LESS SATURATED IDEALS

MOTI GITIK AND SAHARON SHELAH

(Communicated by Andreas R. Blass)

ABSTRACT. We prove the following:

(1) If κ is weakly inaccessible then NS_κ is not κ^+-saturated.
(2) If κ is weakly inaccessible and $\theta < \kappa$ is regular then NS_κ^θ is not κ^+-saturated.
(3) If κ is singular then $NS_{\kappa^+}^{cf\kappa}$ is not κ^{++}-saturated.

Combining this with previous results of Shelah, one obtains the following:

(A) If $\kappa > \aleph_1$ then NS_κ is not κ^+-saturated.
(B) If $\theta^+ < \kappa$ then NS_κ^θ is not κ^+-saturated.

0. INTRODUCTION

By a classical result of R. Solovay [So] every stationary subset S of a regular cardinal κ can be split into κ disjoint stationary subsets. Or, in other terminology, $NS_\kappa \upharpoonright S$, the nonstationary ideal over κ restricted to a stationary subset S of κ, is not κ-saturated.

A natural question is if it is possible to replace κ-saturated ness by κ^+-saturatedness, i.e. instead of κ completely disjoint stationary subsets to have κ^+ with pairwise nonstationary intersection. K. Namba [Na] proved that NS_κ cannot be κ^+-saturated over a measurable κ. J. Baumgartner, A. Taylor and S. Wagon [Ba-Ta-Wa] improved this to a greatly Mahlo. S. Shelah [Sh1] showed that $NS_{\kappa^+} \upharpoonright S$ is not κ^{++}-saturated if for some regular $\theta < \kappa$, $\theta \neq \text{cf } \kappa$, $S \cap \{\alpha < \kappa^+ | cf\alpha = \theta\}$ is stationary. Actually a much more general result is proved there. For $\kappa = \aleph_1$, it is possible to have NS_κ κ^+-saturated, as was shown by J. Steel and R. Van Wesep [St-V] from AD_R, by H. Woodin [Wo1] from AD, by M. Foreman, M. Magidor and S. Shelah [Fo-Ma-Sh] from a supercompact and by S. Shelah [Sh6] from a Woodin cardinal. Also $NS_{\kappa^+} \upharpoonright S$ can be κ^{++}-saturated for a regular κ and $S \subseteq \{\alpha < \kappa^+ \mid \text{cf } \alpha = \kappa\}$, as was shown by T. Jech and H. Woodin [Je-Wo] building on K. Kunen's [Ku] construction of a saturated ideal. If κ is singular, then, by M. Foreman [Fo] and T. Jech, H. Woodin [Je-Wo], $NS_{\kappa^+} \upharpoonright S$ can be κ^{++}-saturated for $S \subseteq \{\alpha < \kappa^+ \mid \text{cf } \alpha = \text{cf } \kappa\}$.

If κ is an inaccessible then T. Jech and H. Woodin [Je-Wo] showed that $NS_\kappa \upharpoonright$ Regulars can be κ^+-saturated, and in [Gi2] was shown the consistency of

$NS_\kappa \restriction S$ κ^+-saturated for $S \subseteq \kappa$ such that $S \cap \{\alpha < \kappa \mid \mathrm{cf}\ \alpha = \theta\}$ is stationary for every regular $\theta < \kappa$.

The purpose of the present paper is to show that NS_κ and $NS_\kappa^\theta = NS_\kappa \restriction \{\alpha \mid \mathrm{cf}\ \alpha = \theta\}$ cannot be κ^+-saturated where κ is weakly inaccessible and $\theta < \kappa$. And for singular κ, $NS_{\kappa^+}^{\mathrm{cf}\,\kappa}$ cannot be κ^{++}-saturated.

