Notre Dame Journal of Formal Logic
Volume 34, Number 1, Winter 1993

Maximal Subgroups of Infinite Symmetric Groups

JAMES E. BAUMGARTNER, SAHARON SHELAH,
and SIMON THOMAS

Abstract We prove that it is consistent that there exists a subgroup of the symmetric group $\mathrm{Sym}(\lambda)$ which is not included in a maximal proper subgroup of $\mathrm{Sym}(\lambda)$. We also consider the question of which subgroups of $\mathrm{Sym}(\lambda)$ stabilize a nontrivial ideal on λ.

1 Introduction The work in this paper was motivated by the following question, which was raised by Peter Neumann. If $\lambda \geq \omega$, does every proper subgroup of $\mathrm{Sym}(\lambda)$ lie in a maximal subgroup of $\mathrm{Sym}(\lambda)$? While a positive answer seems very unlikely, all of the results up to this point have concerned sufficient conditions for a subgroup $G < \mathrm{Sym}(\lambda)$ to lie in a maximal subgroup of $\mathrm{Sym}(\lambda)$. For example, the main theorem in MacPherson and Praeger [3] states that if $G < \mathrm{Sym}(\omega)$ is not highly transitive, then G is contained in a maximal subgroup. In Section 2, we shall prove the following result.

Theorem 1 (F_λ) *There exists a subgroup* $G < \mathrm{Sym}(\lambda)$ *such that the set* $\mathbb{L} = \{H \mid G \leq H < \mathrm{Sym}(\lambda)\}$ *is a well-ordering under inclusion of order-type* 2^λ. *In particular, G is not contained in a maximal subgroup of* $\mathrm{Sym}(\lambda)$.

It is not known whether this theorem can be proved in ZFC. Our extra hypothesis F_λ is the following statement. Let $\mathrm{Sym}_{<\lambda}(\lambda)$ be the group of all permutations π of λ such that $|\mathrm{Mov}(\pi)| < \lambda$, where $\mathrm{Mov}(\pi) = \{\alpha \mid \alpha^\pi \neq \alpha\}$. Let $S(\lambda) = \mathrm{Sym}(\lambda)/\mathrm{Sym}_{<\lambda}(\lambda)$.

(F_λ) If $T < S(\lambda)$ is a subgroup with $|T| < 2^\lambda$, then there exists an element of infinite order $\pi \in S(\lambda) \setminus T$ such that $\langle T, \pi \rangle = T * \langle \pi \rangle$.

Here $*$ denotes the free product. We shall also show that F_λ is consistent with but independent of ZFC.

Another result from [3] states that if I is a nontrivial ideal on λ which contains a set X with $|X| = |\lambda \setminus X| = \lambda$, and $G \leq S_{\{I\}} = \{\pi \in \mathrm{Sym}(\lambda) \mid I^\pi = I\}$,

Received August 12, 1992

then G is contained in a maximal subgroup of $\mathrm{Sym}(\lambda)$. It is also shown in [3] that if $|G| \leq \lambda$, then there exists such an ideal I with $G \leq S_{\{I\}}$. In the third section of this paper, we shall obtain a stronger version of the latter result and also prove the independence of the strongest conceivable version. We shall see that the least size of a subgroup $G \leq \mathrm{Sym}(\lambda)$ which fails to stabilize such an ideal is bounded below by the size $B(\lambda)$ of the smallest family of uniform ultrafilters which cover $[\lambda]^\lambda$. In the final section, we shall prove that it is consistent that $B(\lambda)$ is much bigger than the size of any maximal almost disjoint family $\mathcal{F} \subseteq \mathcal{P}(\lambda)$.

Our notation follows that of Kunen [2]. Thus if $\mathbb{P}$ is a notion of forcing and $p, q \in \mathbb{P}$, then $q \leq p$ means that q is a strengthening of p. The notation $p \| q$ means that p and q are compatible conditions. A subset $X \subset \lambda$ is said to be a moiety if $|X| = |\lambda \setminus X| = \lambda$.

2 The main result Theorem 1 is an immediate consequence of the following result.

Theorem 2.1 *Let S be a group with $|S| = \kappa > \omega$. Suppose that whenever $T < S$ is a subgroup with $|T| < \kappa$, then there exists an element of infinite order $\pi \in S \setminus T$ such that $\langle T, \pi \rangle = T * \langle \pi \rangle$. Then there exists a subgroup $G < S$ such that the set $\mathbb{L} = \{H \mid G \leq H < S\}$ is a well-ordering under inclusion of order-type κ.*

Proof: Let $S = \{g_\alpha \mid \alpha < \kappa\}$. We shall define inductively a sequence of strictly increasing chains of subgroups $\langle H_\beta^\alpha \mid \beta \leq \alpha \rangle$ for $\alpha < \kappa$ such that the following condition is satisfied.

(*) If $\beta \leq \gamma \leq \alpha$, then $H_\beta^\alpha \cap H_\gamma^\gamma = H_\beta^\gamma$.

