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SIMPLE FORCING NOTIONS AND FORCING AXIOMS

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§0. Introduction. In the present paper we are interested in simple forcing notions and Forcing Axioms. A starting point for our investigations was the article [4] in which several problems were posed. We answer some of those problems here.

In the first section we deal with the problem of adding Cohen reals by simple forcing notions. Here we interpret *simple* as *of small size*. We try to establish as weak as possible versions of Martin Axiom sufficient to conclude that some forcing notions of size less than the continuum add a Cohen real. For example we show that **MA**(σ-centered) is enough to cause that every small σ-linked forcing notion adds a Cohen real (see Theorem 1.2) and **MA**(Cohen) implies that every small forcing notion adding an unbounded real adds a Cohen real (see Theorem 1.6). A new almost ω^ω-bounding σ-centered forcing notion $\mathbb{Q}_\odot$ appears naturally here. This forcing notion is responsible for adding unbounded reals in this sense, that **MA**($\mathbb{Q}_\odot$) implies that every small forcing notion adding a new real adds an unbounded real (see Theorem 1.13).

In the second section we are interested in Anti-Martin Axioms for simple forcing notions. Here we interpret *simple* as *nicely definable*. Our aim is to show the consistency of **AMA** for as large as possible class of ccc forcing notions with large continuum. It has been known that **AMA**(ccc) implies **CH**, but it has been (rightly) expected that restrictions to regular (simple) forcing notions might help. This is known under large cardinals assumptions and here we try to eliminate them. We show that it is consistent that the continuum is large (with no real restrictions) and **AMA**(projective ccc) holds true (see Theorem 2.5).

Lastly, in the third section we study the influence of **MA** on Σ_3^1-absoluteness for some forcing notions. We show that **MA**$_{\omega_1}(\mathbb{P})$ implies $\Sigma_3^1(\mathbb{P})$-absoluteness (see Theorem 3.2).

Notation. Our notation is rather standard and essentially compatible with that of [3] and [1]. However, in forcing considerations we keep the convention that *a stronger condition is the greater one*.

For a forcing notion $\mathbb{P}$ and a cardinal κ let **MA**$_\kappa(\mathbb{P})$ be the following statement:

> If $\mathscr{A}_\alpha \subseteq \mathbb{P}$ are maximal antichains in $\mathbb{P}$ (for $\alpha < \kappa$), $p \in \mathbb{P}$
> then there exists a filter $G \subseteq \mathbb{P}$ such that $p \in G$ and $G \cap \mathscr{A}_\alpha \neq \emptyset$ for all
> $\alpha < \kappa$.

For a class $\mathscr{K}$ of forcing notions the sentence $\mathbf{MA}_\kappa(\mathscr{K})$ means $(\forall \mathbb{P} \in \mathscr{K})\mathbf{MA}_\kappa(\mathbb{P})$; $\mathbf{MA}_\kappa$ is the sentence $\mathbf{MA}_\kappa(\mathrm{ccc})$.

For a forcing notion $\mathbb{P}$, the canonical $\mathbb{P}$-name for the generic filter on $\mathbb{P}$ will be called $\Gamma_\mathbb{P}$. The incompatibility relation on $\mathbb{P}$ is denoted by $\perp_\mathbb{P}$ (so $\not\perp_\mathbb{P}$ means "compatible").

$\mathfrak{c}$ stands for the cardinality of the continuum. For a tree $T \subseteq 2^{<\omega}$, $[T]$ is the set of all ω-branches through T.

The family of all sets hereditarily of cardinality $< \chi$ (for a regular cardinal χ) is denoted by $\mathscr{H}(\chi)$.

§1. **Adding a Cohen real.** In this section we obtain several results of the form "a (weak) version of **MA** implies that small forcing notions (of some type) add Cohen reals". As a consequence we answer Problem 5.3 of [4] (see Theorem 1.9 and Corollary 1.10 below).

PROPOSITION 1.1. *Suppose $\mathbb{P}$ is a forcing notion and $\bar{h}$ is a function such that*
  (1) $\mathrm{dom}(\bar{h}) \subseteq \mathbb{P}$, $\mathrm{rng}(\bar{h}) \subseteq 2^{<\omega}$,
  (2) *if $p_1, p_2 \in \mathrm{dom}(\bar{h})$, $p_1 \not\perp_\mathbb{P} p_2$ then either $\bar{h}(p_1) \subseteq \bar{h}(p_2)$ or $\bar{h}(p_2) \subseteq \bar{h}(p_1)$,*
  (3) *if $q \in \mathbb{P}$ then there is $v_0 \in 2^{<\omega}$ such that*

$$(\forall v \in 2^{<\omega}, v_0 \subseteq v)(\exists p' \in \mathrm{dom}(\bar{h}))(p' \not\perp_\mathbb{P} q \ \& \ v \subseteq \bar{h}(p')).$$

Then $\mathbb{P}$ adds a Cohen real.

PROOF. Though this is immediate, we present the proof fully for reader's convenience. Let $\bar{h} : \mathrm{dom}(\bar{h}) \longrightarrow 2^{<\omega}$ be the function given by the assumptions. Define a $\mathbb{P}$-name $\dot{c}$ by

$$\Vdash_\mathbb{P} \dot{c} = \bigcup\{\bar{h}(p) : p \in \mathrm{dom}(\bar{h}) \cap \Gamma_\mathbb{P}\}.$$

First note that, by the properties of $\bar{h}$, for every filter $G \subseteq \mathbb{P}$ the set $\{\bar{h}(p) : p \in \mathrm{dom}(\bar{h}) \cap G\}$ is a chain in $(2^{<\omega}, \subseteq)$. Hence

$$\Vdash_\mathbb{P} \dot{c} \in 2^{\leq\omega}.$$

But really $\dot{c}$ is a name for a member of 2^ω: suppose not. Then we have $q \in \mathbb{P}$, $m \in \omega$ such that

$$q \Vdash_\mathbb{P} \dot{c} \in 2^m.$$

Applying the third property of $\bar{h}$ we get $v_0 \in 2^{<\omega}$ as there. Let $v \in 2^{<\omega}$, $v_0 \subseteq v$ be such that $\mathrm{lh}(v) > m$. We find $p' \in \mathrm{dom}(\bar{h})$ such that $p' \not\perp_\mathbb{P} q$ and $v \subseteq \bar{h}(p')$. Thus $p' \Vdash_\mathbb{P} v \subseteq \bar{h}(p') \subseteq \dot{c}$, a contradiction.

To show that

$$\Vdash_\mathbb{P} \text{"}\dot{c} \text{ is a Cohen real over } \mathbf{V}\text{"}$$

suppose that we have a closed nowhere dense set $A \subseteq 2^\omega$ and a condition $q \in \mathbb{P}$ such that

$$q \Vdash_\mathbb{P} \dot{c} \in A.$$

Take $v_0 \in 2^{<\omega}$ given by condition (3) (for q). Since A is nowhere dense we may choose $v \in 2^{<\omega}$ such that $v_0 \subseteq v$ and $[v] \cap A = \emptyset$. By the choice of v_0, there is a condition $p' \in \mathrm{dom}(\bar{h})$ such that $p' \not\perp_\mathbb{P} q$ and $v \subseteq \bar{h}(p')$ (so $p' \Vdash_\mathbb{P} \dot{c} \notin A$), a contradiction. ⊣

THEOREM 1.2. *Assume* $\mathbf{MA}_\kappa(\sigma\text{-}centered)$. *If* $\mathbb{P}$ *is a* σ*-linked atomless forcing notion of size* κ *then* $\mathbb{P}$ *adds a Cohen real.*

PROOF. We may assume that the partial order $(\mathbb{P}, \leq)$ is separative, i.e.,

  if $p, q \in \mathbb{P}$, $p \not\leq_{\mathbb{P}} q$
  then there is $r \in \mathbb{P}$ such that $q \leq_{\mathbb{P}} r$ and $r \perp_{\mathbb{P}} p$.

Of course we may assume that $\mathbb{P}$ is a partial order on a subset of 2^ω. We are going to show that (under our assumptions) there exists a function $\bar{h}$ as in the assumptions of Proposition 1.1. Since $\mathbb{P}$ is σ-linked there are sets $\mathscr{D}_n \subseteq \mathbb{P}$ such that $\bigcup_{n\in\omega} \mathscr{D}_n = \mathbb{P}$ and any two members of $\mathscr{D}_n$ are compatible in $\mathbb{P}$ (i.e., each $\mathscr{D}_n$ is linked). Let N be a countable elementary submodel of $(\mathscr{H}((\beth_7)^+), \in, <^*)$ such that $\mathbb{P}, \langle \mathscr{D}_n : n \in \omega \rangle, \ldots \in N$.

We define a forcing notion $\mathbb{R} = \mathbb{R}(\mathbb{P})$:
conditions are pairs $r = \langle h, w \rangle = \langle h^r, w^r \rangle$ such that
  (a) h is a finite function, $\mathrm{dom}(h) \subseteq \mathbb{P} \cap N$, $\mathrm{rng}(h) \subseteq 2^{<\omega}$,
  (b) if $p_1, p_2 \in \mathrm{dom}(h)$ then either $p_1 \leq_{\mathbb{P}} p_2$ or $p_2 \leq_{\mathbb{P}} p_1$ or $p_1 \perp_{\mathbb{P}} p_2$,
  (c) if $p_1, p_2 \in \mathrm{dom}(h)$, $p_1 \leq_{\mathbb{P}} p_2$ then $h(p_1) \subseteq h(p_2)$,
  (d) $w \in [\mathbb{P}]^{<\omega}$,
the order is such that $r_1 \leq_{\mathbb{R}} r_2$ if and only if
  (α) $h^{r_1} \subseteq h^{r_2}$,
  (β) $w^{r_1} \subseteq w^{r_2}$,
  (γ) if $q \in w^{r_1}$, $p \in \mathrm{dom}(h^{r_1})$, p, q are compatible in $\mathbb{P}$ and no $p' \in \mathrm{dom}(h^{r_1})$ satisfies $p \leq_{\mathbb{P}} p'$, $p \neq p'$, $p' \not\perp_{\mathbb{P}} q$ then the set

$$J_{p,q}^{r_1,r_2} \overset{\text{def}}{=} \{ h^{r_2}(p_1) : \quad p \leq p_1 \in \mathrm{dom}(h^{r_2}) \ \& \ p_1 \not\perp_{\mathbb{P}} q \\ \& (\forall p_2)(p_1 < p_2 \in \mathrm{dom}(h^{r_2}) \Rightarrow p_2 \perp_{\mathbb{P}} q) \}$$

contains a front of $2^{<\omega}$ above $h^{r_1}(p)$ (i.e., for every $\eta \in 2^\omega$ such that $h^{r_1}(p) \subseteq \eta$ there is k with $\eta \restriction k \in J_{p,q}^{r_1,r_2}$).