Combining this with the previous result of Shelah [Sh1], one gets the following:

Theorem. *Let κ and θ be regular cardinals. Then:*

(1) *If $\kappa > \aleph_1$ then NS_κ is not κ^+-saturated.*
(2) *If $\theta^+ < \kappa$ then NS_κ^θ is not κ^+-saturated.*

The proof is based on a certain combinatorial principle which follows from saturatedness. It was considered independently by both authors. Actually they came to it from different directions. Gitik was dealing with indiscernibles of the Mitchell Covering Lemma and showed it first under $\neg \exists \kappa o(\kappa) = \kappa^{++}$ in [Gi3], and then removed this assumption in [Gi4]. Shelah [Sh2, Sh3, Dz-Sh] went through his club guessing machinery and eventually obtained a better result not appealing to some GCH assumptions used in [Gi4]. He also realized that this leads to inconsistency of saturatedness of NS_κ^θ over small inaccessibles. A minor change in this argument by Gitik led to the full result for inaccessibles. Shelah saw that it gives also the nonsaturatedness of $NS_{\kappa^+}^{\mathrm{cf}\,\kappa}$ for a singular κ.

We are grateful to A. Blass, M. Kojman, P. Matet and the referee of the paper for their remarks and corrections.

1. Main result

Let κ be a regular uncountable cardinal. We denote by NS_κ the nonstationary ideal over κ. For a set $S \subseteq \kappa$ let $NS_\kappa \restriction S$ denote the collection of all subsets of κ having a nonstationary intersection with S. For a regular $\theta < \kappa$ let

$$S_\kappa^\theta = \{\alpha < \kappa \mid \mathrm{cf}\ \alpha = \theta\}\ .$$

We denote $NS_\kappa \restriction S_\kappa^\theta$ simply by NS_κ^θ.

An ideal I over κ is called λ-saturated if there is no sequence $\langle A_\alpha \mid \alpha < \lambda \rangle$ so that

(a) $A_\alpha \subseteq \kappa$,
(b) $A_\alpha \notin I$,
(c) $A_\alpha \cap A_\beta \in I$ whenever $\alpha \neq \beta$.

Our main objective will be the non-κ^+-saturatedness of NS_κ^θ. So from here on, by a saturated ideal over a cardinal λ we will mean a λ^+-saturated ideal.

We will prove the following:

Theorem 1. (1) *If κ is weakly inaccessible then NS_κ cannot be saturated.*
(2) *If κ is weakly inaccessible and $\theta < \kappa$ is regular then NS_κ^θ cannot be saturated.*
(3) *If κ is a singular cardinal then $NS_{\kappa^+}^{\mathrm{cf}\,\kappa}$ cannot be saturated.*

Actually, the proof will show the Theorem of the Introduction for every regular $\kappa > \aleph_2$.

The proof will consist of two parts. The first will introduce a combinatorial principle and show its inconsistency. The second will show that the saturatedness of NS_κ^θ implies the principle.

Definition 2. For regular cardinals κ, θ with $\kappa > \aleph_2$ and $\kappa > \theta^+$ let $\diamondsuit^*_{\text{club}}(\kappa, \theta)$ denote the following: there exists a sequence $\langle S_\alpha \mid \alpha \in S^\theta_\kappa \rangle$ so that for every $\alpha \in S^\theta_\kappa$

(1) $S_\alpha \subseteq \alpha$,
(2) $\sup S_\alpha = \alpha$,
(3) $|S_\alpha| = \theta$,
(4) if α is a limit of ordinals of cofinality $> \theta$ (and $> \aleph_1$ if $\theta = \aleph_0$), then for every $\beta \in S_\alpha$ cf $\beta > \theta$ and if $\theta = \aleph_0$ then cf $\beta > \aleph_1$,
(5) for every club $C \subseteq \kappa$ the set

$$\{\alpha \in S^\theta_\kappa \mid \exists \beta < \alpha \; C \supseteq S_\alpha \backslash \beta\}$$

contains a club intersected with S^θ_κ.

Notice that all but nonstationary many α's in S^θ_κ are limits of ordinals of cofinality $> \theta$.

This is a strengthening of Shelah's club guessing principles; see [Sh2, Sh3]. It turns out that it is too strong. Namely:

Lemma 3. $ZFC \vdash \neg \diamondsuit^*_{\text{club}}(\kappa, \theta)$.

Proof. Suppose otherwise. Let $\langle S_\alpha \mid \alpha \in S^\theta_\kappa \rangle$ be a sequence witnessing $\diamondsuit^*_{\text{club}}(\kappa, \theta)$.

Let us split the proof into two cases according to whether $\theta > \aleph_0$ or $\theta = \aleph_0$.

Case 1. $\theta > \aleph_0$.

