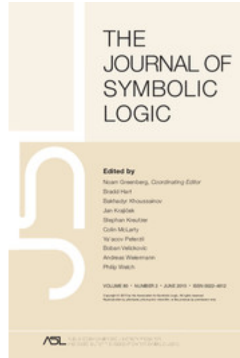
Sh:828



Preserving preservation

Jakob Kellner and Saharon Shelah

The Journal of Symbolic Logic / Volume 70 / Issue 03 / September 2005, pp 914 - 945
DOI: 10.2178/jsl/1122038920, Published online: 12 March 2014

THE JOURNAL OF SYMBOLIC LOGIC
Volume 70, Number 3, Sept. 2005

PRESERVING PRESERVATION

JAKOB KELLNER AND SAHARON SHELAH

Abstract. We prove that the property "P doesn't make the old reals Lebesgue null" is preserved under countable support iterations of proper forcings, under the additional assumption that the forcings are nep (a generalization of Suslin proper) in an absolute way. We also give some results for general Suslin ccc ideals.

§1. Introduction.

Let us consider the following

HYPOTHESIS 1. *Let $(P_\beta, \underset{\sim}{Q}_\beta)_{\beta<\varepsilon}$ be a countable support iteration of proper forcings (ε a limit) such that each P_β ($\beta < \varepsilon$) forces that the set of old reals $X := V \cap 2^\omega$ remains Lebesgue positive. Then P_ε forces this as well.*

The main result of this paper (9.4) is that Hypothesis 1 is true under some additional (relatively mild) requirements on the P_β. It seems that such requirements are needed (this is argued in section 4).

Preservation theorems of this kind have proven to be extremely useful in independence proofs. Hypothesis 1 specifically is used in the proof of the following two theorems of [9]:

> *It cannot be decided in ZFC whether every superposition-measurable function is measurable. (A function $f: \mathbb{R}^2 \to \mathbb{R}$ is superposition-measurable, if for every measurable $g: \mathbb{R} \to \mathbb{R}$ the superposition function $f_g: \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x, g(x))$ is measurable.)*

and (von Weizsäcker's problem):

> *It cannot be decided in ZFC whether for every $f: \mathbb{R} \to \mathbb{R}$ there is a continuous function $g: \mathbb{R} \to \mathbb{R}$ such that $\{ x \in \mathbb{R} : f(x) = g(x) \}$ is of positive outer measure.*

Forcing is a very general method for proving independence results, i.e., results of the form "formula φ is neither provable nor refutable in ZFC". Forcing gives a method for modifying a given set-theoretical universe V to a new universe V' in

Received April 28, 2004; accepted February 28, 2005.
2000 *Mathematics Subject Classification.* 03E40, 03E17.
Jakob Kellner partially supported by FWF grant P17627-N12.
Saharon Shelah supported by the United States-Israel Binational Science Foundation (Grant no. 2002323), publication 828.
The authors thank a referee for proposing numerous enhancements, including a substantial simplification of Lemma 5.11.

0022-4812/05/7003-0012/$4.20

which some formula ψ is guaranteed to hold. In a forcing argument (for example, to violate CH) one typically has to

- *change* the universe (by adding generic objects, in the $\neg$CH example $\aleph_2$ many reals), and
- *preserve* certain properties of the universe (e.g., we have to guarantee that ω_1 and ω_2 remain cardinals).

In the $\neg$CH example, preservation can be guaranteed by the countable chain condition (ccc). If a forcing P is ccc, then it is very "well behaved" in the sense that it preserves many properties of the universe. In particular all cardinalities are preserved.

In a (transfinite) forcing iteration (e.g., to create a world without Suslin trees) one typically has to

- *change* the universe in *successor* stages (pick a tree and kill it),
- *preserve* properties of the universe in *successor* stages (use a ccc forcing),
- make sure that *preservation* still works in *limit* stages, and
- use some *bookkeeping* (make sure that in stage ω_2 all trees are dead).

In the Suslin tree example, preservation at limits is guaranteed by the following iteration (or: preservation) theorem:

The finite support iteration of ccc forcings is ccc.

Historically [14] this was the first theorem of its kind, and it still is of central importance in forcing applications.

However, in many cases finite support iterations are not the right tool. For example, they always add Cohen reals at steps of countable cofinality. While Cohen reals are "harmless" in some respects (Cohen forcing is ccc, and in this sense well behaved), they do change the universe dramatically in some other respects. For example, a Cohen real makes the set of old reals (i.e., $2^\omega \cap V$) Lebesgue null. So if we want to preserve positivity of the old reals we cannot use finite support iterations.

The most popular alternative to finite support iterations of ccc forcings are countable support iterations of proper forcings. A forcing P is proper if (for some large regular χ) for all countable elementary submodels N of $H(\chi)$ and all $p \in P \cap N$ there is a condition $q \leq p$ forcing that G_P is N-generic.

(G_P is N-generic if for all dense P-subsets $D \in N$, $G_P \cap D \cap N$ is nonempty.)

Again, properness implies that P is well behaved in some respects (in particular, ω_1 is not collapsed). Also, we have the following central preservation theorem:

Properness is preserved under countable support iterations.

In addition to this basic theorem, there are numerous additional properties that are preserved in limit steps as well. For example, countable support iterations of proper forcings that are ω^ω-bounding (or that satisfy the Laver or Sacks property) are ω^ω-bounding again (or satisfy the Laver or Sacks property, respectively). Actually, these three properties are instances of a class which we call tools-preserving (see 7.1). All these properties are preserved under countable support iterations of proper forcings (see section 7 for details).

In this paper, we ask: Is the property

(P1) Forcing with P leaves the set of old reals Lebesgue positive preserved under countable support iterations of proper forcings? (I.e., is Hypothesis 1 true)? It seems that the answer is (consistently) no, but a full counterexample is difficult. Instead we give in section 4 a counterexample to the following more general (i.e., stronger) iteration theorem:

HYPOTHESIS 2. *Assume X has positive outer measure. Then the following property is preserved by countable support iterations of proper forcings*:

Forcing with P leaves the set X positive.

So P1 does not seem to be iterable. The Lebesgue version of tools-preserving (9.2) is an iterable property that implies P1. In this paper we show (in sections 6 and 9) that the following property P2 implies Lebesgue-tools-preservation:

(P2) ZFC proves that P is nep and satisfies P1.

So the iteration of forcings satisfying P2 satisfies Lebesgue-tools-preservation and therefore P1.

Non-elementary proper forcing (nep) has been introduced in [13]. It is a generalization of Suslin$^+$ (introduced in [2]), which in turn is a generalization of Suslin proper. For example, Cohen, random, amoeba and Hechler forcing are Suslin ccc, Mathias forcing is Suslin proper, Laver, Miller and Sacks forcing are Suslin$^+$. An introduction to transitive nep forcing and Suslin ccc ideals can be found in [7].

We will investigate not only the Lebesgue ideal, but general Suslin ccc ideals (such as the meager ideal) as well. The case of the meager ideal has already been solved by Goldstern and Shelah in [12, Lemma XVIII.3.11, p. 920].

Annotated contents.

Diagrams of implications (for the general case, as well as for meager and Lebesgue null) can be found on page 944.

Sh:828

§2. Notation and basic results. In this paper, the notion $N \prec H(\chi)$ always means that N is a *countable* elementary submodel.

Forcings are written downwards, i.e., $q < p$ means q is a stronger condition than p. Usually, the symbols for stronger conditions will be chosen lexicographically bigger than those for weaker conditions.

Names for objects in the forcing extension are usually written with a tilde below, such as $\underset{\sim}{\tau}$. The standard name for an object $x \in V$ is denoted by $\check{x}$. The name of the generic filter however (as well as a generic filter itself) will usually be called G (or, e.g., G_P if we want to stress the forcing P).

$\mathrm{ro}(Q)$ denotes the complete Boolean algebra of regular open sets of Q.

We will *fix* a Suslin ccc ideal I (Suslin ccc ideals are defined in 2.2). We will use the phrases "null", "measure 1" or "outer measure 1" for every such I, even if I is not related to a measure. This seems more intuitive than terminology such as "having outer Borel approximation 2^ω". Note that our notation does not mention the ideal I as parameter: we will say "null" instead of e.g., "I-null" (although the notion does of course depend on the ideal I used).

We will mainly be interested in the case that I is the set of Lebesgue null sets.

$\mathbb{C}$ denotes the Cohen algebra and $\mathbb{B}$ the random algebra.

Suslin ccc ideals. We assume that Q^I is a Suslin ccc forcing:

DEFINITION 2.1. A (definition for a) forcing Q is Suslin ccc, if $Q \subseteq 2^\omega$, "$x \in Q$" and "$x \leq_Q y$" are $\underset{\sim}{\Sigma}^1_1$ statements, "x and y are compatible" is Borel, and Q is ccc.

So Q^I is defined using a real parameter r_Q. A candidate is a countable transitive model of some ZFC* $\subseteq$ ZFC containing r_Q (see definition 6.3 for more details on ZFC*).

In addition, we assume that $\underset{\sim}{\eta}^I$ is a hereditarily countable name for a new real (i.e., $\Vdash_{Q^I} \underset{\sim}{\eta}^I \in {}^\omega\omega \setminus V$) such that in all candidates $\{[\![\underset{\sim}{\eta}^I(n) = m]\!], n, m \in \omega\}$ generates $\mathrm{ro}(Q^I)$. (Such a real is sometimes called "generic real".) Note that, e.g., for Cohen forcing the canonical name for the Cohen real has this property; analogously for random forcing.

A Suslin ccc ideal I is an ideal defined from a pair $(Q^I, \underset{\sim}{\eta}^I)$ as above in the following way:

DEFINITION 2.2.

- $A \in \mathrm{BC}$ means A is a Borel code.
- For $A \in \mathrm{BC}$, A^V denotes the evaluation of A in V (i.e., A^V is the Borel set corresponding to the code A).
- A Borel code A is null, or: $A \in I_{\mathrm{BC}}$, if $\Vdash_{Q^I} \underset{\sim}{\eta}^I \notin A^{V[G_{Q^I}]}$.
  A is positive, or: $A \in I^+_{\mathrm{BC}}$, if A is not null.
  A has measure 1 if the code for $2^\omega \setminus A$ is null.
- A subset X of 2^ω is null, or: $X \in I$, if for some $A \in I_{\mathrm{BC}}$, $X \subseteq A^V$.
  $X \subseteq 2^\omega$ is positive, or: $X \in I^+$, if it is not null.
  X is of measure 1 if $2^\omega \setminus X$ is null.
- For an arbitrary set N and a real $r \in 2^\omega$, r is called I-generic over N, or: $r \in \mathrm{Gen}(N)$, if $r \notin A^V$ for all $A \in I_{\mathrm{BC}} \cap N$.
  So $\mathrm{Gen}(N) = 2^\omega \setminus \bigcup\{A^V : A \in I_{\mathrm{BC}} \cap N\}$.

For example, if Q^I is the random algebra $\mathbb{B}$, then I is the ideal of Lebesgue null sets, and if Q^I is Cohen forcing $\mathbb{C}$, then I is the ideal of meager sets.

Note that the notation we use assumes that the ideal I is understood, e.g., we say "positive" instead of "positive with respect to I".

The following can be found, e.g., in [7]:

LEMMA 2.3.

- *I is a σ-complete ccc ideal containing all singletons, and* $\mathrm{ro}(Q^I) \cong$ *Borel/I (as a complete Boolean algebra).*
- *For a Borel code A, the sentences "$q \Vdash_{Q^I} \underset{\sim}{\eta}^I \in A^{V[G_{Q^I}]}$" and "$A \in I_{\mathrm{BC}}$" are $\underset{\sim}{\Delta}^1_2$. (So in particular they are absolute.)*
- *If N is countable, then* $\mathrm{Gen}(N)$ *is a Borel set of measure* 1.
- *If N is a countable elementary submodel of $H(\chi)$ and M the transitive collapse of N, then r is I-generic over N if and only if r is I-generic over M.*
- *Let M be a candidate. r is I-generic over M if and only if there is (in V) a Q^I-generic filter G over M such that $\underset{\sim}{\eta}^I[G] = r$.*
- *If M is a candidate and $q \in Q^I \cap M$, then there is a positive Borel code $B_q \in M$ such that $M \models$ "$[\![\underset{\sim}{\eta}^I \in B_q^{M[G]}]\!]_{\mathrm{ro}(Q^I)} = q$". Such a B_q satisfies*

$$\begin{aligned}\{\, \underset{\sim}{\eta}^I[G] : G \in V \text{ is } Q\text{-generic over } M \text{ and contains } q \,\}\\ = \omega^\omega \setminus \bigcup \{ A^V : A \in M, q \Vdash \underset{\sim}{\eta}^I \notin A \}\\ = \mathrm{Gen}(M) \cap B_q^V.\end{aligned}$$

For example, if we we chose Q^I to be Cohen forcing, then we get the following well known facts: The meager ideal is a σ-complete ccc ideal, $\mathrm{ro}(Q^I)$ is Borel modulo meager, for a Borel code A the statement "A is meager" is absolute, a real c is Q^I-generic over a model M if and only if it is I-generic (i.e., if it avoids all meager Borel sets of M), etc.

For any Suslin ccc ideal I there is a notion analogous to the Lebesgue outer measure. Note however that this generalized outer measure will be a Borel set, not a real number:

DEFINITION 2.4. Let X be a subset of 2^ω.

- A Borel set B is (a representant of) the outer measure of X if B is (modulo I) the smallest Borel superset of X. I.e., $B \supset X$, and for every other Borel set $B' \supset X$, $B \setminus B'$ is null.
- X has outer measure 1, if 2^ω is outer measure of X.

Instead of "$B \supset X$" we could use "$X \setminus B \in I$" in the definition of outer measure.[1]

Clearly, every X has an outer measure (unique modulo I); the outer measure of a Borel set A is A itself; the outer measure of a countable union is the union of the outer measures; etc.

If I is the Lebesgue ideal, then the outer measure of X (according to our definition) is a Borel set B containing X such that $\mathrm{Leb}(B) = \mathrm{Leb}^*(X)$, where $\mathrm{Leb}^*(X) \in \mathbb{R}$ is the outer measure according to the usual definition.

If I is the ideal of meager sets, then the outer measure of a set X is 2^ω minus the union of all clopen sets C such that $C \cap X$ is meager. (This follows from the fact

[1] That makes no difference modulo I, since every null set is contained in a Borel null set.

that every positive Borel set contains (modulo I) a clopen set and that there are only countable many clopen sets).

The random and Cohen algebras on 2^κ. We can add κ many Cohen (or random) reals "simultaneously" using the Cohen algebra $\mathbb{C}_\kappa$ (or random algebra $\mathbb{B}_\kappa$) on 2^κ. We will need these forcings only for the counterexamples 3.3 and 4.1. We briefly recall some well known facts.

Let J be any set. For $i \in J$ and $a \in \{0, 1\}$ define $[i \mapsto a] := \{ x \in 2^J : x(i) = a \}$. A basic clopen set is a finite intersection of such sets $[i \mapsto a]$. These sets form a basis of the topology, and the clopen subsets of 2^J are exactly the finite unions of basic clopen sets.

LEMMA 2.5. *Let $\mathscr{B}_J$ be the σ-algebra on 2^J generated by the (basic) clopen sets.*

- *Every $A \in \mathscr{B}_J$ depends only on a countable $J' \subseteq J$ (i.e., if $x, y \in 2^J$ is such that $x(i) = y(i)$ for all $i \in J'$, then x is in A if and only if y is in A).*
- *$(2^J, \mathscr{B}_J, \mathrm{Leb}^J)$ is a measure space, where Leb^J is the product measure.*
- *For $A \in \mathscr{B}_J$ and $\varepsilon > 0$ there is a $B \supseteq A$ such that $\mathrm{Leb}^J(B) \leq \mathrm{Leb}^J(A) + \varepsilon$ and B is a countable disjoint union of (basic) clopen sets.*

Let $\mathbb{B}_J$ (the random algebra on 2^J) be $\mathscr{B}_J$ factorized by the Leb^J-null sets. The generic filter G on $\mathbb{B}_J$ is determined by the random sequence $\underset{\sim}{r} \in 2^J$ defined by $\underset{\sim}{r}(i) = 0$ if and only if $[i \mapsto 0] \in G$.

