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Toward classifying unstable theories*

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0. Introduction

Having finished [14], an important direction seems to me to try to classify unstable theories; i.e. to find meaningful dividing lines. In [12] the two such were the strict order property and independence property, their disjunction is equivalent to unstability (see [14, 4.7]=[20, 4.7, p. 70]). For theories without the independence property, we know $S(A)$ (and $S_\triangle(A)$) are relatively small (see [12; 9; 14, III, Section 7, 7.3, 7.4, II, Section 4, 4,9, 4.10]). Also for $\lambda > |T|, \{p \in S(A) : p$ does not split over some $B \subseteq A$ of cardinality $< \lambda\}$ is λ-dense (see [14, 7.5]=[20, 7.5, p. 140]).

Later this becomes interesting in the context of analyzing monadic logic (see Baldwin Shelah's [2]; representation Baldwin [1]). By [19] if "no monadic expansion of T has the independence property" is a significant dividing line.

Lately, some model theorists have become interested in finitary versions called UC dimensions, see [11] (good bound for the case of expansion off the real field).

More relevant to the present work is the tree property, which is weaker than the strict order property (in [20, III, p.171]).

In [16] we try to investigate theories without the tree property, hence called simple. This can be looked at as a weakening of stable, so: simple $\Leftrightarrow \kappa_{\rm cdt}(T) < \infty \Leftrightarrow$ failure of the tree property $\Leftrightarrow$ suitable local ranks $< \infty$ are parallel to stable. We try to proceed parallel to (parts of) Chs. II, III of [14], forking being generalized in some ways; but here instead of showing the number of ultrafilters of the Boolean Algebras of formulas $\varphi(x, \bar a)$ over A is small ($\leqslant |A|^{|T|}$) we show that it can be decomposed to few subalgebras satisfying a strong chain condition. In this context we also succeed to get averages; but the Boolean algebras we get were derived from normal ones with a little twist. We did not start with generalizing the rest of [14] like supersimple (i.e. $\kappa_{\rm cdt}(T) = \aleph_0$, equivalently suitable rank is $< \infty$). The test problem in [16] was trying

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to characterize the class of pairs

$$SP(T) = \left\{ (\lambda, \kappa): \text{ every model of } T \text{ of cardinality } \lambda \text{ has a } \kappa\text{-saturated elementary extensions of cardinality } \lambda \right\}.$$

For simplicity we consider there only $\lambda = \lambda^{|T|} > 2^{|T|+\kappa}, \kappa > |T|$ and $(\exists\mu)(\mu = \mu^{<\kappa} \leqslant \lambda \leqslant 2^{\mu})$ (if this fails, see [22, 23]). So by [16] for non simple T, such (λ, κ) is in $SP(T)$ iff $\lambda = \lambda^{<\kappa}$. If $\mu = \mu^{<\mu} < \lambda = \lambda^{<\lambda}$, after sutiable forcing preserving $\mu = \mu^{<\mu}$, not collapsing cardinals and making $2^{\mu} = \lambda$, we have and deduce suitable generalization of MA, $\kappa < \mu < \lambda < 2^{\mu} \Rightarrow (\lambda, \kappa) \in SP(T)$.

It seems much better to use just the cardinal arithmetic assumptions (not the generalizations of MA). This calls to investigate problem of $\mathscr{P}^{-}(n)$-amalgamation (see [18, 20, XII, Section 5]). For the case of $n = 3$ this means that

$(*)_3$ if $p_0(\bar{x}, \bar{y}), p_1(\bar{x}, \bar{z}), p_2(\bar{y}, \bar{z})$, complete types over A, each saying the two sequences of variables are "independent" in suitable ways (like nonforking) then we extend the union of the three (preserving "independence").

Now $(*)_3$ can be proved [16, Claim 7.8, p. 201, (3.5, p. 187)]. But the proof does not work for higher n, naturally counterexamples for the amalgamation should give counterexample to membership in SP. This was carried out by finding counterexamples in a *wider framework*: saturation inside P in [17]; but we could still hope that for the "true" one there is a positive one.

For long, I was occupied elsewhere and did not look into it, but eventually Hrushovski became interested (and through him, others) and we try to explain his relevant research below. Also, it could be asked if simple unstable theories "occur in nature", "are important to algebraic applications". The works cited below gives a positive answer (note that, quite natural, these examples concentrate on the lower part of hierarchy, like strongly minimal or finite Morley rank).

On the one hand, Hrushovski, continuing [17], proves that there are simple theories with bad behavior for $\mathscr{P}(n)$ so in the result above the cardinal arithmetic is not enough.

On the other hand, by the work of Hrushovski and Pillay [8] in specific cases (finite ranks) relevant cases of $(*)_n$ are proved, for $n > 3$ under very specific conditions: for $n = 3$ more generally; but the relationship with [16], 7.8 of $(*)$ was not clarified (in both cases the original rank does not work; the solution in [16] is to use dnwd (= "do not weakly divide"), Hrushovski changes the rank replacing "contradictory" by having small rank; this seems a reasonable approach only for supersimple theories and was carried out only for ones with finite rank, and it gives more information in other respects.

In [5] let $\mathfrak{C}_0$ be the monster model for a strongly minimal theory with elimination of imaginaries, $A \subseteq \mathfrak{C}, A = \operatorname{dcl} A$, such that every $p \in S^m(A)$ with multiplicity 1 is finitely satisfiable in A. Now $\operatorname{Th}(\mathfrak{C}, A)$ is simple (of rank 1) and we can understand PAC in general content. Hrushovski [4] does parallel thing for finite rank.

We turn to the present work. Section 1 deals with the existence of universal models. Note that existence of saturated models can be characterized nicely by stability (see [20]).

By Kojman Shelah [10], the theory of linear order and more generally theories without the strict order property has universal models in "few" cardinals.

By [21] we have a sufficient condition for a consistency of "there is in μ^{++} a model of T universal for models of T of cardinality μ^{+}", we use this condition below.

The main aim is to show that all simple theories behave "better" in this respect than the theory of linear order. Specifically, it is consistent that $\aleph_0 < \lambda = \lambda^{<\lambda}, 2^\lambda > \lambda^{++}$, moreover, there is a club guessing $\langle C_\delta : \delta < \lambda^+, \mathrm{cf}(\delta) = \lambda \rangle$, and every simple T of cardinality $< \lambda$ has a model in λ^{++} universal for λ^+. For this we represent results of [16] and do the things needed specifically for the use of [21]. See Remark 1.4A(2). In Section 2 we start to investigate nonsimple which are 'low'.

1. Simple theories have more universal models

We quote [21, 5.1].

Lemma 1.1. *Suppose*

(A) *T is first order, complete for simplicity with elimination of quantifiers (or just inductive theory with the amalgamation and disjoint embedding property).*

(B) *K_{ap} is a simple λ-approximation system such that every $M \in K_{\mathrm{ap}}$ is a model of T hence every M_Γ, where for $\Gamma \in K_{md}$ we let $M_\Gamma = \bigcup\{M : M \in \Gamma\}$.*

(C) *Every model M of T of cardinality λ^+ can be embedded into M_Γ for some $\Gamma \in K_{md}$.*

Then

(a) *in* [21, 4.9] *in V^P, there is a model of T of cardinality λ^{++} universal for models of T of cardinality λ^+.*

(b) *So in V^P, $univ(\lambda^+, T) \leqslant \lambda^{++} < 2^\lambda$.*

Proof. Straightforward. $\square$

Fact 1.1. (1) *Assume $M \prec N, \bar{a} \in {}^{\omega>}N$, and $\triangle$ a finite set of formulas possibly with parameters from M. Then there is a formula $\psi(\bar{x}, \bar{b}) \in tp(\bar{a}, M, N)$ such that:*

($*$) *for any $\bar{a}' \in M$ realizing $\psi(\bar{x}, \bar{b})$, we can find a $\triangle - 2$-indiscernible sequence $\langle \bar{a}_i : i \leqslant \omega \rangle$ such that: $\bar{a}_0 = \bar{a}', \bar{a}_{\omega+1} = \bar{a}$; hence we can find an indiscernible sequence $\langle a_i' : i < \omega \rangle$ (in $\mathfrak{C}$) such that the $\triangle$-type of $\bar{a}_0'{}^\frown\bar{a}_1'$ is the same as that of $\bar{a}'{}^\frown\bar{a}$.*

(2) *Assume $2^{\theta+|T|} \leqslant \kappa$ and $M \prec N$, moreover,*

$\otimes$ *if $A \subseteq M, |A| \leqslant \kappa, \bar{a} \in {}^\theta N$ then some $\bar{a}' \in M$ realizes $tp(\bar{a}, A, N)$.*

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Then for any $\bar{a} \in {}^{\theta}N$ and $B \subseteq M, |B| \leqslant \theta$, there is $A \subseteq M, |A| \leqslant \kappa$, such that for every $\bar{a}' \in {}^{\theta}M$ realizing $tp(\bar{a}, A, N)$ there is a sequence $\langle \bar{a}_i : i \leqslant \kappa \rangle$ which is 2-indiscernible over $B, \bar{a}_0 = \bar{a}', \bar{a}_\kappa = \bar{a}$, hence there is an indiscernible sequence $\langle \bar{a}'_i : i < \omega \rangle$ such that $\bar{a}'_0 {}^\frown \bar{a}'_1$ realizes the same type as $\bar{a}' {}^\frown \bar{a}$ over B.

Proof. Obvious (notes on combination set theory).

(1) Let $\langle p_i : i < k \rangle$ list the possible Δ-types of sequences of length $\ell g(\bar{a}) + \ell(\bar{a})$, so $k < \omega$. For each p_i, choose $\psi_i(\bar{x}, \bar{b}_i) \in tp(\bar{a}, M, N)$ such that, if possible for no $\bar{a}' \in {}^{\ell g(\bar{a})}M$ realizing $\psi_i(\bar{x}, \bar{b}_i)$ do we have $\bar{a}' {}^\frown \bar{a}$ realizes p_i.

Now $\psi(\bar{x}, \bar{b}) =: \bigwedge_{i<k} \psi_i(\bar{x}, \bar{b})$ is as required.

(2) $\langle p_i : i < k \rangle$ list the complete $2\ell g(\bar{a})$-types over B. Use $p_i \subseteq tp(\bar{a}, M, N), |p_i| \leqslant \theta + |T|$ instead of $\psi_i(\bar{x}, \bar{b}_i), A = \bigcup_{i<k} Dom\ p_i$. $\square$

Theorem 1.2. *If T is a complete simple (f.o.) theory, $|T| < \lambda$, then T satisfies the assumptions of Lemma 1.1 (hence its conclusions).*

Remark 1.2A. (1) We can get results for a theory T of cardinality $\leqslant \lambda$ under stronger assumptions on T.

(2) Though not always necessary, in this section we will assume T is simple.

(3) Also this section is not written in a way focused on Theorem 1.2, but for leisurely relook at simple theories.

Proof. Without loss of generality T has elimination of quantifiers.

We first recapitulate (in Definitions 1.3, 1.4 and 1.8, Remark 1.4A, Claims 1.5 and 1.6, Theorem 1.7 and Observations 1.9 and 1.10) the needed definitions and facts on simple theories from [16] (adding notation and some facts), then say a little more and prove the theorem. So for a while we work in a fixed $\bar{\kappa}$-saturated model $\mathfrak{C}$ of $T, \bar{\kappa}$ big enough. So M, N denotes elementary submodels of $\mathfrak{C}$ of cardinality $< \bar{\kappa}$, A, B, C, D denote subsets of $\mathfrak{C}$ of cardinality $< \bar{\kappa}$ and $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ denote sequences of elements of $\mathfrak{C}$ of length $< \bar{\kappa}$, usually finite. Let $\bar{a}/B = \mathrm{tp}(\bar{a}, B) = \{\varphi(\bar{x}, \bar{b}) : \bar{b} \in {}^{\omega >}B, \varphi \in L(T)$ is first order and $\mathfrak{C} \models \varphi[\bar{a}, \bar{b}]\}$.

Definition 1.3. (1) We say that "$p(\bar{x})$ does not weakly divide over (r, B)" (in short p dnwd (i.e. does not weakly divide) over (r, B); we write over B when $r = p \restriction B$, we write over r if $B = Dom(r)$), where $r = r(\bar{x})$ is a type over B (and $\bar{x}$ may be infinite) when if $\bar{b} \in B$ and $\psi = \psi(\bar{x}^1, \ldots, \bar{x}^n, \bar{y})$ a formula (where $\ell g(\bar{x}^\ell) = \ell g(\bar{x}), \bar{x}^\ell$ with no repetition, $\langle \bar{x}^\ell : \ell = 1, n \rangle {}^\frown \langle y \rangle$ pairwise disjoint) and (see Definition 1.3(2) below) $[r]^\psi$ is finitely satisfiable (in $\mathfrak{C}$) then so is $[r \cup p]^\psi$ (see Definition 1.3(2) below).

(2) If $\psi = \psi(\bar{x}^1, \ldots, \bar{x}^n)$ (possibly with parameters), $q = q(\bar{x})$, then

$$[q]^\psi = \{\psi\} \cup \bigcup_{\ell=1}^{n} q(\bar{x}^\ell).$$

(3) $p(\bar{x})$ divides over A if for some formula $\psi(\bar{x}, \bar{a})$ we have $p \vdash \psi(\bar{x}, \bar{a})$ and for some indiscernible sequence $\langle \bar{a}_\ell : \ell < \omega \rangle$ over $A, \bar{a} = \bar{a}_0$, and $\{\psi(\bar{x}, \bar{a}_i) : i < \omega\}$ is

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$(<\omega)$-contradictory where a set p of formulas is n-contradictory if for any distinct $\varphi_1,\dots,\varphi_n \in p, \{\varphi_1,\dots,\varphi_n\}$ is not realized (in $\mathfrak{C}$), and $(<\omega)$-contradictory means n-contradictory for some n). We write dnd for "does not divide".

(4) The type p fork over the set A if for some $n<\omega$ and formulas $\varphi_\ell(\bar{x},\bar{a}_\ell)$ for $\ell<n$ we have $p \vdash \bigvee_{\ell<n} \varphi_\ell(\bar{x},\bar{a}_\ell)$ and for each $\ell<n$ the formula $\varphi_\ell(\bar{x},\bar{a}_\ell)$ divides over A.

We use "dnf" for "does not fork".

(5) The type p is finitely satisfiable (fs in short) in A (or in $\mathbf{I}$) if every finite subset p' of p is realized by some sequence from A (or a member of $\mathbf{I}$).

