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Countable $\aleph_0$-Indecomposable Mixed Abelian Groups of Finite Torsion-Free Rank

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1. INTRODUCTION

A group is called $\aleph_0$-indecomposable if it does not have a decomposition into a direct sum of $\aleph_0$ non-zero summands. This work is devoted to the study of $\aleph_0$-indecomposable countable mixed abelian groups of finite torsion-free rank in the most general setting. Theorem 1′ proven here has main results of [6, Theorem 2] and [4, Theorem 3] as corollaries: let T be a countable reduced torsion abelian group, and R a torsion-free abelian group of finite rank n. There exists an $\aleph_0$-indecomposable abelian group G with the torsion part T and $G/T \cong R$ if and only if a basic subgroup B of T can be decomposed into a direct sum $B = \bigoplus_{i=0}^{n} B_i$ such that $|B_0| < \aleph_0$,

$$[\mathbf{r}_p(B_i) \mid i = 1,..., n;\ p \in P] \leqslant \mathbf{Rt}(R)$$

and

$$(\forall i = 1,..., n)(\forall p \in P)(\forall m \in N^*): \qquad \mathbf{f}^p_m(B_i) + \mathbf{f}^p_{m+1}(B_i) \leqslant 1$$

(where $\mathbf{r}_p(B_i)$, $\mathbf{f}^p_m(B_i)$ denote the p-rank and the Ulm–Kaplansky invariants of B_i, respectively; $\mathbf{Rt}(R)$ is the Richman type of R; and P is the set of primes).

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Theorems 1 and 1′ solve the last open question from [6, Problem 2]. Theorem 5 from [3] showed that if G is a countable $\aleph_0$-indecomposable abelian group, then already some finite torsion-free rank subgroup H of G with the same torsion part as G is $\aleph_0$-indecomposable. Thus our restriction of torsion-free rank to be finite is only natural.

2. Preliminaries

All groups in this paper are abelian. We will use the notations and terminology of [2]. In addition, N will stand for the set of natural numbers; $N^* = \{0\} \cup N$; P is the set of prime numbers; $\mathbf{f}_k^p(G)$ are the Ulm–Kaplansky invariants ($p \in P$, $k \in N^*$); $\mathbf{C}_F(R)$, $\mathbf{Rt}(R)$ are the characteristic matrix and the Richman type of a finite rank torsion-free group R; $\mathbf{I}^p(T)$ is the p-indexing set of a torsion group T; and if T is a subgroup of G, then $\alpha \equiv \beta \pmod{T}$ means exactly $\alpha - \beta \in T$.

Definition 1. We would say that a group G passes the test if there exists a finite subset A in G such that there is no direct decomposition

$$G = F \oplus G_1$$

with a finite non-zero subgroup F, and $A \subseteq G_1$.

Test Lemma [4]. *Let G be a mixed countable reduced group of finite torsion-free rank. Then G is $\aleph_0$-indecomposable if and only if it passes the test.*

Lemma 1 [4, Lemma 2]. *Let B be a basic subgroup of a reduced countable torsion group T. There exists an $\aleph_0$-indecomposable extension of T by a finite rank torsion-free group R if and only if there exists an $\aleph_0$-indecomposable extension of B by R.*

3. The Richman Type and the p-Indexing Set

This section defines the invariants which will work for us in the proof of the main result.

In 1967 Fred Richman [3] defined the type for finite rank torsion-free groups. Following David Arnold [1], we will call it the Richman type.

Definition 2. Let R be a finite rank torsion-free group and $F \leqslant R$ a free subgroup of the same rank. The Richman type of R is the quasi-isomorphism class of R/F.

The Richman type will play a major role in this paper. We will use every bit of information it delivers.