We define by induction a sequence $\langle E_n \mid n < \omega \rangle$ of clubs of κ. Let E_0 be the set of ordinals below κ which are limits of ordinals of cofinality $> \theta$. If E_n is defined then consider E'_n, the set of limit points of E_n. There exists a club $C \subseteq \kappa$ such that for every $\alpha \in C \cap S^\theta_\kappa$, E'_n contains a final segment of S_α. Set $E_{n+1} = E'_n \cap C$.

Finally let $E = \bigcap_{n<\omega} E_n$. Then E is a club. Let $\delta = \min\left(E \cap S^\theta_\kappa\right)$. For every $n < \omega$, E'_n contains a final segment of S_δ since $\delta \in E_{n+1}$. But cf $\delta = \theta > \aleph_0$. So a final segment of S_δ is contained in E. Pick some $\beta \in E \cap S_\delta$. By Definition 2(4), cf $\beta > \theta$. Since $E'_n \supseteq E_{n+1}$, $\beta \in E'_n$ for every $n < \omega$. So it is a limit point of E_n. Hence $E_n \cap \beta$ is a club of β, for every $n < \omega$. Since cf $\beta > \aleph_0$, also $E \cap \beta$ is a club of β. But cf $\beta > \theta$. Hence there is some $\gamma \in E \cap \beta$ of cofinality θ, which contradicts the minimality of δ.

Case 2. $\theta = \aleph_0$

We define a decreasing sequence of clubs as above of length $\aleph_1$ taking intersections at countable limit stages. Let $\langle E_i \mid i < \omega_1 \rangle$ be such a sequence and let $E = \bigcap_{i<\omega_1} E_i$. As in Case 1 pick δ to be $\min\left(E \cap S^{\aleph_0}_\kappa\right)$. By Definition 2(3), $|S_\delta| = \omega$. Let $\langle s_n \mid n < \omega \rangle$ be a cofinal in δ sequence in S_δ. Since for every $i < \omega_1$ E_{i+1} contains a final segment of S_δ, there will be some $n^* < \omega$ such that for $\aleph_1$ i's we have $E_{i+1} \supseteq \{s_n \mid n \geq n^*\}$. But the sequence $\langle E_i \mid i < \omega_1 \rangle$ is a decreasing sequence. So for every $i < \omega_1$, $E_i \supseteq \{s_n \mid n \geq n^*\}$. Hence there is $\beta \in E \cap S_\delta$. Notice that by Definition 2(4), cf $\beta > \aleph_1$. Now we continue toward the contradiction as in Case 1. □

Lemma 4. *Suppose that κ, θ are regular cardinals and $\theta^+ < \kappa$.*
If NS^θ_κ is saturated then $\diamondsuit^*_{\text{club}}(\kappa, \theta)$ holds.

Remark. Both authors arrived to this statement independently and from different directions. Gitik, dealing with indiscernibles of Mitchell's Covering Lemma, showed this under $\neg\left(o(\alpha) = \alpha^{++}\right)$ and $2^\theta < \kappa$ in [Gi3], and much later in [Gi4] realized that

$\neg\big(o(\alpha) = \alpha^{++}\big)$ is not needed. Shelah went through his club guessing principles [Sh2, Sh3] and got a better result, requiring only $\theta^+ < \kappa$ and not $2^\theta < \kappa$. The proof below follows his lines.

Proof. Let $\diamond'_{\text{club}}(S)$ for S a stationary subset of S_κ^θ be defined as $\diamond^*_{\text{club}}(\kappa, \theta)$ only with S replacing S_κ^θ and with Definition 2(5) replaced by the following:

(5)′ for every club $C \subseteq \kappa$ the set

$$\{\alpha \in S \mid \exists \beta < \alpha \quad C \supseteq S_\alpha \backslash \beta\}$$

is stationary.

It was shown by Shelah [Sh2, Sh3] that $\diamond'_{\text{club}}(S)$ and even stronger principles are true in ZFC. For the benefit of the reader we will present a proof of $\diamond'_{\text{club}}(S)$ below. But first let us use it to complete the proof of the lemma.