CLAIM 1.2.1. $(\mathbb{R}, \leq_{\mathbb{R}})$ *is a partial order.*

PROOF OF THE CLAIM. The relation $\leq_{\mathbb{R}}$ is reflexive as $J_{p,q}^{r,r} = \{h^r(p)\}$ for all relevant p, q. For the transitivity suppose that $r_1 \leq_{\mathbb{R}} r_2$ and $r_2 \leq_{\mathbb{R}} r_3$. Clearly the conditions (α), (β) for the pair r_1, r_3 are satisfied. To get condition (γ) note that if $\{v_0, \ldots, v_{k-1}\}$ is a front in $2^{<\omega}$ above v and $\{v_0^0, \ldots, v_0^{l-1}\}$ is a front in $2^{<\omega}$ above v_0 then $\{v_0^0, \ldots, v_0^{l-1}, v_1, \ldots, v_{k-1}\}$ is a front above v.

CLAIM 1.2.2. $\mathbb{R}$ *is* σ*-centered.*

PROOF OF THE CLAIM. Note that if $r_1, r_2 \in \mathbb{R}$, $h = h^{r_1} = h^{r_2}$ then $\langle h, w^{r_1} \cup w^{r_2} \rangle$ is a common upper bound of r_1, r_2.

CLAIM 1.2.3. *Suppose* $p \in \mathbb{P} \cap N$, $q \in \mathbb{P}$, $r_0 \in \mathbb{R}$ *and* $m \in \omega$. *Then the following sets are dense in* $\mathbb{R}$:
  (1) $I_p^0 \overset{\text{def}}{=} \{ r \in \mathbb{R} : (\forall q \in w^r)[p \not\perp_{\mathbb{P}} q \Rightarrow (\exists p' \in \mathrm{dom}(h^r))(p \leq_{\mathbb{P}} p' \ \& \ p' \not\perp_{\mathbb{P}} q)] \}$,
  (2) $I_q^1 \overset{\text{def}}{=} \{ r \in \mathbb{R} : q \in w^r \}$,

(3) $I_{r_0,m}^2 \overset{\text{def}}{=} \{r \in \mathbb{R} : r \perp_{\mathbb{R}} r_0$ or for every $q \in w^{r_0}$ and $p \in \text{dom}(h^{r_0})$ such that $p \not\perp_{\mathbb{P}} q$ & $(\forall p' \in \text{dom}(h^{r_0}))([p \leq_{\mathbb{P}} p' \& p \neq p'] \Rightarrow p' \perp_{\mathbb{P}} q)$ and for every $v \in 2^m$ such that $h^{r_0}(p) \subseteq v$ there is $p'' \in \text{dom}(h^r)$ with $p \leq_{\mathbb{P}} p''$, $p'' \not\perp_{\mathbb{P}} q$ and $v \subseteq h^r(p'')\}$.

PROOF OF THE CLAIM. (1) Assume $p \in \mathbb{P} \cap N$, $r_0 \in \mathbb{R}$. Let $\langle q^l : l < l^* \rangle$ be an enumeration of $\{q \in w^{r_0} : q \not\perp_{\mathbb{P}} p\}$. Choose conditions p_l (for $l < l^*$) such that

    (1) $p_l \in \mathbb{P} \cap N$,
    (2) for each $p' \in \text{dom}(h^{r_0})$ either $p' \leq_{\mathbb{P}} p_l$ or $p' \perp_{\mathbb{P}} p_l$,
    (3) $p \leq_{\mathbb{P}} p_l$,
    (4) $\langle p_l : l < l^* \rangle$ are pairwise incompatible,
    (5) $p_l \not\perp_{\mathbb{P}} q^l$.

For this we need the assumption that $\mathbb{P}$ is atomless and σ-linked. First take $p_l^+ \in \mathbb{P}$ such that $p, q^l \leq_{\mathbb{P}} p_l^+$ (for $l < l^*$). Next we choose $p_l^{++} \in \mathbb{P}$, $p_l^+ \leq_{\mathbb{P}} p_l^{++}$ such that the clauses (2)–(4) are satisfied (remember that $\mathbb{P}$ is atomless and $\text{dom}(h^{r_0})$ is finite). Let $n_l \in \omega$ be such that $p_l^{++} \in \mathcal{D}_{n_l}$. As N is an elementary submodel of $(\mathscr{H}(\beth_7^+), \in, <^*)$ we find $\langle p_l : l < l^* \rangle \in N$ such that $p_l \in \mathcal{D}_{n_l}$ and the clauses (2)–(4) are satisfied. But now we have (1) too. Moreover this sequence satisfies (5) since $p_l, p_l^{++} \in \mathcal{D}_{n_l}$ and the second condition is stronger than q^l (remember that the sets $\mathcal{D}_{n_l}$ are linked).

Define h^r by

$$\text{dom}(h^r) = \text{dom}(h^{r_0}) \cup \{p_l : l < l^*\}, \quad h^{r_0} \subseteq h^r \quad \text{and}$$

$$h^r(p_l) = \bigcup \{h^{r_0}(p') : p' \leq_{\mathbb{P}} p_l \& p' \in \text{dom}(h^{r_0})\}.$$

First note that all conditions $p' \in \text{dom}(h^{r_0})$ satisfying $p' \leq_{\mathbb{P}} p_l$ are compatible in $\mathbb{P}$ and hence (by (b) for r_0) they are pairwise comparable and thus (by (c) for r_0) the set

$$\{h^{r_0}(p') : p' \leq_{\mathbb{P}} p \& p' \in \text{dom}(h^{r_0})\}$$

is a (finite) chain in $(2^{<\omega}, \subseteq)$. Hence $h^r(p_l) \in 2^{<\omega}$ (actually $h^r(p_l) = h^{r_0}(p_l^*)$ for the $\leq_{\mathbb{P}}$-maximal $p_l^* \in \text{dom}(h^{r_0})$ such that $p_l^* \leq_{\mathbb{P}} p_l$; if there is no such p_l^* then $h^r(p_l) = \langle\rangle$). Consequently h^r satisfies (a). One can easily check that h^r satisfies conditions (b), (c) too and thus $r = \langle h^r, w^{r_0} \rangle \in I_p^0$. Conditions $(\alpha), (\beta)$ for the pair r_0, r are clear. To check the clause (γ) suppose that $p' \in \text{dom}(h^{r_0})$, $q \in w^{r_0}$ are relevant for it. If for each $l < l^*$ either $p_l \perp_{\mathbb{P}} q$ or $p' \not\leq_{\mathbb{P}} p_l$ (which implies $p' \perp_{\mathbb{P}} p_l$) then $J_{p',q}^{r_0,r} = \{h^{r_0}(p')\}$ as the property of p' there is preserved. Otherwise $J_{p',q}^{r_0,r} = \{h^r(p_l) : l < l^*, p_l \not\perp_{\mathbb{P}} q, p' \leq_{\mathbb{P}} p_l\}$. But due to condition (b) for r_0 we have that each condition from $\text{dom}(h^{r_0})$ weaker than any p_l such that $p_l \not\perp_{\mathbb{P}} q$, $p' \leq_{\mathbb{P}} p_l$ is weaker than p'. Consequently $h^r(p_l) = h^{r_0}(p')$ for all relevant p_l and we get $r_0 \leq_{\mathbb{R}} r$.

(2) Let $q \in \mathbb{P}$, $r \in \mathbb{R}$. Take $\langle h^r, w^r \cup \{q\}\rangle$; easily it is a condition in I_q stronger than r.

(3) Assume $r_0 \in \mathbb{R}$, $m \in \omega$. Let $r \in \mathbb{R}$. If r_0, r are incompatible in $\mathbb{R}$ then $r \in I_{r_0,m}^2$ and we are done. So we may assume that $r_0 \leq_{\mathbb{R}} r$.

Let $\langle (q^l, p^l, v^l) : l < l^* \rangle$ list all triples (q, p, v) such that

    $q \in w^r$, $p \in \text{dom}(h^r)$, $q \not\perp_{\mathbb{P}} p$, $h^r(p) \subseteq v \in 2^m$ and there is no $p' \in \text{dom}(h^r)$ with $p <_{\mathbb{P}} p'$, $q \not\perp_{\mathbb{P}} p'$.

(It is possible that $l^* = 0$, e.g., if m is too small.)

Now choose conditions p_l^* such that

(1) $p_l^* \in \mathbb{P} \cap N$,
(2) for each $p \in \mathrm{dom}(h^r)$ either $p \leq_{\mathbb{P}} p_l^*$ or $p \perp_{\mathbb{P}} p_l^*$,
(3) $p^l \leq_{\mathbb{P}} p_l^*$,
(4) $\langle p_l^* : l < l^* \rangle$ are pairwise incompatible,
(5) $p_l^* \not\perp_{\mathbb{P}} q^l$.

For this we follow exactly the lines of the respective part of the proof of (1) (so this is another place we use the assumptions on $\mathbb{P}$).

Next define $h^{r_1} = h^r \cup \{(p_l^*, v^l) : l < l^*\}$, $w^{r_1} = w^r$, $r_1 = \langle h^{r_1}, w^{r_1} \rangle$. Similarly as in (1) one checks that $r_1 \in \mathbb{R}$.

The condition r_1 is stronger than r: clauses (α), (β) are clear. For (γ) suppose that $q \in w^r$, $p \in \mathrm{dom}(h^r)$ are relevant for this clause. If $m \geq \mathrm{lh}(h^r(p))$ then each $v \in 2^m$ extending $h^r(p)$ appears as $v^l = h^{r_1}(p_l^*)$ for some $l < l^*$ such that $q^l = q$, $p^l = p$. Hence $J^{r,r_1}_{p,q}$ contains a front above $h^r(p)$. If $m < \mathrm{lh}(h^r(p))$ then the pair (p,q) does not appear as (p^l, q^l). Note that for each $l < l^*$, if $q^l \not\perp_{\mathbb{P}} p$ then $p^l \perp_{\mathbb{P}} p$ (as p^l cannot be stronger than p since $\mathrm{lh}(h^r(p^l)) \leq m$) and hence $p_l^* \perp_{\mathbb{P}} p$. If $q^l \perp_{\mathbb{P}} p$ then we get the same conclusion (though p^l might be weaker than p, the demands (5), (2) of the choice of p_l^* imply that $p_l^* \perp_{\mathbb{P}} p$). Consequently the "maximality" property of p is preserved in $\mathrm{dom}(h^{r_1})$ and $J^{r,r_1}_{p,q} = \{h^r(p)\}$.

To prove that $r_1 \in I^2_{r_0,m}$ suppose that $q \in w^{r_0}$ and $p \in \mathrm{dom}(h^{r_0})$ is maximal (in $\mathrm{dom}(h^{r_0})$) compatible with q. Let $v \in 2^m$ extend $h^{r_0}(p)$. Since $r_0 \leq_{\mathbb{R}} r$ we find $p' \in \mathrm{dom}(h^r)$ stronger than p, maximal (in $\mathrm{dom}(h^r)$) compatible with q and such that v, $h^r(p')$ are comparable (by condition (γ)). If $v \subseteq h^r(p')$ then we are done. So suppose $h^r(p') \subseteq v$. Then for some $l < l^*$ we have $q = q^l$, $p' = p^l$ and $v = v^l$. By the choice of p^l and the definition of $h^{r_1}(p_l)$ we get

$$p \leq_{\mathbb{P}} p' \leq_{\mathbb{P}} p_l^* \quad \& \quad p_l^* \not\perp_{\mathbb{P}} q_l = q \quad \& \quad v = v_l = h^{r_1}(p_l^*).$$

The claim is proved.