Let R be a torsion-free group of finite rank n and $F \leqslant R$ a free subgroup of the same rank. Then the factor R/F can be completely characterized by a $n \times \omega$ matrix C with non-negative integral and infinite entries. Indeed let p be a prime; p-component of R/F is a direct sum

$$(R/P)_p = \bigoplus_{i=1}^{n} \mathbf{Z}(p^{k_i})$$

where $k_i \in N^* \cup \{\infty\}$, and $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_n$. Now we can define the pth column C_p of C:

$$C_p = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}.$$

DEFINITION 3. The matrix C obtained above is said to be a characteristic matrix of R. It will be denoted as $\mathbf{C}_F(R)$.

DEFINITION 4. Two characteristic matrices of the same size are said to be equivalent if almost all their corresponding entries are equal, and those unequal differ by finite amounts.

Any two characteristic matrices of a finite rank torsion-free group R are equivalent.

DEFINITION 2′. Let R be a finite rank torsion-free group. The equivalence class $\mathbf{Rt}(R)$ of the characteristic matrices of R is called the Richman type of R.

For the sake of comparability we will use in this paper the symbol $\aleph_0$ in place of ∞ everywhere in characteristic matrices and the Richman type.

The following invariants for torsion groups will be used as well:

DEFINITION 5. Let T be a torsion group, $B_p = \bigoplus_{i \in \mathbf{I}^p} \langle x_{pi} \rangle$ a basic subgroup of the p-component T_p of T. Denote

$$\mathbf{I}^p_m = \{ i \in \mathbf{I}^p \mid o(x_{pi}) = p^m \}.$$

The set $\mathbf{I}^p = \mathbf{I}^p(T)$ together with its partition into $\mathbf{I}^p_m$ is said to be the p-indexing set of T.

4. Countable Countably-Indecomposable Groups, n-Decomposable for Any Finite n

Now we are ready to prove the main result of this paper.

THEOREM 1. *Let T be a countable reduced torsion group and R a torsion-free group of finite rank n.*

Then there exists an $\aleph_0$-indecomposable extension G of T by R if and only if T can be decomposed into a direct sum $T = F \oplus H$, where F is a finite group, and for every prime p there exists a partition of the p-indexing set $\mathbf{I}^p = \mathbf{I}^p(H)$ into n subsets $\mathbf{S}_1^p, \mathbf{S}_2^p, ..., \mathbf{S}_n^p \subseteq \mathbf{I}^p$ such that

$$[\,|\mathbf{S}_i^p|\;|\; i = 1, ..., n;\; p \in P] \leqslant \mathbf{Rt}(R) \tag{1}$$

and

$$(\forall i = 1, ..., n)(\forall m \in N): \qquad |\mathbf{S}_i^p \cap (\mathbf{I}_m^p \cup \mathbf{I}_{m+1}^p)| \leqslant 1. \tag{2}$$

Proof. Due to Lemma 1 we can assume without loss of generality that T is one of its own basic subgroups.

Sufficient Condition. Without loss of generality we can assume that $F = 0$. Assume that the partitions satisfying (1) and (2) exist. Accordingly for any $p \in P$,

$$T_p = \bigoplus_{\substack{i = 1, ..., n \\ 0 \leqslant j < h_i^p}} \langle t_{p,i,j} \rangle,$$

where $h_i^p \in N \cup \{\aleph_0\}$, and $\exp(t_{p,i,j}) = n_{p,i,j}$ is a non-decreasing function of j for any $p \in P$; $i = 1, ..., n$. In fact, due to (2), we can claim even more:

$$n_{p,i,j+1} - n_{p,i,j} \geqslant 2. \tag{3}$$

Since T, R satisfy (1), there exists a free subgroup $\bar{X} = \langle \bar{x}_1, ..., \bar{x}_n \rangle$ of R such that the corresponding characteristic matrix $\mathbf{C}_{\bar{x}}(R) = [k_i^p \mid i = 1, ..., n;\; p \in P] \in \mathbf{Rt}(R)$ satisfies the following inequalities for every $i = 1, ..., n$ and $p \in P$:

$$h_i^p \leqslant k_i^p. \tag{4}$$

Since for every $p \in P$ we have $(R/\bar{X})_p = \bigoplus_{i=1}^n \mathbf{Z}(p^{k_i^p})$, we can choose a system of generators $\{\bar{b}_{p,i,m+1} \mid p \in P;\ i=1,\dots,n;\ m<k_i^p\}$ for R modulo $\bar{X}$ satisfying the following conditions:

$$\begin{gathered}\bar{b}_{p,i,0} \in \bar{X} \leqslant R \\ \dots\dots\dots\dots\dots \\ p\bar{b}_{p,i,m+1} = \bar{b}_{p,i,m} - \sum_{j=1}^{n} n_{p,i,m,j}\bar{x}_j\end{gathered} \tag{5}$$

where $n_{p,i,m,j} \in \mathbf{Z}$.

Now we define an extension G of T by R. G is generated by T, $\{x_i\}_{i=1}^n$, $\{b_{p,i,m+1} \mid p \in P;\ i=1,\dots,n;\ m<k_i^p\}$ freely except:

(a) if $\bar{b}_{p,i,0} = \sum_{\tau=1}^n m_\tau \bar{x}_\tau$, then $b_{p,i,0} = \sum_{\tau=1}^n m_\tau x_\tau$;

(b) if $m<h_i^p$, then $pb_{p,i,m+1} = b_{p,i,m} - \sum_{j=1}^n n_{p,i,m,j}x_j - t_{p,i,m}$;

(c) if $h_i^p \leqslant m < k_i^p$, then $pb_{p,i,m+1} = b_{p,i,m}$.

To prove $\aleph_0$-indecomposability of G we will use the Test Lemma with $A=\{x_1,\dots,x_n\}$.

Assume

$$G = F' \oplus H'$$

where $1<|F'|<\aleph_0$, and $A \subseteq H'$. Denote by Y the projection of G onto H'.

Due to (b), we get for every $m<h_i^p$:

$$\begin{aligned}0 \equiv b_{p,i,0} &\equiv t_{p,i,0} + pb_{p,i,1} \\ &\equiv t_{p,i,0} + pt_{p,i,1} + p^2 t_{p,i,2} \equiv \cdots \\ &\equiv t_{p,i,0} + pt_{p,i,1} + \cdots + p^m t_{p,i,m} \\ &\quad + p^{m+1} b_{p,i,m+1} \qquad (\operatorname{mod}\langle A\rangle).\end{aligned} \tag{6}$$

By multiplying both sides of (6) by $p^{n_{p,i,m}-m-1}$, we get rid of all the terms in the right side except for the last two:

$$p^{n_{p,i,m}-1}t_{p,i,m} + p^{n_{p,i,m}}b_{p,i,m+1} \equiv 0 \qquad (\operatorname{mod}\langle A\rangle). \tag{7}$$

Since $\langle A\rangle \leqslant H'$, the following congruence takes place for every $p \in P$; $i=1,\dots,n$; and $m<h_i^p$:

$$Y(p^{n_{p,i,m}-1}t_{p,i,m}) \equiv p^{n_{p,i,m}-1}t_{p,i,m} \qquad (\operatorname{mod} p^{n_{p,i,m}}G).$$

This implies that $\operatorname{Socle}(G)=\operatorname{Socle}(H')$. Since T is a direct sum of cyclics, we get $G=H'$. Contradiction, therefore G is $\aleph_0$-indecomposable.