We set $H_0^0 = 1$. If λ is a limit ordinal, then we define

$$H_\beta^\lambda = \bigcup_{\beta \leq \alpha < \lambda} H_\beta^\alpha \quad \text{if } \beta < \lambda$$

$$H_\lambda^\lambda = \bigcup_{\alpha < \lambda} H_\alpha^\alpha .$$

Assume that H_β^γ has been defined for all $\beta \leq \gamma \leq \alpha$. Our intention is that, at the end of the construction, we will have that

$$\{H \mid H_0^\kappa \leq H < S\} = \{H_\beta^\kappa \mid \beta < \kappa\}$$

where $H_\beta^\kappa = \bigcup_{\beta \leq \alpha < \kappa} H_\beta^\alpha$. To accomplish this, we take steps to ensure that for all $\beta < \kappa$, if $g \in H_{\beta+1}^\kappa \setminus H_\beta^\kappa$, then $\langle H_0^\kappa, g \rangle = H_{\beta+1}^\kappa$. So suppose that there exist $\beta + 1 \leq \alpha$, $g \in H_{\beta+1}^\alpha \setminus H_\beta^\alpha$ and $h \in H_{\beta+1}^\alpha$ such that $h \notin \langle H_0^\alpha, g \rangle$. By hypothesis, there exist elements of infinite order $\pi_1, \pi_2 \in S \setminus H_\alpha^\alpha$ such that $\langle H_\alpha^\alpha, \pi_1, \pi_2 \rangle = H_\alpha^\alpha * \langle \pi_1 \rangle * \langle \pi_2 \rangle$. Let $\varphi = h\pi_1^{-1} g^{-1} \pi_2^{-1} g$; so that $h = \varphi g^{-1} \pi_2 g \pi_1$. For $0 \leq \gamma \leq \alpha$, define $H_\gamma^{\alpha+1} = \langle H_\gamma^\alpha, \pi_1, \pi_2, \varphi \rangle$. We must check that if $0 \leq \gamma \leq \alpha$, then

(**) $H_\gamma^{\alpha+1} \cap H_\alpha^\alpha = H_\gamma^\alpha .$

There are three possibilities to consider.

Case 1. Suppose that $g \in H_\gamma^\alpha$, and hence also $h \in H_\gamma^\alpha$. Then $H_\gamma^{\alpha+1} = H_\gamma^\alpha * \langle \pi_1 \rangle * \langle \pi_2 \rangle$, and (**) is obvious.

Case 2. Suppose that $h \in H^{\alpha}_{\gamma}$, but $g \notin H^{\alpha}_{\gamma}$. It is easily checked that

$$H^{\alpha+1}_{\gamma} = H^{\alpha}_{\gamma} * \langle \pi_1 \rangle * \langle \pi_2 \rangle * \langle g^{-1} \pi_2 g \rangle.$$

Furthermore, if $z \in H^{\alpha+1}_{\gamma}$, $z = a_1 \cdots a_n$ is the unique reduced sequence expression with respect to the above free product decomposition, and m is the length of the unique reduced sequence expression of z with respect to the decomposition $H^{\alpha}_{\alpha} * \langle \pi_1 \rangle * \langle \pi_2 \rangle$, then $m \geq n$. Hence (**) holds.

Case 3. Suppose that $g, h \notin H^{\alpha}_{\gamma}$. Then the proof that (**) holds is similar to that in Case 2, using the free product decomposition

$$H^{\alpha+1}_{\gamma} = H^{\alpha}_{\gamma} * \langle \pi_1 \rangle * \langle \pi_2 \rangle * \langle \varphi \rangle.$$

Finally, let $\delta = \min\{\xi \mid g_{\xi} \notin H^{\alpha+1}_{\alpha}\}$, and define $H^{\alpha+1}_{\alpha+1} = \langle H^{\alpha+1}_{\alpha}, g_{\delta} \rangle$.

It is now clear that we can perform the construction successfully. This completes the proof of Theorem 2.1.

The following result, which is an easy exercise, establishes the consistency of F_{λ}.

Theorem 2.2 (GCH) *For all* $\lambda \geq \omega$, F_{λ} *holds.*

We now prove the independence of F_{λ} for $cf(\lambda) > \omega$ and for $\lambda = \omega$. We first deal with the case when $\lambda = \omega$.

Theorem 2.3 *Let* $M \models \kappa^{\omega} = \kappa$. *Then there exists a generic extension* $M[G]$ *in which the following are true.*

(i) $2^{\omega} = \kappa$.

(ii) *There exists a subgroup* $T < S(\omega)$ *of cardinality* ω_1 *such that for all* $\pi \in S(\omega) \setminus T$, *there exist* $g, h \in T \setminus 1$ *with* $[g^{\pi}, h] = 1$.

Proof: By first adding κ Cohen reals if necessary, we can suppose that $M \models 2^{\omega} = \kappa$. We now perform an iterated finite support construction M_{α}, $\alpha \leq \omega_1$. We pass from M_{α} to $M_{\alpha+1}$ via a 2-step c.c.c. iteration, say

$$M_{\alpha} \subset M^{0}_{\alpha+1} \subset M_{\alpha+1}.$$

First let

$$\mathbb{P} = \{p \mid p : \omega \to \omega \text{ is a finite injective function}\}.$$

Then $M^{0}_{\alpha+1} = M_{\alpha}[G]$, where G is a generic subset of $\mathbb{P}$. Let $\pi = \bigcup G$ and $\Gamma_{\alpha} = \mathrm{Sym}(\omega)^{M_{\alpha}}$. Clearly $\pi \in \mathrm{Sym}(\omega)$.

Claim 2.4 *If* $g_1, \ldots, g_n \in \Gamma_{\alpha}$, *then* $\bigcap_{1 \leq i \leq n} fix(\pi^{g_i})$ *is an infinite subset of* ω.