Since we have assumed $\mathbf{MA}_\kappa(\sigma\text{-centered})$ we find a filter $H \subseteq \mathbb{R}$ such that

$(\oplus_0)$ $H \cap \{r \in \mathbb{R} : r \perp_{\mathbb{R}} r_0 \text{ or } (r_0 \leq r \ \& \ r \in I^0_p)\} \neq \emptyset$ for $p \in \mathbb{P} \cap N$, $r_0 \in \mathbb{R}$,
$(\oplus_1)$ $H \cap I^1_q \neq \emptyset$ for $q \in \mathbb{P}$ and
$(\oplus_2)$ $H \cap I^2_{r_0,m} \neq \emptyset$ for $r_0 \in \mathbb{R}$, $m \in \omega$.

Put $\bar{h} = \bigcup \{h^r : r \in H\}$. Clearly $\bar{h}$ is a function from a subset of $\mathbb{P} \cap N$ to $2^{<\omega}$. Conditions (b), (c) imply that $\bar{h}$ satisfies the second requirement of the assumptions of Proposition 1.1.

Suppose now that $q \in \mathbb{P}$. Take $p \in \mathbb{P} \cap N$ compatible with q and choose $r_0 \in H \cap I^0_p \cap I^1_q$ (so $q \in w^{r_0}$). Next take $p^* \in \mathrm{dom}(h^{r_0})$ such that

$$p \leq_{\mathbb{P}} p^* \ \& \ p^* \not\perp_{\mathbb{P}} q \ \& \ (\forall p' \in \mathrm{dom}(h_0))([p^* \leq_{\mathbb{P}} p' \ \& \ p^* \neq p'] \Rightarrow p' \perp_{\mathbb{P}} q).$$

Assume that $h_{r_0}(p^*) \subseteq v \in 2^m$.

By $(\oplus_2)$ we find $r \in H \cap I^2_{r_0,m}$. As $r_0, r \in H$, H is a filter, we cannot have $r \perp_{\mathbb{R}} r_0$. Consequently "the second part" of the definition of $I^2_{r_0,m}$ applies to r. Looking at this definition (with p^* as p there) we see that there is $p' \in \mathrm{dom}(h^r)$ with

$$p^* \leq_{\mathbb{P}} p' \ \& \ p' \not\perp_{\mathbb{P}} q \ \& \ v \subseteq h^r(p').$$

So $v_0 = \bar{h}(p^*)$ is as required in (3). Applying Proposition 1.1 we finish the proof of the theorem.   ⊣

REMARK 1.3. Of course, what we have shown in Theorem 1.2 is that $\mathbf{MA}_\kappa(\mathbb{R}(\mathbb{P}))$ implies $\mathbb{P}$ adds a Cohen real, provided $\mathbb{P}$ is atomless σ-linked of size κ.

COROLLARY 1.4. *Assume* $\mathbf{MA}_\kappa$. *If* $\mathbb{P}$ *is a ccc atomless forcing notion of size* κ *then* $\mathbb{P}$ *adds a Cohen real.*   ⊣

PROPOSITION 1.5. *Let* $\mathbb{P}$ *be a ccc forcing notion. Then the following conditions are equivalent*:
  (a) $\Vdash_\mathbb{P}$"*there is an unbounded real in* ω^ω *over* $\mathbf{V}$"
  (b) *there exists a sequence* $\langle \mathscr{A}_n : n \in \omega \rangle$ *of maximal antichains of* $\mathbb{P}$ *such that*
      (i) $\mathscr{A}_0 = \{0\}$,
      (ii) $(\forall n \in \omega)(\forall p \in \mathscr{A}_{n+1})(\exists q \in \mathscr{A}_n)(q \leq_\mathbb{P} p)$,
      (iii) $(\forall n \in \omega)(\forall p \in \mathscr{A}_n)(\|\{q \in \mathscr{A}_{n+1} : p \leq_\mathbb{P} q\}\| = \omega)$,
      (iv) $(\forall q \in \mathbb{P})(\exists n \in \omega)(\|\{p \in \mathscr{A}_n : p \not\perp_\mathbb{P} q\}\| = \omega)$.

PROOF. Easy, left for the reader.   ⊣

THEOREM 1.6. *Suppose that* $\mathbb{P}$ *is a ccc forcing notion such that* $\|\mathbb{P}\| < \mathbf{cov}(\mathscr{M})$ *(i.e., unions of* $\|\mathbb{P}\|$ *many meager sets do not cover* 2^ω*) and*

$$\Vdash_\mathbb{P} \text{``there is an unbounded real over } \mathbf{V}\text{''}.$$

Then

$$\Vdash_\mathbb{P} \text{``there is a Cohen real over } \mathbf{V}\text{''}.$$

PROOF. We are going to apply Proposition 1.1 and for this we will construct a function $\bar{h}$ satisfying (1)–(3) of Proposition 1.1.

Let $\langle \mathscr{A}_n : n \in \omega \rangle$ be a sequence of maximal antichains of $\mathbb{P}$ given by (b) of Proposition 1.5. Take a countable elementary submodel N of $(\mathscr{H}(\beth_7^+), \in, <^*)$ such that $\mathbb{P}, \langle \mathscr{A}_n : n \in \omega \rangle, \ldots \in N$. Consider the following partial order $\mathbb{C}$: **conditions** are finite functions h such that
  (a) $\operatorname{dom}(h) \subseteq \bigcup_{n \in \omega} \mathscr{A}_n$, $\operatorname{rng}(h) \subseteq 2^{<\omega}$,
  (b) if $p_1, p_2 \in \operatorname{dom}(h)$, $p_1 \leq_\mathbb{P} p_2$ then $h(p_1) \subseteq h(p_2)$,
the order is the inclusion; $h_1 \leq_\mathbb{C} h_2$ iff $h_1 \subseteq h_2$.
Clearly $\mathbb{C}$ is (isomorphic to) the Cohen forcing notion.

CLAIM 1.6.1. *Let* $p \in \bigcup_{n \in \omega} \mathscr{A}_n$, $h_0 \in \mathbb{C}$, $q \in \mathbb{P}$, $m \in \omega$. *Then the following sets are dense in* $\mathbb{C}$:

(1) $J_p^0 \overset{\text{def}}{=} \{h \in \mathbb{C} : p \in \operatorname{dom}(h)\}$,

(2) $J_{q,m,h_0}^1 \overset{\text{def}}{=} \{h \in \mathbb{C} : h \perp_\mathbb{C} h_0$ *or for every* $p \in \operatorname{dom}(h_0)$ *such that for some* $n \in \omega$, $p \in \mathscr{A}_n$ *and the set* $\{p' \in \mathscr{A}_{n+1} : p \leq_\mathbb{P} p' \& p' \not\perp_\mathbb{P} q\}$ *is infinite we have: for every* $v \in 2^m$ *extending* $h_0(p)$ *there is* $p' \in \operatorname{dom}(h)$ *with* $p \leq_\mathbb{P} p'$, $p' \not\perp_\mathbb{P} q$ *and* $h(p') = v \}$.

PROOF OF THE CLAIM. (1) Assume $p \in \bigcup_{n \in \omega} \mathscr{A}_n, h \in \mathbb{C}$. Extend h to h' by putting

$$h'(p) = \bigcup \{h(p') : p' \in \operatorname{dom}(h) \& p' \leq_\mathbb{P} p\}.$$

Easily this h' satisfies $h' \in J_p^0$, $h \leq_\mathbb{C} h'$.

(2) Suppose that $q \in \mathbb{P}$, $m \in \omega$, $h_0 \in \mathbb{C}$, $h \in \mathbb{C}$. We may assume that $h_0 \leq_\mathbb{C} h$. Let $\langle (p^l, n^l, v^l) : l < l^* \rangle$ enumerate all triples $p \in \mathrm{dom}(h_0)$, $n \in \omega$, $v \in 2^m$ such that

 (α) $p \in \mathscr{A}_n$ and the set $\{p' \in \mathscr{A}_{n+1} : p \leq_\mathbb{P} p' \ \& \ p' \not\perp_\mathbb{P} q\}$ is infinite.
 (β) $h_0(p) \subseteq v$.
Next (using (α) above) choose $p_l^* \in \mathscr{A}_{n^l+1}$ such that
 (1) $p^l \leq_\mathbb{P} p_l^*$
 (2) $\langle p_l^* : l < l^* \rangle$ are pairwise incompatible
 (3) for each $p \in \mathrm{dom}(h)$, $l < l^*$ either $p \leq_\mathbb{P} p_l^*$ or $p \perp_\mathbb{P} p_l^*$.
Now put $\mathrm{dom}(h') = \mathrm{dom}(h) \cup \{p_l^* : l < l^*\}$, $h'(p_l^*) = v^l$ and $h' \upharpoonright \mathrm{dom}(h) = h$. Easily $h' \in \mathbb{C}$, $h \leq_\mathbb{C} h'$ and $h' \in J^1_{q,m,h_0}$. This finishes the claim.

Since $\|\mathbb{P}\| < \mathbf{cov}(\mathscr{M})$ we find a filter $H \subseteq \mathbb{C}$ such that $H \cap J_p^0 \neq \emptyset$ and $H \cap J^1_{q,m,h_0} \neq \emptyset$ for all $q \in \mathbb{P}$, $m \in \omega$, $h_0 \in \mathbb{C}$ and $p \in \bigcup_{n \in \omega} \mathscr{A}_n$. Put $\bar{h} = \bigcup H$. Then clearly $\bar{h} : \bigcup_{n \in \omega} \mathscr{A}_n \longrightarrow 2^{<\omega}$ is a function satisfying the requirements (1), (2) of Proposition 1.1. To check the third condition there suppose $q \in \mathbb{P}$. Take $n \in \omega$ and $p^* \in \mathscr{A}_n$ such that the set $\{p' \in \mathscr{A}_{n+1} : p^* \leq_\mathbb{P} p' \ \& \ p' \not\perp_\mathbb{P} q\}$ is infinite (possible by the choice of the $\mathscr{A}_k$'s). Since $H \cap J_{p^*}^0 \neq \emptyset$ we find a condition $h_0 \in H$ such that $p^* \in \mathrm{dom}(h_0)$. Suppose that $v \in 2^{<\omega}$, $\bar{h}(p^*) \subseteq v$ and let $m = \mathrm{lh}(v)$. Take $h_1 \in H \cap J^1_{q,m,h_0}$. Since h_0, h_1 cannot be incompatible, $p^* \in \mathrm{dom}(h_0)$, $h(p^*) \subseteq v \in 2^m$ we find $p' \in \mathrm{dom}(h_1)$ such that $p^* \leq p'$, $p' \not\perp_\mathbb{P} q$ and $h_1(p') = v$. Since $h_1(p') = \bar{h}(p')$ we conclude that $v_0 = \bar{h}(p^*)$ is as required in (3) of Proposition 1.1 for q. The theorem is proved. $\dashv$

DEFINITION 1.7. A forcing notion $\mathbb{P}$ is *almost ω^ω-bounding* if

for each $\mathbb{P}$-name $\dot{f}$ for an element of ω^ω and a condition $p \in \mathbb{P}$ there is $g \in \omega^\omega \cap \mathbf{V}$ such that for every $X \in [\omega]^\omega \cap \mathbf{V}$:

$$(\exists p' \geq_\mathbb{P} p)(p' \Vdash_\mathbb{P} (\exists^\infty n \in X)(\dot{f}(n) < g(n))).$$