Necessary Condition. Let G be an $\aleph_0$-indecomposable group with the torsion part T, σ the canonical projection of G onto $R=G/T$, and $r_0(R)=n<\aleph_0$. Then G passes the test, i.e., there is a finite subset A of G such that

$$\neg(\exists F): \qquad [G=F\oplus H, A\subseteq H, 1<|F|<\aleph_0]. \tag{8}$$

There are a finite subgroup $B\leqslant G$ and a maximal linearly independent set $\{x_1,..., x_n\}$ of G such that $A\subseteq B\oplus\langle x_1,..., x_n\rangle$. There exists a direct decomposition

$$G=F'\oplus G' \tag{9}$$

where $B\leqslant F'$, $|F'|<\aleph_0$. In accordance with decomposition (9), $x_i=c_i+y_i$, where $c_i\in F'$, $y_i\in G'$ $(i=1,..., n)$. G' has no direct decomposition $G'=F''\oplus H''$ with $1<|F''|<\aleph_0$ and $\{y_1,..., y_n\}\subseteq H''$. Therefore we can assume without loss of generality that G passes the test with its maximal linearly independent set: $A=\{x_1,..., x_n\}$. Denote $\bar{X}=\langle\bar{x}_1,..., \bar{x}_n\rangle$, where $\bar{x}_i$ stands for the image of x_i under σ $(i=1,..., n)$.

Let $\mathbf{C}_{\bar{X}}(R)=[k_i^p \mid i=1,..., n;\ p\in P]\in\mathbf{Rt}(R)$, i.e., $(R/\bar{X})_p=\bigoplus_{i=1}^n \mathbf{Z}(p^{k_i^p})$. We define elements $\bar{b}_{p,i,m+1}\in R$ $(p\in P;\ i=1,..., n;\ m<k_i^p)$ exactly as in (5).

Now we can choose $b_{p,i,0}\in\langle A\rangle$ such that $\bar{b}_{p,i,0}=\sum_{\tau=1}^n m_\tau\bar{x}_\tau$ implies $b_{p,i,0}=\sum_{\tau=1}^n m_\tau x_\tau$. For every $\bar{b}_{p,i,m+1}$, $0<m<k_i^p$, we choose an inverse image $b_{p,i,m+1}$ under projection σ, and define $t_{p,i,m}$ for every $p\in P$; $i=1,..., n$; $0\leqslant m<k_i^p$ as follows:

$$t_{p,i,m}=b_{p,i,m}-pb_{p,i,m+1}-\sum_{j=1}^n n_{p,i,m,j}x_j. \tag{10}$$

Let $p\in P$. We define

$$M_p=\langle T_p, A, \{b_{p,i,m+1} \mid i=1,..., n; 0\leqslant m<k_i^p\}\rangle.$$

Subgroup M_p is p-pure in G, hence if M_p had a finite non-zero direct summand L, the projection δ of M_p onto L would be extendable to a projection of G onto L. Therefore, M_p has no finite non-zero direct summand L such that the projection $\delta: M_p\to L$ is zero on A.

Of course a finite summand of M_p is a summand of $T(M_p)=T_p$. But when can a projection δ_0 of T_p onto its finite non-zero summand L be extended to a projection $\delta: M_p\to L$ such that $\delta(A)=0$? Exactly when we can define $\delta(b_{p,i,m+1})$ for every $m<k_i^p$ so that it is consistent with $\delta(b_{p,i,0})=0$ and $\delta(A)=0$. In fact when $k_i^p<\aleph_0$, it suffices to define $\delta(b_{p,i,m})$

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for just $m = k_i^p$ (please see (10)), otherwise we do it for all m and finish by König's Graph Lemma. Also due to (10):

$$\begin{aligned} 0 = \delta(b_{p,i,0}) &= \delta(t_{p,i,0}) + p\,\delta(b_{p,i,1}) \\ &= \delta(t_{p,i,0} + pt_{p,i,1}) + p^2\,\delta(b_{p,i,2}) = \cdots \\ &= \delta\left(\sum_{j=1}^{q_i - 1} p^j t_{p,i,j}\right) + p^{q_i}\,\delta(b_{p,i,q}), \end{aligned}$$

so we need p^{q_i} to divide $\delta_0(\sum_{j=1}^{q_i-1} p^j t_{p,i,j})$.