Proof: Fix $t \in \omega$. Let $\mathfrak{D}$ consist of those $q \in \mathbb{P}$ for which there exists $m > t$ such that for all $1 \leq i \leq n$, $g_i^{-1} q g_i(m) = m$. It is enough to show that $\mathfrak{D}$ is a dense subset of $\mathbb{P}$. Let $p \in \mathbb{P}$. For each $1 \leq i \leq n$, there are finitely many r such that $g_i(r) \in \operatorname{dom} p \cup \operatorname{ran} p$. So there exists $m > t$ with

$$\{g_i(m) \mid 1 \leq i \leq n\} \cap [\operatorname{dom} p \cup \operatorname{ran} p] = \varnothing.$$

Let $q < p$ satisfy $q(g_i(m)) = g_i(m)$ for $1 \leq i \leq n$. Clearly $q \in \mathfrak{D}$. This proves Claim 2.4.

Now we explain how to pass from $M^0_{\alpha+1}$ to $M_{\alpha+1}$. Let $\mathfrak{F} = \{\mathrm{fix}(\pi^g) \mid g \in \Gamma_\alpha\}$. By Kunen's A10 [2] (p. 289), there exists a c.c.c. notion of forcing such that the generic extension $M_{\alpha+1}$ has the following property: there exists an infinite subset $S \subset \omega$ such that $|S \backslash F| < \omega$ for all $F \in \mathfrak{F}$. Choose an infinite cycle φ such that $\mathrm{Mov}(\varphi) = S$. Then for each $g \in \Gamma_\alpha$, $|\mathrm{Mov}(\pi^g) \cap \mathrm{Mov}(\varphi)| < \omega$. Hence, when regarded as elements of $S(\omega)$, we have that $[\pi^g, \varphi] = 1$. Now write $\pi_\alpha = \pi$ and $\varphi_\alpha = \varphi$, and let $T = \langle \pi_\alpha, \varphi_\alpha \mid \alpha < \omega_1 \rangle$. Then clearly T satisfies the requirements of the theorem. This completes the proof of Theorem 2.3.

Theorem 2.5 *Suppose that* $M \models$ GCH *and* $cf(\lambda) > \omega$. *Then there exists a generic extension* $M[G]$ *such that* $M[G] \models \neg F_\lambda$.

Proof: Let $\lambda = \omega_\alpha$. For each $i \leq \omega$, let $\mu_i = \omega_{\alpha+i}$. Let $\mathbb{P} = Fn(\mu_\omega, 2)$ be the set of finite functions p from μ_ω to 2, and let $\mathbb{P}_n = Fn(\mu_n, 2)$ for $n < \omega$. Let G be a generic subset of $\mathbb{P}$ and let $G_n = G \cap \mathbb{P}_n$. Note that for $1 \leq n < \omega$, $M[G_n] \models 2^\lambda = \mu_n$: while $M[G] \models 2^\lambda = (\mu_\omega)^+$.

Let $\pi \in \mathrm{Sym}(\lambda)^{M[G]}$, and let $\tilde{\pi}$ be a $\mathbb{P}$-name of π. For each $n < \omega$, let $\pi_n = \{\langle \alpha, \beta \rangle \mid (\exists p \in G_n) p \Vdash \tilde{\pi}(\alpha) = \beta\}$. Then $\pi_n \in M[G_n]$ and $\pi_n \subseteq \pi$. Also $\pi = \bigcup_{n \in \omega} \pi_n$. Since $cf(\lambda) > \omega$, there exists $n < \omega$ such that $|\mathrm{dom}(\pi_n)| = \lambda$. By taking a suitable subset of π_n if necessary, we can suppose that $|\lambda \backslash \mathrm{dom}(\pi_n)| = |\lambda \backslash \mathrm{ran}(\pi_n)| = \lambda$. Hence there exist $\psi, \theta \in \mathrm{Sym}(\lambda)^{M[G_n]}$ such that $\psi \supset \pi_n$ and $\mathrm{Mov}(\theta) = \mathrm{dom}(\pi_n)$. Then $\mathrm{fix}(\psi^{-1}\pi) \supseteq \mathrm{Mov}(\theta)$, so that $[\psi^{-1}\pi, \theta] = 1$.

Let $G = \bigcup_{n \in \omega} \mathrm{Sym}(\lambda)^{M[G_n]}$, and let T be the corresponding subgroups of $S(\lambda)^{M[G]}$. Then $|T| = \mu_\omega < 2^\lambda$, and T witnesses the failure of F_λ in $M[G]$.

3 *Small subgroups of* Sym(λ)

In [3], the following observation is made.

Lemma 3.1 *Let* $G \leq \mathrm{Sym}(\lambda)$. *Then the following are equivalent.*
(i) *For some proper ideal* I *on* λ *which contains a moiety of* λ, $G \leq S_{\{I\}}$.
(ii) *There exists a moiety* A *of* λ *such that for all* $g_1, \ldots, g_n \in G$,

$$\lambda \neq \bigcup_{1 \leq i \leq n} A^{g_i}.$$

If condition (ii) holds, we say that λ is not G-covered.

Definition 3.2

$$C(\lambda) = \min\{|G| \mid G \leq \mathrm{Sym}(\lambda),\ \lambda \text{ is } G\text{-covered}\}.$$

In [3], it is proved that $C(\lambda) > \lambda$. To explain what is going on here, it is useful to introduce three more cardinal functions.