Claim 2.1. For every stationary $S \subseteq S_\kappa^\theta$ and every $\diamond'_{\text{club}}(S)$ sequence $\langle S_\alpha \mid \alpha \in S\rangle$ there exists a stationary $S^* \subseteq S$ such that for every club $C \subseteq \kappa$ the set

$$\{\alpha \in S^* \mid \exists \beta < \alpha \quad C \supseteq S_\alpha \backslash \beta\}$$

contains a club intersected with S^*.

Proof. Suppose otherwise. Let S be a set witnessing this. We define by induction an almost (modulo NS_κ) decreasing sequence $\langle C_\alpha \mid \alpha < \kappa^+\rangle$ of clubs of κ and an almost disjoint sequence $\langle A_\alpha \mid \alpha < \kappa^+\rangle$ of stationary subsets of S. Denote for a club C by $N(C)$ the set of places where C is guessed, i.e.

$$\{\alpha \in S \mid \exists \beta < \alpha \quad C \supseteq S_\alpha \backslash \beta\} .$$

Pick C_0 to be a club such that $S \backslash N(C_0)$ is stationary. Set $A_0 = S \backslash N(C_0)$. Suppose now that $\langle C_\beta \mid \beta < \alpha\rangle$ and $\langle A_\beta \mid \beta < \alpha\rangle$ are defined. We want to define C_α and A_α. First take a club C which is almost contained in every C_β for $\beta < \alpha$. Consider $N(C)$. Obviously $N(C)$ is almost contained in every $N(C_\beta)$ for $\beta < \alpha$. Let C_α be a club subset of C witnessing that $N(C)$ is not good, i.e. $N(C) \backslash N(C_\alpha)$ is stationary. Set $A_\alpha = N(C) \backslash N(C_\alpha)$. This completes the inductive definition.

The existence of the sequence $\langle A_\alpha \mid \alpha < \kappa^+\rangle$ contradicts the saturatedness of NS_κ^θ.   □ (claim)

Now using saturatedness, it is easy to glue together the sets S^* given by Claim 2.1 and to get $\diamond^*_{\text{club}}(\kappa, \theta)$. Just pick a maximal pairwise almost disjoint collection of such S^*'s. Since NS_κ^θ is saturated, it consists of at most κ sets. Then make them completely disjoint using the normality of NS_κ^θ. Finally, we put together the sequences witnessing $\diamond'_{\text{club}}$ and obtain $\diamond^*_{\text{club}}(\kappa, \theta)$. This completes the proof of the lemma.   □

Remark. We can replace NS_κ^θ by any normal ideal over κ concentrating on S_κ^θ in Lemma 4. Only the definition of $\diamond^*_{\text{club}}(\kappa, \theta)$ must be changed in the obvious way. The same proof works. We refer to [Gi4] for generalizations for ideals preserving 2^θ and to Dzamonja, Shelah [Dz-Sh] for further generalizations in this direction.

Let us now present a proof of Shelah's club guessing principle; see [Sh2, Sh3, Br-Ma] for variations of it.

Proposition 5. *Suppose that κ, θ are regular and $\kappa > \max(\theta^+, \aleph_2)$. Then $\diamond'_{\text{club}}(S)$ holds for every stationary $S \subseteq S_\kappa^\theta$.*

Remark. Actually, we will show that there is a sequence $\langle S_\alpha \mid \alpha \in S\rangle$ witnessing $\diamond'_{\text{club}}(S)$ so that for every club $C \subseteq \kappa$, $\{\alpha \in S \mid C \supseteq S_\alpha\}$ is stationary.

Proof. Suppose for simplicity that $\theta^+ \geq \aleph_2$. Otherwise we do the same only with θ^+ replaced by θ^{++}.

Suppose that for some stationary $S \subseteq S_\kappa^\theta$, $\diamondsuit'_{\text{club}}(S)$ fails. We shall define sequences $\langle C_i \mid i < \theta^+ \rangle$ of clubs of κ, $\langle T_\alpha^i \mid i < \theta^+, \ \alpha \in S \rangle$ of trees and $\langle S_\alpha^i \mid i < \theta^+, \alpha \in S \rangle$. However, first for a club C with nonlimit points of cofinality $> \theta$ and an ordinal $\alpha \in S_\kappa^\theta$ which is a limit of ordinals of cofinality $> \theta$, let us define canonically a tree $T_\alpha(C)$. Notice that every club contains a subclub with nonlimit points of cofinality $> \theta$. Also all but nonstationary many α's are limits of ordinals of cofinality $> \theta$. The first level of $T_\alpha(C)$ will consist of a closed cofinal in α sequence of order type θ so that each nonlimit point of it has cofinality $> \theta$. We pick such a sequence to be the least in some fixed well-ordering.

