $|\mathrm{succ}_{T_1}(t)| = \emptyset$ iff $|\mathrm{succ}_{T_0}(t)| = \emptyset$

(in other words we keep the kind of nodes),

**c)** $nor_1(t) = nor_0(t)\restriction\mathscr{P}(\mathrm{succ}_{T_1}(t))$ for all $t \in \mathrm{dom}(nor_1)$.

6. Let $\mathbb{T}_0, \mathbb{T}_1, \ldots, \mathbb{T}_n$ be creatures. We say that the creature $\mathbb{T}$ is built of the creatures $\mathbb{T}_0, \ldots, \mathbb{T}_n$ (we will write it as $\mathbb{T} \in \Sigma(\mathbb{T}_0, \ldots, \mathbb{T}_n)$) if there is a maximal antichain $F$ of $T$ such that for each $t \in F$, for some $i \leq n$:
   – $\{s \in T : t \subseteq s\} = \{t\hat{\ }r : r \in T_i\}$,
   – if $s = t\hat{\ }r \in T$, $r \in T_i$ then $L(s) = L_i(r)$, $R(s) = R_i(r)$ and $nor(s) = nor_i(r)$ (if defined).
7. If $n \geq k$, $H : \mathscr{P}(n+1) \longrightarrow \omega$ is a nice norm on $n+1$ (i.e. it satisfies the conditions listed in (1)) then $S_H(\mathbb{T}_0, \ldots, \mathbb{T}_n)$ is the creature $\mathbb{T}$ such that

$$T = \{\langle i\rangle\hat{\ }t : i \leq n, t \in T_i\}, \quad nor(\langle\rangle) = H, \quad nor(\langle i\rangle\hat{\ }t) = nor_i(t)$$

   and similarly for $L$ and $R$.
   Clearly $S_H(\mathbb{T}_0, \ldots, \mathbb{T}_n) \in \Sigma(\mathbb{T}_0, \ldots, \mathbb{T}_n)$.
8. We define gluing $k$ creatures similarly to the operation $S_H$ above. Thus $S(\mathbb{T}_0, \ldots, \mathbb{T}_{k-1}) = \mathbb{T}$ is a creature such that

$$T = \{\langle i\rangle\hat{\ }t : i < k, t \in T_i\}$$

   and $nor$, $L$, $R$ are defined naturally. Once again, $S(\mathbb{T}_0, \ldots, \mathbb{T}_{k-1}) \in \Sigma(\mathbb{T}_0, \ldots, \mathbb{T}_{k-1})$.
9. For a creature $\mathbb{T}$ we define its upper half $\mathbb{T}^{\mathrm{uh}} = \langle T^{\mathrm{uh}}, nor^{\mathrm{uh}}, L^{\mathrm{uh}}, R^{\mathrm{uh}}\rangle$ by:

$$T^{\mathrm{uh}} = T, \quad L^{\mathrm{uh}} = L, \quad R^{\mathrm{uh}} = R \quad \text{but}$$

$$nor^{\mathrm{uh}}(t)(A) = \max\{0, nor(t)(A) - [\frac{\|\mathbb{T}\|}{2}]\}$$

   whenever $t \in T$, $|\mathrm{succ}_T(t)| > k$ and $A \subseteq \mathrm{succ}_T(t)$. Above, $[x]$ stands for the integer part of $x$.
   [It is routine to check that $\mathbb{T}^{\mathrm{uh}}$ is really a creature and that $\|\mathbb{T}^{\mathrm{uh}}\| = \|\mathbb{T}\| - [\frac{1}{2}\|\mathbb{T}\|]$, $\mathrm{cont}(\mathbb{T}^{\mathrm{uh}}) = \mathrm{cont}(\mathbb{T})$.]
10. For creatures $\mathbb{T}_0, \ldots, \mathbb{T}_n$, the closure of $\{\mathbb{T}_0, \ldots, \mathbb{T}_n\}$ under the operations of shrinking (refining), taking the upper half and building creatures is denoted by $\Sigma^*(\mathbb{T}_0, \ldots, \mathbb{T}_n)$. Thus $\{\mathbb{T}_0, \ldots, \mathbb{T}_n\} \subseteq \Sigma^*(\mathbb{T}_0, \ldots, \mathbb{T}_n)$, if $\mathbb{T} \in \Sigma^*(\mathbb{T}_0, \ldots, \mathbb{T}_n)$ then

$$\mathbb{T}^{\mathrm{uh}} \in \Sigma^*(\mathbb{T}_0, \ldots, \mathbb{T}_n), \quad \mathbb{T} \leq \mathbb{T}' \quad \Rightarrow \quad \mathbb{T}' \in \Sigma^*(\mathbb{T}_0, \ldots, \mathbb{T}_n),$$

   and if $\mathbb{T}'_0, \ldots, \mathbb{T}'_m \in \Sigma^*(\mathbb{T}_0, \ldots, \mathbb{T}_n)$ then

$$\Sigma(\mathbb{T}'_0, \ldots, \mathbb{T}'_m) \subseteq \Sigma^*(\mathbb{T}_0, \ldots, \mathbb{T}_n).$$

   [Note that $\Sigma^*(\mathbb{T}_0, \ldots, \mathbb{T}_n)$ is finite (up to isomorphism).]

Now we may define our forcing notion $\mathbb{Q}_k$:

*Conditions* are sequences $\langle w, \mathbb{T}_0, \mathbb{T}_1, \mathbb{T}_2, \ldots\rangle$ such that $w \in [\omega]^{<\omega}$, $\mathbb{T}_i$ are creatures, $\|\mathbb{T}_i\| \longrightarrow \infty$ and

$$\max(w) < L_0(\langle\rangle) \leq R_0(\langle\rangle) < L_1(\langle\rangle) \leq R_1(\langle\rangle) < \ldots$$

(recall that $\mathbb{T}_i = \langle T_i, nor_i, L_i, R_i\rangle$).