LEMMA 1.8. (1) *Suppose that $\mathbb{P}$ is a ccc forcing notion such that for every integer n the product forcing notion $\mathbb{P}^n$ does not add unbounded real and satisfies the ccc. Then the ω-product $\mathbb{P}^\omega$ with finite support is almost ω^ω-bounding and satisfies the ccc.*

(2) *Finite support iteration of ccc almost ω^ω-bounding forcing notions does not add a dominating real.*

PROOF. (1) Suppose that for each $n \in \omega$ the product forcing notion $\mathbb{P}^n$ satisfies the ccc and does not add unbounded reals. By [3, 23.11] we know that then $\mathbb{P}^\omega$ satisfies the ccc. We have to show that $\mathbb{P}^\omega$ is almost ω^ω-bounding. Let $\dot{f}$ be a $\mathbb{P}^\omega$-name for a function in ω^ω. For each $n, k \in \omega$ choose a maximal antichain $\mathscr{A}_k^n$ of $\mathbb{P}^n$ and mappings $\varphi_k^n : \mathscr{A}_k^n \longrightarrow \mathbb{P}^\omega$ and $g_k^n : \mathscr{A}_k^n \longrightarrow \omega$ such that

$$(\forall q \in \mathscr{A}_k^n)(\varphi_k^n(q) \upharpoonright n = q \ \& \ \varphi_k^n(q) \Vdash_{\mathbb{P}^\omega} \dot{f}(k) = g_k^n(q))$$

(possible as $\mathbb{P}^n \lessdot \mathbb{P}^\omega$). Thus, for each $n \in \omega$, we have a $\mathbb{P}^n$-name $\dot{g}^n$ for a function in ω^ω defined by

$$(\forall k \in \omega)(\forall q \in \mathscr{A}_k^n)(q \Vdash_{\mathbb{P}^n} \dot{g}^n(k) = g_k^n(q)).$$

Since $\mathbb{P}^n$ does not add unbounded reals and satisfies the ccc we find a function $g_n \in \omega^\omega$ such that

$$\Vdash_{\mathbb{P}^n} (\exists m \in \omega)(\forall k \geq m)(\dot{g}^n(k) < g_n(k)).$$

Take $g \in \omega^\omega$ such that $(\forall n \in \omega)(\exists m \in \omega)(\forall k \geq m)(g_n(k) < g(k))$. We claim that

$$\Vdash_{\mathbb{P}^\omega} (\forall X \in [\omega]^\omega \cap \mathbf{V})(\exists^\infty k \in X)(\dot{f}(k) < g(k)).$$

To this end suppose that $X \in [\omega]^\omega$, $p \in \mathbb{P}^\omega$ and $N \in \omega$. Take n such that $p \in \mathbb{P}^n$ and look at the function g_n. By its choice we find a condition $p' \in \mathbb{P}^n$ stronger than p and an integer m_0 such that $p' \Vdash_{\mathbb{P}^n} (\forall k \geq m_0)(\dot{g}^n(k) < g_n(k))$. By the choice of g we find $m_1 \in \omega$ such that $(\forall k \geq m_1)(g_n(k) < g(k))$. Let $k \in X$ be such that $k > m_0 + m_1 + N$. Since $\mathscr{A}_k^n$ is a maximal antichain of $\mathbb{P}^n$ we may take a condition $q \in \mathscr{A}_k^n$ compatible with p'. Let p'' be a common upper bound of p' and $\varphi_k^n(q)$ in $\mathbb{P}^\omega$. Then (p'' is stronger than p and)

$$p'' \Vdash_{\mathbb{P}^\omega} \dot{f}(k) = g_k^n(q) = \dot{g}^n(k) < g_n(k) < g(k)$$

(remember k is above m_0, m_1). Since $k \in X$ is greater than N we finish by standard density arguments.

(2) See [6, Ch VI, 3.6+3.17] or [1, 6.5.3].   ⊣

THEOREM 1.9. *Assume* $\mathbf{MA}_\kappa(ccc$ & *almost* ω^ω-*bounding). Then every atomless ccc forcing notion of size* $\leq \kappa$ *adds a Cohen real.*

PROOF. We assume of course that $\kappa \geq \aleph_1$. Let $\mathbb{P}$ be a ccc forcing notion, $\|\mathbb{P}\| \leq \kappa$. If $\mathbb{P}$ adds an unbounded real then Theorem 1.6 applies (note that the Cohen forcing notion is almost ω^ω-bounding, so our assumption implies $\kappa < \mathbf{cov}(\mathscr{M})$). Thus to finish the proof we need to show that $\mathbb{P}$ adds an unbounded real. This fact is done by the two claims below.

CLAIM 1.9.1. *Assume* $\mathbf{MA}_\kappa(ccc$ & ω^ω-*bounding). Suppose that* $\mathbb{P}$ *is a ccc forcing notion which adds no unbounded real (i.e., it is* ω^ω-*bounding). Then for every* $n \in \omega$ *the product forcing notion* $\mathbb{P}^n$ *adds no unbounded real and satisfies the ccc.*

PROOF OF THE CLAIM. As $\mathbf{MA}_\kappa(ccc$ & ω^ω-bounding) applies to $\mathbb{P}$, this forcing notion has the Knaster property (strong ccc) and consequently all powers of it satisfy the ccc. What might fail is not adding unbounded reals. So suppose that n is the first such that

$$\Vdash_{\mathbb{P}^n} \text{"there is an unbounded real over } \mathbf{V}\text{"}.$$

Clearly $n > 1$. By Proposition 1.5 we find maximal antichains $\mathscr{A}_k \subseteq \mathbb{P}^n$ (for $k < \omega$) satisfying conditions (1)–(4) of clause (b) there.

We may think that $\mathbb{P}$ is an ordering on κ. Let N be an elementary submodel of $(\mathscr{H}(\beth_7^+), \in, <^*)$ such that

$$\mathbb{P}, \leq_{\mathbb{P}}, \langle \mathscr{A}_k : k \in \omega \rangle, \dots \in N, \quad \kappa + 1 \subseteq N \quad \text{and} \quad \|N\| = \kappa.$$

Let $\pi : N \longrightarrow M$ be the Mostowski collapse of N, M a transitive set. Note that $\pi(\mathbb{P}) = \mathbb{P}$, $\pi(\mathscr{A}_k) = \mathscr{A}_k$ etc. Since $\mathbb{P}^{n-1}$ is ccc and adds no unbounded real we may apply our restricted version of $\mathbf{MA}_\kappa$ to it and get an M-generic filter $H \subseteq \mathbb{P}^{n-1}$ in

V. (Note that if $\mathscr{A} \subseteq \mathbb{P}^{n-1}$, $\mathscr{A} \in M$ then $M \models$ "$\mathscr{A}$ is a maximal antichain of $\mathbb{P}^{n-1}$" iff $\mathscr{A}$ is really a maximal antichain of $\mathbb{P}^{n-1}$.) Let

$$\mathscr{A}_k^H \stackrel{\text{def}}{=} \{p \in \mathbb{P} : (\exists \bar{p} \in H)((\bar{p}, p) \in \mathscr{A}_k)\} \in M[H].$$

Then

$$M[H] \models \text{ "}\mathscr{A}_k^H \text{ is a maximal antichain of } \mathbb{P}\text{"}$$

and easily the same holds in **V**. As $\mathbb{P}$ adds no unbounded real, by Proposition 1.5 we find $p \in \mathbb{P}$ such that

$$(\forall k \in \omega)(\|\{p' \in \mathscr{A}_k^H : p \not\perp_{\mathbb{P}} p'\}\| < \omega)$$

and thus

$$(\forall k \in \omega)(\|\{(\bar{p}', p') \in \mathscr{A}_k : \bar{p}' \in H \ \& \ p \not\perp_{\mathbb{P}} p'\}\| < \omega).$$

Since $\mathbb{P}^{n-1}$ adds no unbounded real (and this is true in M too) we find finite sets $A_k \subseteq \mathscr{A}_k$ (for $k \in \omega$) and a condition $\bar{p} \in \mathbb{P}^{n-1}$ such that for each $k \in \omega$

$$M \models \ \bar{p} \Vdash_{\mathbb{P}^{n-1}} \{(\bar{p}', p') \in \mathscr{A}_k : \bar{p}' \in \Gamma_{\mathbb{P}^{n-1}} \ \& \ p \not\perp_{\mathbb{P}} p'\} \subseteq A_k.$$

This means that if $(\bar{p}', p') \in \mathscr{A}_k \setminus A_k$ then either $\bar{p} \perp_{\mathbb{P}^{n-1}} \bar{p}'$ or $p \perp_{\mathbb{P}} p'$. Hence the condition $(\bar{p}, p) \in \mathbb{P}^n$ is a counterexample to the fourth property of $\langle \mathscr{A}_k : k \in \omega \rangle$. The claim is proved.

It follows from Claim 1.9.1 and Lemma 1.8 that (under our assumptions) $\mathbb{P}$ is σ-centered. So now we may use the following claim.

CLAIM 1.9.2. *Every σ-centered atomless forcing notion adds an unbounded real.*

PROOF OF THE CLAIM. Folklore; see e.g., 5.2 of [4]. $\dashv$

COROLLARY 1.10. *It is consistent that $\mathfrak{c} > \aleph_1$, every atomless ccc forcing notion of the size $< \mathfrak{c}$ adds a Cohen real but $\mathbf{MA}_{\omega_1}(ccc)$ fails.* $\dashv$

As we saw in Theorem 1.6, if we assume a small part of $\mathbf{MA}_\kappa$ then each forcing notion adding an unbounded real adds a Cohen real, provided the size of the forcing is at most κ. Therefore it is natural to look for requirements implying that small forcing notions add unbounded reals. The main part of the proof of Theorem 1.9 was to show that $\mathbf{MA}_\kappa(\text{ccc \& almost } \omega^\omega\text{-bounding})$ is such a condition. It occurs however, that we need much less for this. As in Theorem 1.6 the crucial role was played by the Cohen forcing, here we naturally arrive at the forcing notion defined below.

DEFINITION 1.11. We define a forcing notion $\mathbb{Q}_\odot$:
conditions are pairs $\langle a, w \rangle$ such that $w \in [2^\omega]^{<\omega}$ and $a \in [2^{<\omega}]^{<\omega}$,
the order is defined by: $\langle a_0, w_0 \rangle \leq_{\mathbb{Q}_\odot} \langle a_1, w_1 \rangle$ if and only if

$$a_0 \subseteq a_1, w_0 \subseteq w_1 \text{ and } (\forall \eta \in w_0)(\forall l \in \omega)(\eta \restriction l \in a_1 \ \Rightarrow \ \eta \restriction l \in a_0).$$

LEMMA 1.12. (1) $\mathbb{Q}_\odot$ *is an almost ω^ω-bounding σ-centered partial order.*
(2) *Let $\dot{A}$ be the $\mathbb{Q}_\odot$-name for a subset of $2^{<\omega}$ given by*

$$\Vdash_{\mathbb{Q}_\odot} \dot{A} = \bigcup \{a : (\exists w)(\langle a, w \rangle \in \Gamma_{\mathbb{Q}_\odot})\}.$$

Then

(α) $\Vdash_{\mathbb{Q}_\odot} (\forall \eta \in 2^\omega \cap \mathbf{V})(\forall^\infty n \in \omega)(\eta \restriction n \notin \dot{A})$

(β) $\Vdash_{\mathbb{Q}_\odot}$ "*if $T \subseteq 2^{<\omega}$ is a perfect tree from the ground model then* $(\exists^\infty n \in \omega)(T \cap 2^n \cap \dot{A} \neq \emptyset)$".