Now let η be a point from the first level. We want to define the set of its immediate successors, i.e., the set $Suc_{T_\alpha(C)}(\eta)$. If η is a limit point then $Suc_{T_\alpha(C)}(\eta) = \emptyset$. Otherwise, let η^* be the largest point of the first level below η, or 0 if η is the least one. Consider $\eta' = \sup(C \cap \eta + 1)$. If $\eta' \leq \eta^*$ then set $Suc_{T_\alpha(C)}(\eta) = \emptyset$. Suppose that $\eta^* < \eta' \leq \eta$. If cf $\eta' > \theta$ then set $Suc_{T_\alpha(C)}(\eta) = \{\eta'\}$. Such η' will be a leaf, i.e. $Suc_{T_\alpha(C)}(\eta') = \emptyset$. If cf $\eta' \leq \theta$, then, as above, we pick the least closed cofinal in η' sequence of order type of cf η' with nonlimit points of cofinality $> \theta$ and the first element above η^* if it exists. Otherwise, set $Suc_{T_\alpha(C)}(\eta') = \emptyset$. The set of immediate successors of η will consist of this sequence. Using it, we continue to define $T_\alpha(C)$ above η in the same fashion.

Obviously the tree $T_\alpha(C)$ so defined is well founded and of cardinality $\leq \theta$.

Now let us turn to the definition of the sequences. Let C_0 be a a club with nonlimit points of cofinality $> \theta$. For $\alpha \in S$ which is a limit of ordinals of cofinality $> \theta$ set $T_\alpha^0 = T_\alpha(C_0)$ and $S_\alpha^0 =$ the set of all points of all the levels of T_α^0 which are in C_0 and have cofinality $> \theta$. Clearly, for all but nonstationarily many α's in S, S_α^0 is unbounded in α. This will be the case for every $\alpha \in S$ which is a limit of elements of C_0 of cofinality $> \theta$.

From here on we ignore this nonstationary set. In order to have S_α^0 defined for every $\alpha \in S$ we may set S_α^0 to be an unbounded in α sequence of order type θ whenever S_α^0 was not defined or was defined and was bounded in α.

Now let C_1 be a club subset of C_0 with every nonlimit point of cofinality $> \theta$ witnessing the failure of $\langle S_\alpha^0 \mid \alpha \in S \rangle$ to be a $\diamondsuit'_{\text{club}}(S)$ sequence. We define T_α^1 and S_α^1 as above with C_1 replacing C_0. Continue by induction. At limit stages i we take C_i to be a club subset of $\bigcap_{j<i} C_j$ with nonlimit points of cofinality $> \theta$.

Finally, let $D = \bigcap_{i<\theta^+} C_i$. Then D is a club since $\kappa > \theta^+$. Let $\alpha \in D \cap S$ be so that elements of D of cofinality $> \theta$ are unbounded in it. Let us show that the trees $T_\alpha(C_i)$ $(i < \theta^+)$ must stabilize, i.e., starting with some $i_\alpha < \theta^+$, $T_\alpha(C_i) = T_\alpha(C_j)$. Otherwise, pick some $\eta_0 \in Lev_1(T_\alpha^0)$ such that there is no stabilization above it.

Notice that the first level in all the trees is the same. Then η_0 cannot be a limit point since otherwise it will have the empty set of successors. Let η_0^* be the largest point of the first level below η_0, or 0 if η_0 is the least one. Since there is no stabilization above η_0 in the trees and the C_i's are decreasing, the sequence $\langle \eta_{0i} \mid i < \theta^+ \rangle$ where $\eta_{0i} = \sup\left(C_i \cap (\eta_0 + 1)\right)$ will be a nonincreasing sequence of ordinals inside the interval $(\eta_0^*, \eta_0]$. Hence it is eventually constant. So there is some $\eta_1 \in (\eta_0^*, \eta_0)$, cf $\eta_1 \leq \theta$, such that $\eta_{0i} = \eta_1$ starting with some $i(1) < \theta^+$. By the definition of the trees the set of immediate successors of η_1 will be the same in every $T_\alpha(C_i)$ $(i \geq i(1))$. Pick η_2 to be one of them with no stabilization above it.