*The order* is given by
$\langle w, \mathbb{T}_0, \mathbb{T}_1 \ldots\rangle \leq \langle w', \mathbb{T}'_0, \mathbb{T}'_1 \ldots\rangle$ if and only if
for some increasing sequence $n_0 < n_1 < n_2 < \ldots < \omega$

$$w \subseteq w' \subseteq w \cup \bigcup_{i<n_0} \mathrm{cont}(\mathbb{T}_i) \quad \text{and} \quad (\forall i \in \omega)(\mathbb{T}'_i \in \Sigma^*(\mathbb{T}_{n_i}, \ldots, \mathbb{T}_{n_{i+1}-1})).$$

We say that a condition $\langle w', \mathbb{T}'_0, \mathbb{T}'_1 \ldots\rangle \in \mathbb{Q}_k$ is *a pure extension* of a condition $\langle w, \mathbb{T}_0, \mathbb{T}_1 \ldots\rangle \in \mathbb{Q}_k$ if

$$\langle w, \mathbb{T}_0, \mathbb{T}_1 \ldots\rangle \leq \langle w', \mathbb{T}'_0, \mathbb{T}'_1 \ldots\rangle \quad \text{and} \quad w = w'.$$

One can easily check that $(\mathbb{Q}_k, \leq)$ is a partial order and that the relation of pure extension is transitive. The proof that the forcing notion $\mathbb{Q}_k$ has the required properties is broken into several claims. The first claim is of a technical character, but it implies in particular that $\mathbb{Q}_k$ is proper. (The proof of this claim is straightforward and we will omit it.)

**Claim 2.4.1** *If $p \in \mathbb{Q}_k$, $\tau_n$ (for $n < \omega$) are $\mathbb{Q}_k$-names for ordinals then there is a pure extension $q = \langle w, \mathbb{T}_0, \mathbb{T}_1, \ldots\rangle$ of $p$ such that*

1. $\langle\|\mathbb{T}_i\| : i \in \omega\rangle$ *is increasing,*
2. *for each $n$, $v \subseteq R_n(\langle\rangle)$ and $i \leq n$,*
   if *some pure extension of $\langle w \cup v, \mathbb{T}_{n+1}, \mathbb{T}_{n+2} \ldots\rangle$ decides the value of $\tau_i$*
   then $\langle w \cup v, \mathbb{T}_{n+1}, \mathbb{T}_{n+2} \ldots\rangle$ *decides it.*

**Claim 2.4.2** *Suppose that $\mathbb{T} = \langle T, nor, L, R\rangle$ is a creature, $\|\mathbb{T}\| \geq 15$ and $B \in [\omega]^{\omega}$. Then there is a creature $\mathbb{T}' \geq \mathbb{T}$ such that $\|\mathbb{T}'\| \geq \|\mathbb{T}\| - 14$ and*

$$(\forall n \in \omega)(\exists i \leq k)(|\ \mathrm{cont}(\mathbb{T}') \cap [\mu_B(n+i), \mu_B(n+i+1))\ | < 2).$$

*Proof of Claim.* We prove this essentially by the induction on $|T|$ (or the height of $T$). We show how to "eliminate" $\langle\rangle$ and apply the inductive hypothesis to $\mathbb{T}$ above $t$ (for $t \in \mathrm{succ}_T(\langle\rangle)$).

*Case 1:* $k = |\mathrm{succ}_T(\langle\rangle)|$

If, for each $t \in \mathrm{succ}_T(\langle\rangle)$, $\mathrm{succ}_T(t) = \emptyset$ then $|\ \mathrm{cont}(\mathbb{T})\ | = k$ and there is no problem. So there are $t \in \mathrm{succ}_T(\langle\rangle)$ such that $|\mathrm{succ}_T(t)| > k$ (here we use the requirement ($\delta$) of the definition of creatures); above each such $t$ we can apply the inductive hypothesis and shrink suitably the tree $T$. However the problems coming from distinct $t$ could accumulate. Therefore for each such $t$ we first choose a set $A = A^t \subseteq \mathrm{succ}_T(t)$ such that $nor(t)(A) \geq \|\mathbb{T}\| - 7$ and one of the following occurs:

– for some $m$, $(\forall s \in A)(\mu_B(m) \leq L(s) \leq R(s) \leq \mu_B(m+1))$
– there are $1 \leq m_0 < m_1$ such that

$$(\forall s \in A)(\mu_B(m_0) \leq L(s) \leq R(s) \leq \mu_B(m_1)),$$

and if $t_0 \in \mathrm{succ}_T(\langle\rangle)$, $t_0 \neq t$ then

$$\text{either} \quad R(t_0) < \mu_B(m_0 - 1) \quad \text{or} \quad \mu_B(m_1 + 1) < L(t_0).$$

To find such a set $A$ we use the second property of nice norms. First we look at the set

$$A_0 = \{s \in \mathrm{succ}_T(t) : (\exists m \in \omega)(\mu_B(m) \leq L(s) \leq R(s) \leq \mu_B(m+1))\}.$$

If $nor(t)(A_0) \geq nor(t)(\mathrm{succ}_T(t)) - 1 \geq \|\mathbb{T}\| - 1$ then we easily finish: either for some $m$

$$nor(t)(\{s \in \mathrm{succ}_T(t) : \mu_B(m) \leq L(s) \leq R(s) \leq \mu_B(m+1)\}) \geq \|\mathbb{T}\| - 7$$

or we have $m_0' < m_0'' < m_0 < m_1 < m_1' < m_1''$ such that

$$nor(t)(\{s \in \mathrm{succ}_T(t) : \mu_B(m_0) \leq L(s) \leq R(s) \leq \mu_B(m_1)\}) \geq \|\mathbb{T}\| - 7 \qquad \text{and}$$

$$(\exists s_0, s_1 \in \mathrm{succ}_T(t))(\mu_B(m_0') \leq L(s_0) \leq \mu_B(m_0'') \ \&\ \mu_B(m_1') \leq R(s_1) \leq \mu_B(m_1'')).$$

So suppose that $nor(t)(A_0) < \|\mathbb{T}\| - 1$. In this case $nor(t)(\mathrm{succ}_T(t) \setminus A_0) \geq \|\mathbb{T}\| - 1$. For each $s \in \mathrm{succ}_T(t) \setminus A_0$ there is $m \in \omega$ such that

$$L(s) < \mu_B(m) < R(s).$$

Removing 4 extreme points from $\mathrm{succ}_T(t) \setminus A_0$ (the first two and the last two, counting according to the values of $L$) we get the required set $A$ (with $nor(t)(A) \geq \|T\| - 5$).
Now we would like to apply the induction hypothesis above each $t \in \mathrm{succ}_T(\langle\rangle)$ restricting ourselves to successors of $t$ from the suitable set $A^t$ (if applicable). A small difficulty is that we have decreased the norm of the creature above those $t$ (possibly by 7, as the result of restricting to $A^t$). But now we apply the procedure described in the case 2 below and we pass to $B^t \subseteq A^t$ such that

$$nor(t)(B^t) \geq nor(t)(A^t) - 7 \geq \|\mathbb{T}\| - 14.$$

Next we apply the inductive hypothesis above each $s \in B^t$, $t \in \mathrm{succ}_T(\langle\rangle) \cap \mathrm{dom}(nor)$. In this way we construct $\mathbb{T}'$ as required (the point is that restricting to the sets $B^t \subseteq A^t$ causes that what happens above distinct $t \in \mathrm{succ}_T(\langle\rangle)$ is isolated in a sense).