(6) If D is an ultrafilter on $Dom(D) = \mathbf{I}$ (where all members of $\mathbf{I}$ have the same length, say m) then $Av(B,D) =: \{\varphi(\bar{x},\bar{a}) : \{\bar{b} \in Dom(D) : \varphi(\bar{b},\bar{a})\} \in D\}$.

Definition 1.4. We say "$\bar{a}/A$ (or $tp(\bar{a},A)$) weakly divides over B" if $B \subseteq A$ and $tp(\bar{a},A)$ weakly divides over $(tp(\bar{a},B),B)$ (similarly for does not weakly divide).

Remark 1.4A. (1) An equivalent formulation is "$\bar{a}/A$ is an extension of $\bar{a}/B$ with the same degree for most $(\Delta,\aleph_0,k)$"; see Claim 1.5(8) below.

(2) On "divides", "fork", "weakly divide" see [16, Definition 1.1, 1.2 and 2.7(2)] respectively. On the first two see also [14], but there the focus is on stable theories. On "finitely satisfiable" see [14, Ch.VII, Section 4]. We present here most of their properties, ignoring mainly the connections with suitable degrees and indiscernibility and the derived Boolean algebras of formulas (satisfying chain conditions). For stable T the notions of Definition 1.3 collapse becoming equivalent (finitely satisfiable – only when the set is a model, see [14, Ch.III]).

Basic properties are (most can be checked directly, but (0A), (6), (8) and (9) are quoted):

Claim 1.5. (0) (Implications) *If p divides over A then p forks over A.*

(0A) *If p forks over A then p weakly divides over $(p \restriction A, \emptyset)$ (by* [16, 2.11(1), p.184], *in its proof we have relied not only on* [16, 2.10(2) and 2.9(3)], *but also on* [16, 2.4(3)]).

(0B) *If a type p is finitely satisfiable in A,* then *p does not fork over A.*

(1) (Monotonicity) *If $B \subseteq A_1 \subseteq A_2 (\subseteq \mathfrak{C})$, then $\bar{a}/A_2$ does not weakly divide over B.*

(1A) *If p does not divide over $A, A \subseteq A_1$ and $p_1 \subseteq p$ or at least $p \vdash p_1$,* then *p_1 does not divide over A_1.*

(1B) *If p does not fork over $A, A \subseteq A_1$ and $p_1 \subseteq p$ or at least $p \vdash p_1$,* then *p_1 does not fork over A_1.*

(1C) *If p does not weakly divide over (r,A), $A_1 \subseteq A, r_1 = rv(r_1 \subseteq p$* and *$r_1 \vdash r)$ and $p_1 \subseteq p$ or at least $p \vdash p_1$,* then *p_1 does not weakly divide over (r_1,A_1).*

(2) (Local character) *$\bar{a}/A$ does not weakly divide over B* iff *for every finite subsequence $\bar{a}'$ of $\bar{a}$ and finite subset A' of A, $\bar{a}'/(A' \cup B)$ does not weakly divide over B.*

(2A) *The type p does not weakly divide over (r,B)* iff *every finite $p' \subseteq p$ dnwd over (r,B).*

(2B) *The type p does not divide over A* iff *for every finite $p' \subseteq p$ does not divide over A,* iff *some finite conjunction φ of members of p satisfies the requirement in Definition 1.3(1).*

(2C) *The type p does not fork over A* iff *every finite $p' \subseteq p$ does not fork over A.*

(3) (More monotonicity) *Assume $Rang(\bar{a}') = Rang(\bar{a}'')$,* then: *$\bar{a}'/A$ dnwd over B* iff *$\bar{a}''/A$ dnwd over B.*

(3A) *If $B \subseteq A$, Rang $\bar{a}'' \subseteq acl(B \cup \bar{a}')$ and $\bar{a}'/\bar{A}$ dnwd over B,* then *$\bar{a}''/A$ dnwd over B.*

(3B) *Similar to (3), (3A) for "does not divide" and for "does not fork" and for "dnwd over (r,B)".*

(4) (Transitivity) *If $A_0 \subseteq A_1 \subseteq A_2$ and $\bar{a}/A_{\ell+1}$ dnwd over A_ℓ for $\ell = 0,1$,* then *$\bar{a}/A_2$ dnwd over A_0.*

(5) (Extendability) *If $B \subseteq A \subseteq A^+$, p an m-type over A and p does not fork over B,* then *p has an extension $q \in S^m(A^+)$ which does not fork over B (clear or see* [16, 2.11(3)]).

(5A) *If p is finitely satisfiable in A and $(Dom\ p) \cup A \subseteq B$* then *we can extend p to a complete type over B finitely satisfiable in A.*

(6) (Trivial nice behavior) *$\bar{a}/A$ does not fork over A (by* [16, 2.11(2)]).

(6A) *For a set A and an m-type p we have: p does not weakly divide over (p,A) (check).*

(6B) *Every m-type over M is finitely satisfiable in M.*

(7) (Continuity) *If p_i does not weakly divide over (r_i,B_i) for $i < \delta$ and $i < j < \delta \Rightarrow p_i \subseteq p_j \& r_i \subseteq r_j \& B_i \subseteq B_j$],* then *$\bigcup_{i<\delta} p_i$ does not weakly divide over $(\bigcup_{i<\delta} r_i, \bigcup_{i<\delta} B_i)$.*

(7A) *If $\langle A_i : i < \delta \rangle$ is increasing, $\langle B_i : i < \delta \rangle$ is increasing and C/B_i is finite satisfiable in A_i for each $i < \delta$ then $C/\bigcup_{i<\delta} B_i$ is finite satisfiable in $\bigcup_{i<\delta} A_i$. (Why? If $p \subseteq C/\bigcup_{i<\delta} B_i$ is finite then for some j it is over B_j hence $\subseteq C/B_j$ is satisfiable in A_j hence is satisfiable in $\bigcup_{i<\delta} A_i$). if $\langle C_i : i < d \rangle$ is increasing, C_i/B is fs in A, then $U_{i<\delta} C_i$ is fs in A.*

(8) (Degree) *Let $\bar{x}_m = \langle x_\ell : \ell < m \rangle, E_m$ be an ultrafilter on $\Omega_m =: \{(\triangle, k) : \triangle = \triangle(\bar{x}_m) \subseteq L(T)$ finite, $k < \omega\}$ such that for every $(\triangle_0, k_0) \in \Omega_m$ the following set belongs to E_m:*

$$\{(\triangle, k) \in \Omega_m : \triangle \subseteq \triangle_0 \text{ and } k_0 < k\}.$$

If $p(\bar{x})$ is a type over $A, \ell g(\bar{x}) = m$, then *for some complete type $q(\bar{x})$ over A extending p for the E_m-majority of $(\triangle(\bar{x}), k)$ we have $D(q(\bar{x}), \triangle, \aleph_0, k) = D(p(\bar{x}), \triangle, \aleph_0, k)$ (by* [16, 2.2(5), p.182]; *of course, we can use infinite $\bar{x}$).*

In such a case we say $q(\bar{x})$ is an E_m-nonforking extension of $p(\bar{x})$ or $q(\bar{x}) E_m$-does not fork over $p(\bar{x})$. If $p(\bar{x}) \in S^m(A)$ (so $p = q \restriction A$) we may replace "over $p(\bar{x})$" by "over A".

(9) (Additivity) *If for every $\alpha < \alpha^*$ the type $tp(\bar{b}^\alpha, \bar{a} \cup A \cup \bigcup_{\beta<\alpha} \bar{b}^\beta)$ does not divide over $A \cup \bigcup_{\beta<\alpha} \bar{b}^\beta$),* then *$tp_*(\bigcup_{\beta<\alpha^*} \bar{b}^\beta, A \cup \bar{a})$ does not divide over A (by* [16, 1.5, p. 181]).

(10) (Finitely satisfiable is average) *Let* $\ell g(\bar{x}) = m$ *and* $p = p(\bar{x})$ *a type.* Then p *is finitely satisfiable in* $\mathbf{I}$ *iff for some ultrafilter* D *over* $\mathbf{I}$ *we have* $p \subseteq Av(D, Dom\ p)$.

(11) *If* D *is an ultrafilter on* $\mathbf{I}$, then $Av(D,A)$ *belongs to* $S^m(A)$ *and is a finitely satisfiable in* $\mathbf{I}$.

Claim 1.6. (1) (small basis) *If* $p \in S^{\varepsilon}(A)$ *and* $B_0 \subseteq A$, then *for some* B *we have*

(α) $B_0 \subseteq B \subseteq A$,

(β) $|B| \leqslant |\varepsilon| + |T| + |B_0|$,

(γ) p *does not weakly divide over* $(p \restriction B, B)$.

(2) *If* $\bar{a}/(A \cup \bar{b})$ *dnwd over* A *and* $A \subseteq A^+ \subseteq ac\ell[A \cup \{\bar{a}' \in \mathfrak{C} : \bar{a}'$ *realizes* $\bar{a}/A\}]$, then *there is* $\bar{b}'$ *(of the same length as* $\bar{b}$*) such that*

(a) if $\bar{a}' \subseteq A^+$ *and* $\bar{a}'/A = \bar{a}'/A$, then $\bar{b}'{}^\frown\bar{a}'/A = \bar{b}{}^\frown\bar{a}/A$.

(3) (Weak symmetry) *If* $\bar{a}/(A \cup \bar{b})$ *dnwd over* A *and then* $\bar{b}/(A \cup \bar{a})$ *dnd over* A.

(4) *Assume* $A \subseteq B \cap C$*(all* $\subseteq \mathfrak{C}$*) and* C/B *is finitely satisfiable in* A *hence* $A = B \cap C$. Then B/C *dnwd over* $(B/A, A)$.

Proof. (1) By [16, 3.3, p.186].

(2) By [16, 2.13, p.185] we can get clause (a).

(3) By [16, 2.14, p.185] it is dnd.

(4) Straightforward (e.g. use Claim 1.5(10)). □

Theorem 1.7. *If* $M \prec N \prec \mathfrak{C}, \|M\| = \mu, \|N\| = \mu^+, |T| < \kappa, \mu = \mu^{<\kappa}$, then *there are* $M^+ \prec N^+$ *such that* $N \prec N^+, M \prec M^+, \|M^+\| = \mu, \|N^+\| = \mu^+$ *and*

$(*)_1$ *if* $B \subseteq A \subseteq N, B \subseteq M, |A| < \kappa, m < \omega, p \in S^m(A)$ *and* p *dnwd over* $(p \restriction B, \emptyset)$, then p *is realized in* M^+.

$(*)_2$ *if* $B \subseteq A \subseteq N, B \subseteq M, |A| < \kappa, C \subseteq \mathfrak{C}, |C| < \kappa$ *and* $A/(B \cup C)$ *dnwd over* B, then *there is* $C' \subseteq M^+$ *realizing* $C/(B \cup A)$.

Proof. Clearly we can prove $(*)_1, (*)_2$ separately. Now $(*)_2$ is immediate from Claim 1.6(2). As for $(*)_1$, this is proved in [16, Section 4] (read [16, 4.13, 4.14, 4.15, p.193] there, so we use [15, Theorem 3.1] which says that a Boolean algebra of cardinality λ^+ satisfying the κ-c.c., $\lambda^{<\kappa} = \lambda$ is λ-centered, i.e. is the union of $\leqslant \lambda$ ultrafilters, so if $\kappa > 2^{|T|}$ we are done which is enough for our main theorem (Theorem 1.2 when $\lambda > |T|$). Actually repeating the proof of [16], Theorem 3.1 in the circumstances of [16, Section 4] shows that $\kappa > |T|$ is enough). □

Definition 1.8. (1) K^0_λ be

$$\{\bar{M} : \bar{M} = \langle M_i : i < \lambda^+ \rangle \text{ is } \prec\text{-increasing continuous, each } M_i \text{ a model of } T \text{ of cardinality } \lambda \text{ and } |M_0| = \emptyset \text{ (we stipulate such a model } \prec M \text{ for every } M \models T)\}.$$

(2) $\leqslant^0$ is the following partial order on K^0_λ : $\bar{M}^1 \leqslant^0 \bar{M}^2$ if for $i < j < \lambda^+$ we have $M^1_i \prec M^2_i$.

Observation 1.9. (1) If $\langle \bar{M}^\alpha : \alpha < \delta \rangle$ is an $\leqslant^0$-increasing chain (in K^0_λ) and $\delta < \lambda^+$, then it has a lub $\bar{M} : M_i = \bigcup_{\alpha<\delta} M^\alpha_i$.

(2) If M is a model of T of cardinality λ^+, then for some $\bar{M} \in K^0_\lambda, M = \bigcup_{i<\lambda^+} M_i$.

(3) If $\bar{M}, \bar{N} \in K^0_\lambda$ and $\bigcup_{\alpha<\lambda^+} M_\alpha \prec \bigcup_{\alpha<\lambda^+} N_\alpha$, then for some club E of λ^+ for every $\alpha \in E$: $M_\alpha \prec N_\alpha$ and $N_\alpha / \bigcup_{\beta<\lambda^+} M_\beta$ dnwd over M_β.

Proof. (1) Immediate. (2) Use Claims 1.5(2) and 1.6(1). $\square$

Using Theorem 1.7 and Claim 1.5(5A) $\lambda^+ \times \lambda$ we get

Observation 1.10. Assume $\lambda = \lambda^{<\kappa}$. For every $\bar{M} \in K^0_\lambda$ there is an $\leqslant^0$-increasing continuous sequence $\langle \bar{N}_\zeta : \zeta \leqslant \lambda \rangle$, in K^0_λ (so $\bar{N}_\zeta = \langle N^\zeta_\alpha : \alpha < \lambda^+ \rangle$), $\bar{N}_0 = \bar{M}$ such that (letting $N_\zeta = \bigcup_{\alpha<\lambda^+} N^\zeta_\alpha$) and (fixing χ, letting $\bar{a}_{\zeta,\alpha}$ an enumeration of $|N^\zeta_\alpha|$ of length λ) we have: every type definable in $(H(\chi), \in, <^*_\chi)$ from $\langle \bar{N}_\varepsilon : \varepsilon \leqslant \zeta \rangle, \langle N^{\zeta+1}_\alpha : \alpha \leqslant \beta \rangle, \langle \langle \bar{a}_{\varepsilon,\alpha} : \alpha < \lambda^+ \rangle : \varepsilon \leqslant \zeta \rangle, \langle \bar{a}_{\zeta,\alpha} : \alpha \leqslant \beta \rangle$ and finitely many ordinals $< \lambda$ is realized in $N^{\zeta+1}_{\beta+1}$, hence

$(*)_1$ if $\alpha < \lambda^+, \zeta \leqslant \lambda, cf(\zeta) \in \{\lambda, 1\}, cf(\alpha) \in \{\lambda, 1\}$, $B \subseteq A \subseteq N_{\zeta-1}, B \subseteq N^{\zeta-1}_\alpha, |A| < \kappa$, $p \in S^m(A)$ and p dnwd over $(p \restriction B, \emptyset)$, then p is realized in N^ζ_α. (Note: $\lambda - 1 = \lambda$).