Definition 3.3
(i) $A(\lambda)$ is the least cardinal κ such that if $\mathcal{A} \subset \mathcal{P}(\lambda)$ is an almost disjoint family, then $|\mathcal{A}| \leq \kappa$.
(ii) $B(\lambda)$ is the least size $|I|$ of a family of ultrafilters $\mathcal{U}_i \subseteq \mathcal{P}(\lambda)$, $i \in I$, such that
  (a) for all $i \in I$ and $X \in \mathcal{U}_i$, $|X| = \lambda$;
  (b) $\{X \subseteq \lambda \mid |X| = \lambda\} \subseteq \bigcup_{i \in I} \mathcal{U}_i$.
(iii) $D(\lambda)$ is the least size $|J|$ of a family of sets $\{Y_j \mid j \in J\} \subseteq \mathcal{P}(\lambda)$ such that

(a) for all $j \in J$, $|Y_j| = \lambda$;
(b) if $X \subseteq \lambda$ with $|X| = \lambda$, then there exists $j \in J$ with $Y_j \subseteq X$.

Theorem 3.4

$$\lambda < A(\lambda) \leq B(\lambda) \leq C(\lambda) \leq D(\lambda) \leq 2^{\lambda}.$$

Corollary 3.5 *If $G < \mathrm{Sym}(\omega)$ and $|G| < 2^{\omega}$, then ω is not G-covered.*

It is clear that $\lambda < A(\lambda) \leq B(\lambda)$. We prove the other inequalities in the next two lemmas.

Lemma 3.6

$$B(\lambda) \leq C(\lambda)$$

Proof: Suppose $G \leq \mathrm{Sym}(\lambda)$ is such that λ is G-covered. Let $\mathfrak{U}$ be a uniform ultrafilter on λ; i.e., $|X| = \lambda$ for all $X \in \mathfrak{U}$. Suppose that there exists a moiety $X \in \mathfrak{U}$ such that $g[X] \cap X \in \mathfrak{U}$ for all $g \in G$. Then for all $g_1, \ldots, g_n \in G$, $\bigcap_{1 \leq i \leq n} g_i[X] \in \mathfrak{U}$. Let I be the ideal which is dual to the filter

$$\mathfrak{F} = \left\{ Z \in \mathcal{P}(\lambda) \,|\, \text{There exist } g_1, \ldots, g_n \in G \text{ with } \bigcap_{1 \leq i \leq n} g_i[X] \subseteq Z \right\}.$$

Then $G \leq S_{\{I\}}$ and I is a proper ideal which contains a moiety of λ, a contradiction. Hence for each moiety $X \in \mathfrak{U}$, there exists $g \in G$ such that $X \backslash g[X] \in \mathfrak{U}$.

Fix an element $g \in G$ and let

$$S(g) = \{ X \in \mathfrak{U} \,|\, X \backslash g[X] \in \mathfrak{U} \}.$$

If $X_1, \ldots, X_n \in S(g)$, then

$$\bigcap_{1 \leq i \leq n} [X_i \backslash g[X_i]] = \left(\bigcap_{1 \leq i \leq n} X_i \right) \Big\backslash \left(\bigcup_{1 \leq i \leq n} g[X_i] \right) \in \mathfrak{U}.$$

In particular, $\bigcup_{1 \leq i \leq n} g[X_i] = g[\bigcup_{1 \leq i \leq n} X_i]$ must be a moiety of λ, so that $\bigcup_{1 \leq i \leq n} X_i$ is a moiety. Hence $\lambda \backslash \bigcup_{1 \leq i \leq n} X_i = \bigcap_{1 \leq i \leq n} (\lambda \backslash X_i)$ is a moiety. Consequently, there exists a uniform ultrafilter $\mathfrak{U}(g) \supseteq \{ \lambda \backslash X \,|\, X \in S(g) \}$. So every moiety of λ lies in one of the uniform ultrafilters $\{\mathfrak{U}\} \cup \{ \mathfrak{U}(g) \,|\, g \in G \}$. Hence $B(\lambda) \leq |G|$, and so $B(\lambda) \leq C(\lambda)$.

Lemma 3.7

$$C(\lambda) \leq D(\lambda)$$

Proof: Let $\mathfrak{F} \subseteq \mathcal{P}(\lambda)$ satisfy the following:

(a) $|X| = \lambda$ for $X \in \mathfrak{F}$;
(b) if $Y \subseteq \lambda$ with $|Y| = \lambda$, then there exists $X \in \mathfrak{F}$ with $X \subseteq Y$;
(c) $|\mathfrak{F}| = D(\lambda)$.

Clearly we can also suppose that

(d) each $X \in \mathfrak{F}$ is a moiety.

For each $X \in \mathfrak{F}$, let $\pi_X \in \mathrm{Sym}(\lambda)$ be an involution such that $\pi_X[X] = \lambda \backslash X$, and let $G = \langle \pi_X \,|\, X \in \mathfrak{F} \rangle$.

Now let $A \subseteq \lambda$ be a moiety. Then there exists $X \in \mathcal{T}$ with $X \subseteq A$. Thus $\pi_X[A] \supseteq \lambda \backslash X \supseteq \lambda \backslash A$, so that $\lambda = A \cup \pi_X[A]$. Hence λ is G-covered, and so $C(\lambda) \leq D(\lambda)$.

The final result in this section shows that it is consistent that there exists $G < \text{Sym}(\lambda)$ with $|G| < 2^\lambda$ such that λ is G-covered. It also demonstrates the consistency of $B(\lambda) < C(\lambda)$.