*Case 2:* $k < |\mathrm{succ}_T(\langle\rangle)|$
Then one of the following possibilities occurs ($i = 0, 1$):

$(\alpha)_i$ $nor(\langle\rangle)(\{t \in \mathrm{succ}_T(\langle\rangle) : (\exists m)(\mu_B(2m+i) \le L(t) \le R(t)$
$< \mu_B(2m+i+1))\}) \ge \|\mathbb{T}\| - 3$

$(\beta)$ $nor(\langle\rangle)(\{t \in \mathrm{succ}_T(\langle\rangle) : (L(t), R(t)) \cap B \ne \emptyset\}) \ge \|\mathbb{T}\| - 3$

If one of cases $(\alpha)_0$, $(\alpha)_1$ holds then the creature refining $\mathbb{T}$ and determined by the respective set of successors of $\langle\rangle$ can serve as $\mathbb{T}'$. In case $(\beta)$ divide the set

$$A = \{t \in \mathrm{succ}_T(\langle\rangle) : (L(t), R(t)) \cap B \ne \emptyset\}$$

into four disjoint subsets, each containing every fourth member of $A$. One of these subsets (call it $A^*$) has the norm $\ge \|\mathbb{T}\| - 7$. For each $t \in A^*$ apply the inductive hypothesis to the creature given by $\{s \in T : t \subseteq s\}$. Note that either it is of the weight $\ge \|\mathbb{T}\|$ or $\mathrm{succ}_T(t) = \emptyset$ or $|\mathrm{succ}_T(t)| = k$ and for $s \in \mathrm{succ}_T(t)$ we have $\mathrm{succ}_T(s) = \emptyset$. The last two cases are trivial and actually should be considered separately (compare Case 1). In this way we get the required creature $\mathbb{T}'$. $\square$

**Claim 2.4.3** *$\mathbb{Q}_k$ does not have the $S_{k+1}$-localization property.*

*Proof of Claim.* Let $\dot{w}$ be a $\mathbb{Q}_k$ name such that if $G \subseteq \mathbb{Q}_k$ is a generic then $\dot{w}^G = \bigcup\{w : (\exists\langle\mathbb{T}_0, \mathbb{T}_1, \ldots\rangle)(\langle w, \mathbb{T}_0, \mathbb{T}_1, \ldots\rangle \in G)\}$. We claim that the $S_{k+1}$-localization always fails for $\dot{w}^G$, i.e. that if $B \in \mathbf{V} \cap [\omega]^\omega$ then

$$\Vdash_{\mathbb{Q}_k} (\forall^\infty n \in \omega)(\exists i \le k)(|\ \dot{w} \cap [\mu_B(n+i), \mu_B(n+i+1))\ | < 2).$$

Let $p = \langle w, \mathbb{T}_0, \mathbb{T}_1, \ldots\rangle \in \mathbb{Q}_k$ be given. We may assume that

1. $(\forall l \in \omega)(|\ [R_l(\langle\rangle), L_{l+1}(\langle\rangle)) \cap B\ | > 2)$,
2. $(\forall l \in \omega)(\|\mathbb{T}_l\| > 15)$.

For each $\mathbb{T}_i$ take the creature $\mathbb{T}'_i \ge \mathbb{T}_i$ given by claim 2.4.2 for $\mathbb{T}_i$ and $B$. Look at the condition $q = \langle w, \mathbb{T}'_0, \mathbb{T}'_1, \ldots\rangle \in \mathbb{Q}_k$. Clearly $q \ge p$ and if $\mu_B(n) > \max(w)$ then

$$q \Vdash_{\mathbb{Q}_k} (\exists i \le k)(|\dot{w} \cap [\mu_B(n+i), \mu_B(n+i+1))| < 2).$$

This proves the claim. $\square$

The next claim explains why we introduced the operation of taking the upper half of a creature as a part of the definition of (the order of) $\mathbb{Q}_k$

**Claim 2.4.4** *Let $p = \langle w, \mathbb{T}_0, \mathbb{T}_1 \ldots\rangle \in \mathbb{Q}_k$, $m \in \omega$. Suppose that $\tau$ is a $\mathbb{Q}_k$-name for an ordinal. Then there are $n_0$ and a nice norm $H$ on $n_0$ such that $H(n_0) \ge m$ and*

*if $\mathbb{T}' \ge S_H(\mathbb{T}_0^{\mathrm{uh}}, \ldots, \mathbb{T}_{n_0-1}^{\mathrm{uh}})$, $\|\mathbb{T}'\| > 0$, $w \subseteq w' \subseteq L_0(\langle\rangle)$ then there exist $v \subseteq \mathrm{cont}(\mathbb{T}')$ and $\langle\mathbb{T}'_0, \mathbb{T}'_1, \ldots\rangle$ such that*

$$\langle\emptyset, \mathbb{T}'_0, \mathbb{T}'_1, \ldots\rangle \ge \langle\emptyset, \mathbb{T}_{n_0}, \mathbb{T}_{n_0+1}, \ldots\rangle$$

*and $\langle w' \cup v, \mathbb{T}'_0, \mathbb{T}'_1, \ldots\rangle$ decides the value of $\tau$.*

*Proof of Claim.* This is essentially 2.14 of [Sh:207].

Define the function $H : [\omega]^{<\omega} \longrightarrow \omega$ by:

$H(u) \geq 0$ always
$H(u) \geq 1$ if $|u| > 1$ and for each $\mathbb{T}'_i \geq \mathbb{T}^{\text{uh}}_i$, $\|\mathbb{T}'_i\| > 0$ (for $i \in u$), for every $w'$, $w \subseteq w' \subseteq L_0(\langle\rangle)$ there is $v \subseteq \bigcup_{i\in u} \text{cont}(\mathbb{T}'_i)$ such that some pure extension of $\langle w' \cup v, \mathbb{T}_l, \mathbb{T}_{l+1}, \ldots\rangle$ decides the value of $\tau$ (where $l = \max u + 1$).
$H(u) \geq n+1$ if for every $u' \subseteq u$ either $H(u') \geq n$ or $H(u \setminus u') \geq n$ (for $n > 0$).

