ISRAEL JOURNAL OF MATHEMATICS, Vol. 49, Nos. 1–3, 1984

ALMOST DISJOINT ABELIAN GROUPS†

BY

P. C. EKLOF, A. H. MEKLER AND S. SHELAH

Dedicated to the memory of Abraham Robinson on the tenth anniversary of this death

ABSTRACT

Under various set-theoretic hypotheses we construct families of maximal possible size of almost free abelian groups which are pairwise almost disjoint, i.e. there is no non-free subgroup embeddable in two of them. We show that quotient-equivalent groups cannot be almost disjoint, but we show how to construct maximal size families of quotient-equivalent groups of cardinality $\aleph_1$, which are mutually non-embeddable.

Introduction

In this paper we construct large families of abelian groups which are all "close" to being free groups and yet are pairwise non-isomorphic in some strong sense. The strongest sense of non-isomorphic which we consider is that of almost disjointness, i.e. the property of having no non-free subgroups in common. (Precise definitions are given below.) It is possible to construct maximal size families of almost disjoint groups all with the same Γ-invariant, which is an equivalence class of stationary sets (Section 1). If we also require the members of the family to be quotient-equivalent then the family cannot have more than one member (Section 2), but still there are maximal size families of strongly ω_1-free groups of cardinality ω_1 whose members are quotient-equivalent and non-isomorphic in a somewhat weaker — but still very strong — sense than almost disjointness (Section 3).

† First and third authors acknowledge assistance from the US–Israel Binational Science Foundation, Grant No. 1110. First author partially supported by NSF Grant No. MCS-8003691. Second author acknowledges support from the National Science and Engineering Research Council of Canada, Grant No. U0075

Received February 15, 1982 and in revised form October 22, 1982

We say that a subgroup B of an abelian group A is *small* if $|B|<|A|$, and call A *almost free* if every small subgroup of A is free. By a theorem of Shelah, an almost free group of singular cardinality is free (cf. [3; chap. 5]). So from now we consider only groups A of regular uncountable cardinality.

We shall deal with almost free groups A which have the stronger property of being *strongly almost free*, i.e. A is almost free and every small subgroup of A is contained in a small subgroup B such that A/B is almost free. We say that two almost free groups A and A' are *almost disjoint* if whenever H is embeddable as a subgroup of both A and A', then H is free. (For short we say that A and A' "have no non-free subgroup in common"). Obviously almost disjoint groups are non-isomorphic in a very strong way.

There is a natural invariant which can be associated with an almost free group A, namely a certain equivalence class, $\Gamma(A)$, of subsets of $|A|$. (Here two subsets of $|A|$ are equivalent if they coincide on a closed unbounded subset of $|A|$; for more details see Section 0.) A is free if and only if $\Gamma(A)=$ the class of $\emptyset$; but there are 2^κ possibilities for the invariant of a non-free almost free group of cardinality κ.

It is not hard to construct pairwise almost disjoint strongly almost free groups of cardinality κ by taking them to have almost disjoint Γ-invariants (see 0.7 and 0.8), but in section 1 we are concerned with the more difficult problem of constructing families of size 2^κ of strongly almost free groups of cardinality κ all of which have the same invariant Γ yet any two of which are almost disjoint. For $\kappa=\aleph_n$ $(n\in\omega)$ we obtain this result as a theorem of ZFC (Theorem 1.1). In order to obtain such results for cardinals $>\aleph_\omega$ we invoke additional set theoretic hypotheses, viz. GCH and $V=L$ (Theorems 1.7 and 1.8). In particular, Theorem 1.8 gives, under the assumption $V=L$, a complete characterization for all regular κ of the classes of $E\subseteq\kappa$ which can be realized as $\Gamma(A)$ for some strongly almost free A of cardinality κ (answering a question in [9], which gave the characterization for successor κ).

The method used in section 1 to obtain almost disjoint groups A and B of cardinality κ is to construct them as the union of continuous chains, $\{A_\nu:\nu<\kappa\}$, $\{B_\nu:\nu<\kappa\}$ respectively, of free groups such that for all ν, $A_{\nu+1}/A_\nu$ and $B_{\nu+1}/B_\nu$ are almost disjoint. In section 2 we show that this is the only possible method: if A and B are such that there is a stationary set of ν such that $A_{\nu+1}/A_\nu$ and $B_{\nu+1}/B_\nu$ are not almost disjoint then A and B are not almost disjoint. In particular, if A and B are *quotient-equivalent* (i.e. for all ν, $A_{\nu+1}/A_\nu\cong B_{\nu+1}/B_\nu$) then they are not almost disjoint. However in section 3 we construct, for every possible non-free quotient-equivalence class, families of size 2^{ω_1} of strongly

ω_1-free groups of cardinality ω_1 which are mutually non-embeddable and pairwise *almost disjoint for pure subgroups*, i.e. they have no non-free *pure* subgroup in common.

0. Preliminaries

Here we collect together some definitions, conventions and simple lemmas that are required for what follows. The reader may wish to skip this section and consult it only as needed. We refer the reader to [3], chapters 1 and 2 for further details.

0.1. Let κ be an uncountable cardinal. A group A is *κ-free* if every subgroup of cardinality $< \kappa$ is free; it is *strongly κ-free* if it is κ-free and in addition every subset of A of cardinality $< \kappa$ is contained in a subgroup B of cardinality $< \kappa$ such that A/B is κ-free. A is called *almost free* if A is $|A|$-free and *strongly almost free* if it is strongly $|A|$-free.

For $\kappa = \omega$, we define: A is ω-free iff A is strongly ω-free iff A is torsion-free.

0.2. A *smooth* chain of groups is a sequence $\{A_\mu : \mu < \alpha\}$ such that: (1) for all $\mu < \nu < \alpha$, A_μ is a subgroup of A_ν; and (2) for all limit ordinals $\mu < \alpha$, $A_\mu = \bigcup_{\nu<\mu} A_\nu$. If $|A| \leqq \kappa$ a *κ-filtration* of A is a smooth chain of subgroups of A indexed by κ whose union is A, such that every member of the chain has cardinality $< \kappa$. If $|A| = \kappa$ we shall, for convenience, also assume that any κ-filtration of A we consider is strictly increasing. If $|A| \leqq \kappa$ and we write $A = \bigcup_{\nu<\kappa} A_\nu$ we mean that $\{A_\nu \mid \nu < \kappa\}$ is a κ-filtration of A. If $|A| = \kappa$, κ regular, and A is strongly κ-free, then it has a κ-filtration $\{A_\nu \mid \nu < \kappa\}$ such that for all $\nu < \kappa$, $A/A_{\nu+1}$ is κ-free; from now on, we demand of any κ-filtration of a strongly κ-free group that it have this additional property.