Theorem 3.8 *Suppose that $M \models$ GCH and $\lambda > \omega$ is regular. Let $\lambda = \omega_\alpha$ and $\kappa = \omega_{\alpha+\omega}$. Let $\mathbb{P} = Fn(\kappa,2)$ be the partial order of finite functions from κ to 2, and let G be a generic subset of $\mathbb{P}$. Then the following statements are true in $M[G]$.*
(a) $2^\lambda = \kappa^+$
(b) $A(\lambda) = B(\lambda) = \lambda^+$
(c) $C(\lambda) = D(\lambda) = \kappa$.

Proof: The facts that $2^\lambda = \kappa^+$ and $A(\lambda) = \lambda^+$ are included in Theorem 5.6 of Baumgartner [1]. Arguing as in the proof of Theorem 2.5, we easily obtain that $D(\lambda) \leq \kappa$. Thus to prove (c), it is enough to show that $C(\lambda) \geq \kappa$.

So suppose that there exists $\Gamma < \text{Sym}(\lambda)^{M[G]}$ with $\lambda < |\Gamma| = \theta < \kappa$ such that λ is Γ-covered. Then there exists $I \subset \kappa$ of cardinality θ such that $\Gamma \in M[G \cap Fn(I,2)] = N$. Let $\mathbb{Q} = Fn(\lambda,2)$ and let $H \subset \mathbb{Q}$ be generic over N. We shall show that λ is not Γ-covered in $N[H]$, which yields the desired contradiction.

Let $f = \cup\{p \mid p \in H\}$ and let $S = \{\alpha \in \lambda \mid f(\alpha) = 1\}$. Clearly S is a moiety of λ. Let $\pi_1, \ldots, \pi_n \in \Gamma$ and let $\mathfrak{D}$ consist of the $q \in \mathbb{Q}$ satisfying:

(+) There exists $\beta \in \lambda$ and $\gamma_1, \ldots, \gamma_n \in \lambda$ such that

(i) $\pi_i(\gamma_i) = \beta$ for $1 \leq i \leq n$;
(ii) $q(\gamma_i) = 0$ for $1 \leq i \leq n$.

Clearly $\mathfrak{D}$ is dense in $\mathbb{Q}$, and if $q \in \mathfrak{D}$ then $q \Vdash \bigcup_{1 \leq i \leq n} \pi_i[S] \neq \lambda$. Thus λ is not Γ-covered in $N[H]$.

It only remains to compute $B(\lambda)$. We shall do this via the following series of lemmas.

Definition 3.9 A $\mathbb{P}$-name σ is simple if it has the form

$$\sigma = \{\langle \check{\alpha}, q_\alpha \rangle \mid \alpha \in X\}$$

where

(a) $X \subseteq \lambda$ has cardinality λ.
(b) If $\alpha \neq \beta$, then $\text{dom}\, q_\alpha \cap \text{dom}\, q_\beta = \emptyset$.
(c) There exists $n < \omega$ and $f_\sigma : n \to 2$ such that for all $\alpha \in X$.
  (i) $\text{dom}\, q_\alpha = \{\alpha_0, \ldots, \alpha_{n-1}\}$
  (ii) $q_\alpha(\alpha_i) = f_\sigma(i)$ for $i < n$.

Lemma 3.10 *If σ is a simple $\mathbb{P}$-name, then $\Vdash \sigma \in [\lambda]^\lambda$.*

A straightforward Δ-system argument yields the next result.

Lemma 3.11 *Suppose that $G \subseteq \mathbb{P}$ is generic and that $M[G] \models \tau_G \in [\lambda]^\lambda$. Then there exists a simple $\mathbb{P}$-name σ such that $M[G] \models \sigma_G \subseteq \tau_G$.*

Sh:459

Thus it suffices to find λ^+ uniform ultrafilters in $M[G]$ such that σ_G is contained in one of them for each simple $\mathbb{P}$-name σ. We shall also make use of the following well-known result.

Lemma 3.12 *For any cardinal $\theta \geq \omega$, $Fn(2^\theta,2)$ is the union of θ centered subsets.*

Clearly it is enough to show that $B(\lambda) \leq \lambda^+$. Initially we will work inside M. Let $\langle \mathcal{U}_\alpha \mid \alpha < \lambda^+ \rangle \in M$ be a sequence of uniform ultrafilters on λ such that for each $X \in [\lambda]^\lambda \cap M$, there exists $\alpha \leq \lambda^+$ with $X \in \mathcal{U}_\alpha$. Let $\sigma = \{\langle \check{\alpha}, q_\alpha \rangle \mid \alpha \in X\}$ be a simple $\mathbb{P}$-name, and let $\operatorname{dom} q_\alpha = \{\alpha_0, \ldots, \alpha_{n-1}\}$ for each $\alpha \in X$. Then $X \in \mathcal{U}_\gamma$ for some $\gamma < \lambda^+$. Define an equivalence relation $\equiv_\gamma$ on ${}^\lambda\kappa$ by:

$$\psi \equiv_\gamma \theta \text{ iff } \{\alpha \mid \psi(\alpha) = \theta(\alpha)\} \in \mathcal{U}_\gamma.$$

Let $[\psi]_\gamma$ be the $\equiv_\gamma$-class containing $\psi \in {}^\lambda\kappa$, and let ${}^\lambda\kappa/\mathcal{U}_\gamma = \{[\psi]_\gamma \mid \psi \in {}^\lambda\kappa\}$. Then σ determines $p_\sigma \in Fn({}^\lambda\kappa/\mathcal{U}_\gamma,2)$ as follows. For $i < n$, define $h_i \in {}^\lambda\kappa$ by

$$\begin{aligned} h_i(\alpha) &= \alpha_i \text{ if } \alpha \in X \\ &= 0 \ \text{ if } \alpha \in \lambda \setminus X. \end{aligned}$$

Let $\operatorname{dom} p_\sigma = \{[h_0]_\gamma, \ldots, [h_{n-1}]_\gamma\}$ and set $p_\sigma([h_i]_\gamma) = f_\sigma(i)$.

