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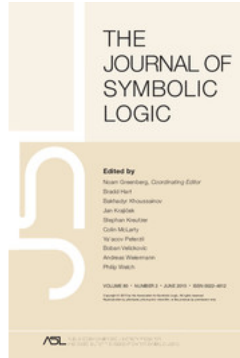

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MORE ON PROPER FORCING

SAHARON SHELAH

§1. A counterexample and preservation of "proper + X".

1.1. THEOREM. *Suppose V satisfies $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$, and for some $A \subseteq \omega_1$, every $B \subseteq \omega_1$ belongs to $L[A]$.*

Then we can define a countable support iteration $\bar{Q} = \langle P_i, \mathbf{Q}_i : i < \beta \rangle$ such that the following conditions hold:

a) *Each $\mathbf{Q}_i$ is proper and $\Vdash_{P_i}$ "$\mathbf{Q}_i$ has power $\aleph_1$".*

b) *Each $\mathbf{Q}_i$ is $\mathfrak{D}$-complete for some simple $\aleph_1$-completeness system.*

c) *Forcing with $P_\alpha = \operatorname{Lim} \bar{Q}$ adds reals.*

PROOF. We shall define $\mathbf{Q}_i$ by induction on i so that conditions a) and b) are satisfied, and $\mathbf{C}_i$ is a $\mathbf{Q}_i$-name of a closed unbounded subset of ω_1. Let $\langle f^*_\xi : \xi < \omega_1 \rangle \in L[A]$ be a list of all functions f which are from δ to δ for some $\delta < \omega_1$, and let $h: \omega_1 \to \omega_1$, $h \in L[A]$, be defined by $h(\alpha) = \operatorname{Min}\{\beta : \beta > \alpha$ and $L_\beta[A] \models$ "$|\alpha| = \aleph_0$"$\}$.

Suppose we have defined $\mathbf{Q}_j$ for every $j < i$; then P_i is defined, is proper (as each $\mathbf{Q}_j$, $j < i$, is proper, and by III 3.2) and has a dense subset of power $\aleph_1$ (by III 4.1).[1] Let $G_i \subseteq P_i$ be generic so clearly there is $B \subseteq \omega_1$ such that in $V[G_i]$ every subset of ω_1 belongs to $L[A, B]$. The following now follows:

FACT. *In $V[G_i]$, every countable $N \prec (H(\aleph_2), \in, A, B)$ is isomorphic to $L_\beta[A \cap \delta, B \cap \delta]$ for some $\beta < h(\delta)$, where $\delta = \delta(N) = \omega_1 \cap N$.*

We shall assume also that $V[G_i]$ has the same reals as V (otherwise we already have an example).

We now define, by induction on $\alpha < \omega_1$, a set T_α such that the following conditions are satisfied:

i) Each $f \in T_\alpha$ is the characteristic function of a closed subset of some successor ordinal $\beta < \alpha$, i.e., $\operatorname{Dom} f = \beta$, and $f^{-1}(\{1\})$ is a closed subset of β and is included in the set of limit points of $\bigcap_{j<i} C_j \cap \omega_1$.

ii) If $f \in T_\alpha$, $\gamma + 1 \leq \operatorname{Dom} f$, then $f \restriction (\gamma + 1) \in T_\alpha$, and even $f \restriction (\gamma + 1) \in T_\beta$ for $\gamma + 1 \leq \beta \leq \alpha$.

iii) If $f \in T_\alpha$, $\operatorname{Dom} f = \beta$, $\beta < \gamma < \alpha$, γ a successor, then $f' = f \cup 0_{[\beta,\gamma)} \in T_\alpha$, i.e.,

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[1] Such references are to [1].

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Dom $f' = \gamma$, and

$$f'(i) = \begin{cases} f(i), & i < \beta, \\ 0, & \beta \leq i < \gamma. \end{cases}$$

iv) If $f, g \in T_\alpha$, $f(i) \neq g(i)$, then $f^{-1}(\{1\}) \cap g^{-1}(\{1\}) - i$ is finite.

v) If $f \in T_\alpha$, $\gamma = \operatorname{Dom} f$, $\gamma + 1 < \alpha$ and the order type of $f^{-1}(\{1\})$ has the form $\xi + 2$, then $f' = f \cup \{\langle \gamma, 1 \rangle\} \in T_\alpha$.