We are looking in fact for a function $f: \mathbf{I}^p \to \{1,..., n\}$ satisfying conditions (1), (2), where $\mathbf{S}_i^p = \{j \in \mathbf{I}^p \mid f(j) = i\}$. So, due to compactness theorem, it suffices to prove the existence of f on $\mathbf{I}^p(\alpha) = \bigcup_{m=1}^{\alpha} \mathbf{I}_m^p$ for any natural number α, i.e., we can divide T_p by $\bigoplus_{i \in \mathbf{I}^p \setminus \mathbf{I}^p(\alpha)} \langle x_i \rangle$.

It suffices now to prove the following statement for any finite p-group A:

Let $A = \bigoplus_{i \in \mathbf{I}^p} \langle x_i \rangle$ be a finite p-group with the p-indexing set $\mathbf{I}^p$; $s_1,..., s_n \in A$; and $k_1^*,..., k_n^*$ positive integers such that there exists no projection Q of A onto some non-zero subgroup of A such that $p^{k_i^*}$ divides $Q(s_i)$ for every $i = 1,..., n$. Then there exists a partition of $\mathbf{I}^p$ into n subsets $\mathbf{S}_1,..., \mathbf{S}_n$ satisfying both

$$(\forall i = 1,..., n): \qquad |\mathbf{S}_i| \leqslant k_i^* \tag{11}$$

and

$$(\forall i = 1,..., n)(\forall m \in N): \qquad |\mathbf{S}_i \cap (\mathbf{I}_m^p \cup \mathbf{I}_{m+1}^p)| \leqslant 1. \tag{12}$$

We will prove it by induction on n. Let $s_n = \sum_{i \in \Delta} v_i x_i$, where Δ is a finite subset of $\mathbf{I}^p$, and $v_i \in \mathbf{Z}$. We can assume without loss of generality that

$$(\forall m \in N): \qquad |\Delta \cap \mathbf{I}_m^p| \leqslant 1 \tag{13}$$

and

$$(\forall i \in \Delta): \qquad v_i = p^{\omega(i)}. \tag{14}$$

Indeed if (13), (14) were not satisfied, we can make some transformations of the basis $\{x_i\}_{i \in \mathbf{I}^p}$ of A so that in the new basis Δ and $\{v_i\}_{i \in \Delta}$ would satisfy the conditions (13), (14).

We can also assume without loss of generality that

$$(\forall i \in \Delta): \qquad \omega(i) < k_n^* \tag{15}$$

because if for some $i_0 \in \Delta$ inequality $\omega(i_0) > k_n^*$ holds, $p^{k_n^*}$ would divide $Q(p^{\omega(i_0)} x_{i_0})$ no matter what Q is, and thus such i_0 could be omitted from Δ.

If we could find $\mathbf{S}_n \subseteq \Delta$ satisfying (11) and (12) and projection Q_n of A

onto $\bigoplus_{i\in \mathbf{I}^p\setminus \mathbf{S}_n}\langle x_i\rangle$ such that $p^{k_n^*}$ divides $Q_n(s_n)$, we have carried the induction. Indeed we would apply the induction hypothesis to $Q_n(\Delta)$; $Q_n(s_1),..., Q_n(s_{n-1})$; $k_1^*,..., k_{n-1}^*$ to obtain $\mathbf{S}_1,..., \mathbf{S}_{n-1}$ satisfying inequalities (11), (12), then we would claim that $\mathbf{I}^p=\mathbf{S}_1\cup\cdots\cup\mathbf{S}_n$, because otherwise we would have found a projection Q of Δ onto $\bigoplus_{i\in\mathbf{I}^p\setminus(\mathbf{S}_1\cup\cdots\cup\mathbf{S}_n)}\langle x_i\rangle$ such that $p^{k_i^*}$ divides $Q(s_i)$ for every $i=1,..., n$.