Lemma 3.13 *Suppose that $\sigma_j = \{\langle \check{\alpha}, q_\alpha^j \rangle \mid \alpha \in X_j\}$ is a simple $\mathbb{P}$-name for $j < k$. Suppose further that:*
(1) $X_j \in \mathcal{U}_\gamma$ for $j < k$;
(2) $p_{\sigma_0}, \ldots, p_{\sigma_{k-1}}$ have a common strengthening $p \in Fn({}^\lambda\kappa/\mathcal{U}_\gamma,2)$.

Then $\Vdash \sigma_0 \cap \cdots \cap \sigma_{k-1} \in [\lambda]^\lambda$.

Proof: For each $j < k$ and $\alpha \in X_j$, let $\operatorname{dom} q_\alpha^j = \{\alpha_0^j, \ldots, \alpha_{n_j-1}^j\}$. Let $Z \in \mathcal{U}_\gamma$ consist of those $\alpha < \lambda$ satisfying

(a) $\alpha \in X_0 \cap \cdots \cap X_{k-1}$.
(b) If $s < t < k$, $l < n_s - 1$ and $m < n_t - 1$, then

$$\alpha_l^s = \alpha_m^t \text{ iff } [h_l^s]_\gamma = [h_m^t]_\gamma.$$

Let $r \in \mathbb{P}$ be arbitrary and $\beta < \lambda$. Then there exists $\alpha \in Z$ such that

(c) $\beta < \alpha < \lambda$.
(d) $\operatorname{dom} r \cap \operatorname{dom} q_\alpha^j = \emptyset$ for all $j < k$.

We define $q = r \cup \bigcup_{j<k} q_\alpha^j$. Assuming that $q \in \mathbb{P}$, we have that $q \leq r$ and that $q \Vdash \alpha \in \sigma_0 \cap \cdots \cap \sigma_{k-1}$. Thus it is enough to show that q is a well-defined function. Suppose that $\alpha_l^s = \alpha_m^t$ for some $s < t < k$. Then, since $[h_l^s]_\gamma = [h_m^t]_\gamma$ and $p_{\sigma_s}, p_{\sigma_t}$ are compatible, we must have $p_{\sigma_s}([h_l^s]_\gamma) = p_{\sigma_t}([h_m^t]_\gamma)$ and hence $q_\alpha^s(\alpha_l^s) = q_\alpha^t(\alpha_m^t)$.

For each $\gamma < \lambda^+$, let $\mathbb{Q}_\gamma = Fn({}^\lambda\kappa/\mathcal{U}_\gamma,2) \in M$. In the remainder of the proof, we will work inside $M[G]$. Notice that the cardinality of $({}^\lambda\kappa/\mathcal{U}_\gamma)^M$ is at most 2^λ in $M[G]$. So by Lemma 3.12, we can express $\mathbb{Q}_\gamma = \bigcup_{\xi<\lambda} A_{\gamma\xi}$ as a union of λ centered sets. Let $S = \{\sigma_G \mid \sigma \text{ is a simple } \mathbb{P}\text{-name}\}$. For each $\gamma < \lambda^+$ and $\xi < \lambda$, define $\mathcal{U}_{\gamma\xi} = \{\sigma_G \mid p_\sigma \in A_{\gamma\xi}\}$. Then

$$S=\bigcup_{\substack{\gamma<\lambda^+\\ \xi<\lambda}} \mathcal{U}_{\gamma\xi}.\ \text{If } (\sigma_0)_G,\dots,(\sigma_{k-1})_G\in\mathcal{U}_{\gamma\xi}$$

then $p_{\sigma_0},\dots,p_{\sigma_{k-1}}$ have a common strengthening in $\mathbb{Q}_\gamma$ and so $\Vdash\sigma_0\cap\cdots\cap\sigma_{k-1}\in[\lambda]^\lambda$. Thus $\mathcal{U}_{\gamma\xi}$ can be completed to a uniform ultrafilter on λ. This completes the proof that $B(\lambda)=\lambda^+$.

4 *Covering families of ultrafilters*

Theorem 4.1 *Let $M\models$ GCH. Let λ and $\kappa>\lambda^{+++}$ be regular cardinals. Then there exists a notion of forcing $\mathbb{P}$, which preserves cofinalities and cardinalities, such that if $G\subseteq\mathbb{P}$ is generic then*

$$M[G]\models\lambda^+=A(\lambda)<B(\lambda)=\kappa=2^\lambda.$$

Definition 4.2 $\mathbb{P}$ consists of all conditions $p=\langle a,h,f,g\rangle$ satisfying
(i) $a\in[\kappa]^{\leq\lambda^{++}}$.
(ii) $h:[a]^2\to\lambda$.
(iii) There exist finite $u\subseteq a$, $v\subseteq\lambda$ such that $f:u\times v\to 2$ and $g:[u]^2\to 2$.
(iv) If $g(\alpha,\beta)=f(\alpha,\gamma)=f(\beta,\gamma)=1$, then $\gamma<h(\alpha,\beta)$.

The order relation is the natural one.