LEMMA 2.6.

- *$\mathbb{B}_J$ is a complete ccc Boolean algebra.*
- *If J and K are disjoint, then $\mathbb{B}_{J \cup K}$ is equivalent to $\mathbb{B}_J * \mathbb{B}_K$.*
- *If J is infinite, then $\underset{\sim}{r}$ codes a sequence of $|J|$ many random reals.*
- *If J is uncountable, then $\mathbb{B}_J$ forces that the set $\underset{\sim}{X}$ of these random reals is a nowhere Lebesgue-null Sierpinski set.*[2]
- *If X is not Lebesgue null then $\mathbb{B}_J$ forces that $\check{X}$ remains positive, and if J is infinite then $\mathbb{B}_J$ forces that $V \cap 2^\omega$ is meager.*

Note that $\mathbb{B}_\kappa$ is not equivalent to either the

product—countable support iteration—finite support iteration

of κ many random forcings, since $\mathbb{B}_\kappa$ does not

add an unbounded real—make the continuum small—add a Cohen real.

A set X in a topological space is called nowhere dense if $\bar{X}^\circ = \emptyset$, and meager if it is the countable union of nowhere dense sets. $\mathbb{C}_J$ (the Cohen algebra on 2^J) is defined as the Borel algebra on 2^J factorized by the ideal of meager sets. Again, the generic filter G on $\mathbb{C}_J$ is determined by the Cohen sequence $\underset{\sim}{c} \in 2^J$, defined by $\underset{\sim}{c}(i) = 0$ if and only if $[i \mapsto 0] \in G$.

LEMMA 2.7.

- *Every Borel set B is equivalent (modulo meager) to an open set (i.e., there is an open set O such that $B \Delta O$ is meager).*
- *$\mathbb{C}_J$ is a complete ccc Boolean algebra.*
- *If J and K are disjoint, then $\mathbb{C}_{J \cup K}$ is equivalent to $\mathbb{C}_J * \mathbb{C}_K$ and $\mathbb{C}_J \times \mathbb{C}_K$.*
- *If J is infinite, then $\underset{\sim}{c}$ codes a sequence of $|J|$ many Cohen reals.*

---

[2]A set of reals is nowhere Lebesgue-null if it has nonempty intersection with every Lebesgue-positive Borel set, and it is a Sierpinski set if it is uncountable and has a countable intersection with any Borel Lebesgue-null set (such a set cannot be measurable).

- *If J is uncountable, then the $\mathbb{C}_J$ forces that the set $\underaccent{\tilde}{X}$ of these Cohen reals is a nowhere meager Luzin set.*[3]
- *If X is non-meager, then $\mathbb{C}_J$ forces that $\check{X}$ remains non-meager, and if J is infinite then $\mathbb{C}_J$ makes $V \cap 2^\omega$ Lebesgue null.*

We can represent $\mathbb{B}_J$ as well as $\mathbb{C}_J$ in a very absolute way. To be able to refer to this property in section 4, we introduce the following definition:

DEFINITION 2.8. (The definition of) a forcing Q is strongly absolute, if the following formulas are upwards absolute between V and every forcing extension of V:

$$\text{“}p \in Q\text{”}, \quad \text{“}q \leq p\text{”} \quad \text{and} \quad \text{“}A \text{ is a maximal antichain”}.$$

The requirement that "A is a maximal antichain" is upwards absolute is very strong and will usually only be satisfied by ccc forcings. For example every Suslin ccc forcing is strongly absolute, but Mathias forcing is not (although it is nicely definable and in particular Suslin proper). Note that for strongly absolute forcings "p and q are compatible" is absolute.

LEMMA 2.9. *For any J (suitable definitions of forcings equivalent to) $\mathbb{B}_J$ and $\mathbb{C}_J$ are strongly absolute.*

Quotient forcings. The following is a basic fact of forcing theory: If $f : P \to Q$ is a complete embedding, then

- Q is equivalent to $P * \underaccent{\tilde}{R}$, where $\underaccent{\tilde}{R}$ contains all $q \in Q$ that are compatible with $f(p)$ for all $p \in G_P$. So in particular
- for every Q-generic filter G_Q over V there is a P-generic filter G_P over V and an $\underaccent{\tilde}{R}[G_P]$-generic filter G_R over $V[G_P]$ such that $V[G_Q] = V[G_P][G_R]$, and
- for every P-generic filter G_P over V and every $\underaccent{\tilde}{R}[G_P]$-generic filter G_R over $V[G_P]$ there is a Q-generic filter G_Q over V such that $V[G_Q] = V[G_P][G_R]$.

Sometimes it is more convenient to use the following analogon that doesn't mention complete embeddings (which is folklore, but we do not have a reference):

LEMMA 2.10. *Let P and Q be arbitrary partial orders.*

(1) *If G_Q is Q-generic over V, and if in $V[G_Q]$ there is a P-generic filter G_P over V, then there is a forcing $R \in V[G_P]$ and an R-generic filter G_R over $V[G_P]$ such that $V[G_Q] = V[G_P][G_R]$. R can be chosen to be a subset of $\mathrm{ro}(Q)^V$ (and G_R is essentially the same as G_Q).*
(2) *Assume that Q forces that for all $p \in P$ there is a P-generic filter over V containing p. Then there is a P-name $\underaccent{\tilde}{R}$ for a subset of $\mathrm{ro}(Q)^V$ such that the following holds: If G_P is P-generic over V and G_R is $\underaccent{\tilde}{R}[G_P]$-generic over $V[G_P]$, then G_R is $\mathrm{ro}(Q)$-generic over V and $V[G_P][G_R]_{\underaccent{\tilde}{R}[G_P]} = V[G_R]_{\mathrm{ro}(Q)}$.*
(3) *Q forces: If $(2^P)^V$ is countable, then for all $p \in P$ there is a P-generic filter G_P over V containing p.*

PROOF. (1) Assume towards a contradiction that $q \in \mathrm{ro}(Q)$ forces that $\underaccent{\tilde}{G}$ is P-generic but there is no such R in $V[\underaccent{\tilde}{G}]_P$. There is a $p_0 \in P$ such that $q \not\Vdash (p \notin \underaccent{\tilde}{G})$ for all $p \leq p_0$. (Otherwise the set $D := \{ p \in P : q \Vdash (p \notin \underaccent{\tilde}{G}) \}$ is dense, so q forces that there is a $p \in \underaccent{\tilde}{G} \cap D$.) In particular the truth value $q_0 := [\![p_0 \in \underaccent{\tilde}{G}]\!] \wedge q$ is positive. There is a complete embedding f from $P_{\leq p_0}$ to $\mathrm{ro}(Q)_{\leq q_0}$ (just set $f(p) := [\![p \in \underaccent{\tilde}{G}]\!]$).

[3]A set of reals is nowhere meager if it has nonempty intersection with every non-meager Borel set, and it is Luzin if it is uncountable and has countable intersection with every meager Borel set.

So $\mathrm{ro}(Q)_{\leq q_0}$ can be factorized as $P_{\leq p_0} * \underset{\sim}{R}$, where $\underset{\sim}{R}$ is the P-name for the set of all $q \in \mathrm{ro}(Q)_{\leq q_0}$ such that $[\![p \in \underset{\sim}{G}]\!] \wedge q \neq 0$ for all $p \in G_P$. If G_P is $P_{\leq p_0}$-generic over V and G_R is $\underset{\sim}{R}[G_P]$-generic over $V[G_P]$, then G_R is $\mathrm{ro}(Q)_{\leq q_0}$-generic over V, and therefore $V[G_R]_Q$ is a Q-generic extension by a filter containing q.

In $V[G_R]$, $\underset{\sim}{G}[G_R] = G_P$:

$G_R \subseteq \underset{\sim}{R}[G_P]$, so $[\![p \in \underset{\sim}{G}]\!] \wedge q \neq 0$ for every $q \in G_R$.
If $p \notin \underset{\sim}{G}[G_R]$ then $[\![p \in \underset{\sim}{G}]\!] \wedge q = 0$ for some $q \in G_R$, so $p \notin G_P$.
If $p \in \underset{\sim}{G}[G_R]$ then $p' \notin \underset{\sim}{G}[G_R]$ for all $p' \perp p$, so $p' \notin G_P$ for all $p' \perp p$, and $p \in G_P$.

So in $V[\underset{\sim}{G}]$ there is an R as required after all and we get a contradiction.

(2) For $p \in P$ pick a $\mathrm{ro}(Q)$-name $\underset{\sim}{G}$ for a P-generic filter containing p. Then there is a $p' \leq p$ such that $\nVdash (p'' \notin \underset{\sim}{G})$ for all $p'' \leq p'$ (as in the proof of (1)). Choose a maximal antichain $A \subseteq P$ of such p' and call the associated names for filters $\underset{\sim}{G}^{p'}$. For $a \in A$ set $q^a := [\![a \in \underset{\sim}{G}^a]\!]$. So $q^a \nVdash (p'' \notin \underset{\sim}{G}^a)$ for all $p'' \leq a$.

Let $\underset{\sim}{R}$ be the following P-name: If $G_P \cap A = \{a\}$ then let $\underset{\sim}{R}$ consist of those $q \in \mathrm{ro}(Q)_{\leq q^a}$ such that $[\![p \in \underset{\sim}{G}^a]\!] \wedge q \neq 0$ for all $p \in G_P$.

Assume that G_P is P-generic and that $G_P \cap A = \{a\}$. Then $V[G_P]$ is an extension by $P_{\leq a}$, and $\underset{\sim}{R}[G_P]$ is the quotient of the complete embedding $f : P_{\leq a} \to \mathrm{ro}(Q)_{\leq q^a}$. So every $\underset{\sim}{R}[G_P]$-generic G_R over $V[G_P]$ is $\mathrm{ro}(Q)_{\leq q^a}$-generic (and therefore Q-generic) over V.

(3) If only countably many subsets of P are in V, then we can start with any p and can construct a decreasing sequence of length ω meeting all these dense sets. ⊣

§3. Preservation.

Recall that we have fixed a Suslin ccc ideal I and the corresponding notions of positivity.

DEFINITION 3.1. Let $X \subseteq 2^\omega$ be positive with outer measure B, and P a forcing.

- P preserves positivity of X if $\Vdash_P \check{X} \in I^+$.
- P preserves Borel positivity if P preserves the positivity of A^V for all positive Borel codes A (i.e., $\Vdash_P A^V \in I^+$).
- P preserves positivity if P preserves the positivity of X for all positive X.
- P preserves outer measure of X if $\Vdash_P (B^{V[G]}$ is outer measure of $X)$.
- P preserves Borel outer measure if P preserves the outer measure of A^V for all Borel codes A (i.e., $\Vdash_P A^{V[G]}$ is outer measure of A^V).
- P preserves outer measure if P preserves the outer measure of X for all X.

Of special interest is preservation of positivity (or outer measure) of 2^ω (we will also say: "of V"), i.e., of the set of all old reals.

We have already mentioned the following: If I is the ideal of Lebesgue null sets, then the random algebra $\mathbb{B}$ preserves positivity, and the Cohen algebra $\mathbb{C}$ does not preserve positivity. Dually, if I is the ideal of meager sets, then the Cohen algebra $\mathbb{C}$ preserves positivity, and the random algebra $\mathbb{B}$ does not preserve positivity.

It is clear that preserving outer measure of X implies preserving positivity of X (since being null is absolute for Borel codes, and the outer measure of X is a null set if and only if X is null).

Preserving the outer measure of V is equivalent to preserving Borel outer measure: Let A be a Borel set in V. Then in $V[G]$, the outer measure of $X := 2^\omega \cap V$ is the disjoint union of the outer measure of $X \cap A^{V[G]} = A^V$ and the outer measure of $X \setminus A^{V[G]} = (2^\omega \setminus A)^V$. So if the outer measure of A decreases, then the outer measure of V decreases.

So another way to characterize Borel outer measure preserving is:

"No positive Borel set disjoint to V is added".

If in every forcing extension of V the set of old reals $2^\omega \cap V$ has either outer measure 0 or 1 then clearly preservation of positivity of V implies preservation of Borel outer measure. Note that this is the case (for any P) if I is either the Lebesgue null or the meager ideal.

Other than for outer measure, positivity preservation of V and of all Borel sets is not equivalent. A trivial counterexample is the following:

Set $B_0 := \{x \in 2^\omega : x(0) = 0\}$, $B_1 := 2^\omega \setminus B_0$. Let Q^I add a $\underset{\sim}{\eta}^I \in 2^\omega$ such that either $\underset{\sim}{\eta}^I \in B_0$ and $\underset{\sim}{\eta}^I$ is random or $\underset{\sim}{\eta}^I \in B_1$ and $\underset{\sim}{\eta}^I$ is Cohen. $\Vdash_{Q^I} \underset{\sim}{\eta}^I \notin B$ if and only if $\Vdash_{Q^I} \underset{\sim}{\eta}^I \notin B \cap B_0$ and $\Vdash_{Q^I} \underset{\sim}{\eta}^I \notin B \cap B_1$, i.e., if and only if $B \cap B_0$ is Lebesgue null and $B \cap B_1$ is meager. In particular, B_0 and B_1 are positive Borel sets. So $\mathbb{C}$ forces that B_0^V is null and that B_1^V remains positive. Therefore $\mathbb{C}$ preserves positivity of V, but not of Borel sets.

However, preservation of positivity of V does imply Borel positivity preservation if additional requirements are satisfied, for example once again if we know that the outer measure of V in $V[G_P]$ is either 0 or 1. Another sufficient condition is the following (which also is satisfied in case that I is Lebesgue null or meager, for any P):

LEMMA 3.2. *Assume that P preserves positivity of V, and that for every $A, B \in I^+_{\mathrm{BC}}$ there is an $A' \in I^+_{\mathrm{BC}}$ and a Borel (definition of a) function $f : A' \to B$ such that $A' \subseteq A$ and P forces that for all null sets $X \subset B$, $f^{-1}(X)$ is null. Then P preserves positivity of Borel sets.*

PROOF (from [13]). Assume that G_P is P-generic over V and that in $V[G_P]$, B^V is null. In V, let X be a maximal family of positive Borel sets such that for every $A' \in X$ there is a $f_{A'} : A' \to B$ as in the assumption and such that for $A' \neq A'' \in X$, $A' \cap A'' \in I$. X is countable and its union is 2^ω (modulo I). In $V[G_P]$, $A' \cap V \subseteq f^{-1}_{A'}(B \cap V)$ is null for each $A' \in X$. So $2^\omega \cap V = \bigcup_{A' \in X}(A' \cap V)$ is null. ⊣

Borel positivity (or outer measure) preserving generally (consistently) does not imply positivity preserving, not even for Cohen or random.

The standard counterexample is the following:

EXAMPLE 3.3. Assume I is the Lebesgue null ideal and R is $\mathbb{B}_{\omega_1}$. (Or I is the meager ideal and R is $\mathbb{C}_{\omega_1}$.) Let G_R be R-generic over V. Then in $V[G_R]$, $X := V \cap 2^\omega$ is positive and there is a ccc forcing P that preserves Borel outer measure but destroys the positivity of X.

PROOF. We assume that I is meager (the Lebesgue case is analog). Note that in both cases, it is enough to show that P preserves positivity of $(2^\omega)^{V[G_R]}$ (this implies preservation of Borel outer measure).

Assume r is $\mathbb{B}$-generic over V, and $(c_i)_{i \in \omega_1}$ is $\mathbb{C}^{V[r]}_{\omega_1}$-generic over $V[r]$. Then $(c_i)_{i \in \omega_1}$ is $\mathbb{C}^V_{\omega_1}$-generic over V as well. So $\mathbb{B} * \mathbb{C}_{\omega_1}$ can be factored as $\mathbb{C}_{\omega_1} * \underset{\sim}{P}$.

where P is a $\mathbb{C}_{\omega_1}$-name for a ccc forcing. Set $V' := V[(c_i)_{i\in\omega_1}]$, and let G_P be the corresponding P-generic filter over V'.

In V', $X = V \cap 2^\omega$ is not meager, but in $V'[G_P] = V[r][(c_i)_{i\in\omega_1}]$ it clearly is (cf. 2.6 and 2.7). So in V', P does not preserve positivity.