0.3. If κ is a limit ordinal of uncountable cofinality, a subset $C \subseteq \kappa$ is called a *cub* (*closed unbounded*) *set* in κ if $\sup C = \kappa$ and for all $X \subseteq C$, if $\sup X < \kappa$, then $\sup X \in C$. For example, $\lim(\kappa)$, the set of limit ordinals, is a cub in κ. Define $D(\kappa)$ to be the set of equivalence classes of subsets of κ under the equivalence relation of equality on a cub (i.e. if E_1 and E_2 are subsets of κ, E_1 and E_2 are equivalent iff there is a cub C such that $E_1 \cap C = E_2 \cap C$). The equivalence class of $E \subseteq \kappa$ is denoted $\tilde{E}$. $D(\kappa)$ is a Boolean algebra under the ordering induced by inclusion, with smallest element $0 \overset{\text{def}}{=} \tilde{\emptyset}$, and largest element $1 \overset{\text{def}}{=} \tilde{\kappa} = \tilde{C}$ (C any cub). We say $E \subseteq \kappa$ is *stationary* (in κ) if $\tilde{E} \neq 0$, and E is *thin* otherwise. If σ is a limit ordinal of cofinality ω, $E \subseteq \sigma$ is *stationary* in σ iff E contains a terminal segment of σ.

Sh:145

0.4. Let κ be a regular uncountable cardinal. If A is almost free of cardinality κ, and $A = \bigcup_{\nu<\kappa} A_\nu$ is a κ-filtration of A, define $\Gamma_\kappa(A) = \tilde{E}$ where $E = \{\nu \mid A/A_\nu \text{ is not } \kappa\text{-free}\} = \{\nu \mid \exists \mu > \nu (A_\mu / A_\nu \text{ is not free})\}$. Then Γ_κ is a well-defined function from κ-free groups of cardinality κ to $D(\kappa)$ and $\Gamma_\kappa(A) = 0$ iff A is free (cf. [3; lemma 2.1]). In fact we shall always write Γ instead of Γ_κ since, in context, there will be no ambiguity. Thus if $\tilde{E} \in D(\kappa)$, $\Gamma^{-1}(\tilde{E})$ is the class of all κ-free groups A of cardinality κ satisfying $\Gamma(A) = \tilde{E}$.

0.5. Let κ be a regular uncountable cardinal. Two subsets X and Y of κ (or two elements $\tilde{E}_1$, $\tilde{E}_2$ of $D(\kappa)$) are called *almost disjoint* if $X \cap Y$ is not stationary in κ (resp. $\tilde{E}_1 \cap \tilde{E}_2 = 0$). Two functions $f, g : \kappa \to \lambda$ are *almost disjoint* if $\{\nu \mid f(\nu) = g(\nu)\}$ is not stationary in κ. In general, questions about the maximal size of a family of pairwise almost disjoint sets or functions are not decidable in ZFC (cf. [7; §35]). However we have the following well-known results.

0.6 Lemma. *Let κ be a regular uncountable cardinal and let E be a stationary subset of κ contained in* $\lim(\kappa)$.

(i) *E is the disjoint union of κ stationary sets.*

(ii) *Assume $\diamondsuit_\kappa(E)$. Then there is a family of size 2^κ of pairwise almost disjoint subsets of E.*

(iii) *Assume $\diamondsuit^*_\kappa(E)$. Then there is a κ-Kurepa tree on E, i.e. a family $\{f_i \mid i < \kappa^+\}$ of pairwise almost disjoint functions: $E \to \kappa^+$ s.t. for all $i < \kappa^+$ and all $\nu \in E$, $f_i(\nu) \in |\nu|$.*

Proof. For (i) see e.g. [7; thm. 85]. For (ii), suppose $\{S_\nu : \nu \in E\}$ is a diamond sequence (see e.g. [3; p. 21]). Then for every subset $X \subseteq \kappa$, let $E_X = \{\nu \in E : X \cap \nu = S_\nu\}$. It is easy to check that the E_X form the desired family.

(iii) Suppose $\{\{P^\gamma_\nu : 0 < \gamma < |\nu|\} : \nu \in E\}$ is given by $\diamondsuit^*_\kappa(E)$, i.e. for all $\nu \in E$, $0 < \gamma < |\nu|$, $P^\gamma_\nu \subseteq \nu$ and for all $X \subseteq \kappa$, there is a cub $C \subseteq \kappa$ such that for all $\nu \in E \cap C$, $X \cap \nu \in \{P^\gamma_\nu : 0 < \gamma < |\nu|\}$. Then for every subset $X \subseteq \kappa$, for $\nu \in E$ define $f_X(\nu) = \gamma$ if $X \cap \nu = P^\gamma_\nu$ and $f_X(\nu) = 0$ if there is no such γ. Then if $X \neq Y$ — say $\alpha \in (X - Y)$ — there is a cub C such that if $\nu \in C \cap E$, then $X \cap \nu$ and $Y \cap \nu$ are both in $\{P^\gamma_\nu : 0 < \gamma < |\nu|\}$. But then $f_X(\nu) = f_Y(\nu)$ implies $\nu \in (E - C) \cup \{\nu \in E : \nu \leqq \alpha\}$ so f_X and f_Y agree only on a thin set. □

We conclude with a simply observation of how one can use almost disjoint sets to produce almost disjoint groups.

0.7 Lemma. *Let A, B, A' be almost free groups of the same regular uncountable cardinality.*

(i) *If B is a subgroup of A, then $\Gamma(B) \subseteq \Gamma(A)$.*

(ii) *If $\Gamma(A)$ and $\Gamma(A')$ are almost disjoint then A and A' are almost disjoint.*

PROOF. (i) Given a κ-filtration $A = \bigcup_{\nu<\kappa} A_\nu$ define a κ-filtration of B by $B_\nu = B \cap A_\nu$. If B_μ / B_ν is not free, then A_μ / A_ν is not free since B_μ / B_ν is isomorphic to a subgroup of A_μ / A_ν. Hence $\{\nu \mid B/B_\nu \text{ is not } \kappa\text{-free}\} \subseteq \{\nu \mid A/A_\nu \text{ is not } \kappa\text{-free}\}$.