vi) If $f \in T_\alpha$, $\delta + 1 = \operatorname{Dom} f$, δ limit, and $f(i) = 1$ for arbitrarily large $i < \delta$, then $\operatorname{Min}\{\xi : f \restriction \delta = f^*_\xi\}$ is larger than $\operatorname{Min}\{\xi : \delta < \xi \in C_j\}$ (for $j < i$).

vii) If $\delta < \alpha$ is limit, δ a limit point of $\bigcap_{j<i} C_j$, $\xi^* < \omega_1$, and $f \in T_\alpha \cap L_\delta[A \cap \delta]$, then there is $g \in T_\alpha$, $\delta + 1 = \operatorname{Dom} g$, such that for every $\mathscr{I} \in L_{h(\delta)}[A \cap \delta, B \cap \delta]$ (an open dense subset of $T_\delta \cap L_\delta[A \cap \delta]$ (ordered by inclusion)), for some $\gamma < \delta$ we have $g \restriction \gamma \in \mathscr{I}$ and $g \restriction \delta \notin \{f^*_\xi : \xi < \xi^*\}$ and $f = g \restriction \operatorname{Dom} f$.

viii) For $f \in T_\alpha$, if $f(\delta) = 1$, $\delta < \beta$, and $f(\beta) = 1$, then for every $j < i$, for some $\gamma < \beta$, the characteristic function of C_j restricted to δ is f^*_γ; and if δ, $f \restriction \delta$ and β satisfy this then $f \restriction (\delta + 1) \cup 0_{[\delta+1,\beta)} \cup 1_{[\beta,\beta+1)}$ belongs to $T_{\beta+1}$.

Case A. α is limit, or $\alpha = \gamma + 1$, γ limit. Let $T_\alpha = \bigcup_{\beta<\alpha} T_\beta$ or $T_\alpha = \bigcup_{\beta<\gamma} T_\beta$.

Case B. $\alpha < \omega$. Let $T_\alpha = \{f : f$ a function from $\beta < \alpha$ to $\{0, 1\}\}$.

Case C. $\alpha = \beta + 3 > \omega$. Let $T_\alpha = T_{\beta+2} \cup \{f : \operatorname{Dom} f = \beta + 2, f \restriction (\beta + 1) \in T_{\beta+2}$, provided that viii) is satisfied$\}$.

Case D. $\alpha = \delta + 2$, δ limit, $\delta \in \bigcap_{j<i} C_j$. This is the main case. Let $\{f^*_e : e < \omega\}$ be a list of $T_\delta \cap L_\delta[A \cap \delta]$, each appearing $\aleph_0$ times, and $\{\mathscr{I}_e : e < \omega\}$ be a list of all open dense subsets of $T_\delta \cap L_\delta[A \cap \delta]$ which belong to $L_{h(\delta)}[A \cap \delta, B \cap \delta]$ and $\{f \in T_\delta \cap L_\delta[A \cap \delta], f \not\subseteq f^*_\xi\}$ for $\xi < h(\delta)$. We now define, by induction on $n < \omega$, an ordinal $\alpha_n < \delta$ and a finite set $F_n \subseteq \{f \in T_\delta \cap L_\delta[A \cap \delta] : \alpha_n = \operatorname{Dom} f\}$ such that:

$$(*) \qquad (\forall f \in F_n)(\exists g \in F_{n+1})(f \subseteq g) \text{ and}$$
$$(\forall f, g \in F_n)(f \restriction \alpha_{n-1} \neq g_n \restriction \alpha_{n-1} \rightarrow f^{-1}(\{1\}) \cap g^{-1}(\{1\}) \subseteq \alpha_{n-1}).$$

Subcase α. If $n = 0 \bmod 3$ then $\alpha_{n+1} = \alpha_n + 1$ and $F_n = \{f \cup \{\langle \alpha_n, 0 \rangle\} : l < 2, f \in F_n\}$; and if $n = 0$, then $F_n = \varnothing$ and $\alpha_n = 0$.