On the other hand, in $V'[G_P] = V[r][(c_i)_{i\in\omega_1}]$ the set $Y := \{ c_i : i \in \omega_1 \} \subseteq V'$ is a Luzin set, in particular non-meager (cf. 2.7). So in V', P preserves positivity of $(2^\omega)^{V'}$. ⊣

However, if P is (absolutely) Borel positivity preserving *and* nep (for example Suslin proper), then positivity preserving *does* follow, see Theorem 6.1.

Note that in any case, preservation of positivity (or outer measure) is trivially preserved by composition of forcings (or equivalently: in successor steps of iterations). In this paper we investigate what happens at limit stages.

We will restrict ourselves to countable support iterations. Note that for example for finite support iterations, in all limit steps of countable cofinalities Cohen reals are added, so preservation of Lebesgue positivity is never preserved in finite support iterations.

Preservation of positivity is connected to preservation of generics (e.g., random reals) over models:

LEMMA 3.4. *If P is proper and X is positive, then the following are equivalent:*

(1) *P preserves the positivity of X.*
(2) *for all $N \prec H(\chi)$ and $p \in P \cap N$ there is an $\eta \in X$ and an N-generic $q \leq p$ forcing that η is I-generic over $N[G]$.*
(3) *for all $p \in P$ there are unbounded (in 2^ω) many $N \prec H(\chi)$ containing p such that there is an $\eta \in X$ and an N-generic $q \leq p$ forcing that η is I-generic over $N[G]$.*

Here, $A \subseteq \{N \prec H(\chi)\}$ is called unbounded in 2^ω, if for every $x \in 2^\omega$ there is a $N \in A$ such that $x \in N$ (or equivalently, if for all $y \subseteq 2^\omega$ countable there is an $N \in A$ such that $y \subseteq N$).

PROOF. (1) → (2): Assume that $N \prec H(\chi)$, that $q_0 \leq p$ is N-generic, and that G_P is P-generic over V and contains q_0. In $V[G_P]$, $\mathrm{Gen}(N[G_P])$ is a measure 1 set, and X is positive, so $\mathrm{Gen}(N[G_P]) \cap X$ is nonempty. This is forced by some $q \leq q_0$ in G_P.

(2) → (3) is clear.

(3) → (1): Assume p forces that X is null, i.e., that $X \subseteq \underaccent{\tilde}{A}^{V[G_P]}$ for some Borel null code $\underaccent{\tilde}{A}$. According to (3), there is an $N \prec H(\chi)$ containing p and $\underaccent{\tilde}{A}$, and there are $\eta \in X$ and an N-generic $q \leq p$ forcing that η is I-generic over $N[G]$.

If G_P is P-generic over V and contains q, then G_P is P-generic over N as well, and $\underaccent{\tilde}{A}[G_P]$ is a Borel null code in $N[G_P]$. In $V[G_P]$, η is I-generic over $N[G_P]$, so $\eta \notin \underaccent{\tilde}{A}[G_P]^{V[G_P]} \supseteq X$, a contradiction. ⊣

LEMMA 3.5. *If P is proper, then the following are equivalent:*

(1) *P preserves positivity.*
(2) *For all $N \prec H(\chi)$, there is a set A of measure 1 such that for all $p \in N$ and $\eta \in A$ there is an N-generic $q \leq p$ forcing that η is I-generic over $N[G]$.*
(3) *For all p there are unbounded (in 2^ω) many $N \prec H(\chi)$ containing p such that for some measure 1 set A and for all $\eta \in A$ there is an N-generic $q \leq p$ forcing that η is I-generic over $N[G]$.*

PROOF. (1) $\rightarrow$ (2): Since there are only countable many p's in N, it is enough to show that for all $N \prec H(\chi)$ and all $p \in P \cap N$ there is a set A as in (2). Let X be the set of exceptions, i.e., $\eta \in X$ if and only if every N-generic $q \leq p$ forces that η is not I-generic over $N[G]$. We have to show that X is a null set. Otherwise (according to Lemma 3.4) there is an $\eta \in X$ and an N-generic $q \leq p$ forcing that $\eta \in \text{Gen}(N[G])$, a contradiction.

(2) $\rightarrow$ (3) is clear, and (3) $\rightarrow$ (1) follows from Lemma 3.4. $\dashv$

Why are we interested in preservation of I-generics over models instead of preservation of positivity? It is not clear how the iterability of preservation of positivity can be shown directly. On the other hand, in some important cases it turns out that preservation of generics is iterable (e.g., if I is meager, see section 8, or if I is Lebesgue null under additional assumptions, see section 9). However, to be able to apply the according iteration theorems, we will generally need that *all* I-generics are preserved, not just a measure 1 set of them (as in Lemma 3.5).

It seems that preservation of all I-generics really is necessary, more specifically that the statement

> "preservation of Lebesgue positivity is preserved in proper countable support iterations"

(and the analog statement for meager) is (consistently) false. A counterexample seems to be difficult, but we can give a counterexample to the following (stronger) statement: Let X be a positive set. Then

> "preservation of positivity of X is preserved under proper countable support iterations".

§4. A counterexample.

EXAMPLE 4.1. Assume that I is the Lebesgue null ideal and that R is $\mathbb{B}_{\omega_1}$. (Or I is the meager ideal and R is $\mathbb{C}_{\omega_1}$.) Then R forces the following: There is a positive set X and a forcing iteration $(P_n, \underset{\sim}{Q}_n)_{n<\omega}$ of ccc forcings such that P_n forces that X remains positive for each $n \in \omega$, but P_ω makes X null (regardless of the kind of limit we use for P_ω).[4]

So if I is the Lebesgue ideal, and we let $(P_n, \underset{\sim}{Q}_n)_{n<\omega}$ be a countable support iteration (i.e., P_ω is the full or inverse limit), then we get the counterexample to

> "preservation of positivity of X is preserved under proper countable support iterations".

If we take the direct limit (i.e., if we interpret $(P_n, \underset{\sim}{Q}_n)_{n<\omega}$ as finite support iteration), then we get nothing new, since we already know that P_ω adds a Cohen real and therefore destroys Lebesgue positivity.

If I is the meager ideal, then the counterexample is interesting for both finite and countable support iteration.

The simplest idea for a counterexample of this kind is the following: Let P be the countable support iteration $\mathbb{B}_{\omega_1} * \mathbb{C} * \mathbb{B}_{\omega_1} * \mathbb{C} * \cdots$.

---

[4]This means that X will be null in any forcing extension V' of $V[G_R]$ that contains P_n-generic filters G_n (over $V[G_R]$) for all n.

So the set X of the $\omega \times \omega_1$ many random reals added by P is null (since the first $n \times \omega_1$ many are made null by the nth Cohen).

Now factor P the following way: First add all the randoms, then the first (former) Cohen, the second, the third, etc. (these reals are not Cohen anymore, of course). One would expect that the first former Cohen will make only the first ω_1 many randoms null, the second only the next ω_1 many, etc. So the set of all randoms will become null only in the limit.

However, it is not clear how to show that this idea actually works, and we will use a slightly different construction:

PROOF OF 4.1. Let I be the ideal of Lebesgue null sets, let J be a set of size $\aleph_1$, and let $R = P_0$ be $\mathbb{B}_J$, the random algebra on 2^J. (Again, the proof for the meager case is analog.)

So we have to construct a directed system $(P_n)_{n<\omega}$ with commuting complete embeddings $f_{n,m}\colon P_n \to P_m$ satisfying the following:

- $P_0 := \mathbb{B}_J$ adds a set X of $\aleph_1$ many random reals.
- P_n is ccc and forces that X is Lebesgue positive (for any n).
- Any limit P_ω makes X null.

Write J as the disjoint countable union of sets J_n of size $\aleph_1$, i.e., $J = \bigcup_{n\in\omega} J_n$, $|J_n| = \omega_1$. Set $J_{\geq n} := \bigcup_{m\geq n} J_m$, and let $\underset{\sim}{r}_0 \in 2^J$ be the random sequence added by P_0.

P_1 is the forcing that adds a Cohen real between the first ω_1 random reals and the rest, i.e.,

$$P_1 := \mathbb{B}_{J_0} * \mathbb{C} * \mathbb{B}_{J_{\geq 1}}.$$

Generally, we define

$$P_n := \mathbb{B}_{J_0} * \mathbb{C} * \cdots * \mathbb{B}_{J_{n-1}} * \mathbb{C} * \mathbb{B}_{J_{\geq n}}.$$

We define the P_n-name $\underset{\sim}{r}_n \in 2^J$ to be the concatenation of the random sequences for all the random algebras used in P_n.

LEMMA 4.2. *There is a complete embedding $f_{n,n+1}$ from P_n to P_{n+1} which leaves the random sequence invariant.*[5]

Assuming this lemma, the rest of the proof is straightforward:

As usual, we interpret $\underset{\sim}{r} \restriction J_n$ as a sequence of ($\aleph_1$ many) random reals. Let $\underset{\sim}{X}_n$ be the set of these reals. Set $\underset{\sim}{X} := \bigcup_{n\in\omega} \underset{\sim}{X}_n$; $\underset{\sim}{X}_{<n} := \underset{\sim}{X}_1 \cup \underset{\sim}{X}_2 \cup \cdots \cup \underset{\sim}{X}_{n-1}$; and $\underset{\sim}{X}_{\geq n} := \underset{\sim}{X} \setminus \underset{\sim}{X}_{<n}$. P_n forces that $\underset{\sim}{X}_{\geq n}$ is a Sierpinski set (in particular positive), and that $\underset{\sim}{X}_{<n}$ is null, and we are done. ⊣

For the proof of Lemma 4.2, we will need the following fact:

LEMMA 4.3. *Assume that $f_0\colon P \to Q$ is a complete embedding and that P forces $\underset{\sim}{S}$ to be strongly absolute. Then f_0 can be extended to a complete embedding $f_1\colon P * \underset{\sim}{S}^{V[G_P]} \to Q * \underset{\sim}{S}^{V[G_Q]}$ defined by $f_1((p,\underset{\sim}{\tau})) = (f_0(p), f_0^*\underset{\sim}{\tau})$.*[6]

---

[5]This means the following: Let G_{n+1} be P_{n+1}-generic over V, and set $G_n := f^{-1}_{n,n+1}(G_{n+1})$ (which is P_n-generic over V, since $f_{n,n+1}$ is complete). Then $\underset{\sim}{r}_n[G_n]_{P_n} = \underset{\sim}{r}_{n+1}[G_{n+1}]_{P_{n+1}}$.

[6]f_0^* is the following mapping from P-names onto Q-names: $f_0\colon P \to Q$ is complete. So if G_Q is Q-generic over V, then $G_P := f_0^{-1}[G_Q]$ is P-generic over V. So for every P-name $\underset{\sim}{\tau}$ there is a Q-name $f_0^*\underset{\sim}{\tau}$ such that $f_0^*\underset{\sim}{\tau}[G_Q]_Q = \underset{\sim}{\tau}[G_P]_P$. ($f_0^*$ can also be defined recursively over the rank of the names.)

For the definition of strongly absolute, see 2.8. In particular $\underset{\sim}{S}$ is a forcing definition using arbitrary P-names as parameters, and $\underset{\sim}{S}^V$ is its "evaluation" in the universe V. Note that $\mathbb{B}_J$ and $\mathbb{C}_J$ are strongly absolute (see 2.9).[7]

PROOF OF LEMMA 4.3. Assume $(p', \underset{\sim}{\tau}') \leq (p, \underset{\sim}{\tau})$. Then $f_0(p') \leq f_0(p)$, and $f_0(p')$ forces that $p' \in G_P$ (by definition of G_P). So if G_Q is Q-generic over V and contains $f_0(p')$, then in $V[G_P]$ we have $\underset{\sim}{\tau}'[G_P] \leq \underset{\sim}{\tau}[G_P]$ (since this is forced by p'). Since "$q \in \underset{\sim}{S}$" and "$q' \leq q$" are upwards absolute between $V[G_P]$ and $V[G_Q]$, in $V[G_Q]$

$$f_0^* \underset{\sim}{\tau}'[G_Q] = \underset{\sim}{\tau}'[G_P] \leq \underset{\sim}{\tau}[G_P] = f_0^* \underset{\sim}{\tau}[G_Q].$$

This is forced by $f_0(p')$, so $(f_0(p'), f_0^* \underset{\sim}{\tau}') \leq (f_0(p), f_0^* \underset{\sim}{\tau})$.

A similar argument shows that f_1 preserves incompatibility.

Finally assume towards a contradiction that $\tilde{A} \subseteq P * \underset{\sim}{S}^{V[G_P]}$ is predense, and that there is a $(q, \sigma) \in Q * \underset{\sim}{S}^{V[G_Q]}$ incompatible to all $f_1((p, \underset{\sim}{\tau}))$ for $(p, \underset{\sim}{\tau}) \in \tilde{A}$. Let G_Q be a Q-generic filter over V containing q. Since G_P is P-generic, the set

$$A := \{ \underset{\sim}{\tau}[G_P] : (p, \underset{\sim}{\tau}) \in \tilde{A}, p \in G_P \}$$

is (in $V[G_P]$) a predense subset of $S^{V[G_P]}$. Since $\underset{\sim}{S}$ is strongly absolute, A is (in $V[G_Q]$) a predense subset of $S^{V[G_Q]}$. So there is a $(p, \underset{\sim}{\tau}) \in \tilde{A}$ with $p \in G_P$ such that $\underset{\sim}{\sigma}[G_Q]$ is compatible with $\underset{\sim}{\tau}[G_P] = f_0^* \underset{\sim}{\tau}[G_Q]$. Now pick a $q' \leq q \in Q$ forcing that $\underset{\sim}{\sigma}'[G_Q] \in S^{V[G_Q]}$ is a witness for this compatibility. Then $(q', \underset{\sim}{\sigma}') \leq f_1((p, \underset{\sim}{\tau})), (q, \underset{\sim}{\sigma})$, and we get a contradiction. ⊣

PROOF OF LEMMA 4.2. First note that $\mathbb{B}_{J_{\geq n}}$ is equivalent to $\mathbb{B}_{J_n} * \mathbb{B}_{J_{\geq n+1}}$. So we have to find a complete embedding from

$$P_n = \overbrace{\mathbb{B}_{J_0} * \mathbb{C} * \cdots * \mathbb{C} * \mathbb{B}_{J_{n-1}} * \mathbb{C} * \mathbb{B}_{J_n} *}^{P} \quad \mathbb{B}_{J_{\geq n+1}}$$

to

$$P_{n+1} = \underbrace{\mathbb{B}_{J_0} * \mathbb{C} * \cdots * \mathbb{C} * \mathbb{B}_{J_{n-1}} * \mathbb{C} * \mathbb{B}_{J_n} * \mathbb{C}}_{Q} * \mathbb{B}_{J_{\geq n+1}}.$$

It is clear that the identity (lets call it f_0) is a complete embedding between P and Q (the two blocks marked above). (Generally, for all R, $\underset{\sim}{S}$ the identity is a complete embedding from R into $R * \underset{\sim}{S}$.) Therefore we can apply Lemma 4.3 to get a complete embedding $f_1 : P_n \to P_{n+1}$. It is clear that f_1 leaves the random J-sequence invariant. ⊣

§5. True preservation.

Preservation of all generics (not just a measure-1-set of them) is closely related to preserving "true positivity", a notion using the stationary ideal on $[\kappa]^{\aleph_0}$.

[7]You should not be confused by the following fact: If f_0 is a complete embedding of P into Q and G_Q is a Q-generic filter over V and G_P the corresponding P-generic filter, then in $V[G_Q]$ the partial order $S^{V[G_P]}$ generally can *not* be completely embedded into $S^{V[G_Q]}$. For example $\mathbb{B} \times \mathbb{B}$ adds an unbounded real. So if P is the trivial partial order and Q and $\underset{\sim}{S}$ are both $\mathbb{B}$, then in $V[G_Q]$ there cannot be a complete embedding of $\underset{\sim}{S}^{V[G_P]} = \mathbb{B}^V$ into $\underset{\sim}{S}^{V[G_Q]} = \mathbb{B}^{V[G_Q]}$. However the f_1 defined in the lemma clearly *is* a complete embedding from $P * \underset{\sim}{S}^{V[G_P]} = \mathbb{B}^V$ into $Q * S^{V[G_Q]} = \mathbb{B} * \mathbb{B}^{V[G_Q]}$ (since a random real over $V[G_Q]$ is random over V as well).