(ii) If B is embeddable in both A and A', then by (i) $\Gamma(B) \subseteq \Gamma(A)$ and $\Gamma(B) \subseteq \Gamma(A')$. Thus by hypothesis $\Gamma(B) = 0$, so B is free. □

In section 2 we show that for A strongly κ-free of cardinality κ, every $E \subseteq \Gamma(A)$ is realizable as $\Gamma(B)$ for some subgroup B of A (Corollary 2.3).

An immediate consequence of 0.7 (ii), 0.6 (ii) and the methods of [2] is the following.

0.8 COROLLARY. *Assume $V = L$. If κ is a regular uncountable cardinal which is not weakly compact, there are 2^κ pairwise almost disjoint strongly almost free groups of cardinality κ.* □

This result will be improved in the next section (cf. Theorem 1.8).

1. Constructing almost disjoint groups

In this section we shall show how to construct the maximal number of pairwise almost disjoint groups with the same value of Γ. The first result, for groups of cardinality $\aleph_n$ ($n \in \omega$), is a theorem of ZFC. For the later results we need to assume some extra set theoretic hypothesis just to insure that non-free almost free groups exist for cardinals $> \aleph_\omega$. Theorems 1.5–1.7 deal with cardinals $< \aleph_{\omega^2}$ and assume only GCH. Theorem 1.8 (which does not depend on 1.5–1.7) assumes $V = L$ and deals with arbitrarily large cardinals.

1.1 THEOREM. *For every $n \in \omega$ and every $\tilde{E} \in D(\omega_n)$ with $\tilde{E} \neq 0$ there exist 2^{ω_n} pairwise almost disjoint strongly ω_n-free groups in $\Gamma^{-1}(\tilde{E})$.*

Before giving the proof we give two elementary lemmas, the first of which provides the initial step of the inductive construction, and the second of which provides the combinatorial fact which is the key to the inductive step. ($\mathbf{Z}^{(\kappa)}$ denotes the direct sum of κ copies of $\mathbf{Z}$.)

1.2 LEMMA. *There exists a family $\{H_i : i < 2^{\aleph_0}\}$ of $2^{\aleph_0}$ countable torsion-free groups such that for all $i \neq j$, $H_i \oplus \mathbf{Z}^{(\omega)}$ and $H_j \oplus \mathbf{Z}^{(\omega)}$ are almost disjoint.*

Sh:145

PROOF. Let $\{S_l \mid l < 2^{\aleph_0}\}$ be a family of subsets of ω such that for all $i \neq j$, $S_i \cap S_j$ is finite. For each $i < 2^{\aleph_0}$ let H_i be the rank one torsion-free group of type

$$t_i = (k_1^i, k_2^i, \cdots)$$

where $k_n^i = 1$ if $n \in S_i$ and $k_n^i = 0$ otherwise (cf. [5; 85]). Now let $F = \mathbf{Z}^{(\omega)}$ and suppose A is embeddable in both $H_i \oplus F$ and $H_j \oplus F$ for some $i \neq j$. If $e_l : H_l \to H_l \oplus F$ and $\pi_l : H_l \oplus F \to F$ are the canonical injection and projection, respectively ($l = i$ or j), then $e_i^{-1}(A) \oplus \pi_i(A) \cong A \cong e_j^{-1}(A) \oplus \pi_j(A)$. Let u be a non-zero element of $e_i^{-1}(A)$; then under the preceding isomorphisms u is carried to an element of the form $v + w$ where $v \in e_j^{-1}(A)$, $w \in \pi_j(A)$. Assume the type of u (in $e_i^{-1}(A)$) is not equivalent to $(0, 0, \cdots, 0, \cdots)$; then we must have $w = 0$ and type of u = type of v (in $e_j^{-1}(A)$). But this is impossible since $S_i \cap S_j$ is finite. We conclude that $e_i^{-1}(A)$, and hence A, is free. □

The following is a standard result (cf. [7; p. 431]).

1.3 LEMMA. *Let $n \in \omega$. There is a family $\{f_\alpha \mid \alpha < 2^{\omega_{n+1}}\}$ of pairwise almost disjoint functions: $\omega_{n+1} \to 2^{\omega_n}$; in fact, if $\alpha \neq \beta$ $\exists \sigma < \omega_{n+1}$ such that for all $\nu > \sigma$, $f_\alpha(\nu) \neq f_\beta(\nu)$.*

PROOF. Enumerate the subsets of ω_{n+1} of power $\leqq \omega_n$ as a sequence $\{Y_i : i < 2^{\omega_n}\}$. For every $X \subseteq \omega_{n+1}$ define a function $f_X : \omega_{n+1} \to 2^{\omega_n}$ by: $f_X(\nu) = i$, where $Y_i = X \cap \nu$. If $X_1 \neq X_2$, choose $\sigma \in (X_1 - X_2) \cup (X_2 - X_1)$; then $f_{X_1}(\nu) \neq f_{X_2}(\nu)$ if $\nu > \sigma$. Since a subset of ω_{n+1} of cardinality $\leqq \omega_n$ is thin, this family is almost disjoint in the sense of 0.5. □

Finally, before proving the theorem we recall a definition and a theorem due to P. Hill [6]. For each $n \in \omega$ Hill defined a class of groups $\mathcal{F}_n$. The class $\mathcal{F}_0$ consists of all countable torsion-free groups. If $\mathcal{F}_n$ has been defined, then $G \in \mathcal{F}_{n+1}$ iff $G = \bigcup_{\nu<\mu} G_\nu$ (smooth) where $\mu \leqq \omega_{n+1}$ and for all $\nu < \mu$, G_ν is free and $G_{\nu+1}/G_\nu \in \mathcal{F}_n$. Hill proved that every element of $\mathcal{F}_n$ is ω_n-free. Mekler [9] showed how to construct elements of $\mathcal{F}_n$ which are strongly ω_n-free.