Subcase β. If $n = 1 \bmod 3$, then $\alpha_{n+1} = \alpha_n + 1$; $F_{n+1} = F_n$ if $[\operatorname{Dom} f^*_{(n-1)/3} > \alpha_n$ or $(\exists g \in F_n)(f^*_{(n-1)/3} \subseteq g)]$; otherwise

$$F_{n+1} = \{f \cup 0_{[\alpha_n,\alpha_{n+1})} : f \in F_n\} \cup \{f_{(n-1)/3} \cup 0_{[\beta,\alpha_{n+1})} : \beta = \operatorname{Dom} f_{(n-1)/3}\}.$$

Subcase γ. If $n = 2 \bmod 3$, $(n-2)/3 = m^2 + k$, $k \leq 2m$, then every $f \in F_{n+1}$ belongs to $\mathscr{I}_k$. Note that we have to take care of (*); hence let $F_n = \{f^n_e : e < |F_n|\}$, and define α^n_e and g^n_e by induction on $e : \alpha^n_0 = \alpha_n$; if α^n_e is defined, chose g^n_e, $f^n_e \cup 0_{[\alpha_n,\alpha^n_e)} \subseteq g^n_e \in \mathscr{I}_k$, and $\alpha^n_{e+1} = \operatorname{Dom} g^n_e$. Now let $\alpha_{n+1} = \alpha^n_{|F_n|}$ and $F_{n+1} = \{g^n_e \cup 0_{[\alpha^n_{e+1},\alpha_{n+1})} : e < |F_n|\}$.

Note that only in Case D, Subcase γ, do we have a free choice, and we eliminate it by choosing the first candidate for F_{n+1} by the canonical well-ordering of $L[A]$. So we have finished defining the F_n's and we let

$$T_{\delta+2} = T_\delta \cup \{f : \operatorname{Dom} f = \delta + 1 \text{ and } \textit{either } f = f' \cup 0_{[\gamma,\delta+1)}, \text{ where}$$
$$f' \in T_\delta, \gamma = \operatorname{Dom} f', \textit{ or } (\forall n > k)[f \restriction \alpha_n \in F_n] \text{ for some } k < \omega,$$
$$f(\delta) = 1 \text{ iff } \delta = \sup f^{-1}(\{1\})\}.$$

It is easy to check that $T_{\delta+2}$ is as required. (Case β in the definition of F_n enables us to satisfy demand vii).)

Case E. $\alpha = \delta + 2$, δ limit, $\delta \notin \bigcap_{j<i} C_j$. Let $T_\alpha = T_\delta \cup \{f{:}\mathrm{Dom}\, f = \delta + 1, (\exists g \in T_\delta) f \restriction ((\delta + 1) - \mathrm{Dom}\, g)$ is zero$\}$.

So we have defined T_α for $\alpha < \omega_1$, and let $Q_i \in V[G_i]$ be $\bigcup_{\alpha<\omega_1} T_\alpha$ ordered by inclusion (really we should have written T^i_α, β_i, etc.); and it is easy to see that $\mathbf{Q}_i$ is as required (in a) and b)).

So $\bar{Q} = \langle P_i, \mathbf{Q}_i{:}i < \omega^2\rangle$ is defined, and it is easy to see that we can replace (in $V[G_i]$) B_i by $\bar{C}^i = \langle C_j{:}j < i\rangle$. Let $G \subseteq P_{\omega^2}$ be generic, and C_i the interpretation of $\mathbf{C}_i$. Let f_i be the characteristic function of C_i, and $C = \bigcap_{i<\omega^2} C_i$, $\{\alpha_\zeta{:}\zeta < \omega_1\}$ an enumeration of C (in increasing order). We shall suppose that forcing by P_{ω^2} does not add reals, and shall deduce that $\langle f_i{:}i < \omega^2\rangle \in V$, which is clearly false, as $\Vdash_{Q_0}$ "$C_0 \notin V$".

By the assumption the $\langle f_i \restriction \alpha_0{:}i < \omega^2\rangle$ belong to V, and we shall show how to compute $\langle f_i \restriction \alpha_\zeta{:}i < \omega^2\rangle$ for every ζ, by induction; as the computation is done in V we get the desired contradiction. More formalistically, there is a function F in V such that

$$\langle f_i \restriction \alpha_{\zeta+1}{:}i < \omega^2\rangle = F(\langle f_i \restriction \alpha_\zeta{:}i < \omega^2\rangle).$$

So suppose $\langle f_i \restriction \alpha_\zeta{:}i < \omega^2\rangle$ is given, and let, for $i < \omega^2$,

$$\beta_i = \mathrm{Min}\, C_i - (\alpha_\zeta + 1), \qquad \xi_i = \mathrm{Min}\{\xi{:}f_i \restriction \alpha_\zeta = f^*_\xi\}.$$