Deal with it as we did with η_0. We will obtain η_3 and η_4. Continue in the same fashion. This process will produce an infinite decreasing sequence of ordinals.

Hence, there is $i_\alpha < \theta^+$ such that $T_\alpha^i = T_\alpha^{i_\alpha}$ for every $i \geq i_\alpha$. Then there are $i^* < \theta^+$ and a stationary $S^* \subseteq S$ such that for every $\alpha \in S^*$ we have $i_\alpha = i^*$ and $\sup S_\alpha^{i^*} = \alpha$. But this contradicts the choice of C_{i^*+1}. $\square$

2. Some open problems on the nonstationary ideal

The following is probably the most interesting problem left open on this subject:

Problem 1. Let κ be a regular uncountable cardinal. Can $NS_{\kappa^+}^\kappa$ be saturated?

By Shelah [Sh1] the answer is negative if we replace $NS_{\kappa^+}^\kappa$ by $NS_{\kappa^+}^\theta$ for some $\theta < \kappa$.

Now for $\kappa = \aleph_0$, the models with $NS_{\aleph_1}$ saturated were constructed by J. Steel and R. Van Wesep [St-V], by H. Woodin [Wo1] using AD_R and AD, then by M. Foreman, M. Magidor and S. Shelah [Fo-Ma-Sh] from a supercompact and by S. Shelah [Sh6] from a Woodin cardinal. By J. Steel [St] "There are a saturated ideal over $\aleph_1$ and a measurable" implies an inner model with a Woodin cardinal. However, the following basic question remains open:

Problem 2. Is $GCH + NS_{\aleph_1}$ is saturated consistent?

A very tight connection of $\neg CH$ with saturatedness of $NS_{\aleph_1}$ was established by H. Woodin. He showed that "$NS_{\aleph_1}$ is saturated and there is a measurable" implies $\neg CH$.

If we relax our assumptions and consider saturatedness of $NS_\kappa \upharpoonright S$ for a stationary set S, then by T. Jech and H. Woodin $NS_{\kappa^+} \upharpoonright S$ can be saturated for every κ. Also $NS_\kappa \upharpoonright$ Regulars can be saturated for inaccessible κ. An almost huge is used for the first result and a measurable for the second. By [Gi2], $NS_\kappa \upharpoonright S$ can be saturated over an inaccessible κ with S having a stationary intersection with every S_κ^θ for θ regular less than κ; $o(\kappa) = \kappa$ is used for this and it is necessary. But in order to obtain the same over the first inaccessible, supercompacts are used.

Problem 3. How strong is $NS_\kappa \upharpoonright S$ saturated for the first inaccessible κ and $S \subseteq S_\kappa^\theta$, $\theta < \kappa$ regular?

One can try to get simultaneously many S's for which $NS_\kappa \upharpoonright S$ is saturated. So the following is natural: we call NS_κ densely saturated if for every stationary $S \subseteq \kappa$ there is $S^* \subseteq S$ such that $NS_\kappa \upharpoonright S^*$ is saturated.

Problem 4. Can NS_κ be densely saturated over an inaccessible κ?

Further weakenings lead to notions of presaturatedness and precipitousness introduced by J. Baumgartner-A. Taylor [Ba-Ta] and T. Jech-K. Prikry [Je-Pr] respectively.

An ideal I over κ is called presaturated if forcing with it preserves κ^+. It is precipitous if the generic ultrapower is well-founded. Saturatedness implies presaturatedness and presaturatedness implies precipitousness. It turned out that for every κ NS_κ can be precipitous. With presaturatedness the situation is less clear. Namely, NS_κ can be presaturated for $\kappa = \aleph_1$ or κ an inaccessible, but by [Sh1] these are the only cases. See Jech-Magidor-Mitchell-Prikry [Je-Ma-Mi-Pr, Gi1, Gi2, Gi3, Gi4], M. Foreman, M. Magidor, S. Shelah [Fo-Ma-Sh] for the consistency results. We do not know the answer to the following

Problem 5. Can $NS_{\aleph_2}^{\aleph_1}$ be presaturated?