PROOF OF 1.1. We shall prove by induction on $n \in \omega$ that there are 2^{ω_n} strongly ω_n-free elements of $\mathcal{F}_n$, $\{H_i : i < 2^{\omega_n}\}$ with the property that if $i \neq j$, then $H_i \oplus \mathbf{Z}^{(\omega_n)}$ and $H_j \oplus \mathbf{Z}^{(\omega_n)}$ are almost disjoint. The initial case, $n = 0$, is Lemma 1.2, so assume the result is true for n (with the family $\{H_i : i < 2^{\omega_n}\}$ as above) and we shall prove it for $n + 1$. Let $\tilde{E} \in D(\omega_{n+1}) - \{0\}$. We shall assume E consists of limit ordinals. Let $\{f_\alpha \mid \alpha < 2^{\omega_{n+1}}\}$ be a family of almost disjoint functions as in Lemma 1.3. For each $\alpha < 2^{\omega_{n+1}}$ we shall define by transfinite induction a continuous chain $\{A_\nu^{(\alpha)} : \nu < \omega_{n+1}\}$ of free groups of cardinality ω_n such that

$A^{(\alpha)} \stackrel{\text{def}}{=} \bigcup_{\nu<\omega_{n+1}} A_\nu^{(\alpha)}$ is strongly ω_{n+1}-free and $\Gamma(A^{(\alpha)}) = \tilde{E}$. (Here '$(\alpha)$' is an index, and does *not* denote direct sum of α copies.)

Suppose that $A_\mu^{(\alpha)}$ has been constructed for all $\mu < \nu$. If ν is a limit ordinal, let $A_\nu^{(\alpha)} = \bigcup_{\mu<\nu} A_\mu^{(\alpha)}$. If $\nu = \tau + 1$ and $\tau \notin E$, let $A_\nu^{(\alpha)} = A_\tau^{(\alpha)} \oplus F$, where F is the free group of rank ω_n; if $\tau \in E$ choose $A_\nu^{(\alpha)} \supseteq A_\tau^{(\alpha)}$ such that

$$A_\nu^{(\alpha)}/A_\tau^{(\alpha)} \cong H_{f_\alpha(\tau)}$$

and moreover such that if $\mu < \tau$ and $\mu \notin E$ then $A_\nu^{(\alpha)}/A_\mu^{(\alpha)}$ is free; this is possible by lemma 5.5 of [9]. Clearly, by construction, $A^{(\alpha)}$ is strongly ω_{n+1}-free and $\Gamma(A^{(\alpha)}) = \tilde{E}$ so it remains only to observe that if $\alpha \neq \beta$ then $A^{(\alpha)} \oplus F$ and $A^{(\beta)} \oplus F$ are almost disjoint, where $F = \mathbf{Z}^{(\omega_{n+1})}$.

Let $F = \bigcup_{\nu<\omega_{n+1}} F_\nu$ be an ω_{n+1}-filtration of F by direct summands; define $\tilde{A}_\nu^{(\alpha)} = A_\nu^{(\alpha)} \oplus F_\nu$, $\tilde{A}_\nu^{(\beta)} = A_\nu^{(\alpha)} \oplus F_\nu$. This defines ω_{n+1}-filtrations of $A^{(\alpha)} \oplus F$ and $A^{(\beta)} \oplus F$ respectively. Now suppose, in order to obtain a contradiction, that there are embeddings $\phi_\alpha : B \to A^{(\alpha)} \oplus F$, $\phi_\beta : B \to A^{(\beta)} \oplus F$ and suppose B is not free. Let $B = \bigcup_{\nu<\omega_{n+1}} B_\nu$ be an ω_{n+1}-filtration; then $S =^{\text{def}} \{\nu \mid B_\nu$ is not ω_{n+1}-pue in $B\}$ is a stationary subset of ω_{n+1} (cf. [3; lemma 2.1 (2)]). There is a cub C in ω_{n+1} such that for all $\mu \in C$, $\phi_\alpha(B_\mu) = \phi_\alpha(B) \cap \tilde{A}_\mu^{(\alpha)}$ and $\phi_\beta(B_\mu) = \phi_\beta(B) \cap A_\mu^{(\beta)}$. Choose $\tau \in C \cap S$ such that $f_\alpha(\tau) \neq f_\beta(\tau)$. Let $\mu \in C$ such that $\mu > \tau$ and B_μ/B_τ is not free. By choice of C, for $i = \alpha$ or β, ϕ_i induces an embedding

$$B_\mu/B_\tau \to \tilde{A}_\mu^{(i)}/\tilde{A}_\tau^{(i)}.$$

But

$$\tilde{A}_\mu^{(i)}/\tilde{A}_\tau^{(i)} = (\tilde{A}_{\tau+1}^{(i)}/A_\tau^{(i)}) \oplus (\tilde{A}_\mu^{(i)}/\tilde{A}_{\tau+1}^{(i)})$$

since $A_\mu^{(i)}/A_{\tau+1}^{(i)}$ is free ($\tau + 1 \notin E$). So we have embeddings of B_μ/B_τ into $H_{f_\alpha(\tau)} \oplus \mathbf{Z}^{(\omega_n)}$ and into $H_{f_\beta(\tau)} \oplus \mathbf{Z}^{(\omega)}$, which is impossible since $f_\alpha(\tau) \neq f_\beta(\tau)$. □

The methods of 1.1 combined with those of [9] yield:

1.4 Theorem. *Suppose that for some α there are 2^{ω_α} strongly ω_α-free groups of cardinality ω_α, $\{H_i \mid < 2^{\omega_\alpha}\}$, such that for $i \neq j$, $H_i \oplus \mathbf{Z}^{(\omega_\alpha)}$ and $H_j \oplus \mathbf{Z}^{(\omega_\alpha)}$ are almost disjoint. Then for every $n \in \omega - \{0\}$ and every stationary $E \subseteq \{\sigma \in \lim(\omega_{\alpha+n}) : \mathrm{cf}(\sigma) \geqq \omega_\alpha\}$, there are $2^{\omega_{\alpha+n}}$ pairwise almost disjoint strongly $\omega_{\alpha+n}$-free groups in $\Gamma^{-1}(\tilde{E})$.* □

In particular, we shall show that, assuming GCH, the hypothesis of Theorem 1.4 holds for $\alpha = \omega n + 1$ for all $n \in \omega$. We make use of the following result of Shelah.

If for all $\beta < \alpha$, we have $\beta + \lambda < \alpha$, we shall say that $\lambda\omega$ *divides* α. Suppose $A = \{a_{\rho\nu} : \rho < \omega, \nu < \omega_m\} \subseteq \alpha$, where $a_{\rho\nu} < a_{\gamma\mu}$ if $\nu < \mu$ or $\nu = \mu$ and $\rho < \gamma$; we say $A^* \subseteq A$ is *big* if there is a cub $C \subseteq \omega_m$ and a function $f : C \to \omega$ such that $A^* = \{a_{\rho\nu} : \rho \geqq f(\nu),\ \nu \in C\}$.