Sh:828

From this section only Definition 5.9 is needed for the proof of main result 9.4.[8]

DEFINITION 5.1. Let $\mathcal{I}$ be arbitrary and $C \subseteq [\mathcal{I}]^{\aleph_0}$ a family of countable subsets of $\mathcal{I}$.

- C is called unbounded, if for all $A \in [\mathcal{I}]^{\aleph_0}$ there is a $B \in C$ such that $B \supseteq A$.
- C is a club set (or: club), if C is unbounded and closed under increasing countable unions.
- The club filter is the family of subsets of $[\mathcal{I}]^{\aleph_0}$ containing a club set.
- A set $S \subseteq [\mathcal{I}]^{\aleph_0}$ is stationary if every club set C meets S, i.e., $C \cap S \neq \emptyset$. (Or equivalently, if the complement of S, $[\mathcal{I}]^{\aleph_0} \setminus S$, is not in the club filter.)

First we recall some basic facts:

LEMMA 5.2. *Let $\mathcal{I}$ and $\mathcal{H}_1 \subseteq \mathcal{H}_2$ be arbitrary.*

(1) *(Jech) The club filter on $[\mathcal{I}]^{\aleph_0}$ is closed under countable intersections.*
(2) *(Menas) $C \subseteq [\mathcal{I}]^{\aleph_0}$ contains a club if and only if there is an $f : [\mathcal{I}]^2 \to [\mathcal{I}]^{\aleph_0}$ such that $C(f) \subseteq C$, where*
$$C(f) := \{\, x \in [\mathcal{I}]^{\aleph_0} : (\forall i \neq j \in x)\ f(\{i, j\}) \subseteq x \,\}.$$
(3) *If $C \subseteq [\mathcal{H}_1]^{\aleph_0}$ is club, then $C^{\mathcal{H}_2} := \{\, B \in [\mathcal{H}_2]^{\aleph_0} : B \cap \mathcal{H}_1 \in C \,\}$ is club.*
(4) *If $C \subseteq [\mathcal{H}_2]^{\aleph_0}$ is club, then $C^{\mathcal{H}_1} := \{\, B \cap \mathcal{H}_1 : B \in C \,\}$ contains a club.*
(5) *If $C_0 \subseteq [\mathcal{I}]^{\aleph_0}$ is club and P an arbitrary forcing, then there is a $C_1 \subseteq C_0$ club and a name $\underaccent{\tilde}{C}'$ such that P forces that $\underaccent{\tilde}{C}' \subseteq [\mathcal{I}]^{\aleph_0}$ is club and that $\underaccent{\tilde}{C} \cap V = C_1$.*
(6) *(Shelah) A forcing P is proper if and only if for arbitrary $\mathcal{I}$ and $S \subseteq [\mathcal{I}]^{\aleph_0}$ stationary, P forces that S remains stationary.*
(7) *The set of countable elementary submodels of $H(\chi)$ contains a club of $[H(\chi)]^{\aleph_0}$.*
(8) *Assume $[\mathcal{I}]^{\aleph_0} \in H(\chi)$. Then the following are equivalent:*
    - *$C \subseteq [\mathcal{I}]^{\aleph_0}$ contains a club.*
    - *For all $N \prec H(\chi)$ containing $\mathcal{I}$ and C, $N \cap \mathcal{I} \in C$.*
    - *For club many $N \prec H(\chi)$, $N \cap \mathcal{I} \in C$.*
(9) *Assume $[\mathcal{I}]^{\aleph_0} \in H(\chi)$. Then the following are equivalent:*
    - *$S \subseteq [\mathcal{I}]^{\aleph_0}$ is stationary.*
    - *There is an $N \prec H(\chi)$ containing $\mathcal{I}$ and S such that $N \cap \mathcal{I} \in S$.*
    - *For stationary many $N \prec H(\chi)$, $N \cap \mathcal{I} \in S$.*

Note that if C is club in V, then generally C will not be club any more in a forcing extension $V[G_P]$, even if P is proper.

PROOF. We refer to Kanamori's *Higher Infinite* [6] or Jech's *Millennium Edition* [5]. The proof of (1) is straightforward (see 25.2 or 8.22). (2) is proven in [6, 25.3] or [5, 8.26]. (3) is trivial. (4) and (5) follow from (2) (for the latter, set $C_1 := C(f)^V$ and $\underaccent{\tilde}{C} := C(f)^{V[G_P]}$).

For (6) see, e.g., *Proper and improper forcing* [12].

For (7), consider the family of countable subsets of $H(\chi)$ closed under some fixed Skolem function.

[8]This proof only needs implication (3) → (1) of Lemma 9.3, i.e., the fact that a strongly preserving forcing is tools-preserving. However we do use the notion of true preservation to show implication (3) → (1) of 5.11, which in turn is used in the proof that Lebesgue-tools-preservation is *equivalent* to strong preservation.

(8): If $C \in N$ is club, then clearly $\mathcal{J} \cap N \in C$. If $\tilde{C} \subseteq \{N \prec H(\chi)\} \subset [H(\chi)]^{\aleph_0}$ is club, then $\tilde{C}^{\mathcal{J}} = \{ N \cap \mathcal{J} : N \in \tilde{C} \} \subseteq [\mathcal{J}]^{\aleph_0}$ contains a club according to (4).

(9) follows directly from (8), since S is stationary if and only if $[\mathcal{J}]^{\aleph_0} \setminus S$ does not contain a club. ⊣

Assume $\mathcal{J}$ is an arbitrary index-set, $S \subseteq [\mathcal{J}]^{\aleph_0}$ is stationary and $\bar{\eta} = (\eta_s : s \in S)$ is a sequence of reals. Pick any $\mathcal{H} \supset \mathcal{J} \cup 2^\omega$ (think of $\mathcal{H}$ to be a $H(\chi)$). For $C \subseteq [\mathcal{H}]^{\aleph_0}$, we define[9]

$$S(C) := \{ s \in S : \exists N \in C : \ N \cap \mathcal{J} = s \;\&\; \eta_s \in \mathrm{Gen}(N) \}, \quad \text{and}$$
$$\bar{\eta}(C) := \{ \eta_s : s \in S(C) \}.$$

So we get $S(C)$ the following way: Take an $N \in C$ (which will be a countable elementary submodel of $H(\chi)$), and let s be the intersection of N with $\mathcal{J}$ (so s is a countable subset of $\mathcal{J}$). If s is an element of S, and if η_s is I-generic over N, then put s into $S(C)$.

Definition 5.3. Assume $\mathcal{H} \supset \mathcal{J} \cup 2^\omega$.

- $\bar{\eta}$ is truly positive, if $\bar{\eta}(C)$ is positive for every club set $C \subseteq [\mathcal{H}]^{\aleph_0}$.
- B is the true outer measure of $\bar{\eta}$, if it is the smallest Borel set containing any of the $\bar{\eta}(C)$, i.e., if the following holds: B is Borel, there is a $C \subseteq [\mathcal{H}]^{\aleph_0}$ club such that $\bar{\eta}(C) \subseteq B$, and for no club $C' \subseteq [\mathcal{H}]^{\aleph_0}$ there is a Borel B' such that $\bar{\eta}(C') \subseteq B'$ and $B \setminus B' \notin I$.

Lemma 5.4.

(1) *The above notions do not depend on $\mathcal{H}$ (provided that $\mathcal{H} \supset \mathcal{J} \cup 2^\omega$).*
(2) *The true outer measure always exists.*
(3) *The following are equivalent:*
  - *$\bar{\eta}$ is truly positive.*
  - *$\bar{\eta}(C) \neq \emptyset$ for every club set $C \subseteq [\mathcal{H}]^{\aleph_0}$.*
  - *for all $x \in H(\chi)$ there is an $N \prec H(\chi)$ containing $x, \mathcal{J}, S$ and $\bar{\eta}$ such that $N \cap \mathcal{J} = s \in S$ and $\eta_s \in \mathrm{Gen}(N)$.*

Proof. (1) Assume that $\mathcal{J} \cup 2^\omega \subseteq \mathcal{H}_1 \subseteq \mathcal{H}_2$ and that $C \subseteq [\mathcal{H}_1]^{\aleph_0}$ is club.

By definition $s \in S(C)$ if and only if for some $N \in C$, $s = N \cap \mathcal{J} \in S$ and $\eta_s \in \mathrm{Gen}(N)$.

In particular $s \in S(C^{\mathcal{H}_2})$ if and only if for some $N' \in [\mathcal{H}_2]^{\aleph_0}$, $N := N' \cap \mathcal{H}_1$ is in C, $s = N' \cap \mathcal{J}$ is in S and $\eta_s \in \mathrm{Gen}(N')$.

So since N and N' contain the same elements of $\mathcal{J}$ and 2^ω, $S(C) = S(C^{\mathcal{H}_2})$.

The same argument works with $C \subseteq [\mathcal{H}_2]^{\aleph_0}$ and $C^{\mathcal{H}_1}$.

For general $\mathcal{H}_1, \mathcal{H}_2$, apply the argument to the pairs $\mathcal{H}_1, \mathcal{H}_1 \cup \mathcal{H}_2$ and $\mathcal{H}_2, \mathcal{H}_1 \cup \mathcal{H}_2$.

(2) The family $\{ \bar{\eta}(C) : C \text{ club} \}$ is semi-closed under countable intersections: If $(C_i)_{i \in \omega}$ is a countable sequence of club sets, and $C' := \bigcap C_i$ its intersection, then C' is club, and $\bar{\eta}(C') \subseteq \bigcap \bar{\eta}(C_i)$.

Let X be the family of Borel sets B such that for some club set C, $B \supset \bar{\eta}(C)$. So X is closed under countable intersections. Therefore X contains a minimal element (modulo I), since I is a ccc-ideal.

[9] $\mathrm{Gen}(N)$ was defined in 2.2.

(3) Without loss of generality $\mathscr{H} = H(\chi)$. Assume $\bar{\eta}$ is not truly positive. Then for some club set C and Borel null set B, $\bar{\eta}(C) \subseteq B$. Set

$$C' := \{ N \prec H(\chi) : N \in C, B \in N \}.$$

C' is club. For any $N \in C'$ and any I-generic η over N, η is not in B. So $\bar{\eta}(C') \subseteq 2^\omega \setminus B$. But $\bar{\eta}(C') \subseteq \bar{\eta}(C) \subseteq B$, so $\bar{\eta}(C') = \emptyset$. The rest is similar. ⊣

DEFINITION 5.5. Let P be a forcing.

- P preserves true positivity if for all $\bar{\eta}$ truly positive, P forces that $\bar{\eta}$ remains truly positive.
- P preserves true outer measure, if for any $\bar{\eta}$ with true outer measure A^V, P forces that $A^{V[G_P]}$ remains the true outer measure of $\bar{\eta}$.

LEMMA 5.6.

(1) *If P is true outer measure preserving, then it is true positivity preserving.*
(2) *If P is true positivity preserving, then it is proper and positivity preserving.*
(3) *If P is true outer measure preserving, then it is outer measure preserving.*

It seems that true positivity preserving generally does not imply true outer measure preserving. (But the equivalence holds if I is the ideal of meager sets, see Lemma 8.1; or if I is the ideal of Lebesgue null sets and P is weakly homogeneous, see Lemma 9.1).

PROOF. (1) is clear since a sequence $\bar{\eta}$ is truly positive if and only if its true outer measure is not 0.

True positivity preservation implies properness because of 5.2 (6).

So for (2) and (3) it is enough to show the following: If X is positive (or: has true outer measure B) then there is a truly positive $\bar{\eta}$ (or: an $\bar{\eta}$ with true outer measure B) such that $\{\eta_s : s \in S\} \subseteq X$. Let $\mathscr{I}$ be 2^ω.

For (2), pick for each $N \prec H(\chi)$ an $\eta \in X \cap \mathrm{Gen}(N)$. (Recall that $\mathrm{Gen}(N)$ is a measure 1 set.) Then $\bar{\eta}$ is truly nonempty (cf. 5.4 (3)).

For (3), set $\beta := 2^{\aleph_0}$. As cited in Kanamori [6, 25.6 (a)] or Jech [5, 38.10 (i)], $[\mathscr{I}]^{\aleph_0}$ can be partitioned into $2^{\aleph_0}$ many stationary sets, i.e., $[\mathscr{I}]^{\aleph_0} = \bigcup_{\alpha\in\beta} S_\alpha$. Enumerate all positive Borel subsets of B as $(B_\alpha : \alpha \in \beta)$. For each $N \prec H(\chi)$ let α be such that $N \in S_\alpha$ and pick an $\eta \in B_\alpha \cap \mathrm{Gen}(N)$. Assume towards a contradiction that the true outer measure of $\bar{\eta}$ is $B' \subset B$ and that $B_\alpha = B \setminus B'$ is positive. Since C is club and S_α stationary, there is an $N \in C \cap S_\alpha$. So $\eta_N \in B_\alpha \cap \bar{\eta}(C)$, a contradiction. ⊣

As announced, the "true" notions are closely related to preservation of generics:

DEFINITION 5.7. P preserves generics, if for all $N \prec H(\chi)$, $p \in N$ and $\eta \in \mathrm{Gen}(N)$ there is a $q \leq p$ N-generic forcing that $\eta \in \mathrm{Gen}(N[G_P])$.

NOTES.

- Instead of "for all N", we can equivalently say "for club many N". (This follows from the proof of Lemma 5.8.)
- Of course the notion does not depend on χ, provided χ is regular and large enough (in relation to $|P|$).
- It is clear that preservation of generics is preserved under composition and implies properness.

LEMMA 5.8. *P preserves generics if and only if P is true positivity preserving.*

PROOF. $\rightarrow$: Assume otherwise, i.e., assume that $\bar{\eta}$ is truly positive, and $p \Vdash \bar{\eta}(\underaccent{\tilde}{C}) = \emptyset$ for some name $\underaccent{\tilde}{C}$ of a club set in $H(\chi)^{V[G]}$.

In V, the set

$$S^* := \{ N \prec H(\chi) : N \cap \mathscr{I} = s \in S, \eta_s \in \mathrm{Gen}(N) \}$$

is stationary. (Otherwise, the complement of S^* would witness that $\bar{\eta}$ is truly empty.) Pick some $\chi' \gg \chi$. According to 5.2 (9), there is an $N' \prec H(\chi')$ containing $\bar{\eta}, S^*, \chi, p, P, \underaccent{\tilde}{C}$ such that $N := N' \cap H(\chi) \in S^*$ (and such that P preserves generics for N', if we assume preservation for club many N only). So $N' \cap \mathscr{I} = N \cap \mathscr{I} = s \in S$, and there is an N'-generic $q \leq p$ forcing that $\eta_s \in \mathrm{Gen}(N'[G])$ (since N and N' contain the same subsets of P). Let G be a P-generic filter over V containing q. In $V[G]$, $N'[G] \cap \mathscr{I} = N'[G] \cap \mathscr{I} = s$ (since G is $N'[G]$-generic), and $N' \cap \underaccent{\tilde}{\mathscr{H}}[G] \in \underaccent{\tilde}{C}[G]$ (since $\underaccent{\tilde}{C} \in N'[G]$ is club). So $\eta_s \in \bar{\eta}(\underaccent{\tilde}{C})$, a contradiction.

$\leftarrow$: Assume towards a contradiction that $N' \prec H(\chi')$, p, η is a counterexample. Without loss of generality there is a $\chi \in N'$ such that $|P| \ll \chi \ll \chi'$. Set

$$S := \{ N \prec H(\chi) : N \text{ is counterexample for } p \text{ and some } \eta \}.$$

This set is stationary, since $S \in N'$ and $N' \cap H(\chi) \in S$.

For each $N \in S$, pick an η_N witnessing the counterexample. Then $\bar{\eta}$ is truly positive: If $N \in C \cap S$, then $\eta_N \in \bar{\eta}(C)$.

Let G be a P-generic filter over V containing p. In $V[G]$, set

$$C_{\text{gen}} := \{ N \prec H^V(\chi) : G \text{ is } N\text{-generic} \}.$$

(Note that the elements of C_{gen} are generally not in V, only subsets of V.)

C_{gen} contains a club:

$N \prec H^V(\chi)$ is guaranteed if N is closed under a Skolem functions of $H^V(\chi)$.