$(*)_2$ if $\alpha \leqslant \beta < \gamma < \lambda^+, \gamma$ is non limit, $B \subseteq A \subseteq A^+ \subseteq M_\gamma, |A^+| < \kappa, \bar{a} \in M_\gamma, tp(\bar{a}, A \cup M_\alpha)$ dnwd over B , then either for some $\bar{a}' \in N_\gamma$ we have: $tp(\bar{a}, A^+ \cup M_\alpha) = tp(\bar{a}', A \cup M_\alpha)$ and $tp(\bar{a}', A \cup N_\beta)$ dnwd over B or there is no such $\bar{a}' \in \mathfrak{C}$.

$(*)_3$ for γ non limit, $\zeta \leqslant \lambda, cf(\zeta) \in \{1, \lambda\}$ we have N^ζ_γ is κ-saturated (so when $\kappa = \lambda$ it is saturated).

Definition 1.11. Let A, B, C be given ($\subseteq \mathfrak{C}$).

(0) $A \leqslant^{-1}_B C$ means that for every $\bar{b} \supseteq B, \bar{b}/(A \cup C)$ dnwd over $(\bar{b}/A, A)$.

(1) $A \leqslant^0_B C$ means that for every $\bar{c} \subseteq C, \bar{c}/(A \cup B)$ dnwd over $(\bar{c}/A, \emptyset)$.

(2) $A \leqslant^1_B C$ means there is an increasing continuous sequence $\langle A_\alpha : \alpha \leqslant \beta \rangle$ such that $A = A_0, A \cup C = A_\beta$ and

$$\alpha \text{ an even ordinal } < \beta \Rightarrow A_\alpha \leqslant^0_B A_{\alpha+1},$$

$$\alpha \text{ an odd ordinal } < \beta \Rightarrow A_\alpha \leqslant^{-1}_B A_{\alpha+1}.$$

(3) $A \leqslant^2_B C$ means that for some $C', C \subseteq C'$ and $A \leqslant^1_B C'$.

(4) $A \leqslant^3_B C$ means that for some increasing continuous sequence $\langle A_\alpha : \alpha \leqslant \beta \rangle$ we have $A = A_0$, $A \cup C = A_\beta$ and $A_\alpha \subseteq^2_B A_{\alpha+1}$.

(5) $A \leqslant^S_B C$ means that for every $\bar{b} \subseteq B_1$ $tp(\bar{b}, A \cup C)$ is finitely satisfiable in A.

(6) $A \leqslant^t_B C$ means that $A \leqslant^S_C B$.

Claim 1.12. (0) $A \leqslant^e_B A$ (*for* $e = -1, 0, 1, 2, 3$).

(1) $A \leqslant^e_B C$ *iff* $A \leqslant^e_{A\cup B} A \cup C$ (*for* $e = -1, 0, 1, 2, 3$). (*Why? For* $e = -1, 0$ *by Claim* 1.5(3*A*) *for* $e = 1, 2, 3$ *use previous case*).

(2) *If* $A \subseteq B_1 \subseteq B \cup A$ *and* $A \leqslant^e_B C$ *then* $A \leqslant^e_{B_1} C$ (*for* $e = -1, 0, 1, 2, 3,$). (*Why? By part* (1) *and for* $e = -1$ *trivially, for* $e = 0$ *by Claim* 1.5(1*C*) *and for* $e = 1, 2, 3$ *use earlier cases*).

(3) *For* $e = 1, 3$ *we have* $\leqslant^e_B$ *is a partial order.* (*Why? Read their definition*).

(4) *If* $e = 1, 3$ *and* $\langle A_\alpha : \alpha \leqslant \beta\rangle$ *is increasing continuous and* $A_\alpha \leqslant^e_B A_{\alpha+1}$ *for* $\alpha < \beta$, then $A_0 \leqslant^e_B A_\beta$. (*Why? Check*).

(5) *For* $(e^1, e^2) \in \{(-1, 1), (0, 1), (1, 2), (2, 3)\}$, *we have* $A \leqslant^{e^1}_B C$ *implies* $A \leqslant^{e^2}_B C$. (*Why? Read the definition*).

(6) *If for every* $\bar{b} \subseteq B, \bar{b}/(A \cup C)$ *is finitely satisfiable in* A, then $A \leqslant^0_B C$ *and* $A \leqslant^{-1}_C B$ (*Why? By Claim* 1.6(4) *and* 1.5(1C) (*and Definition* 1.11(1))).

(7) *If* $A \leqslant^2_B C$ *and* $C' \subseteq C$, then $A \leqslant^2_B C'$. (*Why? Read Definition* 1.11(2)).

(8) $A \leqslant^0_B C$ *if* $A \leqslant^{-1}_C B$. (*Why? Read the definitions*).

Claim 1.13. *Let* $M \prec N$ *and* $M \subseteq A$. *Then the following are equivalent:*

(a) $M \leqslant^3_N A$,

(b) *there are* $M_0 \prec M_1 \prec M_2$ *such that*

(i) $M = M_0$

(ii) *the type* $tp_*(N, M_1)$ *is finitely satisfiable in* M_0 *and the type* $tp_*(M_2, M_1 \cup N)$ *is finitely satisfiable in* M_1,

(iii) *for some elementary map* $f, f(A) \subseteq M_2$ *and* $f \restriction N$ = *identity.*

(c) *Like* (b) *with* $\|M_2\| \leqslant |T| + |A| + \|N\|$.

(d) $M \leqslant^2_N A$.

Remark 13A. (1) Clause (ii) of (b) implies $M_0 \leqslant^0_N M_1 \leqslant^{-1}_N M_2$.

(2) An equivalent formulation of (b) is

(b)* for some M_0, M_1, M_2, f we have $M = M_0 \leqslant^s_{f(N)} M_1 \leqslant^t_{f(N)} M_2, f \restriction M_0 = \mathrm{id}_{M_0}$, $f(A) \subseteq M_2$.

(3) Another formulation is

(b)** like $(b)^*$ but $f \supseteq id_{A\cup N}$.

Proof. (c)$\Rightarrow$(b): Trivial.

(b)$\Rightarrow$(c): By the Lowenkeim Skolem argument.

(b)$\Rightarrow$(d): By Claim 1.12(6) clearly $M_1 \leqslant^{-1}_N M_2$ and similarly $M_0 \leqslant^{-1}_{M_1} N$, hence by Theorem 1.2(8) we have $M_0 \leqslant^0_N M_1$. Hence by Claim 1.12(5), $M_e \leqslant^1_N M_{e+1}$ (for $e = 0, 1$), so by Claim 1.12(3) $M_0 \leqslant^1_N M_2$, hence by Definition 1.11(3) (and clause (iii) of Claim 1.13(b)), $M = M_0 \leqslant^2_N A$ as required.

(d)$\Rightarrow$(a): Trivial (by Claim 1.12(5)).

So the only (and main) part left is:

(a)$\Rightarrow$(b): We know $M \leqslant^3_N A$, by Claim 1.12(1) without loss of generality $M \subseteq A$, hence there is an increasing continuous sequence $\langle A_\varepsilon : \varepsilon \leqslant \zeta\rangle$ such that $A_0 = M, A_\zeta = A$

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and $A_\varepsilon \leqslant^2_N A_{\varepsilon+1}$. By the definition of $\leqslant^2_N, \leqslant^1_N$ there is an increasing continuous sequence $\langle B_{\varepsilon,i} : i \leqslant i_\varepsilon \rangle$ such that $B_{\varepsilon,0} = A_\varepsilon, A_{\varepsilon+1} \subseteq B_{\varepsilon,i_\varepsilon}$ and $B_{\varepsilon,i} \leqslant^{\ell(\varepsilon)}_N B_{\varepsilon,i+1}$ (where for $i < i_\varepsilon$ we have $\ell(\varepsilon) \in \{-1,0\}$ and $\varepsilon = \ell(\varepsilon)$ mod 2). Let $\theta = 2^{|T|} + |N| + \sum_{\varepsilon<\zeta}(|i_\varepsilon|) + |B_{\varepsilon,i_\varepsilon}|)^+$ and choose regular $\mu = \mu^\theta$.

We choose by induction on $\alpha < \mu^+, M_\alpha, N_\alpha$ such that:

(i) $M_\alpha \prec \mathfrak{C}$ is increasing continuous,

(ii) $\|M_\alpha\| = \mu, M \subseteq M_0$,

(iii) f_α is an elementary mapping, $Dom(f_\alpha) = N, Rang(f_\alpha) = N_\alpha$ and $f_\alpha \restriction M = id_M$,

(iv) $tp_*(N_\alpha, M_\alpha)$ is finitely satisfiable in M,

(v) $N_\alpha \subseteq M_{\alpha+1}$,

(vi) $M_{\alpha+1}$ is θ^+-saturated.

There is no problem to carry the definition. (First choose M_α as follows: if $\alpha = 0$ so as to satisfy (i) and (ii), if α is a limit ordinal, as $\bigcup_{\beta<\alpha} M_\beta$, and if $\alpha = \beta + 1$, so as to satisfy (i), (ii) and (v). Second choose f_α, N_α satisfying (iii) and (iv) which exists by Claim 1.5(10) and (11)).

By using Theorem 1.7, λ^+ times we can find $\bar{M}^+ = \langle M^+_\alpha : \alpha < \lambda^+ \rangle$ such that:

(A) $\bar{M}^+$ is an increasing continuous sequence of elementary submodels of $\mathfrak{C}$,

(B) $\|M^+_\alpha\| \leqslant \mu, M_\alpha \prec M^+_\alpha$,

(C)$_1$ if $\alpha < \beta < \mu^+, B_1 \subseteq M_\alpha$ and $cf(\alpha) = \mu, B_1 \subseteq B_2 \subseteq M_\beta, |B_2 \cup C| \leqslant \theta, C \subseteq \mathfrak{C}$ and C/B_2 dnwd over $(C/B_1, \emptyset)$ (equivalently, for every finite $\bar{c} \subseteq C, \bar{c}/B_2$ dnwd over $(\bar{c}/B_1, \emptyset)$) then C/B_2 is realized in M_α,

(C)$_2$ similarly, but we replace the dnwd assumption by "$B_2/(B_1 \cup C)$ dnwd over $(B_2/B_1, B_1)$". [Note: we use $(*)_1$ from Theorem 1.7 for $(C)_1$ and $(*)_2$ from Theorem 1.7 for $(C)_2$].

Now let $M = \bigcup_{\alpha<\mu^+} M_\alpha, M^+ = \bigcup_{\alpha<\mu^+} M^+_\alpha$; and let $E = \{\delta < \mu^+ : \delta$ a limit ordinal and $(M^+_\delta, M_\delta) \prec (M^+, M)\}$. Clearly E is a club of μ^+ and

$(*)$ $\delta \in E \Rightarrow tp(M^+_\delta, M)$ is finitely satisfiable in M_δ.

Choose $\delta \in E$ of cofinality μ. Now we choose g_ε by induction on $\varepsilon \leqslant \zeta$ such that

(α) g_ε an elementary mapping,

(β) $Dom(g_\varepsilon) = N \cup A_\varepsilon$,

(γ) g_ε is increasing continuous in ε,

(δ) $g_\varepsilon \restriction N = f_\delta$,

(ε) $Rang(g_\varepsilon \restriction A_\varepsilon) \subseteq M^+_\delta$.

If we succeed, then we get the desired conclusion (i.e. prove clause (b)). (Why? First note that in clause (b) we can omit $f \restriction N = id$ by $f \restriction M =$ the identity if in clause (ii) we use $f(N)$; we call this (b)$'$. Now (b)$'$ holds with $M, M_\delta, M^+_\delta, g_\zeta$ here standing to M_0, M_1, M_2, f there). So it is enough to carry the induction on ε. For $\varepsilon = 0$ let $g_\varepsilon = f_\delta$, and for ε a limit ordinal let $g_\varepsilon = \bigcup_{\xi<\varepsilon} g_\xi$; lastly for ε a successor ordinal say $\varepsilon = \xi + 1$, we choose $g_{\varepsilon,i}$ by induction on $i \leqslant i_\varepsilon$ such that

(α)$'$ $g_{\varepsilon,i}$ an elementary mapping,

(β)$'$ $Dom(g_{\varepsilon,i}) = N \cup B_{\varepsilon,i}$,

(γ)$'$ $g_{\varepsilon,i}$ is increasing continuous in i,

$(\delta)'$ $g_{\varepsilon,0} = g_\varepsilon$,
$(\varepsilon)'$ *Rang* $(g_\varepsilon) \subseteq M_\delta^+$.

If we succeed then $g_{\varepsilon,i_\varepsilon} \restriction A_{\varepsilon+1}$ is as required. So it is enough to carry the induction on i. For $i=0$ let $g_{\varepsilon,i} = g_\varepsilon$, for i limit let $g_{\varepsilon,i} = \bigcup_{j<i} g_{\varepsilon,j}$ and for i a successor ordinal say $j+1$, use clause $(C)_1$ in the choice of M_α^+ if j even, remembering Definition 1.11(1) and use clause $(C)_2$ in the choice of M_α^+ if j is odd remembering Definition 1.11(0). □

Claim 1.13A. If $M \prec N, M \leqslant_N^2 A_\ell$ for $\ell = 1,2$, then there are M^+, f_1, f_2 such that $M \prec M^+, M \leqslant_N^2 M^+$ and for $\ell = 1,2$, f_ℓ is an elementary mapping, $\mathrm{Dom}(f_\ell) = N \cup A_\ell$, $f_\ell \restriction N = id_N$, $f_\ell(A_\ell) \subseteq M^+$.

Proof. Same proof as Claim 1.13 (just shorter). □

Definition 1.14. (1) Let $K_0^{pr} = \{(M,N) : M \prec N < \mathfrak{C}\}$ and $(M_1,N_1) \leqslant^* (M_2,N_2)$ iff $((M_e,N_e) \in K_0^{pr}$ for $e = 1,2$ and $M_1 \prec M_2, N_1 \prec N_2$ and $M_1 \leqslant_{N_1}^2 M_2$ (equivalently, $M_1 \leqslant_{N_1}^3 M_2$ (by Claim 1.13)).