1.5 THEOREM (Shelah [11]). *Assume* GCH *and let* $\lambda = \aleph_{\omega n}$ *for some fixed* $n \in \omega - \{0\}$.

(1) *There is a stationary* $S \subseteq \lambda^+$ *such that either*:

(a) S *is sparse in* λ^+ *(i.e. for all limit ordinals* $\sigma < \lambda^+$, $S \cap \sigma$ *is not stationary in* σ*) and consists of elements of uncountable cofinality; or*

(b) $S = \{\alpha < \lambda^+;\ \mathrm{cf}(\alpha) \neq \omega$ *and* $\lambda\omega$ *divides* $\alpha\}$, *and there are sets* $\{A_\alpha : \alpha \in S\}$ *such that: the order type of* A_α *is* $\omega\ \mathrm{cf}(\alpha)$; *and for all* $\nu < \lambda^+$ *there is* $\{A^*_\alpha(\nu) : \alpha \in \nu \cap S\}$ *where each* $A^*_\alpha(\nu)$ *is a big subset of* A_α, *and for* $\alpha < \beta$ *in* $\nu \cap S$, $A^*_\alpha(\nu) \cap A^*_\beta(\nu) = \emptyset$.

(2) *If* S *is as in* (1), *then for every* $E \subseteq S$, *there is a strongly almost free group* H *of cardinality* λ^+ *such that* $\Gamma(H) = \tilde{E}$.

PROOF. The result is proved in [11]; the only point in (1) needing additional comment is the assertion about the cofinality of elements of S. We assume knowledge of the details of the proof in [11]; there are two cases. In the first case, if $S^*(\lambda^+)$ is stationary, then by 19(3) of [11] there is an $E \subseteq S^*(\lambda^+)$ such that either $S' = E$ or $S' = F(E)$ is sparse and stationary; but by 14 of [11], if $\nu \in S^*(\lambda^+)$ then $\mathrm{cf}(\nu) \geqq \omega_1$, so $\nu \in F(S^*(\lambda^+))$ implies $\mathrm{cf}(\nu) \geqq \omega_2$. In the second case, if $S^*(\lambda^+)$ is not stationary, then (1) (b) follows immediately from 24 of [11].

As for (2), the proof is by induction on n using the fact that if (2) holds for all $m < n$ then for every regular $\kappa < \aleph_{\omega n}$ there is an almost free non-free group of cardinality κ (by thm. 2.2 of [2]). In Case 1 (a) the construction is well-known (cf. thm. 3.3 of [2]). The idea of using the properties of S in 1 (b) to construct almost free groups is due to Shelah. □

1.6 THEOREM. *Assume* GCH *and let* $\kappa = \aleph_{\omega n+1}$ *for some* $n \in \omega - \{0\}$. *Then there is a family* $\{H_i \mid i < 2^\kappa\}$ *of strongly almost free groups of cardinality* κ *such that for all* $i \neq j$, $\Gamma(H_i)$ *and* $\Gamma(H_j)$ *are almost disjoint and hence* — by Lemma 0.7 — $H_i \oplus \mathbf{Z}^{(\kappa)}$ *and* $H_j \oplus \mathbf{Z}^{(\kappa)}$ *are almost disjoint.*

PROOF. Note first that for the S of 1.5 (1) $\diamondsuit^*_{\lambda^+}(S)$ holds by Conclusion 32 of [11]. Hence there is a family $\{E_i : i < 2^\kappa\}$ of 2^κ pairwise almost disjoint subsets of S (cf. Lemma 0.6 (ii)). The result follows immediately from 1.5 (2). □

Theorems 1.4 and 1.6 immediately yield the following.

1.7 THEOREM. *Assume* GCH. *Then for every regular* $\kappa < \aleph_{\omega^2}$ *there is a family of* 2^κ *pairwise almost disjoint strong almost free groups of cardinality* κ. □

REMARKS. (1) By Theorem 1.4 it is clear that for $\kappa = \omega_{\omega n+m}$, $n \in \omega$, $m \geqq 2$, we can choose the family in 1.7 to belong to $\Gamma^{-1}(\tilde{E})$ for any given $E \subseteq \{\sigma \in \lim(\omega_{\omega n+m}) : \mathrm{cf}\ (\sigma) \geqq \omega_{\omega n}\}$. For $\kappa = \omega_{\omega n+1}$ the situation is less clear, but it seems that an extension of the methods of [11] will allow one to construct a family as in 1.7 belonging to $\Gamma^{-1}(\tilde{S})$ for some $S \subseteq \kappa$.

(2) Using a large cardinal assumption, Magidor and Shelah [10] have constructed a model of ZFC + GCH in which every $\aleph_{\omega^2+1}$-free group is $\aleph_{\omega^2+2}$-free.

For any stationary subset E of λ let $E' = \{\sigma < \lambda : E \cap \sigma$ is stationary in $\sigma\}$ Thus E is sparse iff $E' = 0$.

1.8 THEOREM. *Assume* $V = L$. *Let* λ *be a regular uncountable cardinal and* E *a stationary subset of* λ. *Let* $W = \{\nu < \lambda$: cf (ν) *is weakly compact*$\}$ *and* $R = \{\nu < \lambda : \nu$ *is a regular cardinal*$)$.

(1) *If* λ *is a successor cardinal,* $\lambda = \kappa^+$, *then* $\Gamma^{-1}(\tilde{E}) \neq \emptyset$ *iff* $\tilde{E} \cap \tilde{W} = 0$ *iff* $\Gamma^{-1}(\tilde{E})$ *contains a family of* 2^λ *pairwise almost disjoint strongly almost free groups.*

(2) *If* λ *is an inaccessible cardinal which is not weakly compact then* $\Gamma^{-1}(\tilde{E}) \neq \emptyset$ *iff* $\tilde{E} \cap \tilde{W} = 0$ *and* $\tilde{E}' \cap \tilde{R} = 0$ *iff* $\Gamma^{-1}(\tilde{E})$ *contains a family of* 2^λ *pairwise almost disjoint strongly almost free groups.*

REMARK. The first equivalence in (1) and the necessity of the condition in (2) are proved in [9].

PROOF. The proof is by induction on λ. We shall construct families $\{A_i : i < 2^\lambda\}$ which are pairwise almost disjoint in the *strong sense* that if $i \neq j$ then for any free group F of cardinality λ, $A_i \oplus F$ and $A_j \oplus F$ are almost disjoint. In the proof we shall always mean this strong sense when we say two groups are almost disjoint. (In fact, we shall see in section 2, Corollary 2.4, that this notion is not really stronger.)