G is N-generic means that for every dense subset $D \subseteq P$ in N, $G \cap N \cap D$ is nonempty.

So C_{gen} contains the set of N closed under countably many operations.

Therefore (still in $V[G]$)

$$C_1 := C_{\text{gen}}^{H^{V[G_P]}(\chi)} = \{ \tilde{N} \subseteq H^{V[G_P]}(\chi) \text{ countable} : \tilde{N} \cap V \in C_{\text{gen}} \}$$

contains a club as well, as does the set

$$C := \{ \tilde{N} \prec H^{V[G_P]}(\chi) : G \in \tilde{N} \text{ and } \tilde{N} \in C_1 \}.$$

By the assumption $\bar{\eta}(C) \neq \emptyset$, i.e., for some $\tilde{N} \prec H(\chi)$, we get: $N := \tilde{N} \cap V \in S$ (note that $S \subset V$), and $\eta_N \in \mathrm{Gen}(\tilde{N})$, and G is N-generic and element of $\tilde{N}$. Therefore $N[G] \subseteq \tilde{N}$, and $\eta_N \in \mathrm{Gen}(N[G])$. G contains some $q \leq p$ forcing this all. But we assumed that η_N is a counterexample, therefore no N-generic $q \leq p$ can force that $\eta_N \in \mathrm{Gen}(N)$. $\dashv$

For the analog equivalence to true outer measure preservation we need the notion of interpretation:

DEFINITION 5.9. Let p be a condition in P.

- T is an interpretation of $\underaccent{\tilde}{T}'$ with respect to p, if:
  - T is a positive Borel set,
  - $\underaccent{\tilde}{T}'$ a P-name for a positive Borel set, and
  - for all positive Borel sets $A^V \subset T$, p does not force that $A^{V[G]} \cap \underaccent{\tilde}{T}'$ is null.

- P strongly preserves generics if the following holds:
  For all $N \prec H(\chi)$, $p, T, \underaccent{\tilde}{T}' \in N$ and $\eta \in 2^\omega$ such that T is an interpretation of $\underaccent{\tilde}{T}'$ with respect to p and $\eta \in T \cap \mathrm{Gen}(N)$ there is an N-generic $q \leq p$ forcing that $\eta \in \underaccent{\tilde}{T}' \cap \mathrm{Gen}(N[G_P])$.

NOTES.

- If T is an interpretation of $\underaccent{\tilde}{T}'$ with respect to p, $p \Vdash \underaccent{\tilde}{T}'' \supset \underaccent{\tilde}{T}'$, and $T^* \subseteq T$ is positive, then then T^* is an interpretation of $\underaccent{\tilde}{T}''$.
- Again, instead of "for all N", we can equivalently say "for club many N", and the notion does not depend on χ.

LEMMA 5.10. *For every $p \in P$ and every name $\underaccent{\tilde}{T}'$ for a positive Borel set there is an interpretation T of $\underaccent{\tilde}{T}'$ with respect to p.*

PROOF. Set

$$X := \{ B \in I^+_{\mathrm{BC}} : p \Vdash B^{V[G]} \cap \underaccent{\tilde}{T}' \in I \}.$$

Let Y be a maximal family of pairwise disjoint members of X. Then Y is countable (since I is a ccc ideal) and $\bigcup Y$ is not of measure 1 (since p forces that $\underaccent{\tilde}{T}'$ is positive and that $\bigcup Y^{V[G]} \cap \underaccent{\tilde}{T}' \in I$). Set $T := \omega^\omega \setminus \bigcup Y$. Then T is an interpretation of $\underaccent{\tilde}{T}'$ with respect to p. ⊣

LEMMA 5.11. *The following are equivalent:*

(1) *P preserves true outer measure.*
(2) *P strongly preserves generics.*
(3) *If $p \Vdash \underaccent{\tilde}{T}' \in I^+_{\mathrm{BC}}$, then there is a $T \in I^+_{\mathrm{BC}}$ and a $p' \leq p$ such that:*
    T is an interpretation of $\underaccent{\tilde}{T}'$ with respect to p', and if $N \prec H(\chi)$, $p', T, \underaccent{\tilde}{T}' \in N$, and $\eta \in T \cap \mathrm{Gen}(N)$, then there is an N-generic $q \leq p'$ forcing that $\eta \in \underaccent{\tilde}{T}' \cap \mathrm{Gen}(N[G])$.

PROOF. This is similar to the proof of 5.8.

(1) $\rightarrow$ (2) Assume that $N' \prec H(\chi')$, p, T, $\underaccent{\tilde}{T}'$, η is a counterexample. Without loss of generality there is a $\chi \in N'$ such that $|P| \ll \chi \ll \chi'$. Set

$$S := \{ N \prec H(\chi) : N \text{ is counterexample for } p,\, T,\, \underaccent{\tilde}{T}' \text{ and some } \eta \}.$$

This set is stationary, since $S \in N'$ and $N' \cap H(\chi) \in S$.

For each $N \in S$, pick an η_N witnessing the counterexample. So in particular

$$\eta_N \in T \cap \mathrm{Gen}(N).$$

Let $B \subseteq T$ be a true outer measure of $\bar{\eta}$. B is positive (which just means that $\bar{\eta}$ is truly positive). So there is a $p' \leq p$ forcing that $B \cap \underaccent{\tilde}{T}'$ is positive (since T is an interpretation of $\underaccent{\tilde}{T}'$ with respect to p).

Let G be a P-generic filter over V containing p'. In $V[G]$, set

$$C := \{ \tilde{N} \prec H^{V[G_P]}(\chi) : G, p, T, \underaccent{\tilde}{T}' \in \tilde{N} \text{ and } G \text{ is } \tilde{N} \cap V\text{-generic} \}.$$

C contains a club (as in the proof of 5.8). Assume $\eta_N \in \bar{\eta}(C)$. Then for some $\tilde{N} \in C$

$$N := \tilde{N} \cap H^V(\chi) \in S,\ G \text{ is } N\text{-generic, and } \eta_N \text{ is } I\text{-generic over } \tilde{N}.$$

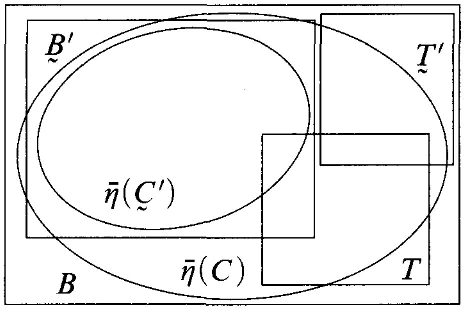


FIGURE 1. T is an interpretation of $\underset{\sim}{T}'$ with respect to p'.

Note that $N[G] \subseteq \tilde{N}$ (since $G \in \tilde{N}$). So η_N is I-generic over $N[G]$ as well. Since $N \in S$ and N, p, T, T', η_N is a counterexample, we know that η_N cannot be in $\underset{\sim}{T}'[G]$. Therefore $\bar{\eta}(C) \subseteq B \setminus \underset{\sim}{T}'[G]$, i.e., the true outer measure of $\bar{\eta}$ decreases (p' forces that $B \cap \underset{\sim}{T}'$ is positive).

(2) $\rightarrow$ (3) This follows from fact 5.10 (for every p, $\underset{\sim}{T}'$ there is an interpretation T of $\underset{\sim}{T}'$ with respect to p).

(3) $\rightarrow$ (1) Assume that $B \supset \bar{\eta}(C)$ is an outer measure of $\bar{\eta}$, and $p \in P$ forces that there is a Borel code $\underset{\sim}{B}'$ and a club set $\underset{\sim}{C}'$ of $H(\chi)^{V[G]}$ such that $\bar{\eta}(\underset{\sim}{C}') \subset \bar{\eta}(\underset{\sim}{C})$, $\underset{\sim}{B}' \subset \underset{\sim}{B}$, $\underset{\sim}{T}' := B \setminus \underset{\sim}{B}' \in \underset{\sim}{I}^+$ and $\underset{\sim}{B}' \supset \bar{\eta}(\underset{\sim}{C}')$.

Without loss of generality, $\eta_s \in B$ for every $s \in S$. Now choose a $p' \leq p$ and an interpretation T of $\underset{\sim}{T}'$ with respect to p' according to (3). Without loss of generality $T \subseteq B$ (cf. Figure 1). In V, the set

$$S^* := \{ N \prec H(\chi) : p', P, T, \underset{\sim}{T}' \in N, N \cap \mathcal{J} = s \in S, \eta_s \in \mathrm{Gen}(N) \cap T \}$$

is stationary. (Otherwise, the complement of S^* contains a club C^*. If $\eta \in \bar{\eta}(C^*)$, then $\eta \notin T$. So $\bar{\eta}(C^*) \subseteq B \setminus T$, and B cannot be true outer measure of $\bar{\eta}$.)

Pick some $\chi' \gg \chi$. There is an $N' \prec H(\chi')$ containing $\bar{\eta}, S^*, p', T, \underset{\sim}{T}', \underset{\sim}{C}'$ etc such that $N := N' \cap H(\chi) \in S^*$. So $N' \cap \mathcal{J} = s \in S$, and there is an N'-generic $q \leq p'$ forcing that $\eta_s \in \mathrm{Gen}(N'[G]) \cap \underset{\sim}{T}'$. Let G be a P-generic filter over V containing q. In $V[G]$, $N'[G] \cap \mathcal{J} = N' \cap \mathcal{J} = s$ (since G is N'-generic), and $N' \cap H(\chi)^{V[G]} \in \underset{\sim}{C}'[G]$ (since $\underset{\sim}{C}'[G] \in N'[G]$ is club). So $\eta_s \in \underset{\sim}{T}'[G] \cap \bar{\eta}(\underset{\sim}{C}[G])$, a contradiction to $\bar{\eta}(\underset{\sim}{C}[G]) \subseteq \underset{\sim}{B}[G]$ and $\underset{\sim}{B}[G] \cap \underset{\sim}{T}'[G] = \emptyset$. $\dashv$

§6. Strong preservation of generics for nep forcings. We already know that preservation of Borel outer measure generally does not imply preservation of positivity. An example was presented in 3.3. Note that the P of this example is very "undefinable". In this section we will show that under some additional assumptions on P, we even get strong preservation:

THEOREM 6.1. *If in all forcing extensions of V, P is nep and Borel outer measure preserving then P strongly preserves generics.*

In particular this requirement will be satisfied if there is a *proof* in ZFC that (a definition of the forcing) P is nep and Borel outer measure preserving. We formulate this as a corollary:

COROLLARY 6.2. *If P is provably nep and provably Borel outer measure preserving then P strongly preserves generics.*

The notion of nep forcing (a generalization of Suslin proper) was introduced in [13]. The most important instances of nep forcings are

(1) nicely definable, proper (but not necessarily Suslin proper) forcings P that are coded as set of reals. We will call such forcings "transitive nep".
(2) nicely definable (long) iterations of forcings as in (1).

Examples for transitive nep forcings are Sacks and Laver. Such forcings typically are Suslin$^+$. In this paper, we will not define the general notion of nep—which would allow us to deal with (2)—since the definition requires technicalities such as non-transitive candidates.[10] Instead, in the following section we will recall the definition of transitive nep.[11]

About transitive nep forcing. In the definition of nep (or Suslin proper) we use candidates, i.e., models of some fixed $\in$-theory ZFC*. Intuitively, we would like to use ZFC (just as we would like to use $N \prec V$ in the definition of proper forcing), but for obvious technical reasons this is not possible. So we will restrict ourselves to a reasonable choice of ZFC*:

DEFINITION 6.3. A recursive theory ZFC$^* \subseteq$ ZFC is called strongly normal, if the following is provable in ZFC:

$$H(\chi) \models \text{ZFC}^* \text{ for all sufficiently large regular } \chi.$$

You can think of ZFC* as ZFC minus the power set axiom plus something like "$\beth_\omega$ exists".[12]

We assume that the forcing P is defined by formulas $\varphi_\in(x)$ and $\varphi_\leq(x, y)$, using a real parameter r_P. Fixing a strongly normal ZFC*, we call M a candidate if it is a countable transitive ZFC* model and $r_P \in M$. So in any candidate $(P^M, \leq^M)$ is defined (but this forcing is generally not equal to $P \cap M$, since the definitions do not have to be absolute). It is important that the transitive collapse of an $N \prec H(\chi)$ (containing r_P) is a candidate (for sufficiently large regular χ).

If M is a candidate, then G is P-generic over M if for all $A \in M$ such that

$$M \models \text{"}A \subseteq P \text{ is a maximal antichain"},$$

$|G \cap A| = 1$. (Note that in this case it is not enough that G meets all dense sets, since incompatibility is generally not absolute).

q is called M-generic is q forces that G is P-generic over M.

---

[10]Note that including forcings of type (2) is not needed for the main result 9.4 of this paper: We will show that forcings of type (1) satisfy a strong, iterable condition, therefore this condition is satisfied by forcings of type (2) anyway.

[11]Usually transitive nep forcings are in fact Suslin$^+$. Nevertheless we define transitive nep here instead of Suslin$^+$ since the definition is actually simpler and better isolates the property needed for the proof. Also, there are examples of transitive nep forcings P that are not Suslin$^+$, to be more exact: whose natural definitions are not Suslin$^+$, e.g., because "$p \in P$" is $\underset{\sim}{\Pi}^1_1$ and not $\underset{\sim}{\Sigma}^1_1$. (It is a different question whether for these examples there are equivalent forcings P' that do have (possibly less natural) Suslin$^+$ definitions.)

[12]For the usual transitive nep forcings we could actually fix this ZFC*. Generally however we should—for technical reasons—not do that, just as we should not fix, e.g., $H(\chi) = H(\beth_\omega^+)$ in the definition of proper forcing.

DEFINITION 6.4. A (definition of a) forcing P is transitive nep (with respect to ZFC*), if

- "$p \in P$" and "$q \leq p$" are upwards absolute between candidates and between candidates and V as well.[13]
- $P \subseteq H(\aleph_1)$ (in V and all candidates), and "$p \in P$" and "$q \leq p$" are absolute between the universe and $H(\chi)$ (for large regular χ).
- If M is a candidate and $p \in P^M$ then there is an M-generic $q \leq p$.

So transitive nep is a direct generalization of Suslin proper. Since (transitive collapses of) elementary submodels (containing r_P) are candidates, every transitive nep forcing is proper. There are popular forcings that are transitive nep and not Suslin proper, for example Laver, Miller or Sacks (all these forcings are Suslin$^+$ and even satisfy an effective version of Axiom A, see [7] for a proof).

If P is nep and M a candidate, then $M \models$"$p \Vdash \varphi(\underset{\sim}{\tau})$" iff $M[G] \models$"$\varphi(\underset{\sim}{\tau}[G])$" for every filter G containing p which is P-generic over M *and* V.[14]

When we say "P is nep" we mean

(a suitable definition of) P is nep with respect to some strongly normal ZFC*.

In practice the choice of ZFC* is immaterial (similar to the choice of χ in the definition of proper forcing). If you believe this you can skip the following explanation, continue at the proof of Theorem 6.1, let $S(\kappa)$ be some regular $\kappa' \gg \kappa$ and ignore the arguments why certain models are in fact candidates.

Why do we use strongly normal here, and not just normal, i.e.,

$H(\chi) \models$ ZFC* for large regular χ?

Normal would definitely be enough to imply proper. The point in using strongly normal is that we can assume without loss of generality that not only the candidate M satisfies ZFC*, but also, e.g., all forcing extensions $M[G]$ (for forcings that are small in M). (This is of course not possible with a ZFC* that is just normal. For example if $V = L$ then there is a normal ZFC* containing $V = L$, but ZFC* fails to be normal in any nontrivial forcing extension.)[15]

Let's explain that in more detail:

First note that we are dealing with two forcings, Q^I and P. Q^I is Suslin ccc. This implies that Q is Suslin proper (and therefore transitive nep) with respect to any ZFC* that contains a certain strongly normal sentence φ.[16] Each two strongly normal theories are compatible (i.e., the union is strongly normal as well), and a forcing remains nep if we strengthen ZFC* (since then there are fewer candidates).