PROOF. (1) Clearly if $a_0 = a_1$, $\langle a_0, w_0 \rangle, \langle a_1, w_1 \rangle \in \mathbb{Q}_\odot$ then $\langle a_0, w_0 \cup w_1 \rangle \in \mathbb{Q}_\odot$ is a common upper bound of $\langle a_0, w_0 \rangle, \langle a_1, w_1 \rangle$. This implies that $\mathbb{Q}_\odot$ is σ-centered. Next note that

$\langle a_0, w_0 \rangle \perp_{\mathbb{Q}_\odot} \langle a_1, w_1 \rangle$ if and only if
either there are $\eta \in w_0$, $l \in \omega$ such that $\eta \restriction l \in a_1 \setminus a_0$
or the symmetrical condition holds (interchanging 0 and 1).

Hence if $a_0, a_1 \subseteq 2^{\leq l_0}$, $\{\eta \restriction l_0 : \eta \in w_1\} = \{\eta \restriction l_0 : \eta \in w_2\}$ then

$$\langle a_0, w_0 \rangle \perp_{\mathbb{Q}_\odot} \langle a_1, w_1 \rangle \text{ iff } \langle a_0, w_0 \rangle \perp_{\mathbb{Q}_\odot} \langle a_1, w_2 \rangle.$$

Since the product space $(2^\omega)^n$ is compact we may conclude that

if $\mathscr{A} \subseteq \mathbb{Q}_\odot$ is a maximal antichain, $n \in \omega$, $a \in [2^{<\omega}]^{<\omega}$ then there is a finite set $A = A^{a,n} \subseteq \mathscr{A}$ such that for every $w \subseteq 2^\omega$, $\|w\| = n$ there is $r \in A$ with $\langle a, w \rangle \not\perp_{\mathbb{Q}_\odot} r$.

The above property easily implies that $\mathbb{Q}_\odot$ is almost ω^ω-bounding: suppose that $\dot{h}$ is a $\mathbb{Q}_\odot$-name for an element of ω^ω. For each $k \in \omega$ fix a maximal antichain $\mathscr{A}_k$ such that each member of $\mathscr{A}_k$ decides the value of $\dot{h}(k)$. For $k, n \in \omega$ and $a \in [2^{<\omega}]^{<\omega}$ choose a finite set $A^{a,n,k} \subseteq \mathscr{A}_k$ with the property stated above. Finally put

$$g(k) = 1 + \max\{l \in \omega : (\exists a \subseteq 2^{\leq k})(\exists n \leq k)(\exists r \in A^{a,n,k})(r \Vdash_{\mathbb{Q}_\odot} \dot{h}(k) = l)\}.$$

To show that the function g works for $\dot{h}$ (for the definition of almost ω^ω-bounding) suppose that $X \in [\omega]^\omega$. Assume that

$$r_0 \Vdash_{\mathbb{Q}_\odot} (\forall^\infty n \in X)(g(n) \leq \dot{h}(n)),$$

so we have r_1 and k such that

$$r_1 \Vdash_{\mathbb{Q}_\odot} (\forall n > k)(n \in X \Rightarrow g(n) \leq \dot{h}(n)).$$

Now take $k^* \in X$ such that $k^* > k$ and if $r_1 = \langle a, w \rangle$ then $a \subseteq 2^{\leq k^*}$, $\|w\| = n \leq k^*$. By the definition of A^{a,n,k^*} we find $r \in A^{a,n,k^*}$ compatible with r_1. But each member of A^{a,n,k^*} forces that $\dot{h}(k^*) < g(k^*)$, a contradiction.

(2) Straightforward.   ⊣

THEOREM 1.13. *Assume* $\mathbf{MA}_\kappa(\mathbb{Q}_\odot)$. *Suppose that $\mathbb{P}$ is a forcing notion such that $\|\mathbb{P}\| \leq \kappa$ and $\Vdash_\mathbb{P} 2^\omega \cap \mathbf{V} \neq 2^\omega$ (i.e., the corresponding complete Boolean algebra $\mathrm{RO}(\mathbb{P})$ is not (ω, ω)-distributive). Then $\mathbb{P}$ adds an unbounded real.*

PROOF. Since $\mathbb{P}$ adds new reals we can find a $\mathbb{P}$-name $\dot{r}$ for an element of 2^ω such that $\Vdash_\mathbb{P} \dot{r} \notin \mathbf{V}$. For a condition $q \in \mathbb{P}$ let

$$T^q \overset{\text{def}}{=} \{v \in 2^{<\omega} : q \not\Vdash_\mathbb{P} v \not\subseteq \dot{r}\}.$$

By our assumptions on $\dot{r}$ we know that each T^q is a perfect tree in $2^{<\omega}$. Next fix $\eta_q \in [T^q]$ (for $q \in \mathbb{P}$). Since we have assumed $\mathbf{MA}_\kappa(\mathbb{Q}_\odot)$ we may apply Lemma 1.12 to find a set $A \subseteq 2^{<\omega}$ such that for each $q \in \mathbb{P}$:

(α) $(\forall^\infty n \in \omega)(\eta_q {\restriction} n \notin A)$ and
(β) $(\exists^\infty n \in \omega)(T^q \cap 2^n \cap A \neq \emptyset)$.

Now define a $\mathbb{P}$-name $\dot{K}$ for a subset of ω by:

$$\Vdash_\mathbb{P} \dot{K} = \{n \in \omega : \dot{r}{\restriction}n \in A\}.$$

First note that $\dot{K}$ is a $\mathbb{P}$-name for an infinite subset of ω: Why? Suppose that $q \in \mathbb{P}$ and $N \in \omega$. By the property (β) of A we find $v \in A \cap T^q$ such that $\mathrm{lh}(v) > N$. Then we have a condition $p_v \geq q$ which forces "$v \subseteq \dot{r}$" and thus $p_v \Vdash_\mathbb{P} \mathrm{lh}(v) \in \dot{K}$.

Suppose now that $q \in \mathbb{P}$, $g \in \omega^\omega$ is an increasing function and $N_0 \in \omega$. Take $N_1 > N_0$ such that $(\forall n \geq N_1)(\eta_q {\restriction} n \notin A)$ and a condition $p_{\eta_q {\restriction} g(N_1)}$ such that $q \leq_\mathbb{P} p_{\eta_q {\restriction} g(N_1)}$ and $p_{\eta_q {\restriction} g(N_1)} \Vdash_\mathbb{P} \eta_q {\restriction} g(N_1) \subseteq \dot{r}$ (remember that $\eta_q \in [T^q]$). Now note that

$$p_{\eta_q {\restriction} g(N_1)} \Vdash_\mathbb{P} \dot{K} \cap [N_1, g(N_1)) = \emptyset.$$

Hence we easily conclude that

$$\Vdash_\mathbb{P} \text{"the increasing enumeration of } \dot{K} \text{ is an unbounded real over } \mathbf{V}\text{"}$$

finishing the proof. $\dashv$

REMARK 1.14. The forcing notion $\mathbb{Q}_\odot$ makes the ground model reals meager in a "soft" way: it does not add a dominating real (see Lemma 1.12). However it adds an unbounded real (just look at $\{n \in \omega : \dot{A} \cap 2^n \neq \emptyset\}$, for $\dot{A}$ as in Lemma 1.12(2)). Consequently it adds a Cohen real (by [7]; note that $\mathbb{Q}_\odot$ is a Borel ccc forcing notion). Hence we may put together Theorems 1.6 and 1.13 and we get the following corollary.

COROLLARY 1.15. *Assume* $\mathbf{MA}_\kappa(\mathbb{Q}_\odot)$. *Then every ccc forcing notion of size* κ *adding new reals adds a Cohen real.* $\dashv$

§2. **Anti-Martin axiom.** In this section we are interested in axioms which are considered as strong negations of Martin Axiom. They originated in Miller's problem if it is consistent with $\neg\mathbf{CH}$ that for any ccc forcing notion of the size $\leq \mathfrak{c}$ there exists an ω_1-Lusin sequence of filters (cf. [5]). The question was answered negatively by Todorcevic (cf. [8]). However under some restrictions (on forcing notions and/or dense sets under consideration) suitable axioms can be consistent with $\neg\mathbf{CH}$. These axioms were considered by van Douwen and Fleissner, who were interested in the axiom for projective ccc forcing notions, but they needed a weakly compact cardinal for getting the consistency (cf. [9]). Cichoń preferred to omit the large cardinal assumption and restricted himself to Σ_2^1 ccc forcing notions and still he was able to obtain interesting consequences (see [2]). Here we show how to omit the large cardinal assumption in getting Anti-Martin Axiom for projective ccc forcing notions. This answers Problem 6.6(2) of [4].

DEFINITION 2.1. For a forcing notion $\mathbb{P}$ and a cardinal κ let $\mathbf{AMA}_\kappa(\mathbb{P})$ be the following sentence:

there exists a sequence $\langle G_i : i < \kappa \rangle$ of filters on $\mathbb{P}$ such that for every maximal antichain $\mathscr{A} \subseteq \mathbb{P}$ for some $i_0 < \kappa$ we have

$$(\forall i \geq i_0)(G_i \cap \mathscr{A} \neq \emptyset).$$

For a class $\mathscr{K}$ of forcing notions the axiom $\mathbf{AMA}_\kappa(\mathscr{K})$ is "for each $\mathbb{P} \in \mathscr{K}$, $\mathbf{AMA}_\kappa(\mathbb{P})$ holds true".

DEFINITION 2.2. (1) For two models N, M and an integer n, $M \prec_{n+1} N$ means:

for every Π_n formula $\varphi(x, \bar{y})$ and every sequence $\bar{m} \subseteq M$,
if $N \models \exists x \varphi(x, \bar{m})$ then $M \models \exists x \varphi(x, \bar{m})$.

(Thus $M \prec N$ if and only if $(\forall n > 0)(M \prec_n N)$.)

(2) If $\mathbb{P}_0$, $\mathbb{P}_1$ are ccc forcing notions, $n > 0$ then $\mathbb{P}_0 \lessdot_n \mathbb{P}_1$ means $\mathbb{P}_0 \lessdot \mathbb{P}_1$ (i.e., $\mathbb{P}_0$ is a complete suborder of $\mathbb{P}_1$) and

$$\Vdash_{\mathbb{P}_1} (\mathscr{H}(\aleph_1)^{\mathbf{V}[\Gamma_{\mathbb{P}_1} \cap \mathbb{P}_0]}, \in) \prec_n (\mathscr{H}(\aleph_1), \in).$$

Instead of $\lessdot$ we may write $\lessdot_0$.

DEFINITION 2.3. Let κ be a cardinal number.

(1) $\mathscr{C}_\kappa$ is the class of all ccc forcing notions of size $\leq \kappa$.