By demand i) in the definition of the T^i_α's, $C_i \subseteq \bigcap_{j<i} C_j$. So clearly $\beta_j \le \beta_i$, and $\beta_i \in C_j$ for $j \le i$. Also by demand vi) on the T_α's, $\beta_j < \xi_i$ for $j < i$, and by demand viii) on the T_α's, $\xi_j < \beta_i$ for $j < i$. We can conclude that $\mathrm{Sup}\{\beta_i{:}i < \omega n\} = \mathrm{Sup}\{\xi_i{:}i < \omega n\}$; but from $\langle f_i \restriction \alpha_\zeta{:}i < \omega^2\rangle$ we can compute $\gamma_n = \mathrm{Sup}\{\xi_i{:}i < \omega n\}$. As $\beta_i \in C_j$ for $j < i$, $\gamma_n \in C_j$ when $j < \omega n$, and clearly $\gamma_n < \gamma_{n+1}$, we have $\gamma = \bigcup_{n<\omega} \gamma_n \in \bigcap_{j<\omega^2} C_j$. By the definition of the α_ζ's, $\gamma = \alpha_{\xi+1}$. As we know $T^0_\gamma \cap L_\delta[A]$, and we know $\{\gamma_n{:}n < \omega\} \subseteq C_0$; $f_0 \restriction \delta$ is uniquely determined (by demand iv)). Similarly we continue to reconstruct $f_i \restriction \gamma$ by induction on i, thus finishing the proof.

1.2. Remarks. (1) We could weaken the demands on V (in 1.1) to $V \models \mathrm{CH}$, provided that we also waive the requirement $\Vdash_{P_i}$"$|\mathbf{Q}_i| = \aleph_1$". For this it suffices to start with a forcing which makes those demands true, and such a forcing notion exists by Jensen and Solovay [2].

(2) The ω^2 in 1.1 is best possible.

(3) Alternatively, we can weaken the demand on V to: CH and

(*) There is a sequence $\langle f_\delta{:}\delta < \omega_1, \delta \text{ limit}\rangle$, f_δ a function from δ to δ, such that for every $f{:}\omega_1 \to \omega_1$ for a closed unbounded set of $\delta < \omega_1$,

$$(\exists \alpha < \delta)(\forall \beta)\,[\alpha < \beta < \delta \to f(\beta) < f_\delta(\beta)].$$

For this we need some forcing like our P_i preserving CH + (*), which seems to be a demand on V, and we must make some changes in the proof

(4) We can improve 1.1 in the following way. Let ε be a countable limit ordinal such that $(\forall \alpha < \varepsilon)\ (\alpha + \alpha < \varepsilon)$ (equivalently ε has the form ω^α (ordinal exponentiation)). Then we can construct a CS iteration $\bar{Q} = \langle P_i, \mathbf{Q}_i{:}i < \omega\varepsilon\rangle$ such that:

a)′ Each $\mathbf{Q}_i$ is α-proper for $\alpha < \varepsilon$ and $\Vdash_{P_i}$ "$\mathbf{Q}_i$ has power $\aleph_1$".

b)′ Each $\mathbf{Q}_i$ is $\bar{\mathfrak{D}}$-complete for some simple $\aleph_1$-completeness system.

c)′ Forcing with $P_\alpha = \mathrm{Lim}\, \bar{Q}$ adds reals.

We again assume $G_i \subseteq P_i$ generic is given; hence $\langle C_j : j < i \rangle$, and by induction on α we define T^i_α, so that in the definition of T^i_α we use A and $\langle C_j \cap \alpha : j < i \rangle$ only (and the list $\{f^*_\xi : \xi < \omega_1\} \in L[A]$), so that a variant of i)–viii) holds. The changes are:

iv)′ If $f, g \in T^i_\alpha, f(i) \neq g(i)$, then $f^{-1}(\{1\}) \cap g^{-1}(\{1\}) - i$ has order-type $< \varepsilon$.

vii)′ In addition to vii), if $\langle \delta_\zeta : \zeta \leq \zeta^* \rangle$ is an increasing sequence of limit points of $\bigcap_{j<i} C_j$, $\langle \delta_\xi : \xi \leq \zeta \rangle \in L_{\delta_{\zeta+1}}[A \cap \delta_{\zeta+1}]$, $f \in T^i_{\delta_0} \cap L_{\delta_0}[A]$, $f_m \in T^i_{\zeta^*+1}$ for $m < \omega$ and $m^* < \omega, \zeta^* < \varepsilon$, then there is $g \in T^i_{\zeta^*+2}$, $f \subseteq g$, $\operatorname{Dom} g = \zeta^* + 1$, such that the following conditions hold:

(α) For every $\mathscr{I} \in L_{h(\delta)}[A \cap \delta, B \cap \delta]$ (an open dense subset of $T^i_\delta \cap L_\delta[A \cap \delta]$ (ordered by inclusion)), for some $\gamma < \delta, g \restriction \gamma \in \mathscr{I}$, where $\delta \in \{\delta_\zeta : \zeta \leq \zeta^*\}$.