[13] I.e., if M_2, M_1 are candidates, $M_2 \models$"M_1 is a candidate", and $M_1 \models p \in P$, then $M_2 \models p \in P$ and $V \models p \in P$, and the same for $\leq$.

[14] More formally this reads: p forces: If G is M-generic, then $M[G] \models$"$\varphi(\underset{\sim}{\tau}[G])$".

[15] We can still formulate the theorem for forcings that are nep with respect to not necessarily strongly normal theories, but then we have to use two theories ZFC** and ZFC* and have to assume something like the following:
A forcing extension of a ZFC**-candidate is a ZFC*-candidate; P is nep with respect to ZFC*; ZFC** implies that every small forcing R forces that P is nep with respect to ZFC*.
So the formulation of the theorem gets messy, while there is no gain in practice, where ZFC* is strongly normal anyway.

[16] φ is the completeness theorem for Keisler logic. This follows from the proof that Suslin ccc implies Suslin proper in [4], see [7] for a discussion.

So we can assume without loss of generality that Q *and* P are nep with respect to the same ZFC*.

If $2^R \in H(\kappa)$ then $\Vdash_R H^V(\kappa)[G] = H^{V[G]}(\kappa)$ and $r \Vdash (H^{V[G]}(\kappa) \models \varphi)$ if and only if $H(\kappa) \models (r \Vdash \varphi)$.

So if P is nep (or Borel outer measure preserving) in V, then it is nep (or preserving) in $H(\chi)$ (for sufficiently large regular χ).

Let R be any forcing notion. We assumed that R forces that P is nep and preserving. Also ZFC* is strongly normal, so R forces that $H^{V[G_R]}(\chi)$ satisfies

($\times$) ZFC* plus P is nep and Borel outer measure preserving

for every regular $\chi \geq \underset{\sim}{\chi}_R$. Clearly we can find a χ'_R such that R forces that $\underset{\sim}{\chi}_R < \chi'_R$, and we can do that for all $R \subseteq H(\kappa)$. So for all κ there is a regular $S(\kappa) \gg \kappa$ such that

($\times_2$) $H(\chi)$ thinks that R forces ($\times$) for all regular $\chi \geq S(\kappa)$ and $R \subset H(\kappa)$.

Proof of Theorem 6.1. The proof is very similar to the proof of "preserving a little implies preserving much" in [13] (or its version in [7]). The point of the proof is that we mix "internal" forcing extensions (i.e., by M-generic filters in V) for Q^I and R (a collapse) with external forcing extensions for P, and use absoluteness to compare what P forces in the different internal models.

We will prove the theorem for the case that P is transitive nep. If you know the general definition of nep, you will see that the same proof works for general nep as well.[17]

Recall that the Suslin ccc ideal I was defined by a Suslin ccc forcing Q^I and a generic real $\underset{\sim}{\eta}^I$ (see 2.2).

We have to show the following: if T is an interpretation of $\underset{\sim}{T}''$ with respect to p, then for all (or just: cofinally many) $N \prec H(\chi)$ containing $p, T, \underset{\sim}{T}''$ and for all $\eta^* \in \mathrm{Gen}(N) \cap T$ the following holds:

($*$) there is an N-generic $q \leq p$ forcing that $\eta^* \in \underset{\sim}{T}'' \cap \mathrm{Gen}(N[G_P])$.

P is nep with respect to a strongly normal ZFC*. Set $\chi'_0 := \omega_1$, $\chi'_1 := S(\chi'_0)$, $\chi'_2 := S(\chi'_1)$, and $\chi'_3 := S(\chi'_2)$.

There are cofinally many such $N_0 \prec H(\chi_3)$ containing $P, p, T, \underset{\sim}{T}''$. We fix such an N_0. So it is enough to show ($*$) for N_0. Let $i\colon N_0 \to M_0$ be the transitive collapse. For $i \in \{1, 2\}$, set $\chi_i := i(\chi'_i)$, and set $\underset{\sim}{T}' := i(\underset{\sim}{T}'')$. Note that i doesn't change p, T or $\underset{\sim}{\eta}^I$, since these objects are hereditarily countable (T is a Borel code, i.e., a real number). Not surprisingly, ($*$) for N_0 is equivalent to the following:

($*_2$) there is an M_0-generic $q \leq p$ forcing that $\eta^* \in \underset{\sim}{T}' \cap \mathrm{Gen}(M_0[G_P])$.

This is straightforward: A filter G is M_0-generic if and only if it is N_0-generic (since i doesn't change the elements of P). Also, the evaluation of a name $\underset{\sim}{s}$ of a real number is absolute: If G is M_0-generic, then $\underset{\sim}{s}[G] = i(\underset{\sim}{s})[G]$. To see this, pick in N_0 maximal antichains A_n deciding $\underset{\sim}{s}(n)$. Fix n. If G is N_0-generic, then G chooses

[17] You just have to use the ord-collapse instead of the transitive collapse, and keep in mind that the evaluation of names $\underset{\sim}{\tau}[G]$ has to be redefined for non-transitive candidates. And you have to formulate awkward requirements on ZFC* if you allow ZFC* to be a $(\in, \kappa_P)$-theory.

an element of $p \in A_n \cap N_0$ forcing that $\underset{\sim}{s}[G](n) = m$. M_0 thinks that p_n forces that $i(\underset{\sim}{s})(n) = m$, so $i(\underset{\sim}{s})[G](n) = m$.

So in particular $\underset{\sim}{T}''[G] = \underset{\sim}{T}'[G]$, and $N_0[G]$ and $M_0[G]$ see the same Borel null sets, i.e., $\mathrm{Gen}(N_0[G]) = \mathrm{Gen}(M_0[G])$.

From now on for the rest of the proof we fix the above M_0, p, T, $\underset{\sim}{T}'$.

We formulate $(*_2)$ as a property of η^*:

DEFINITION 6.5. Let M be a candidate containing p, T and $\underset{\sim}{T}'$. η^* is absolutely generic over M, or: $\eta^* \in \mathrm{Gen}^{\mathrm{abs}}(M)$, if $\eta^* \in T$ and there is an M-generic $q \leq_P p$ forcing that $\eta^* \in \underset{\sim}{T}' \cap \mathrm{Gen}(M[G_P])$.

(Note that the definition of absolutely generic does not only depend on M, but on p, T and $\underset{\sim}{T}'$ as well. However, these parameters are fixed.)

Using this notion, $(*_2)$ reads as follows:

$$\mathrm{Gen}(M_0) \cap T = \mathrm{Gen}^{\mathrm{abs}}(M_0).$$

For any candidate M, $\mathrm{Gen}(M)$ is a measure 1 set. So if $\mathrm{Gen}(M) \cap T = \mathrm{Gen}^{\mathrm{abs}}(M)$ then $\mathrm{Gen}^{\mathrm{abs}}(M)$ is a measure 1 set in T. As the first step in our proof we show that $\mathrm{Gen}^{\mathrm{abs}}(M)$ is at least nonempty:

LEMMA 6.6. *Assume that P is Borel outer measure preserving, M is a candidate, and M thinks that T is an interpretation of $\underset{\sim}{T}'$ with respect to p, and A is a positive Borel (code for a) subset of T. Then* $\mathrm{Gen}^{\mathrm{abs}}(M) \cap A \in I^+$.

PROOF. In M, A is an interpretation of $\underset{\sim}{T}'$ with respect to p, since $A \subseteq T$ is positive. So without loss of generality $A = T$, i.e., we just have to show that $\mathrm{Gen}^{\mathrm{abs}}(M)$ is positive.

Pick (in M) a $p' \leq p$ forcing that $T \cap \underset{\sim}{T}'$ is positive. Let $q \leq p'$ be M-generic, and G a P-generic filter over V containing q. So in $M[G]$ (and therefore by absoluteness in $V[G]$) $T^{V[G]} \cap \underset{\sim}{T}'[G]$ is positive. $T^{V[G]}$ is the outer measure of T^V (since P preserves outer measure). So $T^V \cap \underset{\sim}{T}'[G]$ is positive. Also, $\mathrm{Gen}(M[G])$ is of measure 1. Therefore

$$X := \mathrm{Gen}(M[G]) \cap T^V \cap \underset{\sim}{T}'[G]$$

is positive in $V[G]$. Clearly $X \subseteq \mathrm{Gen}^{\mathrm{abs}}(M)^V$. So in V, $\mathrm{Gen}^{\mathrm{abs}}(M)$ has to be positive. ⊣

Let (in M_0, for $i \in \{1,2\}$) R_i be the collapse of $H^{M_0}(\chi_i)$ to $\aleph_0$, i.e., the set of finite partial functions from ω to $H^{M_0}(\chi_i)$. Fix an $\eta^* \in \mathrm{Gen}(M_0) \cap T$. We have to show that $\eta^* \in \mathrm{Gen}^{\mathrm{abs}}(M_0)$. $\eta^* \in \mathrm{Gen}(M_0)$ means that there is (in V) a Q^I-generic filter G_Q over M_0 such that $\underset{\sim}{\eta}^I[G_Q] = \eta^*$. Pick (again in V) an R_2-generic filter G_{R_2} over $M_0[G_Q]$. Set $M' := M_0[G_Q][G_{R_2}]$. So we get (in V) the following forcing extensions:

$$M_0 \to M_0[G_Q] \to M' := M_0[G_Q][G_{R_2}]$$

M' sees all relevant information about $H_1 := H(\chi_1)^{M_0}$ (in particular M' knows that H_1 is a candidate). So it is enough to show that M' thinks that η^* is absolutely generic for H_1:

LEMMA 6.7. $M' \models \eta^* \in \mathrm{Gen}^{\mathrm{abs}}(H_1)$.

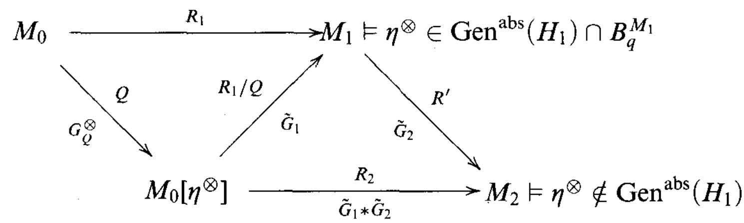


FIGURE 2. The models used in the proof of Lemma 6.7.

If we assume this, Theorem 6.1 follows immediately: $Q^I * R_2 \subseteq H^{M_0}(\chi_2)$. M_0 is the transitive collapse of $N_0 \prec H(S(\chi_2'))$, so according to $(\times_2)$ M' is a candidate and P is transitive nep in M'. Clearly M' knows that H_1 is a candidate. So according to the Lemma there is a $p' \in M'$ such that

$$M' \vDash \text{“}p' \leq p \text{ is } H_1\text{-generic, and } p' \Vdash_P \eta^* \in \underset{\sim}{T}' \cap \mathrm{Gen}(H_1[G_P])\text{”}.$$

Let (in V) $q \leq_P p'$ be M'-generic. We claim that q witnesses $(*_2)$, i.e., that

$$q \text{ is } M_0\text{-generic and forces that } \eta^* \in \mathrm{Gen}(M_0[G_P]) \cap \underset{\sim}{T}'.$$

So let G_P be a P-generic filter over V containing q. Then G_P is M'-generic.

$M'[G_P]$ thinks that G_P (i.e., $G_P \cap P^{M'}$) is H_1-generic, since this is forced by $p' \in G_P$. Being H_1-generic is absolute (it just says that $|G_P \cap A| = 1$ for all maximal antichains in H_1). So G_P really is H_1-generic. Analogously η^* is I-generic over $H_1[G_P]$.

Recall that $H_1 = H(\chi_1)^{M_0}$, the definition of P is absolute between the universe and $H(\chi)$ and χ_1 is sufficiently large. Therefore $P^{H_1} = P^{M_0}$ and H_1 contains exactly the same subsets of P as M_0 does. So G_P is M_0-generic as well. Also, $H_1[G_P]$ contains exactly the same reals (in particular Borel codes) as $M_0[G_P]$. (This can again be seen by deciding $\underset{\sim}{r}(n)$ by a maximal antichain.) Therefore η^* is I-generic over $M_0[G_P]$. This is all forced by some $q \leq p$, so we are finished.

PROOF OF LEMMA 6.7. We already know that η^* is I-generic over M_0. Using the facts that H_1 and M_0 see the same subsets of P and that H_1 is countable in M' we get $M' \vDash \eta^* \in \mathrm{Gen}(H_1) \cap T$.

Assume towards a contradiction that $M' \vDash$ “$\eta^* \notin \mathrm{Gen}^{\mathrm{abs}}(H_1)$”. Since $M' = M_0[G_Q][G_R]$ this is forced by some $q \in G_Q$ and $r \in R_2$. However since R_2 is homogeneous and the sentence $\eta^* \notin \mathrm{Gen}^{\mathrm{abs}}(H_1)$ only contains parameters in $M_0[G_Q]$ we can assume that $r = 1$, i.e.,

(□) $$M_0 \vDash \text{“}q \Vdash_Q (\underset{\sim}{\eta}^I \in T, \Vdash_{R_2} \underset{\sim}{\eta}^I \in \mathrm{Gen}(H_1) \setminus \mathrm{Gen}^{\mathrm{abs}}(H_1))\text{”}.$$

Fix a positive Borel code $B_q \in M_0$ such that

$$\{\underset{\sim}{\eta}^I[G] : G \in V \text{ is an } M_0\text{-generic filter containing } q\} = \mathrm{Gen}(M_0) \cap B_q^V$$

(see 2.3). Without loss of generality $B_q \subseteq T$, since (in M_0) $q \Vdash \underset{\sim}{\eta}^I \in T$.

Choose in V an R_1-generic filter G_{R_1} over M_0, and set $M_1 := \tilde{M}_0[G_{R_1}]$. M_1 knows that H_1 is a candidate, and that $B_q^{M_1} \subseteq T$ is positive. M_1 knows that P preserves Borel outer measure (because of $(\times_2)$), and H_1 thinks that T is an interpretation of $\underset{\sim}{T}'$ with respect to p (since H_1 is the collapse of $H^V(\chi_1)$). So we can apply

Lemma 6.6 in M_1 (i.e., M_1 is the universe V and H_1 the candidate M) and find a $\eta^\otimes \in \mathrm{Gen}^{\mathrm{abs}}(H_1) \cap B_q^{M_1}$. In particular $\eta^\otimes$ is I-generic over H_1 and therefore over M_0. So in M_1 there is a M_0-generic filter $G_Q^\otimes$ such that $\eta^*[G_Q^\otimes] = \eta^\otimes \in B_q^V$. Therefore $q \in G_Q$, and we can factorize R_1 as $R_1 = Q * R_1/Q$ such that $G_{R_1} = G_Q^\otimes * \tilde{G}_1$. (See Figure 2 for a diagram of the forcing extensions we are going to construct).

In $M_0[G_Q^\otimes] = M_0[\eta^\otimes]$ we look at the forcing $R_2 = R_2^{M_0}$ (the finite partial functions from ω to $H^{M_0}(\chi_1) = H_1$). $\chi_2 \gg \chi_1$ and R_1/Q is a subforcing of $\mathrm{ro}(R_1)$, where $R_1 \subset H_1$. So according to 2.10 (2) and (3), R_2 can be factorized as $R_2 = (R_1/Q) * R'$. We already have the (R_1/Q)-generic filter $\tilde{G}_1$ (over $M_0[G_Q^\otimes]$), now choose (as always in V) a R'-generic filter $\tilde{G}_2$ over M_1. Set $G_{R_2} = \tilde{G}_1 * \tilde{G}_2$. So G_{R_2} is R_2-generic over $M_0[G_Q^\otimes]$. Set $M_2 := M_0[\eta^\otimes][G_{R_2}]$.