So, we want to partition Δ into $\mathbf{S}_n\cup\Delta_1\cup\cdots\cup\Delta_{k_n'}$ so that

$$|\mathbf{S}_n|=k_n'\leqslant k_n^* \tag{16}$$

and we can define a projection Q_n by

$$\begin{aligned}(\forall i\in\mathbf{S}_n):&\quad Q_n(p^{\omega(i)}x_i)=-\sum_{j\in\Delta_i}p^{\omega(j)}x_j\\ \text{and}\qquad (\forall i\in\mathbf{I}^p\setminus\mathbf{S}_n):&\quad Q_n(x_i)=x_i\end{aligned} \tag{17}$$

and therefore $Q_n(s_n)=0$. (We use the convention $\sum_{j\in\Delta_i}p^{\omega(j)}x_j=0$ if $\Delta_i=\varnothing$.)

Since a projection cannot decrease the height or increase the order of an element, we have natural constraints for any $j\in\Delta_i$:

$$\omega(i)\leqslant\omega(j) \tag{18}$$

and

$$\exp(x_i)-\omega(i)\geqslant\exp(x_j)-\omega(j). \tag{19}$$

Now the construction of the desired partition is clear. Denote by Ω the set of pairs $\{(\exp(x_i)-\omega(i),\ -w(i))\mid i\in\Delta\}$ with component-wise order (i.e., $(\alpha, b)\leqslant(\alpha_1, b_1)$ iff $\alpha\leqslant\alpha_1$ and $b\leqslant b_1$). Then define the subsets of the partition as follows:

$$\mathbf{S}_n=\{i\in\Delta\mid(\exp(x_i)-\omega(i),\ -\omega(i))\text{ is maximal in }\Omega\} \tag{20}$$

$$\Delta_i=\left\{j\in\Delta\mid j\notin\mathbf{S}_n;\ \omega(j)\geqslant\omega(i);\ \exp(x_j)-\omega(j)\leqslant\exp(x_i)-\omega(i);\ j\notin\bigcup_{u<i}\Delta_u\right\}.$$

All there is left to prove is that $\mathbf{S}_n$ satisfies inequalities (11), (12).

Assume $|\mathbf{S}_n|>k_n^*$. Due to (15) and the pigeonhole principal, this implies that $\omega(i)=\omega(j)$ for some $i, j\in\mathbf{S}_n$; $i\neq j$. Then since both pairs $(\exp(x_i)-\omega(i),\ -\omega(i))$ and $(\exp(x_j)-\omega(j),\ -\omega(j))$ are maximal in Ω, we get $\exp(x_i)=\exp(x_j)$ in contradiction with (13). Hence $\mathbf{S}_n$ satisfies inequality (11).

Let $i, j\in\Delta$; $i\neq j$. Since both pairs $(\exp(x_i)-\omega(i),\ -\omega(i))$,

$(\exp(x_j) - \omega(j), -\omega(j))$ are maximal in Ω, we can assume without loss of generality that

$$\exp(x_i) - \omega(i) \geqslant \exp(x_j) - \omega(j) + 1$$
$$-\omega(i) \leqslant -\omega(j) - 1$$

therefore $\exp(x_i) \geqslant \exp(x_j) + 2$. $\mathbf{S}_n$ satisfies inequality (12).

COROLLARY 1. [6, *Theorem* 2].

COROLLARY 2. [4, *Theorem* 3].

A set-theoric criterion delivered by Theorem 1 can be translated into the language of group-theoretic invariants:

THEOREM 1′. *Let T be a countable reduced torsion group, and R a torsion-free group of finite rank n. There exists an $\aleph_0$-indecomposable extension of T by R if and only if a basic subgroup B of T can be decomposed into a direct sum $B = \bigoplus_{i=0}^{n} B_i$ with $|B_0| < \aleph_0$,*

$$[r_p(B_i) \mid i = 1,..., n; p \in P] \leqslant \mathbf{Rt}(r)$$

and

$$(\forall i = 1,..., n)(\forall p \in P)(\forall m \in N^*): \qquad \mathbf{f}_m^p(B_i) + \mathbf{f}_{m+1}^p(B_i) \leqslant 1.$$

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