(β) For every $m < \omega$, $g^{-1}(\{1\}) \cap f_m^{-1}(\{1\}) - \{\delta_\zeta : \zeta \leq \zeta^*\}$ is a bounded subset of δ_{ζ^*}.

(γ) For every $m < m^*$, $g^{-1}(\{1\}) \cap f_m^{-1}(\{1\}) - \{\delta_\zeta : \zeta \leq \zeta^*\} \subseteq \operatorname{Dom} f$.

In the proof of Case D, we use the canonical well-ordering of $H(\aleph_1)^{L[A]}$ on our assignments (for the existence of $g \in T^i_{\delta+2}$, $\operatorname{Dom} g = \delta + 1$), and construct a witness, preserving and using vii)′.

1.3. THEOREM. (1) *Suppose (D, R) is a smooth strong covering model, $\bar{Q} = \langle P_i, \mathbf{Q}_i : i < \delta \rangle$ a countable support iteration of proper forcing notion (or at least P_α / P_β is proper for $\beta < \alpha < \delta$, β nonlimit) and each P_i is (D, R)-preserving for $i < \delta$. Then* $\operatorname{Lim} \bar{Q}$ *is (D, R)-preserving. (See* VI, §1, *for definitions, and* VI, §2, *for applications.)*

(2) *Suppose $P * \mathbf{Q} \in N_0$, P, $\mathbf{Q}$ are proper and $P * \mathbf{Q}$ is ω^ω-bounding: $N_0 \prec N_1 \prec (H(\lambda), \in)$ (λ big enough), $N_0 \in N_1$, $\|N_e\| \leq \aleph_0$, and $p \in P$ is (N_e, P)-generic for $e = 0, 1$ and $\mathbf{q} \in N_1$ is a P-name of a member of $\mathbf{Q}$, $(p, \mathbf{q})$ is $(N_0, \mathbf{Q})$-generic and for some F for every predense $\mathscr{I} \subseteq P$, $\mathscr{I} \in N_0$, $F(\mathscr{I}) \subseteq \mathscr{I} \cap N_0$ is predense above p (in P) and $F(\mathscr{I})$ is finite.*

Then there is $\mathbf{q}'$ such that $(p, \mathbf{q}) \leq (p, \mathbf{q}')$, $(p, \mathbf{q}')$ is $(N_1, P * \mathbf{Q})$-generic and for some function F', for every predense $\mathscr{I} \subseteq P * \mathbf{Q}$, $\mathscr{I} \in N_0$, $F(\mathscr{I})$ is predense above $(p, \mathbf{q}')$ (in $P * \mathbf{Q}$) and $F(\mathscr{I})$ is finite.

PROOF. (1) The proof is very similar to the proof of VI.1.6, so we mention only the changes. Instead of choosing $\langle N_e : e < \omega \rangle \in \mathrm{SQS}^1_\omega(\lambda)$, we just choose $N_1 \prec (H(\lambda), \in)$ such that $\langle x_n : n < \omega \rangle$, $\langle P_e, \mathbf{Q}_e : e < \omega \rangle$, $\mathbf{f}$, $\langle q^n_e : e < n < \omega \rangle$ and $\langle t_{n,m} : m \leq n < \omega \rangle$ belong to it. We now replace a), b) by

a)′ $p \restriction n \leq r^n$; r^n is (N_1, P_n)-generic.

b)′ For some $T_n \in D$, $r^n \Vdash$ "$\mathbf{f}_n \in \operatorname{Lim} T_n$" and $x_{2n} R T_n$, $T_n \subseteq T_{n+1}$.

Toward the end we know that some $t \in \mathscr{I} \cap N_1$ (not $\mathscr{I} \cap N_{8^n+2^n}$) belongs to the generic subset of P_n, and we let $\mathscr{I} \cap N_1 = \{t_k : 0 < k < \omega\}$.

Then, later, T_{n+1} does not necessarily belong to N_1; in (*), q' is also $(N_1[G_n], \mathbf{Q}_n[G_n])$-generic.

(2) The proof is essentially included in the proof of (1).