As $H$ is monotonic it is enough to find $u$ such that $H(u) \geq m$. The existence of the $u$ can be proved by induction on $m$, for all sequences $\langle w, \mathbb{T}_0, \mathbb{T}_1, \ldots\rangle$.
Let us start with the case $m = 1$. Suppose that $\mathbb{T}'_i \geq \mathbb{T}^{\text{uh}}_i$, $\|\mathbb{T}'_i\| > 0$ (for $i \in \omega$). For each $i \in \omega$ choose a creature $\mathbb{T}^*_i \geq \mathbb{T}_i$ such that

$$\text{cont}(\mathbb{T}^*_i) = \text{cont}(\mathbb{T}'_i) \quad \text{and} \quad \|\mathbb{T}^*_i\| \geq \frac{1}{2}\|\mathbb{T}_i\|$$

(possible by the definition of the upper half of a creature). Then for each $w' \subseteq L_0(\langle\rangle)$:

$$\langle w', \mathbb{T}^*_0, \mathbb{T}^*_1, \mathbb{T}^*_2, \ldots\rangle \in \mathbb{Q}_k$$

and thus we find $n(w') \in \omega$, $v(w') \in [\omega]^{<\omega}$ such that

$$v(w') \subseteq \bigcup_{n<n(w')} \text{cont}(\mathbb{T}^*_n) = \bigcup_{n<n(w')} \text{cont}(\mathbb{T}'_n) \quad \text{and}$$

some pure extension of

$$\langle w' \cup v(w'), \mathbb{T}^*_{n(w')}, \mathbb{T}^*_{n(w')+1}, \ldots\rangle$$

(and so of $\langle w' \cup v(w'), \mathbb{T}_{n(w')}, \mathbb{T}_{n(w')+1}, \ldots\rangle$) decides the value of $\tau$.
Let $M(\mathbb{T}'_0, \mathbb{T}'_1, \mathbb{T}'_2, \ldots)$ be the first $M$ such that for every $w' \subseteq L_0(\langle\rangle)$ there is $v \subseteq \bigcup_{n<M} \text{cont}(\mathbb{T}'_n)$ such that

some pure extension of $\langle w' \cup v, \mathbb{T}_M, \mathbb{T}_{M+1}, \ldots\rangle$ decides the value of $\tau$.

The space

$$\{\langle \mathbb{T}'_0, \mathbb{T}'_1, \mathbb{T}'_2, \ldots\rangle : (\forall i \in \omega)(\mathbb{T}'_i \geq \mathbb{T}^{\text{uh}}_i)\}$$

equipped with the natural (product) topology is compact and the function

$$M : \langle \mathbb{T}'_0, \mathbb{T}'_1, \mathbb{T}'_2, \ldots\rangle \longrightarrow M(\mathbb{T}'_0, \mathbb{T}'_1, \mathbb{T}'_2, \ldots)$$

is continuous. Hence the function $M$ is bounded, say by $n_0$. Clearly $H(n_0) \geq 1$. Now suppose that we always can find a set of the norm $\geq m \geq 1$. Thus we find an increasing sequence $\langle l_i : i \in \omega\rangle$ such that $H([l_i, l_{i+1})) \geq m$ for each $i$. Consider the space of all increasing $\psi \in \omega^\omega$ such that $(\forall i \in \omega)(\psi(i) \in [l_i, l_{i+1}))$ - it is a compact space. For each $\psi$ from the space we may consider $\langle w, \mathbb{T}_{\psi(0)}, \mathbb{T}_{\psi(1)}, \ldots\rangle$ and the respective function $H_\psi$. By the induction hypothesis we find $n = n_\psi$ such that $H_\psi(n) \geq m$. But $H_\psi(n) \leq H(\{\psi(i) : i < n\})$. By the compactness we find one $n$ such that $m \leq H(\{\psi(i) : i < n\})$ for each $\psi$. Hence we conclude that $H(l_{n+1}) \geq m + 1$. □

**Main Claim 2.4.5**

1. *The forcing notion $\mathbb{Q}_k$ has the $S_k$-localization property.*
2. *Moreover the following stronger condition is satisfied by $\mathbb{Q}_k$:*
   $(S^*_k)_{\mathbb{Q}_k}$ *Suppose that $N$ is a countable elementary submodel of $\langle \mathscr{H}(\beth^+_7), \in, <^*\rangle$, $p = \langle w, \mathbb{T}_0, \mathbb{T}_1, \dots\rangle \in \mathbb{Q}_k \cap N$. Assume that $Y \in [\omega]^\omega$ is such that*

$$(\forall X \in [\omega]^\omega \cap N)((X, Y) \in S_k).$$

*Then there is a condition $q \geq p$ which is $(N, \mathbb{Q}_k)$-generic and such that*

$$q \Vdash_{\mathbb{Q}_k} (\forall X \in N[\dot{G}_{\mathbb{Q}_k}] \cap [\omega]^\omega)((X, Y) \in S_k).$$

*Proof of Main Claim.* 1) Since in the next section we will need the property $(S^*_k)_{\mathbb{Q}_k}$, we will present the proof of it fully below. Here we sketch the proof of the $S_k$-localization property for readers not interested in the stronger property (needed for iterations). So suppose that $\dot{X}$ is a $\mathbb{Q}_k$-name for an element of $[\omega]^\omega$ and $p = \langle w, \mathbb{T}_0, \mathbb{T}_1, \dots\rangle \in \mathbb{Q}_k$. We may assume that $\|\mathbb{T}_n\| \geq n$. We inductively define integers

$$0 = b_0 < b_1 < b_2 < \dots \quad \text{and} \quad 0 = n_0 < n_1 < n_2 < \dots$$

and nice norms $H_m$ on $[n_m, n_{m+1})$. Suppose we have defined $b_m, n_m$.