(2) We inductively define subclasses $\mathscr{C}_\kappa^n$ of $\mathscr{C}_\kappa$ (for $n \leq \omega$):

$\mathscr{C}_\kappa^0 = \mathscr{C}_\kappa$,

$\mathscr{C}_\kappa^{n+1}$ is the class of all $\mathbb{P} \in \mathscr{C}_\kappa^n$ such that for every $\mathbb{P}^* \in \mathscr{C}_\kappa^n$

$$\mathbb{P} \lessdot \mathbb{P}^* \quad \Rightarrow \quad \mathbb{P} \lessdot_{n+1} \mathbb{P}^*,$$

$\mathscr{C}_\kappa^\omega = \bigcap_{n < \omega} \mathscr{C}_\kappa^n$.

LEMMA 2.4. *Let κ be a cardinal such that $\kappa^\omega = \kappa$, $n \leq \omega$.*

(1) *If $\mathbb{P}_0, \mathbb{P}_1 \in \mathscr{C}_\kappa^n$, $\mathbb{P}_0 \lessdot \mathbb{P}_1$ then $\mathbb{P}_0 \lessdot_n \mathbb{P}_1$.*

(2) *Suppose $\delta < \kappa^+$, $\mathrm{cf}(\delta) \geq \omega_1$ and $\mathbb{P}_i \in \mathscr{C}_\kappa^n$ (for $i < \delta$) are such that $i < j < \delta \Rightarrow \mathbb{P}_i \lessdot \mathbb{P}_j$. Then $\mathbb{P}_\delta \overset{\mathrm{def}}{=} \bigcup_{i<\delta} \mathbb{P}_i \in \mathscr{C}_\kappa^n$ and if $\mathbb{P} \in \mathscr{C}_\kappa$, $\mathbb{P}_i \lessdot_n \mathbb{P}$ for every $i < \delta$ then $\mathbb{P}_\delta \lessdot_n \mathbb{P}$.*

(3) *If $\mathbb{P} \in \mathscr{C}_\kappa$ then there is $\mathbb{P}^* \in \mathscr{C}_\kappa^n$ such that $\mathbb{P} \lessdot \mathbb{P}^*$.*

(4) *If $\mathbb{P} \in \mathscr{C}_\kappa$ then there are functions $F_k : \prod_{i<\omega} \mathbb{P} \longrightarrow \mathbb{P}$ (for $k \in \omega$) such that for every $\mathbb{Q} \subseteq \mathbb{P}$: if $\mathbb{Q}$ is closed under all F_k then $\mathbb{Q} \lessdot_n \mathbb{P}$.*

PROOF. The proof is by induction on n. For $n = 0$ there is nothing to do. (For (4) consider functions $F_0, F_1 : \prod_{i \in \omega} \mathbb{P} \longrightarrow \mathbb{P}$ such that if $\langle p_i : i \in \omega \rangle \subseteq \mathbb{P}$ is an antichain which is not maximal then $F_0(p_i : i < \omega)$ is a condition incompatible with all p_i; if $p_i \in \mathbb{P}$ ($i \in \omega$) and $p_0 \not\perp_{\mathbb{P}} p_1$ then $F_1(p_i : i \in \omega)$ is a condition stronger than both p_0 and p_1.) So suppose that (1)–(4) hold true for n and we are proving them for $n + 1$.

(1) By the definition.

(2) Suppose that $\mathbb{P} \in \mathscr{C}_\kappa$, $\mathbb{P}_i \lessdot_{n+1} \mathbb{P}$ for each $i < \delta$. By the inductive hypothesis we know that $\mathbb{P}_\delta \lessdot_n \mathbb{P}$, $\mathbb{P}_\delta \in \mathscr{C}_\kappa^n$ and hence (by the definition of $\mathscr{C}_\kappa^{n+1}$) we have $\mathbb{P}_i \lessdot_{n+1} \mathbb{P}_\delta$ for each $i < \delta$. Suppose that $G \subseteq \mathbb{P}$ is a generic filter over $\mathbf{V}$. Then for $i < \delta$:

$(*)$ $\qquad\qquad (\mathscr{H}(\aleph_1)^{\mathbf{V}[G \cap \mathbb{P}_i]}, \in) \prec_{n+1} (\mathscr{H}(\aleph_1)^{\mathbf{V}[G \cap \mathbb{P}_\delta]}, \in)$ and

$(**)$ $(\mathscr{H}(\aleph_1)^{\mathbf{V}[G\cap \mathbb{P}_i]}, \in) \prec_{n+1} (\mathscr{H}(\aleph_1)^{\mathbf{V}[G]}, \in).$

Let $\varphi(x, \bar{y})$ be a Π_n-formula and $\bar{y}_0 \subseteq \mathscr{H}(\aleph_1)^{\mathbf{V}[G\cap\mathbb{P}_\delta]}$. Take $i < \delta$ such that $\bar{y}_0 \subseteq \mathscr{H}(\aleph_1)^{\mathbf{V}[G\cap\mathbb{P}_i]}$ (remember $\mathrm{cf}(\delta) > \omega$). If $(\mathscr{H}(\aleph_1)^{\mathbf{V}[G]}, \in) \models \exists x \varphi(x, \bar{y}_0)$ then $(\mathscr{H}(\aleph_1)^{\mathbf{V}[G\cap\mathbb{P}_i]}, \in) \models \exists x \varphi(x, \bar{y}_0)$ (by $(**)$) and $\mathscr{H}(\aleph_1)^{\mathbf{V}[G\cap\mathbb{P}_\delta]}, \in) \models \exists x \varphi(x, \bar{y}_0)$ (by $(*)$). This shows $\mathbb{P}_\delta \lessdot_{n+1} \mathbb{P}$. To prove that $\mathbb{P}_\delta \in \mathscr{C}_\kappa^{n+1}$ suppose that $\mathbb{P} \in \mathscr{C}_\kappa^n$, $\mathbb{P}_\delta \lessdot \mathbb{P}$. Then for each $i < \delta$ we have $\mathbb{P}_i \lessdot \mathbb{P}$, $\mathbb{P}_i \in \mathscr{C}_\kappa^{n+1}$ and consequently $\mathbb{P}_i \lessdot_{n+1} \mathbb{P}$. By the previous part we get $\mathbb{P}_\delta \lessdot_{n+1} \mathbb{P}$ finishing (2)

(3) Let $\mathbb{P} \in \mathscr{C}_\kappa$. By a bookkeeping argument we inductively build sequences $\langle \mathbb{P}_i : i \le \kappa \rangle$ and $\langle (p_i, \varphi_i, \dot{\tau}_i) : i < \kappa \rangle$ such that for all $i < j < \kappa$:

(a) $\mathbb{P}_i \in \mathscr{C}_\kappa^n$, $\mathbb{P} \lessdot \mathbb{P}_0$, $\mathbb{P}_i \lessdot \mathbb{P}_j$, $\mathbb{P}_\kappa = \bigcup_{i<\kappa} \mathbb{P}_i$,

(b) $p_i \in \mathbb{P}_i$,

(c) φ_i is a Π_n-formula, $\dot{\tau}_i$ is a $\mathbb{P}_i$-name for a finite sequence of elements of $\mathscr{H}(\aleph_1)$,

(d) $\langle (p_i, \varphi_i, \dot{\tau}_i) : i < \kappa \rangle$ lists all triples $(p, \varphi, \dot{\tau})$ such that $\varphi = \varphi(x, \bar{y})$ is a Π_n-formula, $\dot{\tau}$ is a (canonical) $\mathbb{P}_\kappa$-name for a finite sequence (of a suitable length) of members of $\mathscr{H}(\aleph_1)$, $p \in \mathbb{P}_\kappa$,

(e) if i is limit, $\mathrm{cf}(i) = \omega$ then $\mathbb{P}_i \in \mathscr{C}_\kappa^n$ is such that $\bigcup_{\ell<i} \mathbb{P}_\ell \lessdot \mathbb{P}_i$,

(f) if i is limit, $\mathrm{cf}(i) > \omega$ then $\mathbb{P}_i = \bigcup_{\ell<i} \mathbb{P}_\ell \in \mathscr{C}_\kappa^n$,

(g) *if* there is $\mathbb{P}^* \in \mathscr{C}_\kappa^n$ such that $\mathbb{P}_i \lessdot \mathbb{P}^*$ and for some $p^* \in \mathbb{P}^*$ we have $p^* \not\perp_{\mathbb{P}^*} p_i$ and
$$p^* \Vdash_{\mathbb{P}^*} (\mathscr{H}(\aleph_1), \in) \models \exists x \varphi_i(x, \dot{\tau}_i)$$
then $\mathbb{P}_{i+1}$ is an example of such $\mathbb{P}^*$.

The construction is fully described by the above conditions (and easy to carry out; remember about the inductive hypothesis and the assumption that $\kappa^\omega = \kappa$). Clearly $\mathbb{P}_\kappa \in \mathscr{C}_\kappa^n$ (by the inductive assumption (2)) and $\mathbb{P} \lessdot \mathbb{P}_\kappa$. We have to show that actually $\mathbb{P}_\kappa \in \mathscr{C}_\kappa^{n+1}$. Suppose not. Then we find $\mathbb{P}^* \in \mathscr{C}_\kappa^n$ such that $\mathbb{P}_\kappa \lessdot \mathbb{P}^*$ but $\mathbb{P}_\kappa \not\lessdot_{n+1} \mathbb{P}^*$. The second means that there are a condition $p^* \in \mathbb{P}^*$ and a Π_n-formula φ and a $\mathbb{P}_\kappa$-name $\dot{\tau}$ for a sequence of elements of $\mathscr{H}(\aleph_1)$ such that

$$p^* \Vdash_{\mathbb{P}^*} \text{``}(\mathscr{H}(\aleph_1), \in) \models \exists x \varphi(x, \dot{\tau}) \text{ but } (\mathscr{H}(\aleph_1)^{\mathbf{V}[\Gamma_{\mathbb{P}^*} \cap \mathbb{P}_\kappa]}, \in) \models \neg \exists x \varphi(x, \dot{\tau})\text{''}.$$

Take $p \in \mathbb{P}_\kappa$ such that $p^* \not\perp_{\mathbb{P}^*} p$ and there is no condition $p' \in \mathbb{P}_\kappa$ such that $p \le_{\mathbb{P}_\kappa} p'$ and $p' \perp_{\mathbb{P}^*} p^*$. Let $i < \kappa$ be such that $(p, \varphi, \dot{\tau}) = (p_i, \varphi_i, \dot{\tau}_i)$. Condition (g) of the construction implies that for some $p^+ \in \mathbb{P}_{i+1}$ we have $p^+ \not\perp_{\mathbb{P}_\kappa} p$ and

$$p^+ \Vdash_{\mathbb{P}_{i+1}} (\mathscr{H}(\aleph_1), \in) \models \exists x \varphi(x, \dot{\tau}).$$

Since $\mathbb{P}_{i+1} \lessdot_n \mathbb{P}_\kappa$ (the inductive hypotheses (2), (1)) we get

$$p^+ \Vdash_{\mathbb{P}_\kappa} (\mathscr{H}(\aleph_1), \in) \models \exists x \varphi(x, \dot{\tau}).$$

The choice of p implies $p^+ \not\perp_{\mathbb{P}^*} p^*$ and this provides a contradiction as

$$p^+ \Vdash_{\mathbb{P}^*} (\mathscr{H}(\aleph_1)^{\mathbf{V}[\Gamma \cap \mathbb{P}_\kappa]}, \in) \models \exists x \varphi(x, \dot{\tau}).$$

(4) Let $F_k^0 : \prod_{i<\omega} \mathbb{P} \longrightarrow \mathbb{P}$ (for $k \in \omega$) be functions such that if $\mathbb{Q} \subseteq \mathbb{P}$, $\mathbb{Q}$ is closed under all F_k^0 then $\mathbb{Q} \lessdot_n \mathbb{P}$ (they are given by the inductive hypothesis (4)). Let $A_{i,j,k} \subseteq \omega \setminus \{0\}$ be disjoint infinite sets (for $i, j, k \in \omega$). For a Π_n-formula $\varphi(x, y_0, \dots, y_{\ell-1})$ and $m \in \omega$ we choose a function $F^{\varphi,m} : \prod_{i<\omega} \mathbb{P} \longrightarrow \mathbb{P}$ satisfying the condition described below.