Theorem 1.1 gives us the initial steps of the induction.

(1) Suppose first that $\lambda = \kappa^+$. By inductive hypothesis, if κ is regular and not weakly compact, there is a family $\{B_\sigma^{(\kappa)} : \sigma < \kappa^+\}$ of pairwise almost disjoint strongly almost free groups of cardinality κ. Thus for every regular cardinal $\rho \leqq \kappa$ which is not weakly compact there is a family $\{C_\sigma^\rho : \sigma < \kappa\}$ of pairwise almost disjoint strongly almost free groups which are of cardinality $\geqq \rho$ and $\leqq \kappa$. (Let $\{C_\sigma^\rho : \sigma < \kappa\}$ be the union of the families $\{B_\sigma^{(\mu)} : \sigma < \mu^+\}$ where $\rho \leqq \mu \leqq \kappa$ and μ is regular and not weakly compact.) Now by $\square_\kappa$ $E = \amalg_{\gamma \leqq \kappa} E_\gamma$ where each E_γ is sparse and $\nu \in E_\gamma$ implies cf $(\nu) = $ cf (γ) (cf. [9] or [3; lemma 6.9]). We may

suppose $E \cap W = \emptyset$. By Lemma 0.6 (iii) we have for each γ a λ-Kurepa tree on E_γ, i.e. a family $\{f_i^\gamma : E_\gamma \to \kappa \mid i < \lambda^+\}$ of pairwise almost disjoint functions. For each $\gamma \leqq \kappa$, $i < \lambda^+$ define by induction a smooth chain $\{A_\nu^{i,\gamma} : \nu < \kappa^+\}$ of free groups of cardinality κ such that $A_\mu^{i,\gamma}/A_\nu^{i,\gamma}$ is free if $\nu \notin E_\gamma$; and if $\nu \in E_\gamma$ and cf $(\gamma) = \rho$ and $\sigma = f_i^\gamma(\nu)$,

$$A_{\nu+1}^{i,\gamma}/A_\nu^{i,\gamma} \cong C_\sigma^\rho.$$

(This is possible since cf $(\nu) = \rho$ is not weakly compact and C_σ^ρ is ρ-free: cf. [9; theorem 2.15].) Let $A^{i,\gamma} = \bigcup_{\nu<\lambda} A_\nu^{i,\gamma}$ and $A^i = \bigoplus A^{i,\gamma}$. Then as in the proof of 1.1 we can verify that for $i \neq j$, A^i and A^j are almost disjoint. Moreover $\Gamma(A^i) = \bigcup_{\gamma \leqq \kappa} \tilde{E}_\gamma = \tilde{E}$.

(2) Suppose now that λ is inaccessible and not weakly compact and $\tilde{E}' \cap \tilde{R} = 0$, $\tilde{E} \cap \tilde{W} = 0$. Since the infinite cardinals form a cub in λ we may assume that every member of E is an infinite cardinal: Consider first the following two cases.

Case 2a. E consists only of regular cardinals. Now every singular cardinal is a limit of singular ordinals so $E' \subseteq R$; but then since $\tilde{E}' \cap \tilde{R} = 0$, we have $\tilde{E}' = 0$, i.e. E is sparse. Let $\{f_i : E \to \lambda \mid i < \lambda^+\}$ be a λ-Kurepa tree as in Lemma 0.6 (iii). For each $\kappa \in E$ let $\{B_\sigma^{(\kappa)} : \sigma < \kappa^+\}$ be as in (1). Define by induction a smooth chain of free groups $\{A_\nu^i : \nu < \lambda\}$ such that $|A_\nu^i| = |\nu + \omega|$; if $\nu \notin E$, A_μ^i/A_ν^i is free for all $\mu > \nu$; and if $\nu \in E$

$$A_{\nu+1}^i/A_\nu^i \cong B_{f_i(\nu)}^{(\nu)}.$$

Define $A^i = \bigcup_{\nu<\lambda} A_\nu^i$. Then if $i \neq j$, A^i and A^j are almost disjoint.

Case 2b. E consists only of singular cardinals. We make use of the following version of $\square_\lambda$ for a regular cardinal λ (cf. [1]):

(*) for every singular limit ordinal $\alpha < \lambda$ there is a cub C_α in α of order type $< \alpha$ such that whenever β is a limit point of C_α, then β is a singular limit ordinal and $C_\beta = C_\alpha \cap \beta$.

Now for every $\gamma \leqq \lambda$ let $E_\gamma = \{\alpha \in E$: order type of $C_\alpha = \gamma\}$. Then just as in lemma 6.9 of [3] we can prove that $E_\gamma' \subseteq R$, and since by hypothesis $\tilde{E}' \cap \tilde{R} = 0$ we have $\tilde{E}_\gamma' = 0$, i.e. E_γ is sparse. Thus $E = \coprod_{\gamma \leqq \lambda} E_\gamma$. Then for a λ-Kurepa tree $\{f_i^\gamma : E_\gamma \to \lambda : i < \lambda^+\}$ construct $A^{i,\gamma}$ as in (1). Then we define $A^i = \bigoplus_{\gamma \leqq \lambda} A^{i,\gamma}$ filtered by $A_\alpha^i = \bigoplus\{A_\nu^{i,\gamma} : \nu < \alpha, \gamma < \alpha\}$. Noticing that $\alpha \in E_\gamma$ implies $\alpha > \gamma$ we can check that $\Gamma(A^i) = U\tilde{E}_\gamma = \tilde{E}$. Finally as before we check that if $i \neq j$, A^i and A^j are almost disjoint. This completes Case 2b.

If E is an arbitrary stationary set of cardinals we can write $E = E_a \amalg E_b$ where $E_a = \{\alpha \in E : \alpha$ is regular$\}$ and $E_b = \{\alpha \in E : \alpha$ is singular$\}$. Taking direct sums of the families constructed in the above two cases for E_a and E_b respectively we obtain the desired family for E.