Now set $H_2 := H(\chi_2)^{M_1}$. H_2 thinks that P is nep and that $\eta^\otimes \in \mathrm{Gen}^{\mathrm{abs}}(H_1)$ (by absoluteness in M_1). Also, H_2 is a candidate (by $(\times_2)$, since $H_2 = H^{M_0}[G_{R_1}]$). So for some $p_1 \in H_2$,

$$H_2 \vDash \text{“}p_1 \leq p \text{ is } H_1\text{-generic}, p_1 \Vdash \eta^\otimes \in \mathrm{Gen}(H_1[G_P]) \cap \underset{\sim}{T}'\text{”}.$$

In M_2, there is an H_2-generic $p_2 \leq p_1$ (since M_2 thinks that P is nep and that H_2 is a candidate.) Let G_P be a P-generic filter over M_2 containing p_2. From now on, we work in $M_2[G_P]$. G_P is H_2-generic and contains p_1, so by absoluteness G_P is H_1-generic as well and $\eta^\otimes \in \mathrm{Gen}(H_1[G_P]) \cap \underset{\sim}{T}'$. On the other hand, according to $(\Box)$, $\eta^\otimes \notin \mathrm{Gen}(H_1[G_P]) \cap \underset{\sim}{T}'$, a contradiction. ⊣

Note that if we set $T = \underset{\sim}{T}' = 2^\omega$ this proof gives us “preserving a little implies preserving much” of [13]:

THEOREM 6.8. *If in all forcing extensions of V, P is nep and preserves Borel positivity, then P preserves generics.*

In particular, if P is provably nep and provably preserves Borel positivity, then P preserves generics.

§7. **A general preservation theorem.** For proving the main result 9.4 we will use a general iteration theorem for countable support iterations of proper forcings. It appeared as “Case A” in *Proper and improper forcing* [12, XVIII,§3]. The proof there is not easily digestible, though. A simplified version appeared in Section 5 of Goldstern's *Tools* [2]. This version uses the additional requirement that every forcing of the iteration adds a new real. Note that this requirement is met in many applications anyway (e.g., in the forcings of [9] cited in the introduction).

A proof of the iteration theorem *without* this additional requirement appeared in [8] and was copied into *Set Theory of the Reals* [1] (as “first preservation theorem” 6.1.B), but Schlindwein pointed out a problem in this proof.[18] Another proof (building on the one in [2]) will appear in [3].

The general preservation theorem uses the following setting: Fix a sequence of increasing arithmetical two-place relations R_n on ω^ω. Let R be the union of the

[18]See [10]. In this paper Schlindwein wrote a simple proof for the special case of ω^ω-bounding, however he later found a problem in his own proof [C. Schlindwein, personal communication, April 2005]. He is preparing a new version [11].

Sh:828

R_n. Assume

- $\mathscr{C} := \{ f \in \omega^\omega : f \; R \; \eta$ for some $\eta \in \omega^\omega \}$ is closed,
- $\{ f \in \omega^\omega : f \; R_n \; \eta \}$ is closed for all $n \in \omega$, $\eta \in \omega^\omega$, and
- for every countable N there is an η such that $f \; R \; \eta$ for all $f \in N \cap \mathscr{C}$ (in this case we say "η covers N").

DEFINITION 7.1. Let P be a forcing notion, $p \in P$.

- $\bar{f}^* := f_1^* \dots f_k^*$ is a tools-interpretation of $\bar{f} := \underset{\sim}{f}_1, \dots, \underset{\sim}{f}_k$ under p, if each $\underset{\sim}{f}_i$ is a P-name for an element of $\mathscr{C}$, and there is an decreasing chain $p = p_0 \geq p_1 \geq \cdots$ of conditions in P such that $p_i \Vdash (\underset{\sim}{f}_1 \restriction i = f_1^* \restriction i \& \dots \& \underset{\sim}{f}_k \restriction i = f_k^* \restriction i)$.
- A forcing notion P is tools-preserving, if for all $N \prec H(\chi)$, η covering N, p, n_i, $\bar{f}^*$, $\bar{f} \in N$ such that $\bar{f}^*$ is a tools-interpretation of $\bar{f}$ under p and $f_i^* R_{n_i} \eta$ there is an N-generic $q \leq p$, forcing that η covers $N[G_P]$, and $\underset{\sim}{f}_i \; R_{n_i} \; \eta$ for all $i \leq k$.

Note that if $\bar{f}$ is a tools-interpretation, then $f_i^* \in \mathscr{C}$.

Tools-interpretations differ from the interpretations of Definition 5.9. They obviously deal with functions from ω to ω instead of Borel sets modulo I. But there is another technical difference: For tools-interpretations, we require that there is an decreasing sequence of conditions $p \geq p_1 \geq p_2 \geq \cdots$, not just that for all n, the truth value of $(\forall m \leq n)\; \underset{\sim}{f}(m) = f^*(m)$ is positive.[19]

Now we can formulate the "first preservation theorem" [1, 6.1.B] already mentioned:

THEOREM 7.2. *Assume $(P_i, \underset{\sim}{Q}_i)_{i<\alpha}$ is a countable support iteration of proper, tools-preserving forcings. Then P_α is tools-preserving.*[20]

§8. Preservation of non-meager. In this section I is the ideal of meager sets.

This is the easiest (and already well known) case: strong preservation is equivalent to preservation of generics and is iterable.

We already know that preservation of Borel positivity is equivalent to preservation of Borel outer measure. The same holds for the non-Borel notions as well:

LEMMA 8.1. *Preservation of positivity implies preservation of outer measure, and the same holds for the true version.*

PROOF. Assume towards a contradiction that A is outer measure of X, and that p forces that $\underset{\sim}{B}$ is outer measure of $\check{X}$ and $A \setminus \underset{\sim}{B}$ is positive. Then $A \setminus \underset{\sim}{B}$ contains a nonempty clopen set $D \in V$. So p forces that $D^{V[G_P]} \cap \check{X} = D^V \cap \check{X}$ is null. By positivity preservation $D \cap X$ has to be null, a contradiction.

To show the lemma for the true notion, the same argument works: Assume towards a contradiction that A is true outer measure of $\bar{\eta}$ and that p forces $\bar{\eta}(\underset{\sim}{C}') \cap D$

[19]Given a forcing P and a tools-interpretation f^* of a function $\underset{\sim}{f} \notin V$ under p, we can find a dense subforcing $P' \subset P$ such that for every condition p' of P' there is a $n(p')$ such that p' forces that $f^*(n(p')) \neq \underset{\sim}{f}(n(p'))$. So with respect to P', f^* cannot be a tools-interpretation of $\underset{\sim}{f}$ any more. Definition 5.9 of interpretation on the other hand is invariant under equivalent forcings.

[20]Let us call P densely preserving if there is a dense subforcing Q of P that is tools-preserving. Since tools-interpretations are not absolute, densely preserving does not seem to imply tools-preserving. When iterating forcings that do not necessarily add reals, it actually seems that densely preserving is the property that is preserved and not tools-preserving, see [3]. In practice this distinction is of course not important.

is null for some clopen $D \subseteq A$. Then define

$$S^* := \{ s \in S : \eta_s \in D \}, \quad \text{and}$$
$$\bar{\eta}^* := \bar{\eta} \restriction S^*.$$

The usual argument shows that $\bar{\eta}^*$ is truly positive: Otherwise, let C be club such that $\bar{\eta}^*(C) = \emptyset$. Then C witnesses that A is not true outer measure of $\bar{\eta}$. On the other hand, p forces that $\bar{\eta}^*(\underset{\sim}{C}')$ is null, a contradiction to true positivity preservation. ⊣

It is well known that "preservation of Cohens" is iterable:

THEOREM 8.2. *If $(P_i, \underset{\sim}{Q}_i)_{i<\alpha}$ is a countable support iteration of proper forcings such that $\Vdash_{P_i}$ ($\underset{\sim}{Q}_i$ preserves Cohens) for all $i \in \alpha$, then P_α preserves Cohens.*

PROOF. This is proven as application 3 in [2] or as Theorem 6.3.20 of *Set theory of the reals* [1]. It is an easy application of tools-preservation (7.2):

Let Ω be the set of clopen sets of 2^ω. Set

$$\mathscr{C} = \{ f \in \Omega^\Omega : (\forall U \in \Omega)\ f(U) \subseteq U \}.$$

We define $f\ R_n\ \eta$ by

$$f \in \mathscr{C} \text{ and } \eta \in 2^\omega \text{ and for some } k \leq n,\ \eta \in f(U_k).$$

Then for any $N \prec H(\chi)$, η covers N if and only if η is Cohen over N. Also $\{ f : f\ R_n\ \eta \}$ is clopen, so $f_i\ R_{n_i}\ \underset{\sim}{\eta}$ can be forced by determining $f_i \restriction m$ for some m. Therefore P preserves Cohens if and only if P is tools-preserving. This finishes the proof of 8.2.

Note that in this simple case tools-preservation isn't really needed. It is enough to trivially modify the proof that a countable support iteration of proper forcings is proper (see, e.g., [1, 6.1.3]). In the following we point out the changes that have to be made to this specific proof:

The Lemma now reads:

Suppose $(P_\alpha, \underset{\sim}{Q}_\alpha)_{\alpha<\delta}$ is a countable support iteration such that for all $\alpha < \delta$, $\Vdash_\alpha$ "$\underset{\sim}{Q}_\alpha$ is proper and preserves Cohens". Suppose that $N \prec H(\chi)$ contains $(P_\alpha, \underset{\sim}{Q}_\alpha)$. Then for all $\beta \in N \cap \delta$, for all $\alpha \in N \cap \beta$ and for all $p \in P_\beta \cap N$, whenever $q \leq_\alpha p \restriction \beta$ is (N, P_α)-generic and forces that $\eta^* \in \text{Gen}(N[G_\alpha])$, there is an (N, P_α)-generic condition $r \leq_\alpha p \restriction \beta$ such that $r \restriction \alpha = q$ and $r \Vdash \eta^* \in \text{Gen}(N[G_\beta])$.

The successor step is trivial. In the limit step we enumerate (just like the the $\underset{\sim}{\tau}_n$) a list $\underset{\sim}{T}_n$ of the P_β-names in N of nowhere dense trees. Then we add the following requirement: p_{n+1} determines $\underset{\sim}{T}_n$ up to a level m, and $\eta^* \notin \underset{\sim}{T}_n \restriction m$.

Why can we do this? By induction we already know that there is a P_{α_n}-generic $q_n \leq p_n$ over N that forces $\eta^* \in \text{Gen}(N[G_{\alpha_n}])$. Assume G_{α_n} is P_{α_n}-generic over V and contains q. In $N[G_{\alpha_n}]$, construct T^*_m and an decreasing sequence $p^m_n < p_n$ such that $p^m_n \Vdash T^*_m = \underset{\sim}{T}_n \restriction m$ and $\bigcup_{m\in\omega} T^*_m$ is a nowhere dense tree. So $\eta^* \notin T^*_m$ for some m. T^*_m lives in V (since it is finite). So there is an m, T^*_m and a $p^m_n \leq p_n \in N$ such that $\eta^* \notin T^*_m$. Now choose $p_{n+1} \leq p^m_n$.

So in this case the additional preservation property can be satisfied "locally" (we can once and for all deal with $\underset{\sim}{M}_n$ in step n). ⊣

Applying Theorem 6.1 gives the following result due to Goldstern and Shelah [12, Lemma XVIII.3.11]:

COROLLARY 8.3. *If $(P_i, \underset{\sim}{Q}_i)_{i<\alpha}$ is a countable support iterationsuch that $\underset{\sim}{Q}_i$ is provably nep and provably doesn't make V meager, then P_α doesn't make any old set meager.*

§9. Preservation of Lebesgue-positive. In this section, I is the ideal of Lebesgue null sets.

The outer measure of X as defined in this paper is equivalent to the outer measure in the usual sense, called $\mathrm{Leb}^*(X)$. In particular, preservation of outer measure is equivalent to the preservation of the value of $\mathrm{Leb}^*(X)$, more formally: Assume $\mathrm{Leb}^*(X) = r$. Then P preserves outer measure of X if and only if P forces that $\mathrm{Leb}^*(\check{X}) = \check{r}$.

Also the true outer measure is fully described by the true outer measure as a real, defined by $\text{T-Leb}^*(\bar\eta) := \min\{\mathrm{Leb}^*(\bar\eta(C)) : C \text{ club}\}$ (note that T-Leb* really is a minimum). Then P is true outer measure preserving if and only if P preserves T-Leb*. (This follows from the proof of the next lemma).

LEMMA 9.1. *If P is weakly homogeneous*[21] *and preserves (true) positivity then P preserves (true) outer measure.*

PROOF. For the "untrue" version, this is [1, Lemma 6.3.10]. The same proof works for true outer measure as well: Assume that B is a true outer measure of $\bar\eta$, that $\mathrm{Leb}(B) = r_1$ and that p forces that $\underset{\sim}{B}' \supseteq \bar\eta(\underset{\sim}{C}')$ and $\mathrm{Leb}(\underset{\sim}{B}') < r_2 < r_1$, r_2 rational. We have to show that there is a truly positive $\bar\eta^*$ that fails to be truly positive after forcing with P.

So p forces that there is a sequence $\underset{\sim}{I}_n$ of clopen sets such that $\bigcup \underset{\sim}{I}_n \supseteq \bar\eta(\underset{\sim}{C}')$ and $\Sigma\,\mathrm{Leb}(\underset{\sim}{I}_n) < r_2$. Let $p_n, h(n), I_n^*$ be such that for all $m \leq h(n)$,

$$p_n \Vdash \mathrm{Leb}\Big(\bigcup_{m>h(n)} \underset{\sim}{I}_m\Big) < \frac{1}{n} \;\&\; (\forall k < m)\; \underset{\sim}{I}_k = I_k^*.$$

So $\mathrm{Leb}(\bigcup I_m^*) \leq r_2$, and $B \setminus \bigcup I_m^*$ is not null. Therefore

$$S^* := \{\, s \in S : \eta_s \notin \bigcup I_m^* \,\}$$

is is stationary (otherwise, the complement of S^* would witness that B is not the true outer measure or $\bar\eta$). Define $\bar\eta^* := \bar\eta \restriction S^*$. So $\bar\eta^*$ is truly positive.

$$\begin{aligned} &p_n \Vdash \mathrm{Leb}\Big(\bigcup \underset{\sim}{I}_m \setminus \bigcup I_m^*\Big) < \frac{1}{n}, \quad \text{and}\\ &p_n \Vdash \bar\eta^*(\underset{\sim}{C}') \subseteq \bigcup \underset{\sim}{I}_m \setminus \bigcup I_m^*, \quad \text{i.e.,}\\ &p_n \Vdash \mathrm{Leb}^*(\bar\eta^*(\underset{\sim}{C}')) < \frac{1}{n}. \quad \text{So}\\ &p_n \Vdash \text{T-Leb}^*(\bar\eta^*) \leq \frac{1}{n}. \end{aligned}$$

Since the last statement does not contain any names except standard-names, and since P is weakly homogeneous, we get $1_P \Vdash \text{T-Leb}^*(\bar\eta^*) \leq 1/n$ for all n, i.e., $\Vdash \text{T-Leb}^*(\bar\eta^*) = 0$. So the truly positive $\bar\eta^*$ becomes null after forcing with P. ⊣

Now we are going to show that strong preservation is equivalent to the Lebesgue version of tools-preservation (see Definition 7.1).

[21] So if φ only contains standard-names, then $(p \Vdash_P \varphi)$ implies $(1_P \Vdash_P \varphi)$.

We list the clopen subsets of 2^ω as $(I_i)_{i\in\omega}$, and interpret a function $f \in \omega^\omega$ as a sequence of clopen sets. We set

$$\mathscr{C} := \{ f : \forall i \ \mathrm{Leb}(I_{f(i)}) < 2^{-i} \}, \quad \text{and}$$
$$f \ R_n \ \eta \quad \text{if and only if} \quad f \in \mathscr{C}, \eta \in 2^\omega, \text{and for all } l > n, \eta \notin I_{f(l)}.$$

For $f \in \mathscr{C}$, the set $\mathscr{N}_f := \bigcap_{n\in\omega} \bigcup_{i>n} I_{f(i)}$ is a null set. In fact, every null set is contained in a $\mathscr{N}_f$ for some $f \in \mathscr{C}$ (see, e.g., [1, 2.3.10] or [2]). $f \ R \ \eta$ just means $\eta \notin \mathscr{N}_f$.

η is random over a model N if and only if η is not element of any null set coded by a real in N. So η is random over N if and only if η covers N.

For reference we baptize this version of tools-preservation:

Definition 9.2. P is called Lebesgue-tools-preserving if it is tools-preserving for the R_n defined as above.