Let $\tau_m$ be a $\mathbb{Q}_k$-name for an integer such that

$$\Vdash_{\mathbb{Q}_k} |[b_m, \tau_m) \cap \dot{X}| > 2.$$

Modifying the tail (above $n_m$) of $p$ we may assume that if $n_m \leq n$, $w \subseteq w' \subseteq R_n(\langle\rangle)$ and some pure extension of $\langle w', \mathbb{T}_{n+1}, \mathbb{T}_{n+2}, \dots\rangle$ decides the value of $\tau_m$ then $\langle w', \mathbb{T}_{n+1}, \mathbb{T}_{n+2}, \dots\rangle$ does it already (see 2.4.1). Applying 2.4.4 we find $n_{m+1} > n_m$ and a nice norm $H_m$ on $[n_m, n_{m+1})$ such that

> $H_m([n_m, n_{m+1})) \geq m + 1$ and if $\mathbb{T}' \geq S_{H_m}(\mathbb{T}^{\mathrm{uh}}_{n_m}, \dots, \mathbb{T}^{\mathrm{uh}}_{n_{m+1}-1})$, $w \subseteq w' \subseteq L_{n_m}(\langle\rangle)$ then there is $v \subseteq \mathrm{cont}(\mathbb{T}')$ such that some pure extension of
>
> $$\langle w' \cup v, \mathbb{T}_{n_{m+1}}, \mathbb{T}_{n_{m+1}+1}, \dots\rangle$$
>
> decides the value of $\tau_m$ and thus $\langle w' \cup v, \mathbb{T}_{n_{m+1}}, \mathbb{T}_{n_{m+1}+1}, \dots\rangle$ does it.

Let $b_{m+1}$ be an integer larger than all possible values forced to $\tau_m$ in the condition above.

Now for each $l \in \omega$ we put

$$\mathbb{T}^*_l = S\left(S_{H_{lk}}(\mathbb{T}^{\mathrm{uh}}_{n_{lk}}, \dots, \mathbb{T}^{\mathrm{uh}}_{n_{lk+1}-1}), \dots, S_{H_{(l+1)k-1}}(\mathbb{T}^{\mathrm{uh}}_{n_{(l+1)k-1}}, \dots, \mathbb{T}^{\mathrm{uh}}_{n_{(l+1)k}-1})\right),$$

and then

$$q = \langle w, \mathbb{T}^*_0, \mathbb{T}^*_1, \dots\rangle, \qquad B = \{b_0, b_1, b_2, \dots\}.$$

Check that $q \geq p$ and

$$q \Vdash_{\mathbb{Q}_k} \text{“the set } B \quad S_k\text{–localizes } \dot{X}\text{”}$$

(or see the end of the proof of 2) below).

2) The construction of the condition $q$ required in $(S_k^*)_{\mathbb{Q}_k}$ is similar to that in 1). Here, however, we have to take care of all names for elements of $[\omega]^\omega$ from the model $N$ (as well as names for ordinals – to ensure the genericity).
Let $\langle \sigma_n : n \in \omega \rangle$ enumerate all $\mathbb{Q}_k$-names from $N$ for ordinals and let $\langle \dot{A}_n : n \in \omega \rangle$ list all names (from $N$) for infinite subsets of $\omega$. Of course, both sequences are *not* in $N$ but all their initial (finite) segments are there.

Now we inductively define sets $B_n = \{b_0^n, b_1^n, b_2^n, \dots\} \in [\omega]^\omega$ and conditions $q_n = \langle w, \mathbb{T}_0^n, \mathbb{T}_1^n, \mathbb{T}_0^n, \dots \rangle \in \mathbb{Q}_k$ such that $B_n, q_n \in N$:
To start with we put $B_0 = \omega$, $q_0 = p = \langle w, \mathbb{T}_0, \mathbb{T}_1, \dots \rangle$.
Arriving at stage $n > 0$ we have defined $B_{n-1}$, $q_{n-1} \in N$. We define $B_n$, $q_n$ applying the following procedure *inside* the model $N$ (so the result will be there; compare this procedure with that in part (1)):

Let $\tau_0$ be a $\mathbb{Q}_k$-name for an integer such that $\Vdash_{\mathbb{Q}_k} (\forall i < n)(|\tau_0 \cap \dot{A}_i| > 2)$. We modify $q_{n-1}$ (passing to a pure extension of it) and we assume that

$(*)_{\tau_0, \sigma_0, \dots, \sigma_{n-1}}$ if $v \subseteq R_i^{n-1}(\langle\rangle)$, $i \in \omega$ and there exists a pure extension of $\langle w \cup v, \mathbb{T}_{i+1}^{n-1}, \mathbb{T}_{i+2}^{n-1}, \dots \rangle$ deciding the value of one of $\tau_0, \sigma_0, \dots, \sigma_{n-1}$ then $\langle w \cup v, \mathbb{T}_{i+1}^{n-1}, \mathbb{T}_{i+2}^{n-1}, \dots \rangle$ decides it already

(see 2.4.1). Next, by Claim 2.4.4, we find $n_0$ and a nice norm $H_0$ on $n_0$ such that

$H_0(n_0) \geq 1$ and if $\mathbb{T}' \geq S_{H_0}((\mathbb{T}_0^{n-1})^{\mathrm{uh}}, \dots, (\mathbb{T}_{n_0-1}^{n-1})^{\mathrm{uh}})$, $\|\mathbb{T}'\| > 0$, $w \subseteq w' \subseteq L_0^{n-1}(\langle\rangle)$ then there exists $v \subseteq \mathrm{cont}(\mathbb{T}')$ such that $\langle w' \cup v, \mathbb{T}_{n_0}^{n-1}, \mathbb{T}_{n_0+1}^{n-1}, \mathbb{T}_{n_0+2}^{n-1}, \dots \rangle$ decides the value of $\tau_0$.