Let $\langle p_m : m < \omega \rangle \subseteq \mathbb{P}$. For $k < \ell$ we try to define a $\mathbb{P}$-name $\dot{\tau}_k$ for a real in ω^ω by

$$(\forall m \in A_{i,j,k})(p_m \Vdash_\mathbb{P} \dot{\tau}_k(i) = j).$$

If this definition is correct then we ask if these reals encode (in the canonical way) elements of $\mathscr{H}(\aleph_1)$ (which we identify with the names $\dot{\tau}_k$ themselves). If yes then we ask if

$$p_0 \Vdash_\mathbb{P} (\mathscr{H}(\aleph_1), \in) \models \exists x \varphi(x, \dot{\tau}_0, \dots, \dot{\tau}_{\ell-1}).$$

If the answer is positive then we fix a $\mathbb{P}$-name $\dot{\tau}$ for (a real encoding) a member of $\mathscr{H}(\aleph_1)$ such that

$$p_0 \Vdash_\mathbb{P} (\mathscr{H}(\aleph_1), \in) \models \varphi(\dot{\tau}, \dot{\tau}_0, \dots, \dot{\tau}_{\ell-1}).$$

This name can be represented similarly as names $\dot{\tau}_k$ (for $k < \ell$) so we have a sequence $\langle q_m : m < \omega \rangle \subseteq \mathbb{P}$ encoding it. Finally we want $F^{\varphi,m}$ to be such that if the above procedure for $\langle p_m : m < \omega \rangle$ works then $F^{\varphi,m}(p_m : m < \omega) = q_m$.

Now take all the functions F^0_k, $F^{\varphi,m}$; it is easy to check that they work.

Lastly note that the case $n = \omega$ follows immediately from the lemma for $n < \omega$. (For (3) construct an increasing sequence $\langle \mathbb{P}_i : i < \omega_1 \rangle$ such that $\mathbb{P} \lessdot \mathbb{P}_0$ and if $\lambda < \omega_1$ is limit, $k < \omega$ then $\mathbb{P}_{\lambda+k} \in \mathscr{C}^k_\kappa$.)   ⊣

THEOREM 2.5. *Suppose that θ, κ are cardinals such that $\aleph_1 \leq \theta = \mathrm{cf}(\theta) \leq \kappa = \kappa^\omega$. Then there exists a ccc forcing notion $\mathbb{P}$ such that*

$$\Vdash_\mathbb{P} \mathfrak{c} = \kappa \ \& \ \mathbf{AMA}_\theta(\text{projective ccc}).$$

PROOF. The forcing notion $\mathbb{P}$ which we are going to construct will be essentially a finite support iteration of length $\kappa \cdot \theta$ of ccc forcing notions. One could try to force with "all possible ccc orders" in the iteration. However some care is necessary to make sure that several notions (including "being a maximal antichain") are sufficiently absolute for intermediate stages. Therefore we use forcing notions from the class $\mathscr{C}^\omega_\kappa$. So we inductively build sequences $\langle \mathbb{P}_i : i \leq \kappa \cdot \theta \rangle$ and $\langle (\varphi_i, \psi_i, \dot{\tau}_i) : i < \kappa \cdot \theta \rangle$ such that for all $i < j < \kappa \cdot \theta$:

(1) $\mathbb{P}_i \in \mathscr{C}^\omega_\kappa$, $\mathbb{P}_{\kappa \cdot \theta} = \bigcup_{i < \kappa \cdot \theta} \mathbb{P}_i \in \mathscr{C}^\omega_\kappa$,

(2) $\mathbb{P}_i \lessdot \mathbb{P}_j$,

(3) $\langle (\varphi_i, \psi_i, \dot{\tau}_i) : i < \kappa \cdot \theta \rangle$ lists with cofinal repetitions all triples $(\varphi, \psi, \dot{\tau})$ such that φ is a formula with $n + 1$ variables, ψ is a formula with $n + 2$ variables and $\dot{\tau}$ is a $\mathbb{P}_{\kappa \cdot \theta}$-name for a sequence of length n of elements of $\mathscr{H}(\aleph_1)$,

(4) if $\dot{\tau}_i$ is a $\mathbb{P}_i$-name and

$$\Vdash_{\mathbb{P}_i} \text{ "} \langle \varphi_i(x, \dot{\tau}_i), \psi_i(x_0, x_1, \dot{\tau}_i) \rangle \text{ defines in } (\mathscr{H}(\aleph_1), \in) \text{ a ccc partial order } \dot{\mathbb{Q}}_i \text{"}$$

then $\mathbb{P}_i * \dot{\mathbb{Q}}_i \lessdot \mathbb{P}_{i+1}$.

It is easy to carry the construction (use a bookkeeping argument, remembering $\kappa^\omega = \kappa$ plus Lemma 2.4). We want to show that $\mathbb{P} = \mathbb{P}_{\kappa \cdot \theta}$ has the required properties. Easily $\Vdash_\mathbb{P} \mathfrak{c} = \kappa$. Now suppose that $G \subseteq \mathbb{P}$ is a generic filter over $\mathbf{V}$ and work in $\mathbf{V}[G]$.

Assume that $\mathbb{Q}$ is a projective ccc forcing notion and thus it is definable in $(\mathscr{H}(\aleph_1), \in)$. Thus we have formulas $\varphi(x, \bar{y})$ and $\psi(x_0, x_1, \bar{y})$ and a sequence $\bar{r} \subseteq \mathscr{H}(\aleph_1)$ such that

$$\mathbb{Q} = \{x \in \mathscr{H}(\aleph_1) : (\mathscr{H}(\aleph_1), \in) \models \varphi(x, \bar{r})\}$$
$$\leq_{\mathbb{Q}} = \{(x_0, x_1) \in \mathscr{H}(\aleph_1) \times \mathscr{H}(\aleph_1) : (\mathscr{H}(\aleph_1), \in) \models \psi(x_0, x_1, \bar{r})\}.$$

Let $\dot{\tau}$ be a $\mathbb{P}$-name for $\bar{r}$. We may assume that

$$\Vdash_{\mathbb{P}} \text{ ``}\langle \varphi(x, \dot{\tau}), \psi(x_0, x_1, \dot{\tau})\rangle \text{ defines (in } (\mathscr{H}(\aleph_1), \in)) \text{ a ccc partial order''}.$$

There is an increasing cofinal in $\kappa \cdot \theta$ sequence $\langle i_j : j < \theta \rangle$ such that $\dot{\tau}$ is a $\mathbb{P}_{i_0}$-name and $(\varphi_{i_j}, \psi_{i_j}, \dot{\tau}_{i_j}) = (\varphi, \psi, \dot{\tau})$. Since $\mathbb{P}, \mathbb{P}_{i_j} \in \mathscr{C}_{\kappa}^{\omega}$ we have that

$$\Vdash_{\mathbb{P}} \text{ ``}(\mathscr{H}(\aleph_1)^{\mathbf{V}[\Gamma_{\mathbb{P}} \cap \mathbb{P}_{i_j}]}, \in) \prec (\mathscr{H}(\aleph_1), \in)\text{''}$$

and hence the formulas $\langle \varphi(x, r), \psi(x_0, x_1, r)\rangle$ define (in $(\mathscr{H}(\aleph_1)^{\mathbf{V}[G \cap \mathbb{P}_{i_j}]}, \in)$) the partial order $\mathbb{Q} \cap \mathscr{H}(\aleph_1)^{\mathbf{V}[G \cap \mathbb{P}_{i_j}]}$. The incompatibility relation in this partial order is expressible in $(\mathscr{H}(\aleph_1), \in)$ and thus it is the restriction of $\perp_{\mathbb{Q}}$. Consequently $\mathbb{Q} \cap \mathscr{H}(\aleph_1)^{\mathbf{V}[G \cap \mathbb{P}_{i_j}]}$ is ccc in $\mathbf{V}[G]$ and hence in $\mathbf{V}[G \cap \mathbb{P}_{i_j}]$. Hence in $\mathbf{V}[G \cap \mathbb{P}_{i_{j+1}}]$ we have a filter $G_j^* \subseteq \mathbb{Q} \cap \mathscr{H}(\aleph_1)^{\mathbf{V}[G \cap \mathbb{P}_{i_j}]}$ generic over $\mathbf{V}[G \cap \mathbb{P}_{i_j}]$ (here we apply Condition 4 of the construction). Look at the sequence $\langle G_j^* : j < \theta \rangle$. Let $\mathscr{A} \subseteq \mathbb{Q}$ be a maximal antichain. It is countable and hence for sufficiently large $j < \theta$ we have $\mathscr{A} \in \mathbf{V}[G \cap \mathbb{P}_{i_j}]$. Moreover the antichain can be coded as a one real and the fact that it is a maximal antichain in the partial order defined by $\langle \varphi, \psi \rangle$ is expressible in $(\mathscr{H}(\aleph_1), \in)$. Applying $\mathbb{P}_{i_j} \in \mathscr{C}_{\kappa}^{\omega}$ we get that

$$\mathbf{V}[G \cap \mathbb{P}_{i_j}] \models \mathscr{A} \text{ is a maximal antichain in } \mathbb{Q} \cap \mathscr{H}(\aleph_1)^{\mathbf{V}[G \cap \mathbb{P}_{i_j}]}.$$

Consequently for sufficiently large $j < \theta$ we have

$$G_j^* \cap \mathscr{A} \neq \emptyset.$$

This finishes the proof.   ⊣

REMARK 2.6. In Lemma 2.4 and Theorem 2.5 we used $\mathscr{H}(\aleph_1)$ as we were mainly interested in $\mathbf{AMA}_{\omega_1}$ and projective ccc forcing notions. But we may replace it by $\mathscr{H}(\chi)$ for any uncountable regular cardinal χ such that $\sum_{\alpha < \chi} \kappa^{|\alpha|} = \kappa$. Then in Lemma 2.4(2) we consider $\delta < \kappa^+$ such that $\mathrm{cf}(\delta) \geq \chi$ and in Theorem 2.5 we additionally assume that $\theta \geq \chi$.

§3. Absoluteness and embeddings.

In this section we answer positively Problem 4.4 of [4] (see Theorem 3.2) and we give a negative answer to Problem 3.3 of [4] (see Corollary 3.5).