Another direction is to consider $\diamond_\kappa$. Obviously $\diamond_\kappa \longrightarrow (NS_\kappa$ is not saturated). One can try to prove $\diamond_\kappa$ in $ZFC+$ instances of GCH, of course. Since by Foreman and Woodin [Fo-Wo] GCH can fail everywhere, so can $\diamond_\kappa$. H. Woodin [Cu-Wo] showed the consistency of $\neg\diamond_\kappa$ over the first inaccessible. But GCH fails on a club of κ. He started from a supercompact. R. Jensen proved that at least a measurable is needed for this.

Problem 6. Is it consistent that κ is an inaccessible, $\neg\diamond_\kappa$ and GCH?

By [Sh5], it is possible to have a stationary subset S of an inaccessible κ with $\neg\diamond_S$ and GCH.

Problem 7. Is it consistent that $2^\kappa = \kappa^+$ and $\neg\diamond_{S^{\mathrm{cf}\,\kappa}_{\kappa^+}}$ for a singular κ? The same for $\neg\diamond_{\kappa^+}$?

It is known that large cardinals are needed for this; see [Sh5]. By Gregory [Gr] and [Sh4], $\diamond_{S^\kappa_\lambda}$ holds for a cardinal $\lambda = 2^\mu = \mu^+$ and regular $\kappa < \mu$ provided $\mu^\kappa = \mu$ or μ is singular, κ differs from cf μ, and for every $\delta < \mu$, $\delta^\kappa < \mu$. This was continued in [Sh9]. Thus it was shown that for κ above $\beth_\mu$, $\diamond_{\kappa^+}$ is equivalent to $2^\kappa = \kappa^+$. Also see [Sh5] and Dzamonja, Shelah [Dz-Sh2] for related results. On the other hand, Shelah showed that it is consistent that GCH and $\neg\diamond_{S^{\aleph_1}_{\aleph_2}}$; see King-Steinhorn [Ki-St].

REFERENCES

[Ba-Ta]   J. Baumgartner and A. Taylor, *Saturation properties of ideals in generic extensions* I, II, Trans. Am. Math. Soc. **270** (1982), 557-574 and 271 (1982), 587-609. MR **83k:**03040a,b

[Ba-Ta-Wa]   J. Baumgartner, A. Taylor and S. Wagon, *On splitting stationary subsets of large cardinals*, J. Sym Logic **42** (1977), 203-214. MR **58:**21619

[Br-Ma]   M. Burke and M. Magidor, *Shelah's pcf theory and its applications*, Ann. Pure and Ap. Logic **50** (1990), 207-254. MR **92f:**03053

[Cu-Wo]   J. Cummings and H. Woodin, *Generalized Prikry Forcing* (to appear).

[Dz-Sh]   M. Džamonja and S. Shelah, *On squares, outside guessing of clubs and $I_{<f}[\lambda]$*, Fund. Math. **148** (1995), 165–198. CMP 96:04.

[Dz-Sh2]   M. Džamonja and S. Shelah, *Saturated filters at filters at successors of singulars, weak reflection* (to appear; Shelah's paper 545).

[Fo]   M. Foreman, *More saturated ideals*, Cabal Seminar 79-81,Lecture Notes in Math. 1019, Springer, 1983, pp. 1-27. MR **86j:**03002

[Fo-Ma-Sh]   M. Foreman, M. Magidor and S. Shelah, *Martin's Maximum, saturated ideals and nonregular ultrafilters,* I, Ann. of Math. **127** (1988), 1-47. MR **89f:**03043

[Fo-Wo]   M. Foreman and H. Woodin, *GCH can fail everywhere*, Ann. of Math. **133** (1991), 1-36. MR **91k:**03130

[Gi1]   M. Gitik, *The nonstationary ideal on $\aleph_2$*, Israel Journal of Math. (1984), 257-288. MR **86i:**03061

[Gi2]   M. Gitik, *Changing cofinalities and the nonstationary ideal*, Israel Journal of Math. **56** (1986), 280-314. MR **89b:**03086

[Gi3]   M. Gitik, *Some results on the nonstationary ideal*, Israel Journal of Math. **92** (1995), 61-112. CMP 96:03

[Gi4]   M. Gitik, *Some results on nonstationary ideal 2*, to appear, Israel J. of Math.