Let $\mathbb{T}_0^n = S_{H_0}((\mathbb{T}_0^{n-1})^{\mathrm{uh}}, \dots, (\mathbb{T}_{n_0-1}^{n-1})^{\mathrm{uh}})$ and let $b_0^n$ be greater than all possible values of $\tau_0$ (i.e. the values forced in the condition on $H_0$ above). Let $\tau_1$ be a $\mathbb{Q}_k$-name for an integer such that $\Vdash_{\mathbb{Q}_k} (\forall i < n)(|\,[b_0^n, \tau_1) \cap \dot{A}_i\,| > 2)$. We modify "the tail" of $q_{n-1}$ and we assume $(*)_{\tau_1, \sigma_0, \dots, \sigma_{n-1}}$ (for $i \geq n_0$, $q_{n-1}$). Next we choose $n_1 > n_0$ and a nice norm $H_1$ on $[n_0, n_1)$ such that

$H_1([n_0, n_1)) \geq 2$ and if $\mathbb{T}' \geq S_{H_1}((\mathbb{T}_{n_0}^{n-1})^{\mathrm{uh}}, \dots, (\mathbb{T}_{n_1-1}^{n-1})^{\mathrm{uh}})$, $\|\mathbb{T}'\| > 0$, $w \subseteq w' \subseteq L_{n_0}^{n-1}(\langle\rangle)$ then there exists $v \subseteq \mathrm{cont}(\mathbb{T}')$ such that $\langle w' \cup v, \mathbb{T}_{n_1}^{n-1}, \mathbb{T}_{n_1+1}^{n-1}, \dots \rangle$ decides the value of $\tau_1$.

Let $\mathbb{T}_1^n = S_{H_1}((\mathbb{T}_{n_0}^{n-1})^{\mathrm{uh}}, \dots, )\mathbb{T}_{n_1-1}^{n-1})^{\mathrm{uh}})$ and let $b_1^n$ be greater than all possibilities for $\tau_1$ in the above property.
We continue in this fashion and we determine integers $n_0 < n_1 < n_2 < \dots$, $b_0^n < b_1^n < b_2^n < \dots$ and nice norms $H_0, H_1, H_2 \dots$ and we define creatures $\mathbb{T}_i^n = S_{H_i}((\mathbb{T}_{n_{i-1}}^{n-1})^{\mathrm{uh}}, \dots, ((\mathbb{T}_{n_i-1}^{n-1})^{\mathrm{uh}})$. Finally we let $B_n = \{b_0^n, b_1^n, b_2^n, \dots\}$ and $q_n = \langle w, \mathbb{T}_0^n, \mathbb{T}_1^n, \dots \rangle$ (actually we should have taken more care while getting $(*)_{\tau_i, \sigma_0, \dots, \sigma_{n-1}}$ in constructing $\mathbb{T}_i^n$: this is necessary for $\lim\limits_{i \to \infty} \|\mathbb{T}_i^n\| = \infty$).

Suppose now that $Y \in [\omega]^\omega$ is a set $S_k$-localizing $[\omega]^\omega \cap N$. Then for each $n \in \omega$ we have

$$(\exists^\infty i)(\forall m < k)(|\,[\mu_Y(i+m), \mu_Y(i+m+1)) \cap B_n| \geq 2).$$

We inductively define increasing sequences $\langle i_n : n \in \omega\rangle$, $\langle l_n : n \in \omega\rangle$ of integers and a sequence $\langle \mathbb{T}^*_n : n \in \omega\rangle$ of creatures:

Let $i_0$ be such that

$$(\forall m < k)(|\,[\mu_Y(i_0+m), \mu_Y(i_0+m+1)) \cap B_1| \geq 2).$$

Let $j^0_m \in \omega$ (for $m < k$) be such that $\mu_{B_1}(j^0_m)$ is the first element of

$$[\mu_Y(i_0+m), \mu_Y(i_0+m+1)) \cap B_1.$$

Let $\mathbb{T}^*_0 = S(\mathbb{T}^1_{j^0_0+1}, \mathbb{T}^1_{j^0_1+1}, \ldots, \mathbb{T}^1_{j^0_{k-1}+1})$. Note that the creatures $\mathbb{T}^1_j$ were obtained as results of the operation $S_{H_j}$ (for some norms $H_j$) and hence their roots (i.e. $\langle\rangle$) are $> k$-splitting points, thus no danger can appear in this procedure. Finally we choose $l_0$ such that $\mathbb{T}^*_0 \in \Sigma^*(\mathbb{T}_0, \ldots, \mathbb{T}_{l_0-1})$.

Assuming that we have defined $i_n$, $l_n$, $\mathbb{T}^*_n$, take $i_{n+1} > i_n + k$ such that

$$(\forall m < k)(|\,[\mu_Y(i_{n+1}+m), \mu_Y(i_{n+1}+m+1)) \cap B_{n+2}| \geq 2)$$

and if $\mu_{B_{n+2}}(j) \in [\mu_Y(i_{n+1}), \mu_Y(i_{n+1}+1))$ then $\mathbb{T}^{n+2}_j \in \Sigma^*(\mathbb{T}_{l_n}, \ldots, \mathbb{T}_m)$ (for some $m > l_n$) and $\|\mathbb{T}^{n+2}_j\| \geq n+1$. Now, let $j^{n+1}_m$ (for $m < k$) be such that $\mu_{B_{n+2}}(j^{n+1}_m)$ is the first element of $[\mu_Y(i_{n+1}+m), \mu_Y(i_{n+1}+m+1)) \cap B_{n+2}$. Finally, $\mathbb{T}^*_{n+1} = S(\mathbb{T}^{n+2}_{j^{n+1}_0+1}, \ldots, \mathbb{T}^{n+2}_{j^{n+1}_{k-1}+1})$ and $l_{n+1}$ is such that

$$\mathbb{T}^*_{n+1} \in \Sigma^*(\mathbb{T}_{l_n}, \ldots, \mathbb{T}_{l_{n+1}-1}).$$

The condition $q$ is $\langle w, \mathbb{T}^*_0, \mathbb{T}^*_1, \ldots\rangle$. Clearly $q \geq p$. To show that it is $(N, \mathbb{Q}_k)$-generic suppose that $\sigma \in N$ is a $\mathbb{Q}_k$-name for an ordinal, say $\sigma = \sigma_n$, and let $q' = \langle w', \mathbb{T}'_0, \mathbb{T}'_1, \ldots\rangle \geq q$ be a condition deciding $\sigma$. Look at the construction of $q_{n+1} = \langle w, \mathbb{T}^{n+1}_0, \mathbb{T}^{n+1}_1, \ldots\rangle$. Because of $(*)_{\tau_0,\sigma_0,\ldots,\sigma_n}$, if $v \subseteq M^{n+1}_i(\langle\rangle)$ and there is (in $N$) a pure extension of $\langle w \cup v, \mathbb{T}^{n+1}_{i+1}, \mathbb{T}^{n+1}_{i+2}, \ldots\rangle$ deciding $\sigma_n$ then $\langle w \cup v, \mathbb{T}^{n+1}_{i+1}, \mathbb{T}^{n+1}_{i+2}, \ldots\rangle$ decides it already.