Finally, for the converse implication, if $\Gamma^{-1}(\tilde{E}) \neq \emptyset$, then $\tilde{E} \cap \tilde{W} = 0$ by [9; thm. 1.13]; and $\tilde{E}' \cap \tilde{R} = 0$ by [9; thm. 1.15]. □

2. Quotient-equivalent groups

The method of constructing almost disjoint groups used in the first section is to construct $A = \bigcup_{\nu<\kappa} A_\nu$ and $B = \bigcup_{\nu<\kappa} B_\nu$ so that for "almost all" ν, $A_{\nu+1}/A_\nu$ and $B_{\nu+1}/B_\nu$ are almost disjoint. We shall see (Theorem 2.1) that this is the only way of constructing almost disjoint groups. In particular quotient-equivalent groups cannot be almost disjoint.

If A and B are strongly almost free of cardinality κ we say they are *quotient-equivalent* if they have κ-filtrations $A = \bigcup_{\nu<\kappa} A_\nu$, $B = \bigcup_{\nu<\kappa} B_\nu$ such that for all $\nu \in \kappa$, $A_{\nu+1}/A_\nu \cong B_{\nu+1}/B_\nu$, or, equivalently, if for any κ-filtrations $A = \bigcup_{\nu<\kappa} A_\nu$, $B = \bigcup_{\nu<\kappa} B_\nu$, there is a cub C such that for all $\nu \in C$, $A_{\nu+1}/A_\nu \oplus F \cong B_{\nu+1}/B_\nu \oplus F$, for some free group F of cardinality $< \kappa$. The logical significance of quotient-equivalence is discussed in [4], and the construction of non-isomorphic quotient-equivalent groups is discussed in [4] and [3; chap. 11].

It is clear that if A and B are quotient-equivalent then $\Gamma(A) = \Gamma(B)$.

It may be helpful to study the proofs in this section first for the simpler case that $A_{\sigma+1}/A_\sigma \cong Q^{(2)} = \{m/2^n \in Q : m, n \in \mathbf{Z}\}$ whenever $A_{\sigma+1}/A_\sigma$ is not free.

2.1 THEOREM. *Let A and A' be strongly κ-free groups of regular uncountable cardinality κ with κ-filtrations $A = \bigcup_{\nu<\kappa} A_\nu$, $A' = \bigcup_{\nu<\kappa} A'_\nu$ such that there is a stationary $E \subseteq \kappa$ such that for all $\nu \in E$, $A_{\nu+1}/A_\nu$ and $A'_{\nu+1}/A'_\nu$ are not almost disjoint. Then A and A' are not almost disjoint.*

PROOF. We may assume that for all $\nu < \kappa$, $A_{\nu+2}/A_{\nu+1}$ and $A'_{\nu+2}/A'_{\nu+1}$ are free of rank $= |A'_{\nu+1}| = |A_{\nu+1}|$. For each $\nu \in E$ choose a non-free group B_ν which is embeddable in both $A_{\nu+1}/A_\nu$ and $A'_{\nu+1}/A'_\nu$. Without loss of generality we may regard B_ν as a subgroup of both $A_{\nu+1}/A_\nu$ and $A'_{\nu+1}/A'_\nu$.

We shall define by induction on $\nu < \kappa$ subgroups $H_\nu \subseteq A_\nu$, $H'_\nu \subseteq A'_\nu$ and maps $f_\nu : H_\nu \to H'_\nu$ satisfying for all ν

(1) f_ν is an isomorphism and for all $\mu < \nu$, $H_\mu \subseteq H_\nu$ and $f_\nu \mid H_\mu = f_\mu$;

(2) if $\nu \in E$ the natural map $\theta_\nu : H_{\nu+1}/H_\nu \to A_{\nu+1}/A_\nu$ maps $H_{\nu+1}/H_\nu$ onto B_ν with kernel a torsion subgroup of $H_{\nu+1}/H_\nu$; and similarly for $\theta' : H_{\nu+1}/H'_\nu \to B_\nu \subseteq A'_{\nu+1}/A'_\nu$;

(3) if $\nu \in \lim(\kappa)$, for all $x \in A_\nu$, $x' \in A'_\nu$, $d \in \mathbf{Z}$, there exist $h \in H_\nu$, $h' \in H'_\nu$ such that $f_\nu(h) = h'$ and $d \mid (x - h)$ in A_ν and $d \mid (x' - h')$ in A'_ν, (where $d \mid y$ in G means $\exists g \in G$ s.t. $dg = y$).

Suppose that in fact we can carry out this construction for all $\nu < \kappa$. Then let $H = \bigcup_{\nu<\kappa} H_\nu$, $H' = \bigcup_{\nu<\kappa} H'_\nu$ and $f = \bigcup_{\nu<\kappa} f_\nu$. Clearly (by (1)) H (resp H') is a subgroup of A (resp. A') and $f: H \to H'$ is an isomorphism. Moreover H is not free since for $\nu \in E$, $H_{\nu+1}/H_\nu$ is not free (by (2)). (In fact note that there is a cub $C \subseteq \kappa$ such that for $\nu \in C$, $H \cap A_\nu = H_\nu$ and $H' \cap A'_\nu = H'_\nu$; for $\nu \in E \cap C$, $H_{\nu+1}/H_\nu \cong B_\nu$ and $H'_{\nu+1}/H'_\nu \cong B_\nu$.)

Therefore it remains to describe the construction of H_ν, H'_ν and f_ν, which will be done by induction, starting with $H_0 = 0$, $H'_0 = 0$, $f_0 \equiv 0$. Suppose that the construction has been carried out for all $\nu < \sigma$; we consider four cases.

Case 1: σ is a limit of limit ordinals.

Let $H_\sigma = \bigcup_{\nu<\sigma} H_\nu$, $H'_\sigma = \bigcup_{\nu<\sigma} H'_\nu$, $f_\sigma = \bigcup_{\nu<\sigma} f_\nu$. then clearly (1)–(3) hold. (In particular (3) holds because for every $x \in A_\sigma$, $x' \in A'_\sigma$ there exists $\nu \in \lim(\kappa)$, $\nu < \sigma$ such that $x \in A_\nu$, $x' \in A'_\nu$.)

Case 2: $\sigma = \nu + 1$, $\nu \in E$.