The intuitive meaning is that we are adjoining the sets $A_\alpha=\{\gamma<\lambda\mid f(\alpha,\gamma)=1\}$ for $\alpha<\kappa$. The function h gives a vague promise that $A_\alpha\cap A_\beta\subseteq h(\alpha,\beta)$. But h is unreliable, and should only be taken seriously when $g(\alpha,\beta)=1$.

Definition 4.3
(a) $q=\langle a_1,h_1,f_1,g_1\rangle\underset{pr}{\leq}p=\langle a_0,h_0,f_0,g_0\rangle$ iff $q\leq p$, $f_0=f_1$ and $g_0=g_1$.
(b) $q=\langle a_1,h_1,f_1,g_1\rangle\underset{ap}{\leq}p=\langle a_0,h_0,f_0,g_0\rangle$ iff $q\leq p$, $a_0=a_1$ and $h_0=h_1$.

Lemma 4.4 *If $q\leq p$, then there exists $r\in\mathbb{P}$ such that $q\underset{ap}{\leq}r\underset{pr}{\leq}p$.*

An easy Δ-system argument yields the next result.

Lemma 4.5 *If $p\in\mathbb{P}$, then $\{q\in\mathbb{P}\mid q\underset{ap}{\leq}p\}$ satisfies the c.c.c.*

Lemma 4.6 *If $p\in\mathbb{P}$ and $\tilde{\tau}$ is a $\mathbb{P}$-name of an ordinal, then there exists $q\in\mathbb{P}$ such that*
(i) $q\underset{pr}{\leq}p$;
(ii) *if $r\leq q$ and $r\Vdash\tilde{\tau}=\gamma$, then there exists $r'\parallel r$ such that $r'\underset{ap}{\leq}q$ and $r'\Vdash\tilde{\tau}=\gamma$.*

Proof: We define inductively p_i, and also r_j,γ_j for successor j such that:

(i) $p_0=p$;
(ii) $p_i\underset{pr}{\leq}p$ and the chain $\{p_k\mid k\leq i\}$ is strictly decreasing and continuous;
(iii) $r_j\underset{ap}{\leq}p_j$ and $r_j\Vdash\tilde{\tau}=\gamma_j$;
(iv) if $j_1<j_2$ then $r_{j_1}\nparallel r_{j_2}$.

Suppose that the construction can be continued for all $i < \omega_1$. Then there exists $p^* \in \mathbb{P}$ with $p^* \underset{pr}{\leq} p_i$ for all $i < \omega_1$. Notice that for each successor $j < \omega_1$, there exists $r_j^* \in \mathbb{P}$ such that $r_j^* \leq r_j$ and $r_j^* \underset{ap}{\leq} p^*$. But then $\{r_j^* \mid j < \omega_1$ is a successor$\}$ is an uncountable antichain, which contradicts Lemma 4.5.

So where does the inductive construction break down? Since $\{q \in \mathbb{P} \mid q \underset{pr}{\leq} p\}$ is λ^{+++}-closed, the construction cannot fail at a limit stage. Thus we can suppose that p_i has been constructed, but that it is impossible to construct p_{i+1}, r_{i+1}, γ_{i+1}. We claim that $q = p_i$ satisfies our requirements. Suppose not. Then there exists γ and $r \leq p_i$ with $r \Vdash \tilde{\tau} = \gamma$ such that there is no $r' \underset{ap}{\leq} p_i$ satisfying $r' \parallel r$ and $r' \Vdash \tilde{\tau} = \gamma$. Let $r_{i+1} = r \underset{ap}{\leq} p_{i+1} \underset{pr}{\leq} p_i$, and let $\gamma_{i+1} = \gamma$. Then (iv) must fail, and so there exists $j \leq i$ with $r_j \parallel r_{i+1}$. In particular, $\gamma_j = \gamma_{i+1} = \gamma$ and $r_j \Vdash \tilde{\tau} = \gamma$. But now there exists $r_j^* \underset{ap}{\leq} p_i$ with $r_j^* \leq r_j$ and $r_j^* \parallel r$, which is a contradiction.

Using the fact that $\{q \in \mathbb{P} \mid q \underset{pr}{\leq} p\}$ is λ^{+++}-closed for each $p \in \mathbb{P}$, we easily obtain the following result.

Lemma 4.7 *If $\tilde{\tau}_i, i < \lambda^{++}$, are $\mathbb{P}$-names for ordinals and $p \in \mathbb{P}$, then there exists $q \underset{pr}{\leq} p$ such that if $i < \lambda^{++}$ and $r \leq q$ with $r \Vdash \tilde{\tau}_i = \gamma$, then there exists $r' \parallel r$ such that $r' \underset{ap}{\leq} q$ and $r' \Vdash \tilde{\tau}_i = \gamma$.*

Lemma 4.8 *$\mathbb{P}$ preserves all cardinals and cofinalities less than or equal to λ^{+++}.*

Proof: For example, suppose that $p \Vdash \tilde{f}: \lambda^{++} \to \lambda^{+++}$. Let $q \underset{pr}{\leq} p$ satisfy the conclusion of Lemma 4.7 with respect to the $\mathbb{P}$-names $\tilde{f}(\check{\alpha})$, $\alpha < \lambda^{++}$. Since $\{r \in \mathbb{P} \mid r \underset{ap}{\leq} q\}$ satisfies the c.c.c., we see that $q \Vdash \tilde{f}$ is not a cofinal map in λ^{+++}.

An easy Δ-system argument (which makes use of the assumption that $M \models$ GCH) yields the next result.