(2) We define $(M_1,N_1) \leqslant_{fs} (M_2,N_2)$ similarly replacing "$M_1 \leqslant_{N_1}^2 M_2$" by "$N_1/M_2$ is finitely satisfiable in M_1".

Claim 1.15. (1) *$\leqslant^*$ is a partial order on K_0^{pr}.*

(2) *If $\langle (M_\alpha, N_\alpha) : \alpha \leqslant \beta \rangle$ is increasing continuous and $(M_\alpha,N_\alpha) \leqslant^* (M_{\alpha+1},N_{\alpha+1})$ for* $\alpha < \beta$, then $(M_0,N_0) \leqslant^* (M_\beta,N_\beta)$.

(3) *If $M \prec N$ and $M \leqslant_N^2 A$ then for some (M_1,N_1) we have $A \subseteq M_1$ and $(M,N) \leqslant^*$ $(M_1,N_1) \in K_0^{pr}$.*

Proof. (1) If $(M_0,N_0) \leqslant^* (M_1,N_1) \leqslant^* (M_2,N_2)$ then
  (i) $M_0 \subseteq M_1 \subseteq M_2$ and $N_0 \subseteq N_1 \subseteq N_2$,
  (ii) $M_0 \leqslant_{N_0}^2 M_1$ and
  (iii) $M_1 \leqslant_{N_1}^2 M_2$,
  (iv) $M_\ell \subseteq N_\ell$.
By Claim 1.12(2) and (1) and clause (iii) above
  (v) $M_1 \leqslant_{N_0}^2 M_2$,
by Claim 1.12(5) we have (by (ii) and (v) respectively)
  (ii)$'$ $M_0 \leqslant_{N_0}^3 M_1$,
  (v)$'$ $M_1 \leqslant_{N_0}^3 M_2$,
hence by Claim 1.13
  (vii) $M_0 \leqslant_{N_0}^2 M_2$,
hence $(M_0,N_0) \leqslant^* (M_2,N_2)$ holds by (i), (iv) and (vii).

(2) Similarly using Claim 1.12(4) and (1.13).

(3) Use Claim 1.13 (see Claim 1.13A(3)) so there are $M_0 \prec M_1 \prec M_2$ such that $M = M_0, N/M_1$ fs in $M_0, M_2/(M_1 \cup N)$ fs in M_1 and $A \subseteq M_2$. So by 1.12(6) $M_0 \leqslant_N^{-1} M_1$,

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$M_1 \leqslant^0_N M_2$ hence (see Claim 1.12(5) $M_0 \leqslant^3_N M_0 \leqslant^3_N M_1$ hence (see Claim 1.12(3)) $M_0 \leqslant^3_N M_2$ hence for any $N^*, M_2 \cup N \subseteq N^* \prec \mathfrak{C}$ we have $(M,N) \leqslant^* (M_2, N^*) \in K^{pr}_0$. $\square$

Definition 1.16. $K^{pr}_1 = \{(M,N)$: the pair $(M,N) \in K^{pr}_0$ and if $(M,N) \leqslant^* (M',N') \in K^{pr}_0$ then N/M' is fs in $M\}$.

Claim 1.17. *If $(M,N) \in K^{pr}_0$ then for some (M',N') we have*
(a) $(M,N) \leqslant (M',N') \in K^{pr}_0$,
(b) $\|N'\| \leqslant \|N\| + |T|$,
(c) $(M',N') \in K^{pr}_1$ *i.e. if $(M',N') \leqslant^* (M'',N'') \in K^{pr}_0$ then N'/M'' is fs in M'.*

Proof. Let $\mu = \|N\| + |T|$, assume the conclusion fails. We now choose by induction on $\alpha < \mu^+, (M_\alpha, N_\alpha)$ such that

(i) $(M_0, N_0) = (M, N)$,
(ii) $(M_\alpha, N_\alpha) \in K^{pr}_0, \|N_\alpha\| \leqslant \mu$,
(iii) $\beta < \alpha \Rightarrow (M_\beta, N_\beta) \leqslant^* (M_\alpha, N_\alpha)$
(iv) for limit δ we have $(M_\delta, N_\delta) = \left(\bigcup_{\alpha<\delta} M_\alpha, \bigcup_{\alpha<\delta} N_\delta\right)$,
(v) $M_{\alpha+1}/N_\alpha$ is not fs in M_α.

For $\alpha = 0$ see (i) for α limit see (iv) and Claim 1.15(2) if $\alpha = \beta + 1$ find (M_α, N_α) satisfying $(M_\beta, N_\beta) \leqslant^* (M_\alpha, N_\alpha) \in K^{pr}_1$ and satisfying (v) exists as we assume that the condition fails. By Lowenheim–Skolem argument without loss of generality $\|N_\alpha\| \leqslant \mu$ and by Claim 1.15(1) also clause (iii) holds. For a club of $\delta < \mu^+$ we get contradiction to clause (v). $\square$

Fact 1.18. (1) *If $(M,N) \in K^{pr}_1$ and $(M',N') \in K^{pr}_0$ and $(M,N) \leqslant^* (M',N')$ then $(M,M') \leqslant^* (N,N')$.*

(2) *If $(M,N) \in K^{pr}_1$ and $M \leqslant^2_N A$ then A/N is* fs *in M.*

Proof. (1) By Definition 1.16 we know N/M' is fs in M hence by Claim 1.13, (b)$\Rightarrow$(d) we know $M \leqslant^2_{M'} N$ which give the desired conclusion.

(2) By Claim 1.13A. $\square$

Claim 1.19. (1) *If $(M_\alpha, N_\alpha) \in K^{pr}_1$ for $\alpha < \delta$ and $\langle (M_\alpha, N_\alpha) : \alpha < \delta \rangle$ is $<^*$-increasing,* then *for $\alpha < \delta$*

$$(M_\alpha, N_\alpha) \leqslant^* \left(\bigcup_{i<\delta} M_i, \bigcup_{i<\delta} N_i \right) \in K^{pr}_1.$$

(2) *If $(M_\alpha, N_\alpha) \in K^{pr}_1$ and $\langle (M_\alpha, N_\alpha) : \alpha \leqslant \delta \rangle$ is $<^*$-increasing then $\left(\bigcup_{\alpha<\delta} M_\alpha, \bigcup_{\alpha<\delta} N_\alpha\right) \in K^{pr}_1$ and $\left(\bigcup_{\alpha<\delta} M_\alpha, \bigcup_{\alpha<\delta} N_\alpha\right) \leqslant^* (M_\delta, N_\delta)$.*

Proof. We prove both together by induction on δ.

(1) By the induction hypothesis without loss of generality $\langle (M_\alpha, N_\alpha) : \alpha < \delta \rangle$ is increasing continuous.

Clearly $(M_\alpha, N_\alpha) \leqslant^* \left(\bigcup_{i<\delta} M_i, \bigcup_{i<\delta} N_i\right) \in K^{pr}_0$ (see Claim 1.15(2)). Suppose $\left(\bigcup_{i<\delta} M_i, \bigcup_{i<\delta} N_i\right) \leqslant^* (M,N)$. So by Claim 1.15(2), for $\alpha < \delta, (M_\alpha, N_\alpha) \leqslant^* (M,N)$, but

$(M_\alpha, N_\alpha) \in K_1^{pr}$ hence N_α/M is fs in M_α. But this implies $\bigcup_{\alpha<\delta} N_\alpha/M$ is fs in $\bigcup_{\alpha<\delta} M_\alpha$ by Claim 1.5(7A).

(2) As we are proving by induction on δ; without loss of generality $\langle (M_\alpha, N_\alpha) : \alpha < \delta\rangle$ is $\leqslant^*$-increasing continuous, so by part (1), $(M_\alpha, N_\alpha) \leqslant^* (\bigcup_{i<\delta} M_i, \bigcup_{i<\delta} N_i) \in K_1^{pr}$ for $\alpha < \delta$. Now for $\alpha < \delta, (M_\alpha, N_\alpha) \leqslant^* (M_\delta, N_\delta)$ and $(M_\alpha, N_\alpha) \in K_1^{pr}$ clearly N_α/M_δ is fs in M_α hence by Claim 1.5(7A), $\bigcup_{\alpha<\delta} N_i/M_\delta$ is fs in $\bigcup_{i<\delta} M_i$, hence by Claim 1.13 $\bigcup_{i<\delta} M_i \leqslant^3_{\bigcup_{\alpha<\delta} N_i} \bigcup_{i<\delta} N_i$ hence $(\bigcup_{i<\delta} M_i, \bigcup_{i<\delta} N_i) \leqslant^* (M_{\delta+1}, N_{\delta+1})$. □

Now we want to apply Lemma 1.1. Toward this (for λ as there) we define:

Definition 1.20. (1) $K^0_{\mathrm{ap}} = K^0_{\mathrm{ap}} = K^0_{\mathrm{ap}}[T] = K^0_{\mathrm{ap}}[T, \lambda]$ is the set of models M of T with universe $\subseteq \lambda^+$ and cardinality $< \lambda$ such that $M \cap \lambda \neq \emptyset$ and $0 < \alpha < \lambda^+$ implies $M \restriction (\lambda \times \alpha) \prec M$. For such M let $Dom(M) = \{\alpha < \lambda^+ : [\lambda \times \alpha, \lambda \times \alpha + \lambda) \cap M \neq \emptyset\}$. We now define $<_{K^0_{\mathrm{ap}}}$ by $M \leqslant_{K^0_{\mathrm{ap}}} N$ if (both are in K^0_{ap} and $M \prec N$ and) for every $\alpha \in (0, \lambda^+), M \restriction (\lambda \times \alpha) <^2_{M\restriction(\lambda\times\alpha+\lambda)} N \restriction (\lambda \times \alpha)$.

Observation 1.21. So $M \leqslant_{K^0_{\mathrm{ap}}} N$ iff both are in $K^0_{\mathrm{ap}}, M < N$ and for $\alpha \in (0, \lambda^+)$ we have $(M \restriction (\lambda \times \alpha), M \restriction (\lambda \times \alpha + \lambda)) \leqslant^* (N \restriction (\lambda \times \alpha), N \restriction (\lambda \times \alpha + \lambda))$.

Claim 1.22. (1) $\leqslant_{K^0_{\mathrm{ap}}}$ *is a partial order on* K^0_{ap}. (*Why? By Claim* 1.15(1)).

(2) *If* $\langle M_i : i < \delta\rangle$ *is* $\leqslant_{K^0_{\mathrm{ap}}}$*-increasing,* $\sum_{i<\delta} \|M_i\| < \lambda$, then $M_i \leqslant_{K^0_{\mathrm{ap}}} \bigcup_{j<\delta} M_j \in K^0_{\mathrm{ap}}$. (*Why? By Claim* 1.15(2)).

Claim 1.23. *Let* $\bar{N}^\zeta, N_\zeta$ *(for* $\zeta \leqslant \lambda$*) be as in Observation* 1.10. *Let* $E \subseteq \lambda^+$ *be a thin enough club of* $\lambda^+, \{\varepsilon(\alpha) : 1 \leqslant \alpha < \lambda^+\}$ *enumerate* E, H *a 1-to-1 map from* N_λ *onto* λ^+ *mapping* $N^\lambda_{\varepsilon(\alpha)}$ *onto* $\lambda \times \alpha$. *Let* $N^*_\alpha = H(N^\lambda_{\varepsilon(\alpha)}), N^* = \bigcup_{\alpha<\lambda} N^*_\alpha$.

(1) *If* $M \in K^0_{\mathrm{ap}}$ *then there is a lawful* f*(see* [21, 4.1]), *which is an elementary embedding of* M *into* N^* *such that for* $\alpha \in Dom(M) \backslash \{0\}$ *we have,* $f(M \restriction (\lambda \times \alpha)) <^2_{f(M\restriction(\lambda\times\alpha+\lambda))} N^* \restriction (\lambda \times \alpha)$.

Proof. Straightforward. □

But we want more, not only universality but also homogeneity.

Definition 1.24. $K^1_{\mathrm{ap}} = K^1_{\mathrm{ap}}[T, \lambda]$ is the set of $M \in K^0_{\mathrm{ap}}$ such that for every $\alpha \in (0, \lambda^+)$, if $\neg(M \subseteq \lambda \times \alpha)$ then $(M \restriction (\lambda \times \alpha), M) \in K^{pr}_1$.

Let $\leqslant_{K^1_{\mathrm{ap}}}$ be $\leqslant_{K^0_{\mathrm{ap}}} \restriction K^1_{\mathrm{ap}}$.

Claim 1.25. (1) $\leqslant_{K^1_{\mathrm{ap}}}$ *is a partial order on* K^1_{ap}. (*Why? By Claim* 1.22 *and Definition* 1.2).

(2) *If* $\langle M_i : i < \delta\rangle$ *is* $\leqslant_{K^1_{\mathrm{ap}}}$*-increasing,* $\sum_{1<\delta} \|M_i\| < \lambda$, then $M_i \leqslant_{K^1_{\mathrm{ap}}} \bigcup_{j<\delta} M_j \in K^1_{\mathrm{ap}}$. (*Why? By* 1.19(1)).

(3) K^n_{ap} *is dense in* $(K^0_{\mathrm{ap}}, \leqslant_{K^0_{\mathrm{ap}}})$ (*Why? By* 1.17 *and* 1.19 *used repeatedly*).

Claim 1.26. *Let $\bar{N}^{\zeta}, N_{\zeta}$ (for $\zeta \leqslant \lambda$) be as in Observation* 1.10. *Let $E \subseteq \lambda^+$ be a thin enough club of λ^+, $\{\varepsilon(\alpha) : 1 \leqslant \alpha < \lambda^+\}$ enumerate E, H a 1-to-1 map from N_λ onto λ^+ mapping $N^{\lambda}_{\varepsilon(\alpha)}$ onto $\lambda \times \alpha$. Let $N^*_\alpha = H(N^{\lambda}_{\varepsilon(\alpha)})$, $N^* = \bigcup_{\alpha<\lambda} N^*_\alpha$.*

(1) *If $M \in K^1_{\mathrm{ap}}$ then there is a lawful f(see* [21, 4.1]) *which is an elementary embedding of M into N^* such that for $\alpha \in Dom(M)\backslash\{0\}$, $f(M \restriction (\lambda \times \alpha)) <^2_{f(M\restriction(\lambda\times\alpha+\lambda))} N^* \restriction (\lambda \times \alpha)$.*

(2) *If $M_0 \leqslant_{K^1_{\mathrm{ap}}} M_1$ and (M_0, f_0) is as in part* (1), then *we can find $f_1, f_0 \subseteq f_1$ such that (M_1, f_1) is as in part* (1). *Moreover, if $f_0 \cup (f_1 \restriction (M_1 \restriction (\lambda \times \alpha))$ has been determined we can continue.*

Amalgamation Claim 1.27. *Assume $M_0 \leqslant_{K^0_{\mathrm{ap}}} M_\ell$ for $\ell = 1,2$ and (for simplicity) $|M_1| \cap |M_2| = |M_0|$.* Then *there is $M \in K^0_{\mathrm{ap}}$ such that $M_1 \leqslant_{K^0_{\mathrm{ap}}} M$ and $M_2 \leqslant_{K_{\mathrm{ap}}} M$.*

Proof. Follows from Claims 1.26(1) and (2) □

Claim 1.28. *$(K_{\mathrm{ap}}, \leqslant^*)$ is a smooth λ-approximation family (see* [21, Section 4]).