It is clear that Lebesgue-tools-preserving implies preservation of generics and therefore preservation of positivity. Lebesgue-tools-preservation is preservation of generics plus "side functions". It turns out that this is equivalent to strong preservation:

Lemma 9.3. *The following are equivalent:*

(1) *P is Lebesgue-tools-preserving.*
(2) *P is Lebesgue-tools-preserving for $k = 1$ and $n_1 = 0$.*
(3) *P strongly preserves randoms.*

Proof. (2) $\rightarrow$ (1): Assume $N \prec H(\chi)$, p, η, $f_1^*, \dots, f_k^*$, $\underset{\sim}{f}_1, \dots, \underset{\sim}{f}_k$ and $n_1, \dots, n_k$ are as in the definition of Lebesgue-tools-preserving.
Set $n^* := \max(k, n_1, \dots, n_k)$. $p_{n^*} \leq p$ forces that $f_i^* \restriction n^* = \underset{\sim}{f}_i \restriction n^*$. Let $g^* \in \omega^\omega$ be such that $I_{g^*(m)} = \bigcup_{i=1\dots k} I_{f_i^*(n^*+m)}$, and $\underset{\sim}{g}$ the name of a function in ω^ω such that p forces that $I_{\underset{\sim}{g}(m)} = \bigcup_{i=1\dots k} I_{\underset{\sim}{f}_i(n^*+m)}$. So for all m,

$$p \Vdash \mathrm{Leb}(I_{g^*(m)}) < k 2^{-(n^*+m)} < 2^{-m},$$

i.e., $p \Vdash \underset{\sim}{g} \in \mathscr{C}$.

g^* is a tools-interpretation of $\underset{\sim}{g}$ under p_{n^*} (this is clear if we assume that the list I_m contains no repetitions; otherwise we just have to choose $\underset{\sim}{g}(m)$ accordingly).

$\eta \notin I_{g^*(m)}$ for all m; i.e., $g^* \ R_0 \ \eta$.

Since we assume (2) we can find an N-generic $q \leq p_{n^*}$ forcing that η is random over $N[G]$ and that $\underset{\sim}{g} \ R_0 \ \eta$.

This means that q forces that $\eta \notin I_{\underset{\sim}{f}_i(m)}$ for all $i \leq k$ and $m > n^*$. And for $n_i \leq m \leq n^*$, p_{n^*} forces that $I_{\underset{\sim}{f}_i(m)} = I_{f_i^*(m)}$ and therefore that $\eta \notin I_{\underset{\sim}{f}_i(m)}$. So q forces that $\underset{\sim}{f}_i \ R_{n_i} \ \eta$.

(2) $\rightarrow$ (3): We show the equivalent property (3) of Lemma 5.11. So fix p and N and assume $p \Vdash \underset{\sim}{T}' \in I^+_{\mathrm{BC}}$. We want to show that there is a T and a $p' \leq p$ such that T is an interpretation of $\underset{\sim}{T}'$ with respect to p', and for every $\eta^* \in T \cap N$ there is an N-generic $q \leq p$ forcing that $\eta^* \in \mathrm{Gen}(N[G]) \cap \underset{\sim}{T}'$.

Since every positive set contains a positive closed set we can assume without loss of generality that p forces that $\underset{\sim}{T}'$ is closed and that the measure of $\underset{\sim}{T}'$ is at least some rational number r.

Any measurable $A \subseteq 2^\omega$ can be approximated from the outside by countable unions of clopen sets. If A is closed (i.e., compact), then any open cover of A has a finite sub-cover. So for any $\varepsilon > 0$ there is a clopen set $C \supseteq A$ such that $\mathrm{Leb}(C \setminus A) < \varepsilon$. In particular there is a sequence of clopen sets

$$2^\omega = A_0 \supseteq A_1 \supseteq \cdots \supseteq A$$

such that $\mathrm{Leb}(A_n \setminus A) = \mathrm{Leb}(A_n) - \mathrm{Leb}(A) < 2^{-n}$ and $A = \bigcap A_n$. Set $B_n := A_n \setminus A_{n+1}$. Then the B_n are a disjoint sequence of clopen sets, $A = 2^\omega \setminus \bigcup B_n$ and

$$\mathrm{Leb}(B_n) = \mathrm{Leb}(A_n) - \mathrm{Leb}(A_{n+1}) \leq \mathrm{Leb}(A_n) - \mathrm{Leb}(A) < 2^{-n}.$$

So the sequence (B_n) is coded by an $f \in \mathscr{C}$. Also $\Sigma\, \mathrm{Leb}(B_n) = 1 - \mathrm{Leb}(A)$.

Applying this to $\underset{\sim}{T}'$ we get that p forces that there is the according $\underset{\sim}{f}$ for $\underset{\sim}{T}'$. Pick an $N' \prec H(\chi')$ containing p and $\underset{\sim}{f}$, and let $G \in V$ be an N'-generic filter. Then $f^* := \underset{\sim}{f}[G]$ is a tools-interpretation of $\underset{\sim}{f}$, witnessed by a decreasing sequence p_n of elements of G.

Let $(B_n^*)_{n\in\omega}$ be the sequence of clopen sets corresponding to f^*. B_n^* is disjoint to B_m^* for $m \neq n$ (since p forces this for $\underset{\sim}{B}$). Also p forces that $\Sigma\, \mathrm{Leb}(\underset{\sim}{B}_n) \leq 1 - \mathrm{Leb}(\underset{\sim}{T}') < 1 - r$, and therefore $\Sigma\, \mathrm{Leb}(B_n^*) \leq 1 - r$. So $T := 2^\omega \setminus \bigcup B_n^*$ is positive. T is an interpretation of $\underset{\sim}{T}'$ with respect to p: Assume $A \subseteq T$ has measure $s > 2^{-n}$. Then $p_n \leq p$ forces that $\mathrm{Leb}(\underset{\sim}{T}' \setminus \bigcup_{m<n} \underset{\sim}{B}_m) =< 2^{-n}$ and that $\bigcup_{m<n} \underset{\sim}{B}_m = \bigcup_{m<n} B_m^*$. A is disjoint to $\bigcup_{n<m} B_m^*$, so $A \cap \underset{\sim}{T}'$ has to be positive.

Assume $N \prec H(\chi)$ contains $p, P, T, \underset{\sim}{T}'$ and $\eta \in T \cap \mathrm{Gen}(N)$. $\eta \in T$ means $\eta \notin B_n^*$ for all n, i.e., $f^*\ R_0\ \eta$. So by (2) there is a $q \leq p$ N-generic forcing that $\eta \in \mathrm{Gen}(N[G])$ and that $\eta\ R_0\ \underset{\sim}{f}$. That again means that $\eta \in \underset{\sim}{T}'$, and we are finished.

(3) $\to$ (2): Fix an $N \prec H(\chi)$, a tools-approximation f^* of $\underset{\sim}{f}$ under p such that $\underset{\sim}{f}, f^*$, and p are in N, and an $\eta \in \mathrm{Gen}(N)$ such that $f^*\ R_0\ \eta$.

So the appropriate $p_2 \leq p$ forces that $f^*(0) = \underset{\sim}{f}(0)$ and $f^*(1) = \underset{\sim}{f}(1)$.

Set $T := 2^\omega \setminus \bigcup_{m>1} I_{f^*(m)}$ and $\underset{\sim}{T}' := 2^\omega \setminus \bigcup_{m>1} I_{\underset{\sim}{f}(m)}$.

Then T is an interpretation of $\underset{\sim}{T}'$ with respect to p_2:

Assume $A \subseteq T$ is a positive Borel set. Pick $N \in \omega$ such that $\Sigma_{n \geq N} 2^{-n} < \mathrm{Leb}(A)$. $p_N \leq p_2$ forces that $f^*(i) = \underset{\sim}{f}(i)$ for all $i < N$. So p_N forces that $A \cap \bigcup_{m<N} I_{\underset{\sim}{f}(m)}$ is empty, and that $\mathrm{Leb}(\bigcup_{m \geq N} I_{\underset{\sim}{f}(m)}) < \mathrm{Leb}(A)$, and therefore that $A \cap \underset{\sim}{T}'$ is positive.

So by (3) we know that there is an N-generic $q \leq p_2$ forcing that η is random over $N[G]$ and that $\eta \in \underset{\sim}{T}'$. $\eta \in \underset{\sim}{T}'$ means that for all $m > 1$, $\eta \notin I_{\underset{\sim}{f}(m)}$. Since $q \leq p_2$, q forces that η is not in $I_{\underset{\sim}{f}(0)} = I_{f^*(0)}$ or $I_{\underset{\sim}{f}(1)} = I_{f^*(1)}$ either. So q forces that $\underset{\sim}{f}\ R_0\ \eta$. ⊣

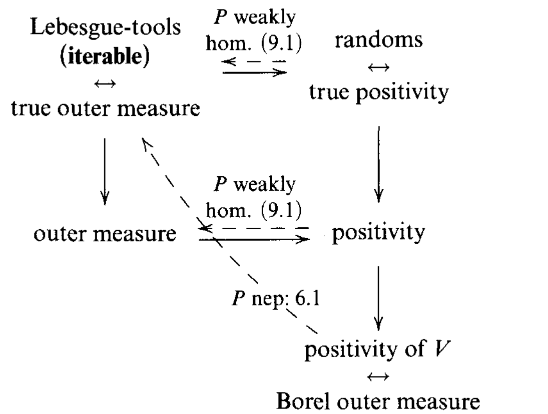
Using this Lemma, Theorem 6.1 and the fact that strong preservation implies preservation we get:[22]

COROLLARY 9.4. *Assume that $(P_i, \underset{\sim}{Q}_i)_{i<\alpha}$ is a countable support iteration such that for all i, $\underset{\sim}{Q}_i$ is provably nep and provably preserves Lebesgue positivity of V. Then P_α preserves Lebesgue positivity (of all old positive sets).*

[22]Compare that to Zapletal [15, Cor 5.4.10]: Assume that there is a proper class of measurable Woodin cardinals. If P is a forcing adding a single real which has a definition satisfying a (very general) syntax and preserves Lebesgue positivity of V (or in fact positivity with respect to similar ideals), then the countable support iteration of P (or arbitrary length) preserves positivity as well.

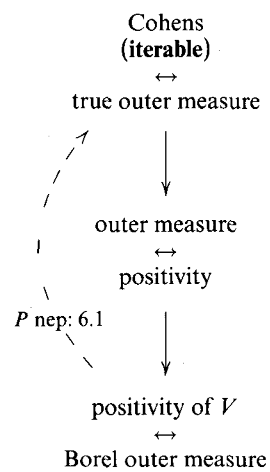
The diagram of implications. So in the cases of the Lebesgue null and the meager ideal we have implications of preservation of the following notions:

I is the Lebesgue null ideal:

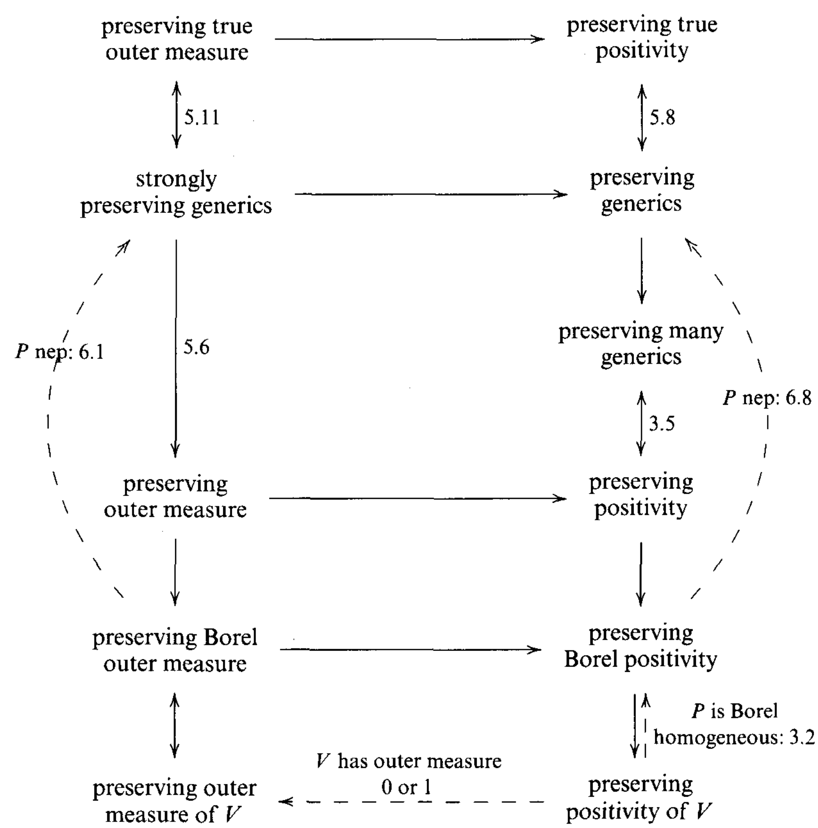


For the definition of Lebesgue-tools-preserving, see 9.2.
For nep see section 6.

I is the meager ideal:



For general Suslin ccc ideals we get:



Sh:828

Preservation of (Borel) positivity and outer measure is defined in 3.1, the true notions in 5.5, and (strong) preservation of generics in 5.7 and 5.9. For "P nep" see section 6.

REFERENCES

[1] Tomek Bartoszynski and Haim Judah, ***Set theory: On the structure of the real line***, A K Peters, Wellesley, MA, 1995.

[2] Martin Goldstern, *Tools for Your Forcing Construction*, ***Set theory of the reals*** (Haim Judah, editor), Israel Mathematical Conference Proceedings, vol. 6, American Mathematical Society, 1993, pp. 305–360.

[3] Martin Goldstern and Jakob Kellner, *New reals: Can live with them, can live without them*, preprint, see `http://arxiv.org/math.LO/0505471`.

[4] Jaime Ihoda (Haim Judah) and Saharon Shelah, *Souslin forcing*, this Journal, vol. 53 (1988), pp. 1188–1207.

[5] Thomas Jech, ***Set theory***, Monographs in mathematics, Springer-Verlag, 2002, 3rd millennium ed.

[6] Akihiro Kanamori, ***The higher infinite***, Perspectives in Mathematical Logic, Springer-Verlag, 1994.

[7] Jakob Kellner, *Preserving non-null with Suslin$^+$ forcings*, preprint, see `http://arxiv.org/math.LO/0211385`.

[8] Miroslav Repicky, *Goldstern-Judah-Shelah preservation theorem for countable support iterations*, ***Fundamenta Mathematicae***, vol. 144 (1994), pp. 55–72.

[9] Andrzej Roslanowski and Saharon Shelah, *Measured creatures*, ***Israel Journal of Mathematics***, accepted, `math.LO/0010070`.

[10] Chaz Schlindwein, *A short proof of the preservation of the ω^ω-bounding property*, ***Mathematical Logic Quarterly***, vol. 50 (2004), no. 1, pp. 29–32.

[11] ———, *Understanding preservation theorems: omega-omega bounding*, preprint, see `http://arxiv.org/math.LO/0505645`.

[12] Saharon Shelah, ***Proper and improper forcing***, Perspectives in Mathematical Logic, Springer-Verlag, 1998.

[13] ———, *Properness without elementaricity*, ***Journal of Applied Analysis***, vol. 10 (2004), pp. 168–289, `math.LO/9712283`.

[14] R. M. Solovay and S. Tennenbaum, *Iterated Cohen extensions and Souslin's problem*, ***Annals of Mathematics. Second Series***, vol. 94 (1971), pp. 201–245.

[15] Jindřich Zapletal, *Descriptive set theory and definable forcing*, ***Memoirs of the American Mathematical Society***, vol. 167, 3 (2004), no. 793.

INSTITUT FÜR DISKRETE MATHEMATIK UND GEOMETRIE
TECHNISCHE UNIVERSITÄT WIEN
1050 WIEN, AUSTRIA
E-mail: kellner@fsmat.at
URL: http://www.logic.univie.ac.at/~kellner

EINSTEIN INSTITUTE OF MATHEMATICS
EDMOND J. SAFRA CAMPUS, GIVAT RAM
THE HEBREW UNIVERSITY OF JERUSALEM
JERUSALEM, 91904, ISRAEL
and
DEPARTMENT OF MATHEMATICS
RUTGERS UNIVERSITY
NEW BRUNSWICK, NJ 08854, USA
E-mail: shelah@math.huji.ac.il
URL: http://www.math.rutgers.edu/~shelah