Lemma 4.9 *$\mathbb{P}$ is λ^{++++}-c.c.; and hence $\mathbb{P}$ preserves all cardinals and cofinalities.*

Lemma 4.10

$$\Vdash A(\lambda) = \lambda^{+}.$$

Proof: Suppose that $p \Vdash$ "$\langle \tilde{T}_i \mid i < \lambda^{++} \rangle$ is an almost disjoint family in $\mathcal{P}(\lambda)$." For each $i < j < \lambda^{++}$, let $\tilde{\tau}_{ij} = \sup(\tilde{T}_i \cap \tilde{T}_j)$. Then $p \Vdash \tilde{\tau}_{ij} < \lambda$. Choose $q \underset{pr}{\leq} p$ satisfying the conclusion of Lemma 4.7 with respect to the $\mathbb{P}$-names $\tilde{\tau}_{ij}, i < j < \lambda^{++}$. Using Lemma 4.5, we see that there exists $\beta_{ij} < \lambda$ such that $q \Vdash \tilde{T}_i \cap \tilde{T}_j \subseteq \beta_{ij}$.

Since $M \models$ GCH, $\lambda^{++} \to (\lambda^{+})^2_{\lambda}$. Hence there exists $H \subset \lambda^{++}$ with $|H| = \lambda^{+}$ and $\beta < \lambda$ such that for all distinct $i,j \in H$, $q \Vdash \tilde{T}_i \cap \tilde{T}_j \subseteq \beta$. Let $G' \ni q$ be generic and $T_i = (\tilde{T}_i)_{G'}$. Then in $M[G']$, $\{T_i \setminus \beta \mid i \in H\}$ is a collection of λ^+ nonempty pairwise disjoint subsets of λ, which is a contradiction.

Definition 4.11 For each $\alpha < \kappa$, $\tilde{A}_\alpha = \{\langle \check{\gamma}, \langle a,h,f,g\rangle\rangle \mid f(\alpha,\gamma) = 1\}$.

Lemma 4.12
(i) $\Vdash |\tilde{A}_\alpha| = \lambda$.
(ii) *If* $p = \langle a,h,f,g\rangle$ *and* $g(\alpha,\beta) = 1$, *then* $p \Vdash \tilde{A}_\alpha \cap \tilde{A}_\beta \subseteq h(\alpha,\beta) < \lambda$.

Lemma 4.13

$$\Vdash B(\lambda) = \kappa = 2^\lambda.$$

Proof: Suppose not, and let $\theta = \lambda^{++++}$. Then there exists a $\mathbb{P}$-name $\tilde{\mathfrak{D}}$ for a uniform ultrafilter on λ, distinct ordinals $\alpha_i < \kappa$ for $i < \theta$, and conditions $p_i \in \mathbb{P}$ such that $p_i \Vdash \tilde{A}_{\alpha_i} \in \tilde{\mathfrak{D}}$. Let $p_i = \langle a_i, h_i, f_i, g_i\rangle$. We can suppose that $\alpha_i \in a_i$ for each $i < \theta$.

Since $M \models$ GCH, we can also suppose that the following hold.

(i) $\{a_i \mid i < \theta\}$ forms a Δ-system with root A; and the h_i are pairwise compatible functions.
(ii) $\{u_i \mid i < \theta\}$ forms a Δ-system with root U, $\{v_i \mid i < \theta\}$ forms a Δ-system with root V; and the f_i, g_i are pairwise compatible functions. Since $|A| \leq \lambda^{++}$, we can also suppose that
(iii) $\alpha_i \notin A$ for all $i < \theta$.

Fix $i < j < \theta$. Since $\alpha_i, \alpha_j \notin A$, we can form a condition $q = \langle a,h,f,g\rangle \leq p_i, p_j$ such that $g(\alpha_i,\alpha_j) = 1$ and $h(\alpha_i,\alpha_j)$ is given a sufficiently large value. But then

$$q \Vdash \tilde{A}_{\alpha_i} \cap \tilde{A}_{\alpha_j} \subseteq h(\alpha_i,\alpha_j) < \lambda,$$

which is a contradiction.

This completes the proof of Theorem 4.1. The following problems remain open.

Question 4.14 Suppose that $G < \mathrm{Sym}(\lambda)$ and $|G| < 2^\lambda$. Is G contained in a maximal subgroup of $\mathrm{Sym}(\lambda)$?

Question 4.15 Does $C(\lambda) = D(\lambda)$?

Question 4.16 Is it consistent that $C(\omega_1) = \omega_2 < 2^{\omega_1}$?

Acknowledgments Professor Baumgartner's research was partially supported by NSF Grant DMS-8906964. Professor Shelah's research was supported by the BSF and the Fund for Basic Research administered by the Israel Academy of Sciences and Humanities. Professor Thomas's research was partially supported by NSF Grant DMS-8902139.

REFERENCES

[1] Baumgartner, J., "Almost disjoint sets, the dense set problem and the partition calculus," *Annals of Mathematical Logic*, vol. 10 (1976), pp. 401–439.

[2] Kunen, K., *Set Theory*, North-Holland, Amsterdam, 1980.

[3] MacPherson, H. and C. Praeger, "Maximal subgroups of infinite symmetric groups," *Journal of London Mathematical Society*, vol. 42 (1990), pp. 85–92.

Department of Mathematics
Dartmouth College
Hanover, NH 03755

Institute of Mathematics
The Hebrew University
Jerusalem, Israel

Department of Mathematics
Rutgers University
New Brunswick, NJ 08903