Proof. Check. □

Claim 1.29. *$(K_{\mathrm{ap}}, \leqslant^*)$ is simple (see* [21, Section 4]).

Proof. Included in the proof of amalgamation (see last clause of Claim 1.26(2)).

Claim 1.30. *If M is a model of T of cardinality λ^+,* then *for some $\Gamma \in K^{md}_\lambda$, M can be elementarily embedded into M_Γ.*

Proof. Use Observation 1.10 with $M = \bigcup_{\alpha<\lambda} M_\alpha$ so we get $N^*, N^*_\alpha (\alpha < \lambda)$ as in Fact 1.18. Check. □

Proof of 1.2. Use the above claims. □

2. On the strong order properties and finitary versions

Discussion 2.0. By [21], for some nonsimple (first order complete) the answer to the following is yes:

$\bigoplus_T$ if $\lambda = \lambda^{<\lambda} > |T|, 2^\lambda = \lambda^+$, is there a ($\lambda$-complete), λ^+-c.c. forcing notion Q, $\Vdash_Q$ "univ$(\lambda^+, T) \leqslant \lambda^{++} < 2^\lambda$"?

$\bigoplus'_T$ and by [24] if $\lambda = \lambda^{<\lambda} > |T|, 2^\lambda = \lambda^+$ is there a λ-complete λ^+-c.c. forcing notion $Q, \Vdash_Q$ "univ$(\lambda^+, T) = 1, \lambda^+ < 2^\lambda$"?

We know that for theories T with the strict order property the answer is no (by [10], or see [21, Section 3]). We would like to characterize the answer by a natural property of T (hence show that the answer to all reasonable variants is the same, e.g. does not depend on $\lambda, \oplus_T \equiv \oplus'_T$, etc.) So the results we mention above give a lower bound (simple theories $+T_{qef} + T_{trf}$) and an upper bound (failure of the strict order property) to the family of T's with a positive answer. However, we can lower the upper bound. We suggest below a strictly weaker property. From another point of view, a major theme of [14, 20] was to find natural dividing lines for the family of first order theories (so the main ones there were stable, superstable and also NTOP, deepness, NOTOP). Now [16] suggests another one: simplicity. Note that the negation of simple, the tree property has been touched upon in [14] but there were conclusions only for one side. [16] establishes this dividing line by having consequences for both the property and its negation and having "semantical characterization" for T simple: when $|T| \leqslant \kappa < \lambda = \lambda^{<\lambda} < \mu = \mu^\kappa$ we can force by a λ^+-c.c. λ-complete forcing notion Q that $2^\lambda > \mu$ and every model of T of cardinality μ can be extended to a κ^+-saturated one, and the tree property implies a strong negation. Of course, both the inner theory and such "outside", "semantical" characterization are much weaker than those for stable theories.

The strict order property has no such results, only several consequences. We suggest below weaker properties (first the strong order property then the n-version of it for $n < \omega$) which has similar consequences and so may be the right dividing line (for some questions). Remember (this is in equivalent formulations):

Definition 2.1. T has the strict order property if some formula $\varphi(\bar{x}, \bar{y})$ (with $\ell g\bar{x} = \ell g\bar{y}$) define in some model M of T a partial order with infinite chains.

Definition 2.2. (1) A first order complete T has the strong order property if some sequence $\bar{\varphi} = \langle \varphi_n(\bar{x}^n; \bar{y}^n) : n < \omega \rangle$ of formulas exemplifies it which means that for every λ:

$(*)^\lambda_{\bar{\varphi}}$ (a) $\ell g\bar{x}^n = \ell g\bar{y}^n$ are finite, $\bar{x}^n$ an initial segment of $\bar{x}^{n+1}$, $\bar{y}^n$ an initial segment of $\bar{y}^{n+1}$

(b) $T, \varphi_{n+1}(\bar{x}^{n+1}, \bar{y}^{n+1}) \vdash \varphi_n(\bar{x}^n, \bar{y}^n)$,

(c) for $m \leqslant n, \neg(\exists \bar{x}^{n,0} \cdots \bar{x}^{n,m-1})[\bigwedge\{\varphi_n(\bar{x}^{n,k}, \bar{x}^{n,\ell}) : k = \ell + 1 \bmod m\}]$ belongs to T,

(d) there is a model M of T and $\bar{a}^n_\alpha \in M$ (of length $\bar{y}^n$, for $n < \omega, \alpha < \lambda$) such that $\bar{a}^n_\alpha = \bar{a}^{n+1}_\alpha \restriction \ell g\bar{y}^n$ and $M \models \varphi_n[\bar{a}^n_\beta, \bar{a}^n_\alpha]$ for $n < \omega$ and $\alpha < \beta < \lambda$.

(2) The finitary strong order property is defined similarly but $\bar{x}^n = \bar{x}, \bar{y}^n = \bar{y}^n$.

(3) We use the shorthand SOP, FSOP and for the negation NSOP, NFSOP (similarly later for $\mathrm{NSOP_n}$).

Claim 2.3. (1) *The strict order property implies the finitary strong order property which implies the strong order property.*

(2) *There is a first order complete T, which has the strong order property (even the finitary one) but not the strict order property.*

(3) *Also some first order complete T has the strong order property but not the finitary strong order property, i.e. no $\langle \varphi_n(\bar{x}, \bar{y}) : n < \omega \rangle$ exemplifies it (i.e. with $\ell g \bar{x}_n$ constant).*

Proof. (1) Immediate.

(2) For $\ell \leqslant n < \omega$ let $<_{n,\ell}$ be a two-place relation. Let $<_n = <_{n,0}$. Let T_0 say:

(a) $x <_{n,m-1} y \Rightarrow x <_{n,m} y$,

(b) $x <_{n,n} y$,

(c) $\neg(x <_{n,n-1} x)$,

(d) if $\ell + k + 1 = m \leqslant n$ then $x <_{n,\ell} y \,\&\, y <_{n,k} z \Rightarrow x <_{n,m} z$.

We shall now prove that T_0 has the amalgamation property; it also has the joint embedding property (as the latter is easier we leave its checking to the reader).

Now suppose $M_i \models T_0, M_0 \subseteq M_i$ for $i = 0,1,2$ and $M_1 \cap M_2 = M_0$. We define a model M: its universe is $M_1 \cup M_2$ and

$$\begin{aligned} <^M_{n,m} = \{(a,b) \in M \times M : {} & \text{if } m < n \text{ then for some } i \in \{1,2\} \text{ we have} \\ & (a,b) \in <^{M_i}_{n,m} \text{ or } a \in M_i \backslash M_0, b \in M_{3-i} \backslash M_0 \text{ and for some } c \in M_0 \\ & \text{and } \ell, k \text{ we have } m = \ell + k + 1, (a,c) \in <^{M_i}_{n,\ell}, (c,b) \in <^{M_{3-i}}_{n,k}\}. \end{aligned}$$

Now clearly M extends M_1 and M_2: trivially $<^M_{n,m} \restriction M_i = <^{M_i}_{n,m}$. Is M a model of T_0? Let us check.

Clause (a) *holds*: For $x, y \in M_i$ as $M_i \subseteq M$; for $i = 1,2$ and $x \in M_i \backslash M_0, y \in M_{3-\ell} \backslash M_0$, without loss of generality $m < n$; let $c \in M$ witness $(a,b) \in \leqslant^M_{n,m-1}$ i.e. for some ℓ, k we have $\ell + k + 1 = m - 1, (a,c) \in <^{M_i}_{n,\ell}$ and $(c,b) \in <^{3-i}_{n,k}$. Now by clause (a) applied to $M_i, (a,c) \in <^{M_i}_{n,\ell+1}$ now apply the definition to get $(a,b) \in <^M_{n,(\ell+1)+k+1} = <^M_{n,m}$.

Clause (b) *holds*: Check as defining $<^M_{n,m}$ we say: "if $m < n$ then ..." so if $n = m$ there is no requirement.

Clause (c): As $M_i \subseteq M$ and $M_i \models T_0$.

Clause (d): Check by cases, i.e. for some $i \in \{1,2\}$ one of the following cases hold.

(1) $\{x,y,z\} \subseteq M_i$: use "M_i is a model of T_0 and M_i a submodel of M".

(2) $\{x,y\} \subseteq M_i, \{y,z\} \subseteq M_{3-i}$ but not case (1): use the definition of $<^M_{n,m}$.

(3) $y \in M_i \backslash M_0, \{x,z\} \subseteq M_{3-i} \backslash M_0$.

As $x <_{n,\ell} y$ there are ℓ_1, ℓ_2 and $x_1 \in M_0$ such that $x <^{M_{3-i}}_{n,\ell_1} x_1, x_1 \leqslant^{M_i}_{n,\ell_2} y$ and $\ell_1 + \ell_2 + 1 = \ell$.

As $y <_{n,k} z$ there are k_1, k_2 and $z_1 \in M_0$ such that $y \leqslant^{M_i}_{n,k_1} z_1, z_1 \leqslant^{M_{3-i}}_{n,k_2} z, k_1 + k_2 + 1 = k$. In M_i we have $x_1 <^{M_i}_{n,\ell_2} y <^{M_i}_{n,k_1} z_1$, hence $x_1 \leqslant^{M_i}_{n,\ell_2+k_1+1} z_1$ and as $\{x_1, z_1\} \subseteq M_0 \subseteq M_i$, clearly $x_1 <^{M_0}_{n,\ell_2+k_1+1} z_1$. Now in M_{3-i} we have $x <^{M_{3-i}}_{n,\ell_1} x_1 <^{M_{3-i}}_{n,\ell_2+k_1+1} z_1$, hence $x \leqslant^{M_{3-i}}_{n,\ell_1+\ell_2+k_1+2} z_1$ so $x <^{M_{3-\ell}}_{n,\ell_1+\ell_2+k_1+2} z_1 <^{M_{3-\ell}}_{n,k_2} z$, hence $x <^{M_{3-\ell}}_{n,\ell_1+\ell_2+k_1+k_2+3} z$ but $\ell_1 + \ell_2 + k_1 + k_2 + 3 = \ell + k + 1 = m$ so $x <^{M_{3-\ell}}_{n,m} z$ as required.

(4) $y \in M_i \backslash M_0, x \in M_{3-i} \backslash M_0, z \in M_0$. Similar to case (3) but with no x_1.
(5) $y \in M_i \backslash M_0, x \in M_0, z \in M_{3-i} \backslash M_i$. Similar to case (3) but with no z_1.

Let T be the model completion of T^0; easy to check that it exists and has elimination of quantifilters. Let $\varphi_n(x,y) = \bigwedge_{\ell \leqslant n} x <_\ell y$ (remember $x <_\ell y$ means $x <_{\ell,0} y$) now $\langle \varphi_n : n < \omega \rangle$ exemplifies that T has the (finitary) strong order property. On the other hand we shall show that for every $n(*) < \omega$ the theory $T_{n(*)} =: T \restriction \{<_{n,\ell} : \ell \leqslant n \leqslant n(*)\}$ does not have the strict order property (as $T = \bigcup_{n<\omega} T_n$, this clearly implies that T does not have the strict order property). First note that $T_{n(*)}$ has elimination of quantifiers and then check directly.

(3) Let T^0 say:

(a) P_n (for $n < \omega$) are pairwise disjoint (P_n unary predicates),
(b) F_n a partial one-place function from P_{n+1} into P_n,
(c) $<_{n,\ell}$ are two-place relations on P_n for $\ell \leqslant n < \omega$; and let $<_n = <_{n,0}$,
  (α) $x <_{n,m-1} y \to x <_{n,m} y$,
  (β) $P_n(x) \& P_n(y) \to x <_{n,n} y$,
  (γ) $\neg(x <_{n,n-1} x)$,
  (δ) if $\ell + k + 1 = m \leqslant n$ then: $x <_{n,\ell} y \& y <_{n,k} z \to x <_{n,m} z$,
(d) $x <_{n+1,\ell} y \to F_n(x) <_{n,\ell} F_n(y)$.

Again T will be the model completion of T^0 and it has elimination of quantifiers and we shall use $\bar{x}_n = \langle x_i : i < n \rangle, \bar{y}_n = \langle y_i : i < n \rangle$ and $\varphi_n(\bar{x}_n, \bar{y}_n) = \bigwedge_{i<n} F_i(x_{i+1}) = x_i \& \bigwedge_{i<n} F_i(y_{i+1}) = y_i \& \bigwedge_{i<n} x_i <_i y_i$. $\square$

Claim 2.4. (1) *The following are equivalent (for $\lambda \geqslant |T|$):*
(A) *T has the strong order property,*
$(\mathrm{B})_\lambda$ *there is a λ^+-saturated model M of T, a L_{∞,λ^+}-formula $\varphi = \varphi(\bar{x},\bar{y}), \varepsilon = \ell g \bar{x} = \ell g \bar{y} \leqslant \lambda$, possibly with $\leqslant \lambda$ parameters, such that in M, φ defines a partial linear order with a chain of length $\geqslant \beth_2(\lambda)^+$.*
(2) *The following are equivalent ($\lambda \geqslant |T|$):*
$(\mathrm{A})'$ *T has the finitary strong order property,*
$(\mathrm{B})'_\lambda$ *like $(B)_\lambda$ but $\varepsilon < \omega$.*

Proof. (1) $(A) \Rightarrow (B)_\lambda$: Straight: for a given $\bar{\varphi} = \langle \varphi_n(\bar{x}_n, \bar{y}_n) : n < \omega \rangle$, let $\bar{x}, \bar{y}$ be the limit of $\bar{x}_n, \bar{y}_n$ respectively and write $\psi^*(\bar{x},\bar{y}) = \bigvee_m (\exists \bar{z}_0, \ldots, \bar{z}_m)[\bar{x} = \bar{z}_0 \& \bar{y} = \bar{z}_m \& \bigwedge_{\ell<m} \varphi_\omega(\bar{z}_\ell, \bar{z}_{\ell+1})]$ where $\varphi_\omega(\bar{x},\bar{y}) =: \bigwedge_n \varphi_n(\bar{x}_n, \bar{y}_n)$.