Let $i$ be such that $w' \subseteq M^{n+1}_i(\langle\rangle)$. Then there exists a pure extension of the condition $\langle w', \mathbb{T}^{n+1}_{i+1}, \mathbb{T}^{n+1}_{i+2}, \ldots\rangle$ deciding $\sigma_n$, e.g. that one which can be obtained from $q'$ (note that $q'$ and $\langle w', \mathbb{T}^{n+1}_{i+1}, \mathbb{T}^{n+1}_{i+2}, \ldots\rangle$ are compatible). By the elementarity of $N$ there is such an extension in $N$. This implies that $\langle w', \mathbb{T}^{n+1}_{i+1}, \mathbb{T}^{n+1}_{i+2}, \ldots\rangle$ decides $\sigma_n$ (and "the decision" belongs to $N$). The conditions $q'$ and $\langle w', \mathbb{T}^{n+1}_{i+1}, \mathbb{T}^{n+1}_{i+2}, \ldots\rangle$ are compatible, so the values given by them to $\sigma_n$ are the same and we are done (with the genericity).

Now we want to show that

$$q \Vdash_{\mathbb{Q}_k} (\forall X \in N[\dot{G}_{\mathbb{Q}_k}] \cap [\omega]^\omega)((X, Y) \in S_k).$$

Let $\dot{A} \in N$ be a $\mathbb{Q}_k$-name for an infinite subset of $\omega$, say $\dot{A} = \dot{A}_m$. We are going to prove that

$$q \Vdash_{\mathbb{Q}_k} (\exists^\infty n \in \omega)(\forall i < k)(|\,[\mu_Y(n+i), \mu_Y(n+i+1)) \cap \dot{A}_m\,| \geq 2).$$

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Let $q' = \langle w', \mathbb{T}'_0, \mathbb{T}'_1, \ldots \rangle \geq q$. Fix $l > m$. Look at $\mathbb{T}'_l$ – for some $m'$, $m''$, $l < m' < m''$ we have $\mathbb{T}'_l \in \Sigma^*(\mathbb{T}^*_{m'}, \ldots, \mathbb{T}^*_{m''})$ and $w' \subseteq \bigcup_{i<m'} \mathrm{cont}(\mathbb{T}^*_i)$. So we find $n > l$ and $t \in \mathbb{T}'_l$ such that $\mathbb{T}'_l$ "above $t$" comes from $\mathbb{T}^*_n$ by decreasing norms (like in the operation of taking the upper half but possibly with other values subtracted) and refining. Note that neccessarily $|\mathrm{succ}_{\mathbb{T}'_l}(t)| = k$ and $\mathbb{T}'_l$ above each successor of $t$ refines some $\mathbb{T}^{n+1}_{j^n_i+1}$ (for $i < k$) modulo decreasing norms by some values, which does not influence contributions. Next we find $v \subseteq \mathrm{cont}(\mathbb{T}'_l)$ (actually $v = v_0 \cup \ldots \cup v_{k-1}$, $v_i$ is included in the contributions of the part above that successor of $t$ which refines $\mathbb{T}^{n+1}_{j^n_i+1}$) such that $\langle w' \cup v, \mathbb{T}'_{l+1}, \mathbb{T}'_{l+2}, \ldots \rangle$ decides the values of the names $\tau_{j^n_0+1}, \ldots, \tau_{j^n_{k-1}+1}$ (defined in the construction of $\mathbb{T}^{n+1}_j$'s) and forces them to be bounded by $b^{n+1}_{j^n_0+1}, \ldots, b^{n+1}_{j^n_{k-1}+1}$, respectively. By the choice of $\tau_j$'s this means that

$$\langle w' \cup v, \mathbb{T}'_{l+1}, \mathbb{T}'_{l+2}, \ldots \rangle \Vdash_{\mathbb{Q}_k} (\forall i < k)(\forall j < n)(|\, [b^{n+1}_{j^n_i}, b^{n+1}_{j^n_i+1}) \cap \dot{A}_j \,| \geq 2).$$

Since $m < l \leq n$ and $\mu_Y(i_n + i) \leq \mu_{B_{n+1}}(j^n_i) < \mu_{B_{n+1}}(j^n_i + 1) < \mu_Y(i_n + i + 1)$ (for all $i < k$) we get

$$\langle w' \cup v, \mathbb{T}'_{l+1}, \mathbb{T}'_{l+2}, \ldots \rangle \Vdash_{\mathbb{Q}_k} (\forall i < k)(|\, [\mu_Y(i_n + i), \mu_Y(i_n + i + 1)) \cap \dot{A}_m \,| \geq 2).$$

As $l > m$ was arbitrary we are done. The Main Claim and the theorem are proved. □□

*Remark.* More examples of forcing notions distinguishing the localization properties we have introduced in this section will be presented in [RoSh:470]. They are (like the forcing notion $\mathbb{Q}_k$) applications of the general schemata of that paper.

## 3 Cardinal coefficients related to the localizations

In this section we discuss cardinal coefficients related to the localization properties introduced earlier.

Following Vojtáš (cf [Voj]) with any relation $R \subseteq X \times Y$ we may associate two cardinal numbers (the unbounded and the dominating number for $R$):

$$\mathfrak{b}(R) = \min\{|B| : (\forall y \in Y)(\exists x \in B)((x, y) \notin R)\}$$

$$\mathfrak{d}(R) = \min\{|D| : (\forall x \in X)(\exists y \in D)((x, y) \in R)\}.$$

For purposes of applications these cardinals are introduced for relations $R \subseteq X \times Y$ such that $\mathrm{dom}(R) = \mathrm{dom}(cR) = X$ and $\mathrm{rng}(R) = \mathrm{rng}(cR) = Y$. Note that for each such relation we have

$$\mathfrak{b}(R) = \mathfrak{d}(cR^{-1}), \quad \mathfrak{d}(R) = \mathfrak{b}(cR^{-1}), \quad \mathfrak{b}(R^{-1}) = \mathfrak{d}(cR), \quad \mathfrak{d}(R^{-1}) = \mathfrak{b}(cR).$$

All results of the previous sections provide information on dominating numbers $\mathfrak{d}(R)$ for the considered relations. Let

$$\mathfrak{b} = \mathfrak{b}(\leq^*) = \min\{|F| : F \subseteq \omega^\omega \ \& \ (\forall x \in \omega^\omega)(\exists y \in F)(\exists^\infty n \in \omega)(x(n) < y(n))\}$$

$\mathfrak{d} = \mathfrak{d}(\leq^*) = \min\{|F| : F \subseteq \omega^\omega \ \& \ (\forall x \in \omega^\omega)(\exists y \in F)(\forall^\infty n \in \omega)(x(n) \leq y(n))\}$

$$non(\mathbb{K}) = \min\{|X| : X \subseteq 2^\omega \ \& \ X \text{ is not meager }\}.$$

These are three of ten cardinal invariants forming the Cichoń Diagram. For more information on the cardinals related to measure and category see [BJSh:368] or [CiPa].