Note that N_1 has a list $\langle \tau_e : e < \omega \rangle$ of the $P * \mathbf{Q}$-names of ordinals, and there is a sequence $\langle \mathbf{q}_e : e < \omega \rangle$ ($\in N_1$), $\Vdash_P$ "$\mathbf{q}_e \in \mathbf{Q}$ and $q \leq q_e \leq q_{e+1}$" and $(p, q_e) \Vdash$ "$\tau_e = \sigma_e$" for some P-name σ_e (of an ordinal) from N_1.

REMARK. We can replace proper by semi-proper as in Chapter X.

§2. Intermediate forcing. In §1 we showed that just excluding the forcing notions like the one from Example V.5.1 (by demanding $\mathfrak{D}$-completeness for a simple 2-completeness system) is not enough to ensure that the iterated forcing does not add reals. In VIII, §4, on the other hand, we have quite weak restrictions on each $\mathbf{Q}_i$

ensuring $\text{Lim}\langle P_i, \mathbf{Q}_i : i < \alpha\rangle$ does not add reals. However, here we shall represent forcing notions which fall in between (and the corresponding consistency problems).

2.1. Problem. Let $f_\delta : \delta \to \delta$ for any limit $\delta < \omega_1$. Is there $f : \omega_1 \to \omega_1$ such that for every $\delta < \omega_1$, for arbitrarily large $\alpha < \delta$, $f_\delta(\alpha) < f(\alpha)$?

2.2. Definition. For any sequence $\bar{f} = \langle f_\delta : \delta < \omega_1\rangle$, $f_\delta : \delta \to \delta$, let $P^0_{\bar{f}} = \{g : g$ a function from some $\alpha < \omega_1$ into ω_1, such that for every $\delta \leq \alpha$, for arbitrarily large $\beta < \delta$, $f_\delta(\beta) < g(\beta)\}$; ordered by inclusion.

2.3. Problem. Let $C_\delta \subseteq \delta$ be a subset of δ, for $\delta < \omega_1$. Is there a closed unbounded $C \subseteq \omega_1$ such that for no δ, $C_\delta \subseteq C$? Consider in particular the cases when we restrict ourselves to

a) C_δ has order-type ω, $\delta = \text{Sup}\, C_\delta$,

b)$_\xi$ C_δ has order-type ξ, $\delta = \text{Sup}\, C_\delta$ (ξ limit),

c) C_δ has order-type $< \delta$, $\delta = \text{Sup}\, C_\delta$,

d) $C_\delta = \phi \bmod D_\delta$, D_δ a filter on δ, $\delta = \text{Sup}\, C_\delta$, $\bar{D} = \langle D_\delta : \delta < \omega_1\rangle$.

2.4. Definition. For $\bar{C} = \langle C_\delta : \delta < \omega_1\rangle$, $C \subseteq \omega$, let $P^1_{\bar{C}} = \{f : f$ a function from some $\alpha < \omega_1$ to $\{0, 1\}$, and for no $\delta \leq \alpha$ is $C_\delta \subseteq f^{-1}(\{1\})\}$.

2.5. Problem. Let C_δ be an unbounded subset of δ, for $\delta < \omega_1$. Is there a closed unbounded $C \subseteq \omega_1$ such that for every δ, $C \cap C_\delta$ is a bounded subset of δ, when we restrict ourselves as in 2.3?

2.6. Definition. For a sequence $\bar{C} = \langle C_\delta : \delta < \omega_1\rangle$, C_δ an unbounded subset of δ, let $P^2_{\bar{C}} = \{g : g$ a function from some $\alpha < \omega_1$ to ω_1, so for every $\delta \leq \alpha$, $\text{Sup}[C_\delta \cap g^{-1}(\{1\})] < \delta\}$.

2.7. Claim. $P^0_{\bar{f}}$, $P^1_{\bar{C}}$ and $P^2_{\bar{C}}$ (when one of the Cases A–D from 1.1 holds) are proper and $\mathfrak{D}$-complete for some simple $\aleph_1$-completeness system.

Concluding Remark. We shall later conclude that a positive answer is consistent with ZFC + GCH. The point is that though the corresponding forcing notions are not α-proper for many $\alpha < \omega_1$, still a reasonable weakening holds, i.e. for suitable $\langle N_i : i \leq \delta\rangle$ and $p \in N_0 \cap P$ there is a $q \geq p$ such that $q \Vdash_P$ "$\{i: N_i[G] \cap \text{ord} = N_i \cap \text{ord}\}$ is large".

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