$(B)_\lambda \Rightarrow (A)$: Let $\bar{a}_\alpha \in {}^\varepsilon M$ for $\alpha < \beth_2(\lambda)^+$ form a φ-chain. Without loss of generality the order φ defines is strict (i.e. $\vdash \varphi(\bar{x},\bar{x})$) and no parameters (just add them to the $\bar{a}_\alpha$'s). By Erdos Rado theorem without loss of generality for some type $q = q(\bar{x},\bar{y})$ for all $\alpha < \beta < \omega$ the sequence $\bar{a}_\alpha{}^\frown \bar{a}_\beta$ realizes q.

For every $n, \bigcup \{q(\bar{x}_\ell, \bar{x}_k) : k = \ell + 1 \bmod n$ and $k, \ell < n\}$ cannot be realized in M (as, if $\bar{b}_0{}^\frown \cdots {}^\frown \bar{b}_{n-1}$ realizes if we get a contradiction to "$\varphi(\bar{x},\bar{y})$ defines a strict partial order"). By saturation there is $\varphi^0_n(\bar{x},\bar{y}) \in q(\bar{x},\bar{y})$ such that $\{\varphi^0_n(\bar{x}_\ell, \bar{x}_k) : k = \ell + 1 \bmod n$ and $k, \ell < n\}$ is not realized in M. The rest should be clear.

(2) Left to the reader. $\square$

Definition 2.5. Let $n \geqslant 3$. (1) T has the n-strong order property (SOP_n) *if* there is a formula $\varphi(\bar{x}, \bar{y})$ having this property for T which means $\ell g \bar{x} = \ell g \bar{y}$ (allowing parameters change nothing) and there is a model M of T and $\bar{a}_k \in {}^{\ell g \bar{x}}M$ for $k < \omega$ such that

(a) $M \models \varphi[\bar{a}_k, \bar{a}_m]$ for $k < m < \omega$,

(b) $M \models \neg \exists \bar{x}_0 \cdots \bar{x}_{n-1}(\bigwedge\{\varphi(\bar{x}_\ell, \bar{x}_k) : \ell, k < n \text{ and } k = \ell + 1 \bmod n\})$.

(2) "$T, \varphi(\bar{x}, \bar{y})$ have the $SOP_{\leqslant n}$" is defined similarly except that in (b) we replace n by each $m \leqslant n$.

Claim 2.6. *$SOP \Rightarrow SOP_{n+1}$, $SOP_{n+1} \Rightarrow SOP_n$, $SOP_{\leqslant n+1} \Rightarrow SOP_{\leqslant n}$ and $SOP_n \Leftrightarrow SOP_{\leqslant n}$ for any given T(we did not say "for any φ").*

Proof. The first clause is immediate. The second clause is straight too: let $\varphi(\bar{x}, \bar{y}), M, \langle \bar{a}_m : m < \omega \rangle$ exemplify SOP_{n+1} and without loss of generality the sequence $\langle \bar{a}_m : m < \omega \rangle$ is an indiscernible sequence. Does $M \models (\exists \bar{x}_0, \ldots, \bar{x}_{n-1})[\bar{x}_0 = \bar{a}_1 \& \bar{x}_{n-1} = \bar{a}_0 \& \bigwedge\{\varphi(\bar{x}_\ell, \bar{x}_k) : \ell, k < n \text{ and } k = \ell + 1 \bmod n\})$? If the answer is yes we can replace $\bar{a}_1$ by $\bar{a}_2$ (by indiscernability), let $\bar{c}_0, \ldots, \bar{c}_{n-1}$ be as required above on $\bar{x}_0, \ldots, \bar{x}_{n-1}$ and $\bar{b}_0 =: \bar{a}_1, \bar{b}_1 =: \bar{a}_2 (= \bar{c}_0), \bar{b}_2 =: \bar{c}_1, \ldots, \bar{b}_{n-1} =: \bar{c}_{n-2}$, $\bar{b}_n =: \bar{c}_{n-1} = a_0$; now they satisfy the requirement mentioned in (b) of Definition 2.5(1) on $\bar{x}_0, \ldots, x_n$ (for SOP_{n+1}), contradicting clause (b) of Definition 2.5(1). So assume "no" and now $\varphi'(\bar{x}, \bar{y})$ have SOP_n for T where $\varphi'(\bar{x}, \bar{y}) =: \varphi(\bar{x}, \bar{y}) \& \neg(\exists \bar{x}_0, \ldots, x_{n-1})[\bar{x}_0 = \bar{x} \& \bar{x}_1 = \bar{y} \& \bigwedge\{\varphi(\bar{x}_\ell, \bar{x}_k) : \ell, k < n \text{ and } k = \ell \bmod n\}]$.

As for $SOP_n \Leftrightarrow SOP_{\leqslant n}$, the implications $\Rightarrow$ is really included in the proof above (i.e. by it, if $\langle \bar{a}_\ell : \ell < \omega \rangle, \varphi_n$ exemplifies SOP_n and $\langle a_l : l < u \rangle$ is an indiscernible sequence, for some φ_{n-1} we have $\langle \bar{a}_\ell : \ell < \omega \rangle, \varphi_{n-1}$ exemplifies SOP_{n-1} (with $n, n-1$ here corresponding to $n+1, n$ there), and we can define $\varphi_{n-2}, \ldots$ similarly; now $\langle \bar{a}_\ell : \ell < \omega \rangle, \bigwedge_{i \leqslant n} \varphi_i$ exemplifies $SOP_{\leqslant n}$. The implication $\Leftarrow$ is trivial. Now the third clause $SOP_{\leqslant n+1} \Rightarrow SOP_{\leqslant n}$ is trivial (read the definition). $\square$

Claim 2.7. *Let T be complete. If T has SOP_3 then T has the tree property (i.e. is not simple).*

Proof. Let $\kappa = cf(\kappa) > |T|$ and $\lambda > \kappa$ be a strong limit singular cardinal of cofinality κ. Let $J = {}^\kappa\lambda, I = \{\eta \in {}^\kappa\lambda : \eta(i) = 0 \text{ for every } i < \kappa \text{ large enough}\}$. Let $\varphi(\bar{x}, \bar{y})$ exemplify the SOP_3. By definition we can find a model M of T and $\bar{a}_\eta \in M$ (for $\eta \in J$) such that:

$(*)$ $\eta <_{\ell x} \nu$ in $I \Rightarrow M \models \varphi[\bar{a}_\eta, \bar{a}_\nu]$.

Without loss of generality $\|M\| \geqslant \lambda, M$ is κ^+-saturated. So for every $\eta \in {}^\kappa(\lambda \backslash \{0\}) \backslash I$ we can find $\bar{a}_\eta \in M$ such that it realizes $p_\eta = \{\varphi(a_{(\eta \upharpoonright i)^\frown 0_{[i,\kappa)}}, \bar{x}) \& \varphi(\bar{x}, \bar{a}_{(\eta \upharpoonright i)^\frown \langle \eta(i)+1 \rangle ^\frown 0_{[i,\kappa)}}) : i < \kappa\}$. So $|p_\eta| \leqslant \kappa$. But if $\eta_1 <_{\ell x} \eta_2 \in {}^\kappa(\lambda \backslash \{0\})$ then we can find $\nu, \rho \in I$ such that $\eta_1 <_{\ell x} \nu <_{\ell x} \rho <_{\ell x} \eta_2$ and $\varphi(\bar{x}, \bar{a}_\nu) \in p_{\eta_1}, \varphi(\bar{a}_\rho, \bar{x}) \in p_{\eta_2}$ and by $(*)$ we have $M \models \varphi[\bar{a}_\nu, \bar{a}_\rho]$, so $p_{\eta_1} \cup p_{\eta_2}$ is contradictory (by clause (b) of Definition 2.5(1) for "φ

have the SOP_3"). So $\langle p_\eta : \eta \in {}^\kappa(\lambda\backslash\{0\})\rangle$ are pairwise contradictory, $|p_\eta| = \kappa$, and $\lambda^\kappa > \lambda = \lambda^{<\kappa} > 2^{|T|}$ and $\bigcup\{Dom\ p_\eta : \eta \in {}^\kappa(\lambda\backslash\{0\})\}$ has cardinality $\leqslant\lambda$ and $\kappa > |T|$.

By [14, III, 7.7]=[20, III, 7.7, p.141] this implies that T has the tree property. □

Claim 2.8. (1) *The theory $T_n =: T \restriction \{<_{n,\ell} : \ell\leqslant n\}$ from Claim 2.3(2) has SOP_n but not SOP_{n+1}.*

(2) *T^{mc}_{trf}, the model completion of the theory of triangle free graphs has SOP_3 but not SOP_4.*

(3) *For $n\geqslant 3$ the model completion $T^{mc}_n = T^{mc}_{dcf(n)}$ of the theory $T_n = T_{dcf(n)}$ of (directed graphs, no loops or multiple edge for simplicity) with no directed circle of length $\leqslant n$ has SOP_n but not SOP_{n+1}.*

(4) *For odd $n\geqslant 3$, the model completion $T^{mc}_n = T^{mc}_{ocf(n)}$ of the theory $T_n = T_{ocf(n)}$ of graphs with no* odd *circle of length $\leqslant n$, has SOP_n but not SOP_{n+1}.*

(5) *For $n\geqslant 3$, the model completion $T^{mc}_{cf(n)}$ of the theory $T_n = T_{cf(n)}$ of graphs with no circles of length $\leqslant n$, has SOP_3 but not SOP_4.*

(6) *The theory T_{qcf} (see [21]) does not have SOP_3 (but is not simple).*

Remark 2.8A. (1) Note that $\mathrm{univ}(\lambda, T^{\mathrm{mc}}_{\mathrm{cf(n)}}) = univ(\lambda, T_{\mathrm{cf(n)}})$.

(2) For those theories, $D(T^{\mathrm{mc}})$ is an uncountable; they have no universal model in $\lambda < 2^{\aleph_0}$.

Proof. (1) Proved really in Claim 2.3.

(2) This is included in part (5).

(3)–(5) We discuss the existence of model completion later; note that the meaning of T_n depends on the part we are proving.

Let xRy mean (x, y) is an edge; when we say (x, y) is an edge, for graphs we mean $\{x, y\}$ is an edge. Let $\bar{y} = \langle y_\ell : \ell < n\rangle, \varphi(\bar{x}, \bar{y}) = \bigwedge_{\ell<n-1} x_\ell R y_{\ell+1} \& x_{n-1} R y_0$. First we note there $T_n \vdash \neg(\exists\bar{x}_0, \ldots, \bar{x}_{n-1}) \bigwedge\{\varphi(\bar{x}_i, \bar{x}_k) : \ell, k < n, k = \ell + 1 \bmod n\}$, otherwise there are $M \models T_n$ and $\bar{a}_\ell = \langle a_{\ell,0}, \ldots, a_{\ell,n-1}\rangle \in {}^nM$ as forbidden but then $a_{0,0}, a_{1,1}, \ldots, a_{n-1,n-1}$ is a circle, so in all cases this is impossible.

For parts (3) and (4) let M be the following model of T_n; elements a^ℓ_i $(i < \omega, \ell < n), R = \{(a^\ell_i, a^{\ell+1}_j) : i < j < \omega, \ell < n - 1\} \cup \{(a^{n-1}_i, a^0_j) : i < j < \omega\}$ (but for graphs we put all such pairs and the inverted pair as R should be symmetric and irreflexive relation). Clearly for $i < j < \omega, M \models \varphi[\bar{a}_i, \bar{a}_j]$ where $\bar{a}_i = \langle a^0_i, \ldots, a^{n+1}_i\rangle$. Lastly $M \models T$: for part (3) as R is not symmetric the absence of any circle should be clear, $M \models a^{\ell(1)}_{i(1)} R a^{\ell(2)}_{i(1)} \Rightarrow i(1) < i(2)$; for part (4) there are circles but even or long and $M \models a^{\ell(1)}_{i(1)} R a^{\ell(2)}_{i(2)} \Rightarrow \ell(1) = \ell(2) + 1 \bmod n$, so $T_{\mathrm{dcf}(n)}, T_{\mathrm{ocf}(n)}$ (and $T^{\mathrm{mc}}_{\mathrm{ocf}(n)}$) has even or long circles. So $T_{\mathrm{dcf}(n)}, T_{\mathrm{ocf}(n)}$ (and $T^{\mathrm{mc}}_{\mathrm{dcf}(n)}$ and $T^{\mathrm{mc}}_{\mathrm{ocf}(n)}$) has SOF_n.

Let $n = 3$. Now $T_{\mathrm{cf}(3)} = T_{\mathrm{ocf}(3)}$ so we can ignore part (5). Also $T_{\mathrm{dcf}(n)}, T_{\mathrm{ocf}(n)}$ has the amalgamation property and joint embedding property. Thus, it is enough to show that $T^{\mathrm{mc}}_{\mathrm{dcf}(n)}, T^{\mathrm{mc}}_{\mathrm{ocf}(n)}$ fails the SOP_4. As T^{md} has elimination of quantifiers the reader can check directly that T_n does not have SOP_4.

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Let $n > 3$. Though T_n^{mc} does not have elimination of quantifiers, every formula is equivalent to a Boolean combination of formulas of the form $x = y, xRy$ for $m < n, \varphi_m(x,y) =: (\exists x_0,\dots,x_m)[x = x_0 \& y = x_m \bigwedge \bigwedge_{\ell<m} x_\ell R x_{\ell+1}]$ (i.e. the distance from x to y is $\leqslant m$, directed from x to y in the case of di-graphs). For part (5) of Claim 2.8, we should add for $\ell < m < n/2, \ell > 0$ a partial function $F_{m,\ell}$ defined by $F_{m,\ell}(x,y) = z$ iff there are $t_0,\dots,t_m$ with no repetition such that $x = t_0, y = t_m, z = t_\ell$ and $\bigwedge_{\ell<m} t_\ell R t_{\ell+1}$ and lastly $\psi_{m,\ell}(x,y) =: (\exists z)[F_{m,\ell}(x,z) = z]$. Let T_n^2 be the set of obvious (universal) axioms for those relations. Then easily T_n^2 has amalgamation and has model completion, T_n^1 which has elimination of quantifiers (but the closure of a finite set under those functions may be infinite). Moreover, assume $M \models T_n^2, \langle \bar a_m : m < \omega\rangle$ is an indiscernible sequence in $M, \bar a_m = \langle a_\ell^m : \ell < k\rangle$, with $k < \omega$. Then there is $w \subseteq k$ such that $[a_\ell^m = a_\ell^{m+1} \Leftrightarrow \ell \in w]$ and without loss of generality $[\ell_1 < \ell_2 \Rightarrow a_{\ell_1}^m \neq a_{\ell_2}^m]$. Let for $u \subseteq w, M_u$ be the submodel of M generated by $\bigcup_{m\in u} \bar a_m$; note that for parts (3) and (4), $M_u = \bigcup\{M_v : v \subseteq u \text{ and } |v| \leqslant 1\}$ so things are simple. By the indiscernibility (increasing the $\bar a_m$'s e.g. taking ω blocks) without loss of generality

(∗) $M_u \cap M_v = M_{u\cap v}$ and the universe of $M_{\{m\}}$ is the range of $\bar a_m$.