DEFINITION 3.1. Let $\mathbb{P}$ be a forcing notion. We say that $\Sigma_n^1(\mathbb{P})$-absoluteness holds if for every Σ_n^1 formula φ (with parameters in $\mathbf{V}$) and a generic filter $G \subseteq \mathbb{P}$ over $\mathbf{V}$
$$\mathbf{V}[G] \models \varphi \text{ if and only if } \mathbf{V} \models \varphi.$$

Obviously $\Sigma_2^1(\mathbb{P})$-absoluteness holds for any forcing notion $\mathbb{P}$.

THEOREM 3.2. *Assume* $\mathbf{MA}_{\omega_1}(\mathbb{P})$. *Then* $\Sigma_3^1(\mathbb{P})$-*absoluteness holds.*

PROOF. Suppose that φ is a Σ_3^1 sentence (with a parameter $a \in \omega^\omega$). Using the tree representation of Π_2^1-sets we find a tree T (constructible from a) over $\omega \times \omega_1$ such that

$$\varphi \equiv (\exists x \in \omega^\omega)(\forall f \in \omega_1{}^\omega)(\exists n \in \omega)(\langle x \restriction n, f \restriction n \rangle \notin T)$$

$$\equiv (\exists x \in \omega^\omega)(\text{the tree } T(x) \text{ is well founded}).$$

(For $x \in \omega^\omega$, $T(x)$ is the tree on ω_1 consisting of all $\bar{\alpha} \in \omega_1{}^{<\omega}$ such that $\langle x \restriction \mathrm{lh}(\bar{\alpha}), \bar{\alpha} \rangle \in T$.) Moreover, as by $\mathbf{MA}_{\omega_1}(\mathbb{P})$ we know that $\Vdash_\mathbb{P} \omega_1^{\mathbf{V}} = \omega_1$, the tree T represents φ in $\mathbf{V}^\mathbb{P}$ too:

$$\Vdash_\mathbb{P} \text{``}\varphi \equiv (\exists x \in \omega^\omega)(\text{the tree } T(x) \text{ is well founded})\text{''}.$$

Suppose now that $\Vdash_\mathbb{P} \varphi$. Then we have a $\mathbb{P}$-name $\dot{r}$ for a real in ω^ω such that

$$\Vdash_\mathbb{P} \text{``the tree } T(\dot{r}) \text{ is well founded''}.$$

Consequently we have a $\mathbb{P}$-name $\dot{\rho}$ for a function such that

$$\Vdash_\mathbb{P} \text{``}\dot{\rho} : T(\dot{r}) \longrightarrow \mathrm{Ord} \text{ is a rank function''}.$$

For $n \in \omega$, $\bar{\alpha} \in \omega_1{}^n$ put

$$J_n^0 = \{p \in \mathbb{P} : (\exists m \in \omega)(p \Vdash_\mathbb{P} \dot{r}(n) = m)\},$$
$$J_{\bar{\alpha}}^1 = \{p \in \mathbb{P} : \text{ either } p \Vdash_\mathbb{P} \bar{\alpha} \notin T(\dot{r})$$
$$\text{ or } (\exists \xi \in \mathrm{Ord})(p \Vdash_\mathbb{P} \bar{\alpha} \in T(\dot{r}) \ \& \ \dot{\rho}(\dot{r} \restriction n, \bar{\alpha}) = \xi)\}.$$

Clearly these are dense subsets of $\mathbb{P}$. By $\mathbf{MA}_{\omega_1}(\mathbb{P})$ we find a filter G on $\mathbb{P}$ such that $G \cap J_n^0 \neq \emptyset$ for $n \in \omega$ and $G \cap J_{\bar{\alpha}}^1 \neq \emptyset$ for $\alpha \in \omega_1{}^{<\omega}$. Using this filter we may interpret the name $\dot{r}$ to get $r = \dot{r}^G \in \omega^\omega$. Moreover we may interpret the name $\dot{\rho}$ to get a function $\rho = \rho^G : T(r) \longrightarrow \mathrm{Ord}$:

$$\rho(r \restriction n, \bar{\alpha}) = \xi \text{ iff } (\exists p \in G)(p \Vdash_\mathbb{P} \bar{\alpha} \in T(\dot{r}) \ \& \ \dot{\rho}(\dot{r} \restriction n, \bar{\alpha}) = \xi).$$

[Note that this really defines a function from $T(r)$ to ordinals: suppose that $\langle r \restriction n, \bar{\alpha} \rangle \in T$. First we find $p \in G \cap \bigcap_{m<n} J_m^0$; then clearly

$$p \Vdash_\mathbb{P} \text{``}\dot{r} \restriction n = r \restriction n \text{ and } \bar{\alpha} \in T(\dot{r})\text{''}.$$

Thus if $p' \in J_{\bar{\alpha}}^1 \cap G$ then $p' \not\Vdash_\mathbb{P} \bar{\alpha} \notin T(\dot{r})$ and hence for some ordinal ξ we have $p' \Vdash_\mathbb{P} \bar{\alpha} \in T(\dot{r}) \ \& \ \dot{\rho}(\dot{r} \restriction n, \bar{\alpha}) = \xi$. Moreover if ξ_0, ξ_1 are such that for some $p^0, p^1 \in G$ we have

$$p^i \Vdash_\mathbb{P} \bar{\alpha} \in T(\dot{r}) \ \& \ \dot{\rho}(\dot{r} \restriction n, \bar{\alpha}) = \xi_i$$

then (as $p^0 \not\perp_\mathbb{P} p^1$) we cannot have $\xi_0 \neq \xi_1$.]

We claim that ρ is a rank function on $T(r)$. Suppose that $n_0 < n_1$, $\bar{\alpha}_0 \in \omega_1{}^{n_0}$, $\bar{\alpha}_1 \in \omega_1{}^{n_1}$, $\bar{\alpha}_0 \subsetneq \bar{\alpha}_1$ and $\langle r \restriction n_0, \bar{\alpha}_0 \rangle, \langle r \restriction n_1, \bar{\alpha}_1 \rangle \in T$. Take a condition $p \in G \cap \bigcap_{m<n_1} J_m^0$. Then

$$p \Vdash_\mathbb{P} \text{``}\dot{r} \restriction n_1 = r \restriction n_1 \ \& \ \bar{\alpha}_0, \bar{\alpha}_1 \in T(\dot{r})\text{''}.$$

Next choose conditions $p^0, p^1 \in G$ such that

$$p^i \Vdash_\mathbb{P} \text{``}\bar{\alpha}_i \in T(\dot{r}) \ \& \ \dot{\rho}(\dot{r} \restriction n_i, \bar{\alpha}_i) = \rho(\bar{\alpha}_i)\text{''}.$$

Take $p^* \in G$ stronger than p^0, p^1, p. Since $\dot{\rho}$ is (forced to be) a rank function on $T(\dot{r})$ we have

$$p^* \Vdash_{\mathbb{P}} \rho(\bar{\alpha}_0) = \dot{\rho}(\dot{r} \upharpoonright n_0, \bar{\alpha}_0) > \dot{\rho}(\dot{r} \upharpoonright n_1, \bar{\alpha}_1) = \rho(\bar{\alpha}_1).$$

Hence $\rho(\bar{\alpha}_0) > \rho(\bar{\alpha}_1)$ and we may conclude our theorem: the tree $T(r)$ is well founded so $\mathbf{V} \models \varphi$.   ⊣

PROPOSITION 3.3. *Suppose that $\mathbb{Q}$ is a ccc Souslin forcing notion (i.e., $\mathbb{Q}$, $\leq_{\mathbb{Q}}$ and $\perp_{\mathbb{Q}}$ are Σ^1_1-sets), $\dot{r}$ is a $\mathbb{Q}$-name for a function from $2^{<\omega}$ to 2. Let*

$$A[\dot{r}] \overset{\text{def}}{=} \{\eta \in 2^\omega : (\exists p \in \mathbb{Q})(\forall^\infty m \in \omega)(p \Vdash_{\mathbb{Q}} \text{``} \dot{r}(\eta \upharpoonright m) = 1\text{''})\}.$$

Then $A[\dot{r}]$ is an analytic set.

PROOF. For each $v \in 2^{<\omega}$ choose a maximal antichain $\langle p_{v,l} : l \in \omega \rangle$ in $\mathbb{Q}$ and a set $I_v \subseteq \omega$ such that for each $l \in \omega$:

$$l \in I_v \Rightarrow p_{v,l} \Vdash_{\mathbb{Q}} \dot{r}(v) = 0 \quad \text{and} \quad l \notin I_v \Rightarrow p_{v,l} \Vdash_{\mathbb{Q}} \dot{r}(v) = 1.$$

Now note that for each $\eta \in 2^\omega$ we have

$$\eta \in A[\dot{r}] \equiv (\exists p \in \mathbb{Q})(\forall^\infty n \in \omega)(\forall l \in I_{\eta \upharpoonright n})(p_{\eta \upharpoonright n, l} \perp_{\mathbb{Q}} p).\qquad ⊣$$

PROPOSITION 3.4. *For every $A \subseteq 2^\omega$ there exist a σ-centered forcing notion $\mathbb{Q}^A$ and a $\mathbb{Q}^A$-name $\dot{r}$ (for a function from $2^{<\omega}$ to 2) such that $A = A[\dot{r}]$ and $\|\mathbb{Q}\| = \|A\| + \aleph_0$.*

PROOF. The forcing notion $\mathbb{Q}^A$ is defined by
conditions are pairs $\langle r, w \rangle$ such that r is a finite function, $\text{dom}(r) \subseteq 2^{<\omega}$, $\text{rng}(r) \subseteq 2$ and $w \in [A]^{<\omega}$,
the order is such that $\langle r_1, w_1 \rangle \leq \langle r_2, w_2 \rangle$ if and only if $r_1 \subseteq r_2$, $w_1 \subseteq w_2$ and

$$(\forall v \in \text{dom}(r_2) \setminus \text{dom}(r_1))([(\exists \eta \in w_1)(v \subseteq \eta)] \Rightarrow r_2(v) = 1).$$

The $\mathbb{Q}^A$-name $\dot{r}$ is such that

$$\Vdash_{\mathbb{Q}^A} \dot{r} = \bigcup \{r : (\exists w)(\langle r, w \rangle \in \Gamma_{\mathbb{Q}^A})\}.$$

It should be clear that $\mathbb{Q}^A$ is σ-centered, $\|\mathbb{Q}^A\| = \|A\| + \aleph_0$ and

$$\Vdash_{\mathbb{Q}^A} \dot{r} : 2^{<\omega} \longrightarrow 2.$$

Moreover for each $\eta \in 2^\omega$ and $\langle r, w \rangle \in \mathbb{Q}^A$:

$$(\forall^\infty m)(\langle r, w \rangle \Vdash_{\mathbb{Q}^A} \dot{r}(\eta \upharpoonright m) = 1) \quad \text{iff} \quad \eta \in A.$$

Consequently $A = A[\dot{r}]$.   ⊣

COROLLARY 3.5. *If $A \subseteq 2^\omega$ is not analytic then $\mathbb{Q}^A$ cannot be completely embedded into a ccc Souslin forcing. In particular, if $\mathfrak{c} > \aleph_1$ then there is a σ-centered forcing notion of size $\aleph_1$ which cannot be completely embedded into a ccc Souslin forcing notion.*   ⊣

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