**Corollary 3.1**

1. *(see 1.2)* $(\forall k \in \omega)(\mathfrak{d}(R^\forall_k) = \mathfrak{c})$
2. *(see 1.3)* $(\forall k > 0)(\mathfrak{d}(cR^\forall_k) = \mathfrak{b})$
3. *(see 1.4)* $(\forall k \in \omega)(\mathfrak{d}(cR^\exists_k) = \mathfrak{d})$
4. *(see 1.6)* $(\forall k \in \omega)(\mathfrak{d}(R^\exists_k) = \mathfrak{d}(R^\exists_0))$
5. *(see 1.9)* $\mathfrak{b} \leq \mathfrak{d}(R^\exists_0) \leq non(\mathbb{K})$

**Corollary 3.2**

1. Con$(\mathfrak{d}(R^\exists_0) < \mathfrak{d})$
2. Con$(\mathfrak{d} < \mathfrak{d}(R^\exists_0))$

*Proof.* For the first model add $\aleph_2$ Cohen reals to a model of CH. Then in the extension we will have $\mathfrak{d} = \aleph_2$ and $\mathfrak{d}(R^\exists_0) = \aleph_1$ (see 1.10). The second model can be obtained by adding $\aleph_2$ random reals to a model of CH, which results in a model for $\mathfrak{d} = \aleph_1$ and $\mathfrak{d}(R^\exists_0) = \aleph_2$ (see 1.10). $\square$

**Corollary 3.3** *Let $\phi \in \omega^\omega$ be an increasing function. Then:*

1. *(see 2.1)* $\mathfrak{b} = \mathfrak{d}(S_1) \leq \mathfrak{d}(S_2) \leq \ldots \leq \mathfrak{d}(S_k) \leq \ldots \leq \mathfrak{d}(S_+)$,
   $\mathfrak{d}(S_+) \leq \mathfrak{d}(S_{+\epsilon}), \mathfrak{d}(S^\phi_+) \leq \mathfrak{d}$, *and*
   *if $\phi$ is increasing fast enough (e.g. $\phi(n) > 2^{2n}$) then* $\mathfrak{d}(S_{+\epsilon}) \leq \mathfrak{d}(S^\phi_+)$,
2. *(see 2.2)* $\mathfrak{d}(S^\phi_+) \leq non(\mathbb{K})$. $\square$.

**Corollary 3.4** *Let $\phi \in \omega^\omega$ be an increasing function. Then*

$$\text{Con}(\mathfrak{d}(S^\phi_+) = \aleph_1 + non(\mathbb{K}) = \mathfrak{d} = \aleph_2).$$

*Proof.* Start with a model for CH and add to it first $\aleph_2$ Cohen reals (what causes $\mathfrak{d} = \aleph_2$ but keeps $\mathfrak{d}(S^\phi_+) = \aleph_1$) and next add $\aleph_2$ random reals (what preserves $\mathfrak{d} = \aleph_2$, $\mathfrak{d}(S^\phi_+) = \aleph_1$ but causes $non(\mathbb{K}) = \aleph_2$), see 2.3. $\square$

**Corollary 3.5** *Let $k > 0$. Then*

$$\text{Con}(\mathfrak{d}(S_k) < \mathfrak{d}(S_{k+1})).$$

*Proof.* Let $\mathbb{Q}_k$ be the forcing notion from the proof of 2.4. It is proper (see 2.4.1). To get the respective model it is enough to take the countable support iteration of the length $\omega_2$ of the forcing notions $\mathbb{Q}_k$ over a model of CH. As $\mathbb{Q}_k$ does not have the $S_{k+1}$-localization we easily get that in the resulting model we will have $\mathfrak{d}(S_{k+1}) = \aleph_2$. The only problem is to show that the iteration has the $S_k$-localization property (to conclude that in the extension $\mathfrak{d}(S_k) = \aleph_1$). But this is an application of §3 Chapter XVIII of [Sh:f]. We may think of $S_k$ as a relation on $\omega^\omega$ (after canonical mapping). Keeping the notation of [Sh:f] we put:

$S \subseteq \mathscr{S}_{<\aleph_1}(H(\aleph_1)^{\mathbf{V}_1})$ and for $a \in S$, $\mathbf{g}_a \in \omega^\omega$ is such that for each $f \in a \cap \omega^\omega$, $f\ S_k\ \mathbf{g}_a$, $\bar{\mathbf{g}} = \langle \mathbf{g}_a : a \in S\rangle$;
$\alpha^* = 1$; $\bar{R} = \langle R_0\rangle = \langle S_k\rangle$.

Note that $(\bar{R}, S, \mathbf{g})$ strongly covers iff it covers iff $S$ is stationary (for "strongly covers" we are in Possibility B). The property $(S_k^*)_{\mathbb{Q}_k}$ of claim 2.4.1(2) guarantees that the forcing notion $\mathbb{Q}_k$ is $(\bar{R}, S, \mathbf{g})$-preserving. Hence Theorem 3.6 of Chapter XVIII of [Sh:f] applies to this situation and the iteration is $(\bar{R}, S, \mathbf{g})$-preserving. Consequently we are done. □

**Problem 3.6** *Are the following consistent:*

1. $\mathfrak{b} < \mathfrak{d}(S_2) < \mathfrak{d}(S_3) < \ldots \mathfrak{d}(S_k) < \mathfrak{d}(S_{k+1}) < \ldots$ *?*
2. *There exists a sequence $\langle \phi_\alpha : \alpha < \omega_1\rangle \subseteq \omega^\omega$ of increasing functions such that*

$$\alpha < \beta < \omega_1 \ \Rightarrow\ \mathfrak{d}(S_+^{\phi_\alpha}) < \mathfrak{d}(S_+^{\phi_\beta})?$$

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