[Gr]   J. Gregory, *Higher Souslin trees and GCH*, J. Symb. Logic **41** (1976), 663-671. MR **58:**5208

[Je-Wo]   T. Jech and H. Woodin, *Saturation of the closed unbounded filter on the set of regular cardinals*, Trans. Am. Math. Soc. **292 (1)** (1985), 345-356. MR **87d:**03139

[Je-Ma-Mi-Pr] T. Jech, M. Magidor, W. Mitchell and K. Prikry, *Precipitous ideals*, J. Symb. Logic **45** (1980), 1-8. MR **81h:**03097

[Je-Pr]   T. Jech and K. Prikry, *Ideals of sets and the power set operation*, Bul. Am. Math. Soc. **82** (1976), 593-595. MR **58:**21618

[Ki-St]   J. King and C. Steinhorn, *The uniformization property for R'_2*, Israel J. Math. **36** (1980), 248-256. MR **82a:**03051

[Ku]   K. Kunen, *Saturated ideals*, J. of Sym. Logic **43** (1978), 65-76. MR **80a:**03068

[Na]   K. Namba, *On closed unbounded ideal of ordinal numbers*, Comm. Univ. Sancti Pauli **22** (1974), 33-56. MR **50:**4308

[Sh1]   S. Shelah, *Proper forcing*, Lec. Notes in Math. 940, Springer-Verlag, 1982. MR **84h:**03002

[Sh2]   S. Shelah, *Cardinal Arithmetic*, Oxford Univ. Press, 1994. MR **96e:**03001

[Sh3]   S. Shelah, *Appendix: on stationary sets (to "Classification of nonelementary classes II). Abstract elementary classes") in Proc. of the USA-Israel Conference on Classification theory*, Lec. Notes in Math. 1292 (J. Baldwin, ed.), Springer, 1987, pp. 483-485. MR **91h:**03046

[Sh4]   S. Shelah, *On Successors of Singular Cardinals, in Logic Colloquium 78* (M. Boffa, D. van Dalen and K. McAloon, eds.), North Holland, 1979, pp. 357-380. MR **82d:**03079

[Sh5]   S. Shelah, *Diamonds and uniformizations*, J. Symb. Logic **49** (1984), 1022-1033. MR **86g:**03083

[Sh6]   S. Shelah, *Iterated forcing and normal ideals on ω_1*, Is. Jour. of Math. **60** (1987), 345-380. MR **90g:**03050

[Sh7]   S. Shelah, *More on stationary coding, in Around Classification Theory of Models*, Lecture Notes in Math. **1182** (1986), 224-246. MR **90a:**03037

[Sh8]   S. Shelah, *Non structure theory*, Oxford Univ. Press ([Sh-h]) (to appear).

[Sh9]   S. Shelah, *The Generalized Continuum Hypothesis Revisited ([Sh460])* . (to appear).

[So]   R. Solovay, *Real valued measurable cardinals, in Axiomatic Set Theory*, Proceedings of Sym. in Pure Math. XIII **1** (1971), 387-428. MR **45:**55

[St]   J. Steel, *The Core Model Iterability Problem* (to appear).

[St-V]   J. Steel and R. Van Wesep, *Two consequence of determinacy consistent with choice*, Trans. An. Math. Soc. **272 (1)** (1982), 67-85. MR **83i:**03081

[Wo1]   H. Woodin, *Some consistency results yin ZF using AD*, Cabal Seminar, Lect. Notes in Math. 1019, Springer, 1983, pp. 172-199. MR **86j:**03002

SCHOOL OF MATHEMATICAL SCIENCES, SACKLER FACULTY OF EXACT SCIENCES, TEL AVIV UNIVERSITY, RAMAT AVIV 69978, ISRAEL

E-mail address: `gitik@math.tau.ac.il`

HEBREW UNIVERSITY OF JERUSALEM, DEPARTMENT OF MATHEMATICS, GIVAT RAM, JERUSALEM, ISRAEL

E-mail address: `shelah@math.huji.ac.il`