Let $m = n$ for parts (3) and (4) of Claim 2.8, $m = 3$ for part (5). For part (5) note the distance between $a_{\ell_0}^0, a_{\ell_1}^1$ is $> \frac{n}{4}$. (Why? If not there is a path $C^{i,j}$ of length $\leqslant \frac{n}{4}$ for $a_{\ell_0}^i$ to $a_{\ell_1}^j$, now $C^{0,3} \cup C^{1,3} \cup C^{1,4} \cup C^{0,4}$ is a circle of length $\leqslant n$, may cross itself but still there is a too small circle).

We can now define models $N_{\{\ell\}}$ (for $\ell < n+1$), $N_{\{\ell,\ell+1\}} (\ell < n)$ and $N_{\{n,0\}}$ and isomorphisms $h_\ell, g_\ell (\ell < n+1)$ such that

(a) for $\ell < n+1, h_\ell$ an isomorphism from $M_{\{\ell\}}$ onto $N_{\{\ell\}}$,

(b) for $\ell < n, g_\ell$ an isomorphism $M_{\{\ell,\ell+1\}}$ onto $N_{\{\ell,\ell+1\}}$ extending $h_\ell, h_{\ell+1}$,

(c) for $\ell = n, g_\ell$ an isomorphism from $M_{\{n,n+1\}}$ onto $N_{\{n,0\}}$ extending h_n and $h_0 \circ f$ where f is the isomorphism from $M_{\{n+1\}}$ onto $M_{\{0\}}$ taking $\bar a_{n+1}$ onto $\bar a_0$,

(d) $N_\emptyset =: g_\ell(M_\emptyset)$ does not depend on ℓ,

(e) $N_u \cap N_v = N_{u\cap v}$ if u, v are among $\emptyset, \{\ell\}, \{m,m+1\}, \{n,0\} (\ell < n+1, m < n)$.

Now,

⊗ There is a model of T_n^2 extending all $N_{\{\ell,\ell+1\}}, N_{\{n,0\}} (\ell < n)$.[1]

This is enough for showing that T_n^1 lacks the SOP_{m+1}. Lastly the reader can check that $T_{\mathrm{cf}(n)}^{\mathrm{mc}}$ has SOP_3 [choose $k \in (n/3, n), \bar a_\ell = \langle a_{\ell,0}\rangle, \ell_1 < \ell_2 \Rightarrow \varphi_k(a_{\ell,0}, a_{\ell_2,0})$]. □

Theorem 2.9. *Let T be first order complete, $\lambda \geqslant |T|$ and T has the SOP_3. Then*

(1) T is maximal in the Keisler order $\triangleleft_\lambda$, i.e. for a regular filter D on λ and some (= every) model $M \models T$ we have M^λ/D is λ^+-saturated iff *D is a good ultrafilter.*

(2) Moreover, in Definition 2.10 *T is $\triangleleft^\ell$-maximal (see Definition* 2.10 *below).*

We delay the proof.

[1] Why? E.g. define it as a graph by the union (adding no edges) extend it to an existentially complete model of the apropriate T_n, and this has a natural expansion to a model of T_n^2.

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Remark. The order $\triangleleft$ was introduced and investigated by Keisler [9]; it was investigated further in [13, 14, Ch. 6], new version [20, Ch. 6]. The following is a generalization.

Definition 2.10. (1) For models M_0, M_1 we say $M_0 \triangleleft^*_\lambda M_1$ if the following holds: for some model $\mathfrak{B}_0$ in which M_0, M_1 are intepreted (so $M_i = M_i^{\mathfrak{B}_0}$), for every elementary extension $\mathfrak{B}$ of $\mathfrak{B}_0$, which is $(\aleph_0 + |\tau(M_0)| + |\tau(M_1)|)^+$-saturated we have [$M_1^{\mathfrak{B}}$ is λ^+-saturated $\Rightarrow M_0^{\mathfrak{B}}$ is λ^+-saturated].

(2) $M_0 \triangleleft^* M_1$ if for every $\lambda \geqslant \aleph_0 + |\tau(M_0)| + |\tau(M_1)|$ we have $M_0 \triangleleft^*_\lambda M_1$.

(3) Using the superscript ℓ instead of $*$ means in the saturation we use only φ-types for some $\varphi = \varphi(\bar{x}, \bar{y})$ (so any φ is satisfactory, but for each type φ is constant) and omit the saturation demand on $\mathfrak{B}$.

(4) For complete theories T_1, T_2 we say $T_1 \triangleleft^*_\lambda T_2$ if for every model M_1 of T_1 for some model M_2 of T_2, $M_1 \triangleleft^*_\lambda M_2$. Similarly for $T_1 \triangleleft^* T_2, T_1 \triangleleft^\ell_{(\lambda)} T_2$.

Observation 2.11. (1) In 2.10(1) we can just use $\mathfrak{B}_0$ of the form $(H(\chi), \in, <^*_\chi, M_0, M_1)$ with χ strong limit.

(2) $\triangleleft^*_\lambda$ is a partial order, also $\triangleleft^\ell_*, \triangleleft^\ell$ are partial orders; $M \triangleleft^*_\lambda M$ and if M_0 is interpretable in M_1 then $M_0 \triangleleft^*_\lambda M_1$.

(2A) For models of countable vocabulary, similar statements hold for $\triangleleft^*$ (without the countability if $|\tau(M_1)| > |\tau(M_0)| + |\tau(M_2)| + \aleph_0$, we can get a silly situation).[2]

(3) If $\lambda \geqslant \aleph_0 + |\tau(M_0)| + |\tau(M_1)|$ then $M_0 \triangleleft^*_\lambda M_1$ iff for every finite $\tau \subseteq \tau(M_0), M_0 \restriction \tau \triangleleft^*_\lambda M_1$.

(4) $M_1 \triangleleft^\ell_\lambda M_2 \Rightarrow M_1 \triangleleft^*_\lambda M_2$.

(5) Parallel results hold for theories.

(6) Any (complete first order) theory of any infinite linear order is $\triangleleft^\ell$-maximal hence $\triangleleft^*_\lambda$-maximal for every $\lambda \geqslant |T| + \aleph_0$.

(7) All countable stable theories without the f.c.p. (e.g. $T = Th(\omega_1 =)$) are $\triangleleft^\ell$-equivalent.

(8) All countable stable theories with the f.c.p. are equivalent (e.g. $T_{\mathrm{eq}} = Th(\bigcup_n (\{n\} \times n), E)$ where E is equally of first coordinates).

(9) If T_1 is countable unstable, and $T_= = Th(\omega_1 =)$, then $T_= \triangleleft^\ell T_{\mathrm{eq}} \triangleleft T_1, \lambda \geqslant 2^{\aleph_0} \Rightarrow \neg(T_{\mathrm{eq}} \triangleleft^\ell_\lambda T_=)$ and $\lambda = \varphi(\lambda) \geqslant \aleph_0 \Rightarrow \neg(T_1 \triangleleft^\ell_\lambda T_{\mathrm{eq}})$ and $\lambda < 2^{\aleph_0} \Rightarrow T_{\mathrm{eq}} \triangleleft^\ell_\lambda T_=$.

Proof. (1)–(4) Obvious.

(5) The proof of [14, VI, 2.6] = [20, VI, 2.6, p. 337] gives this, too.

(6)–(8) As is [1]. $\square$

Proof of Theorem 2.9(1) and (2). Without loss of generality $\tau(T)$ is finite. Remember if T' has infinite linear order as model then it is $\triangleleft^\ell$-maximal. Let J be a dense linear order, such that

(a) J has a closed interval which is I,

[2] so to overcome this, we may in Definition 2.10(2) replace "every $\lambda > \cdots$" by " every large enough λ"

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(b) for any regular $\mu_1, \mu_2 \leqslant |J|$, J has an interval isomorphic to $(\{1\} \times \mu_1) \cup (\{2\} \times \mu_2)$ ordered by $(i_1, \alpha_1) < (i_2, \alpha_2) \Leftrightarrow (i_1 = 1 \& i_2 = 2) \vee (i_1 = 1 = i_2 \& \alpha_1 < \alpha_2) \vee (i_1 = 2 = i_2 \& \alpha_1 > \alpha_2)$.

Let $\varphi(x, y)$ exemplify the SOP_3. Let M be a model of T and $F : I \to {}^{\ell g \bar{x}}M$ be such that $I \models \eta < \nu \Rightarrow M \models \varphi[F(\eta), F(\nu)]$ and for every $c \in M^{\mathfrak{B}}$ we can find a finite $I' \subseteq I$ such that if $[t_1, t_2 \in (I \setminus I')] \& \bigwedge_{s \in I'} [s <_I t \equiv s <_I t_2]$ then $M^{\mathfrak{B}} \models \varphi[F(t_1), c] \equiv \varphi[F(t_2), c]$ and $M^{\mathfrak{B}} \models \varphi[c, F(t_1)] \equiv \varphi[c, F(t_1)]$. Let $\mathfrak{B}_0 = (H(\chi), \in, <^*_\chi, J, I, F, M)$, and $\mathfrak{B}$ be a model, $\mathbf{j}$ an elementary embedding of $\mathfrak{B}_0$ into $\mathfrak{B}$ such that $M^* = M^{\mathfrak{B}} \restriction L(T)$ is locally λ^+-saturated but $I^{\mathfrak{B}} = \mathbf{j}(I)$ is not λ^+- saturated (for Theorem 2.9(1): $\mathfrak{B}^* = \mathfrak{B}^\lambda/D$).

As $\mathbf{j}(I)$ is not λ^+-saturated, we can find $\lambda_0, \lambda_1 \leqslant \lambda$ and $\alpha_i^\ell \in \mathbf{j}(I)$ (for $i < \lambda_\ell$, $\ell < 2$) such that

(α) $I^{\mathfrak{B}} \models a_i^0 < a_j^0$ for $i < j < \lambda_0$,
(β) $I^{\mathfrak{B}} \models a_i^1 > a_i^1$ for $i < j < \lambda_1$,
(γ) $I^{\mathfrak{B}} \models a_i^0 < a_j^1$ for $i < \lambda_0, j < \lambda_1$,
(δ) $I^{\mathfrak{B}} \models \neg(\exists x)[\bigwedge_{i < \lambda_0, j < \lambda_1} a_i^0 < x < a_j^1]$.

Clearly $\{\varphi(\bar{a}_i^0, \bar{x}), \varphi(\bar{x}, \bar{a}_j^1) : i < \lambda_0, j < \lambda_1\}$ is finitely satisfiable in $\mathbf{j}(M)$. Now as $\mathbf{j}(M)$ is locally λ^+-saturated there is $\bar{a} \in {}^{\ell g \bar{x}}(M^*)$ such that $M \models$ "$\varphi(\bar{a}_i^0, \bar{a}) \& \varphi(\bar{a}, \bar{a}_j^1)$ for $i < \lambda_0, j < \lambda_1$". In $\mathfrak{B}$ we can define

$$I_-^{\mathfrak{B}}[\bar{a}] =: \{\eta \in I^{\mathfrak{B}} ; \text{there is } \nu \in I^{\mathfrak{B}} \text{ such that } I^{\mathfrak{B}} \models \text{“}\eta \leqslant \nu\text{” and } \mathbf{j}(M) \models \varphi(\bar{a}_\nu, \bar{a})\},$$

$$I_+^{\mathfrak{B}}[\bar{a}] =: \{\eta \in I^{\mathfrak{B}} : \text{there is } \nu \in I^{\mathfrak{B}} \text{ such that } I^{\mathfrak{B}} \models \text{“}\nu \leqslant \eta\text{” and } j(M) \models \varphi[\bar{a}, \bar{a}_\nu]\}.$$

Clearly

(a) $I_-^{\mathfrak{B}}[\bar{a}]$ is an initial segment of $\mathbf{j}(I)$ which belongs to $\mathfrak{B}$.
(b) $I_+^{\mathfrak{B}}[t]$ is an end segment of $\mathbf{j}(I)$ which belongs to $\mathfrak{B}$.
(c) For every $i < \lambda_0$

$(*)_0^i$ $a_i^0 \in I_-^{\mathfrak{B}}[\mathbf{j}, t]$.

(d) for every $j < \lambda_1$

$(*)_1^j$ $a_j^1 \notin I_-^{\mathfrak{B}}[t]$.

(e) By the choice of φ

$(*)_3$ $I_-^{\mathfrak{B}}[t] \cap I_+^{\mathfrak{B}}[t] = \emptyset$.

If for some $c \in \mathfrak{B}, \mathfrak{B} \models$ "$c \in \mathbf{j}(J)$ and $(\forall x \in I_-[t])(c \leqslant_I c)$ and $(\forall x \in I_+[t])[c \leqslant_I x]$" we are done. So $\mathfrak{B}$ thinks $(I_-^{\mathfrak{B}}[t'], I_+^{\mathfrak{B}}[t])$ is a Dedekind cut, so let $\mathfrak{B} \models$ "$cf(I_-^{\mathfrak{B}}[t], <_I) = t_-, cf(I_+^{\mathfrak{B}}[t] >_I) = t_+$ and the (outside) cofinalities of t_-, t_+ are μ_1, μ_2 respectively. If μ_1, μ_2 are infinite, we use clause (b) of the choice of J (and the choice of μ). We are left with the case where $\mu_1 = 1 < \mu_2$ (the other case is the same). Use what $\mathfrak{B}$ "thinks" is a (t_1, t_2) Dedekind cut of J to show $\mu_2 \geqslant \mu^+$ a contradiction. $\square$

Theorem 2.12. (1) *The theorems on nonexistence of a universal model in* λ *for linear order from* [10], [21, Section 3] *hold for any theory with* SOP_4.

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(2) *We can use embedding (not necessarily elementary) if $\varphi(\bar{x},\bar{y})$ is quantifier free or even existential.*

Proof. We concentrate on the case λ is regular and part (1). We will concentrate on the new part relative to [10]. Let $\varphi(\bar{x},\bar{y})$ exemplify $SOP_{\leqslant 4}$ (exists by Claim 2.6(1)) in a complete first order theory T. Without loss of generality $\ell g\bar{x} = \ell g\bar{y} = 1$ and $T \vdash \neg\varphi(\bar{x},\bar{x})$.

Let M be a model of T with universe λ, I a linear order, $a_s \in M, M \models \varphi[a_s,a_t]$ for $s <_I t$ (from I).

We do not have a real Dedekind cut (as $\varphi(x,y)$ is not transitive), but we use replacements. Now for every $b \in M$, let $I^-[b] = \{t : M \models \varphi[a_t,b]\}$ and $I^+[b] = \{t : M \models \varphi[b,a_t]\}$. As φ exemplifies also $SOP_{\leqslant 3}$ clearly the following is satisfied:

(*) $s \in I^-[b] \& t \in I^+[t] \Rightarrow s < t$ (if $t < s$ from a counterexample, b,t,s gives a contradiction).

Note: $I^-[a_s] = \{t : t <_I s\}, I^+[a_s] = \{t : s <_I t\}$. Let $P = \{a_s : s \in I\}, <^* = \{(a_s,a_t) : s <_I t\}$ and $J^-[t,\alpha] = \{s \in I : s <_I t, a_s \in \alpha\}, J^+[t,\alpha] = \{s \in I : t <_I s, a_s \in \alpha\}$ (remember: $|M|$, the universe of M, is $\lambda, \alpha = \{\beta : \beta < \alpha\}$). Hence $C =: \{\delta < \lambda : (M,P,<^*,<) \restriction \delta \prec M^+ =: (M,P,<^*,<)\}$ is a club of λ. Clearly possibly shrinking C:

(**) let $\delta \in C, b \in M \cap \delta$;

(i) if $(I^-[b],<_I)$ has cofinality $< \lambda$, then $I^-[b] \cap M_\delta$ is $<_I$-cofinal in it,

(ii) if $(I^+[b],>_I)$ has cofinality $< \lambda$, then $I^+[b] \cap M_\delta$ is $(>_I)$-cofinal in it,

(iii) if there is t such that $I^-[b] \leqslant_I t \leqslant_I I^+[b]$, then there is such $t \in M \cap \delta$.

Now suppose that $\delta_1 < \delta_2$ are in $C, t(*) \in I, a_{t(*)} \in P \backslash \delta_2$.

Case 1: For some $s(*) \in M \cap \delta_2$ we have $(\forall s \in J^-[t(*),\delta_1])(s <_I s(*) <_I t(*))$. Let $b =: a_{s(*)}$. Hence for every $c \in M \cap \delta_1$ if $\varphi(c,a_{t(*)})$ then for every t',t'' satisfying $t' <_I t(*) <_I t'', t' \in I, a_{t'} \in \delta_1, t'' \in I, a_{t''} \in \delta_1$ we have $M^+ \models (\exists x)[x \in P \& \varphi(c,x) \& a_{t'} <^* x <^* a_{t''}]$. Clearly (or see the middle of the proof of Case 2 below) necessarily for arbitrary $<_I$- large $t \in J^-[t(*),\delta_1]$ we have $\varphi[c,a_t]$ but for any such $t, \varphi[a_t,a_{s(*)}]$ i.e. $\varphi[a_t,b]$ hence

$(*)_1$ $(\forall c \in M \cap \delta_1)(\varphi(c,a_{t(*)}) \to (\exists y \in M \cap \delta_1)[\varphi(c,y) \& \varphi(y,b)])$.

Of course,

$(*)_2$ $b \in M \cap \delta_2$,

$(*)_3$ $\varphi[b,a_{t(*)}]$.

Note: Those three properties speak on $M,\delta_1,\delta_2,a_{t(*)},b$ but not on $I,<^*,P,<$.

Case 2: For no $s(*) \in I, a_{s(*)} \in \delta_2$ do we have $(\forall s \in J^-[t(*),\delta_1])[s <_I s(*) <_I t(*)]$, we further assume:

(A) $\{a_s : s \in I, a_s \in \delta_2, s <_I t(*)\}$ is not definable in $M^+ \restriction \delta_2\}$.

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We shall now show that for no $b \in M \cap \delta_2$ do we have $(*)_1, (*)_2$ and $(*)_3$, so assume b is like that and we shall get a contradiction.

We are assuming (A) holds. By $(*)_3$ we have $\varphi[b, a_{t(*)}]$ hence for arbitrarily $<_I$-large $t \in J^-[t(*), \delta_2]$ we have $\varphi[b, a_t]$; choose such t_0. (Why? Otherwise $I^+[b] \cap J^-[t(*), \delta_2)$ is bounded say by some $t^* \in J^-[t(*), \delta_2]$, so $\theta(x, b, a_{t^*}) =: x \in P \& (\exists y)[y \in P \& y \leqslant^* x \& \varphi(b, y) \& t^* <^* x]$ define in M^+ a set which is an end segment of $(P, <^*)$, include $t(*)$ (check) but no $s \in \delta_2, s <^* t(*)$. So in $M^+ \restriction \delta_2$ it defines the set $\{a_s : s \in J^+[t(*), \delta_2]\}$; hence $P(x) \& \neg\theta(x, b, a_{t^*})$ defines in $M^+ \restriction \delta_2$ the set $\{a_s : s \in J^-[t(*), \delta_2]\}$; hence by the assumption of the case, $M^+ \restriction \delta_2$ satisfies

$$(\forall z)[z \in P \& \neg\theta(z, b, a_{t^*}) \rightarrow (\exists y < \delta_1)(y \in P \& z \leqslant^* y \& \neg\theta(z, b, a_{t^*})]$$

contradicting (A) above).

So by the assumption of the case (i.e. that $t_0 < \delta_2$ cannot serve as $s(*)$ and $t_0 <_I t(*)$) for some $t_1 \in J^-[t(*), \delta_1]$ we have $t_0 <_I t_1$ and clearly $t_1 <_I t(*)$; hence $\varphi[a_{t_1}, a_{t(*)}]$. So by $(*)_1$ applied with a_{t_1} standing for c for some $y \in M \cap \delta_1$ we have $\varphi[a_{t_1}, y] \& \varphi[y, b]$. Now b, a_{t_0}, a_{t_1}, y contradicts "$\varphi(x, y)$ exemplifies SOP_4".
Hence we get together

$\bigoplus$ if $\delta_1 < \delta_2 \in C, a_{t(*)} \in P \backslash \delta_2$ (so $t(*) \in I$) then the following conditions are equivalent:
 - (α) for some $s(*) \in \delta_2$, (so necessarily $s(*) \neq t(*)$) we have $(\forall s \in I)[a_s \in \delta_1 \Rightarrow s <_I s(*) \equiv s <_I t(*)])$
 - (β) for some $b \in \delta_2$ the conditions $(*)_1, (*)_2, (*)_3$ above holds for $\varphi(x, y)$ or for $\varphi^-(x, y)$ where $\varphi^-(x, y) = \varphi(y, x)$.

Proof. If clause (α) holds for $s(*) <_I t(*)$ then use Case 1 above. If clause (α) holds and $s(*) <_I t(*)$ fails, then $t(*) <_I s(*)$, inverts the order of I, use φ^- and now apply Case 1 above. So assume $\neg(\alpha)$. We first want to apply Case 2 to prove there is no b satisfying $(*)_1, (*)_2, (*)_3$. For this we need clause (A) there. We claim it holds.

(Why? Assume $\bar{d} \in (M^2 \restriction \delta_2), \psi$ a first order formula (in the vocabulary of M^+), such that for every $e \in M^+ \restriction \delta_2$ we have $M^+ \restriction \delta_2 \models \psi[e, \bar{d}]$ iff $e \in \{a_s : s \in I, a_s \in \delta_2, s <_I t(*)\}$. So $M^+ \models (\exists z)[P(z) \& (\forall y)(y < \delta_1 \& P(y) \Rightarrow y <^* z \equiv \psi[y, \bar{d}])]$ as $z \mapsto a_{t(*)}$ satisfies it, but $M^+ \restriction \delta_2 \prec M^+$; hence there is $z^* \in \delta_2$ satisfying this. So $z^* \in P$ hence for some $s(*)$, $z^* = a_{s(*)}$; so $s(*)$ contradicts the assumption $\neg(\alpha)$.

We will finish the proof of 2.12 later.

Definition 2.13. Let M be a model with universe λ and $\varphi(x, y)$ a formula exemplifying SOP_4 (possibly with parameters) let $\varphi^+(x, y) =: \varphi(x, y), \varphi^-(x, y) =: \varphi(y, x)$. Assume $\bar{C} = \langle C_\delta : \delta \in S \rangle$ is a club system, $S \subseteq \lambda$ stationary, guessing clubs[3] (i.e. for every club E of λ for stationarily many $\delta < \lambda, \delta \in S, C_\delta \subseteq E$)

[3] otherwise dull.

(a) for $x \in |M|$ and $\delta \in S$ let

$$inv_\varphi(x, C_\delta, M) = \{\alpha \in nacc\ C_\delta : \text{letting } \delta_2 = \alpha, \delta_1 = sup(C_\delta \cap \alpha)(\text{well defined}), \text{ forsomeb conditions } (*)_1, (*)_2, (*)_3 \text{ of Case 1 holds for } \varphi^+ \text{ or for } \varphi^-\}$$

(b) $Inv_\varphi(C_\delta, M) = \{inv(x, C_\delta, M) : x \in M\}$:
$INv_\varphi(M, \bar{C}) = \langle Inv_\varphi(C_\delta, M) : \delta \in S\rangle$
$INV_\varphi(M, \bar{C}) = INv_\varphi(M, \bar{C})/id^a(\bar{C})$ where $id^a(\bar{C})$ is defined as follows.

Definition 2.14.

$$id^a(\bar{C}) = \{ S' \subseteq \lambda : \text{for some club } E \text{ of } \lambda \text{ the set of } \delta \in S' \cap S \text{ for which } C_\delta \subseteq E \text{ is not stationary}\}.$$

Observation 2.15. If $M' \cong M''$ are models of T and both have universe λ then $INV_\varphi(M', \bar{C}) = INV_\varphi(M'', \bar{C})$ so $INV_\varphi(M, \bar{C})$ can be defined for any model of cardinality λ.

Proof. Let f be from M' onto M'', so f is a permutation of λ. So $E_0 = \{\delta < \lambda : \delta$ a limit ordinal, f maps δ onto $\delta\}$. Assume $C_\delta \subseteq E$, then for $x \in M' \backslash \delta, \delta \in S$ we have $\mathrm{inv}(x, C_\delta, M') = \mathrm{inv}_\varphi(f(x), C_\delta, M'')$. (Why? Read $(*)_1, (*)_2, (*)_3$). Hence $Inv_\varphi(C_\delta, M') \in Inv_\varphi(C_\delta, M'')$. By Definition of $id^a(\bar{C})$ we are done. □

Observation 2.16. If $\bar{C} = \langle C_\delta : \delta \in S\rangle, S \subseteq \lambda$ stationary, $C_\delta \subseteq \delta = sup(C_\delta), C_\delta$ closed, I a linear order with the set of elements being λ we let

(a) for $x \in \lambda, \delta \in S, inv(x, C_\delta, I) = \{\alpha \in nacc(C_\delta)$: there are $y, z \in \alpha$ satisfying $y <_I x <_I z$ such that $(\forall s)[s < \sup(C_\delta \cap \alpha) \Rightarrow s <_I y \vee z \equiv_I s]\}$,
(b) $Inv(C_\delta, I) = \{inv(x, C_\delta, I) : x \in M\}$,
(c) $INv(I, \bar{C}) = \langle Inv(C_\delta, I) : \delta \in S\rangle$,
(d) $INV(I, \bar{C}) = INv(I, \bar{c})/\mathrm{id}^a(\bar{C})$.

Observation 2.17. $INV(I, \bar{C}) = INV(I', \bar{C})$ if $I \cong I'$, so actually it is well defined for any linear order with cardinality λ.

Observation 2.18. If M is a model with universe λ and $\varphi, \langle a_s : s \in I\rangle$ is as above and, $\emptyset \notin id^a(\bar{C})$, then $INV(I, \bar{C}) \leqslant INV_\varphi(M, \bar{C})$ i.e. for some club E of $\lambda, \delta \in S \& C_\delta \subseteq E \Rightarrow Inv(C_\delta, I) \subseteq \mathrm{Inv}_\varphi(C_\delta, M)$.

Proof. By $\oplus$ above (after case 2). □

Proof of 2.12 (*Conclusion*). As in [10]. □

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Claim 2.19. *For a complete T, the following are equivalent:*

(a) *T does not have* SOP_3,

(b) *if in* $\mathfrak{C}, \langle \bar{a}_i : i < \alpha \rangle$ *is an indiscernible sequence,* α *infinite and* $\{\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y})\}$ *contradictory and for each j for some* $\bar{b}_j$ *we have* $i \leqslant j \Rightarrow \models \varphi[\bar{b}, \bar{a}_i]$ *and* $i > j \Rightarrow \models \psi[\bar{b}, \bar{a}_i]$ *then for* $i < j$ *we have* $(\exists \bar{x})(\varphi(\bar{x}, \bar{a}_j) \& \psi(\bar{x}, \bar{a}_i))$,

(c) *in clause (b) we replace the conclusion: for every finite disjoint* $u, v \subseteq \omega$ *we have* $(\exists \bar{x}) \left(\bigwedge_{i \in u} \varphi(\bar{x}, \bar{a}_i) \& \bigwedge_{j \in v} \psi(\bar{x}, \bar{a}_j) \right)$.

Proof. $(c) \Rightarrow (b)$: Trivial.

$\neg(c) \Rightarrow \neg(b)$: Choose counterexample with $|u \cup v|$ minimal, assume $\alpha > \omega + |u \cup v|$.

$\neg(a) \Rightarrow \neg(b)$: Straight by the Definition of SOP_3, etc.

$\neg(b) \Rightarrow \neg(a)$: Without loss of generality $\langle \bar{a}_i\hat{\ }\bar{b}_i : i < \alpha \rangle$ is an indiscernible sequence. Now we cannot find $\bar{c}_0, \bar{c}_1, \bar{c}_2$ such that $\bar{c}_0\hat{\ }\bar{c}_1, \bar{c}_1\hat{\ }\bar{c}_2, \bar{c}_2\hat{\ }\bar{c}_0$ realizes the same type as $(\bar{a}_0\hat{\ }\bar{b}_0)\hat{\ }(\bar{a}_1\hat{\ }\bar{b}_1)$, so SOP_3 is exemplified. □

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