By hypothesis there is a sequence $\bar{y} = (y_i)_{i<\alpha}$ (resp. $\bar{y}' = (y'_i)_{i<\alpha}$) of elements of $A_{\nu+1}$ (resp. $A'_{\nu+1}$) independent over A_ν (resp. A'_ν), a cardinal $\lambda < \kappa$ and for each $\mu < \lambda$ a term $t_\mu(\bar{v})$, a non-zero integer d_μ and elements $x_\mu \in A_\nu$ (resp. $x'_\mu \in A'_\nu$) such that $B_\nu \cong \tilde{B}_{\nu+1}/A_\nu$ (resp. $\cong \tilde{B}'_{\nu+1}/A'_\nu$) where

$$\tilde{B}_{\nu+1} = \left\langle A_\nu \cup \left\{ \frac{t_\mu(\bar{y}) - x_\mu}{d_\mu} : \mu \in \lambda \right\} \right\rangle \text{(resp. } \tilde{B}'_{\nu+1} = \left\langle A'_\nu \cup \left\{ \frac{t_\mu(\bar{y}') - x'_\mu}{d_\mu} : \mu \in \lambda \right\} \right\rangle\text{)}.$$

(A *term* $t(\bar{v})$ is an expression $\Sigma_{i<\alpha} n_i v_i$, where $\bar{v} = (v_i)_{i<\alpha}$ is a sequence of variable symbols and $n_i \in \mathbf{Z}$ and $n_i = 0$ for almost all i.)

By (3) for each $\mu \in \lambda$ there exists $h_\mu \in H_\nu$, $h'_\mu \in H'_\nu$ such that $f_\nu(h_\mu) = h'_\mu$, $d_\mu \mid (x_\mu - h_\mu)$ in A_ν and $d_\mu \mid (x'_\mu - h'_\mu)$ in A'_ν. Now define

$$H_{\nu+1} = \left\langle H_\nu \cup \left\{ \frac{t_\mu(\bar{y}) - h_\mu}{d_\mu} : \mu \in \lambda \right\} \right\rangle \subseteq A_{\nu+1}$$

and

$$H'_{\nu+1} = \left\langle H_\nu \cup \left\{ \frac{t_\mu(\bar{y}') - h'_\mu}{d_\mu} : \mu \in \lambda \right\} \right\rangle \subseteq A'_{\nu+1}.$$

It follows from the independence of the $\bar{y}$ and $\bar{y}'$ over A_ν and A'_ν respectively and the fact that $f_\nu(h_\mu) = h'_\mu$ that there is a well-defined isomorphism $f_{\nu+1}: H_{\nu+1} \to H'_{\nu+1}$ extending f_ν and satisfying

$$f_{\nu+1}\left(\frac{t_\mu(\bar{y}) - h_\mu}{d_\mu} \right) = \frac{t_\mu(\bar{y}') - h'_\mu}{d_\mu}.$$

It is easy to see that (1)–(3) hold.

Case 3: $\sigma = \nu + m + 1$, $\nu \in \lim(\kappa)$, where $m > 0$ if $\nu \in E$.

Let $\rho = |A_{\nu+1}| = |A'_{\nu+1}|$. Then $A_{\nu+m+1}/A_{\nu+m}$ and $A'_{\nu+m+1}/A'_{\nu+m}$ are free of rank ρ. Choose sets $\{b_i : i < \rho\}$ $\{b'_i : i < \rho\}$) of elements of $A_{\nu+m+1}$ (resp. $A_{\nu+m+1}$) whose cosets mod $A_{\nu+m}$ (resp. $A'_{\nu+m}$) form a basis of $A_{\nu+m+1}/A_{\nu+m}$ ($A'_{\nu+m+1}/A'_{\nu+m}$). Choose a one-to-one correspondence $\Phi_m : \rho \to A_{\nu+m} \times A'_{\nu+m} \times (\mathbf{Z} - \{0\})$. For each $i < \rho$, if $\Phi(i) = (x_i, x'_i, d_i)$, let $h_i = d_i b_i + x_i$ and $h'_i = d_i b'_i + x'_i$. Then the h_i (resp. h'_i) form a linearly independent set over $H_{\nu+m}$ (resp. $H'_{\nu+m}$) and we define

$$H_{\nu+m+1} = \langle H_{\nu+m} \cup \{h_i : i < \rho\}\rangle, \quad H'_{\nu+m+1} = \langle H'_{\nu+m} \cup \{h'_i : i < \rho\}\rangle$$

and extend $f_{\nu+m}$ to $f_{\nu+m+1}$ by sending h_i to h'_i.

Case 4: $\sigma = \nu + \omega$, $\nu \in \operatorname{Lim}(\kappa)$.

Let $H_\sigma = \bigcup_{m\in\omega} H_{\nu+m}$, $H'_\sigma = \bigcup_{m\in\omega} H'_{\nu+m}$, $f_\sigma = \bigcup_{m\in\omega} f_{\nu+m}$. Then clearly (1) and (2) hold and the construction in Case 3 insures that (3) holds as well. □

As an immediate consequence of the proof we obtain

2.2 THEOREM. *Suppose A and A' are strongly almost free groups of cardinality κ which are non-free and quotient-equivalent. Then there is a strongly almost free group H of cardinality κ which is embeddable in both A and A' and quotient-equivalent to both of them (and hence not free).* □

We also obtain the following corollary:

2.3 COROLLARY. *If A is strongly κ-free of cardinality κ and $\tilde{E} \subseteq \Gamma(A)$, then there is a subgroup B of A with $\Gamma(B) = \tilde{E}$.* □

This should be compared with lemma 1.3 of [8] which, for A and E as above, constructs an epimorphism $C \to A$ with $\Gamma(C) = \tilde{E}$.

2.4 COROLLARY. *If A and B are strongly almost free of cardinality κ and almost disjoint, then they are almost disjoint in the strong sense (see proof of Theorem 1.8) that if F is the free group of rank κ, then $A \oplus F$ and $B \oplus F$ are almost disjoint.*

PROOF. Suppose that A and B are strongly κ-free of cardinality κ and almost disjoint. Choose κ-filtrations $A = \bigcup_{\nu<\kappa} A_\nu$ and $B = \bigcup_{\nu<\kappa} B_\nu$ so that for all $\nu < \kappa$, $|A_{\nu+1}| = |B_{\nu+1}|$ and if F' is the free group of rank $|A_{\nu+1}|$ then $A_{\nu+1}/A_\nu \oplus F' \cong A_{\nu+1}/A_\nu$ and $B_{\nu+1}/B_\nu \oplus F' \cong B_{\nu+1}/B_\nu$.

Now to obtain a contradiction, assume that, if F is the free group of rank κ, $A \oplus F$ and $B \oplus F$ are not almost disjoint. Let $F = \bigcup_{\nu<\kappa} F_\nu$ be a κ-filtration such that for all ν, F_ν is a direct summand of F and $|F_{\nu+1}| = |A_{\nu+1}|$. Then

$A \oplus F = \bigcup_{\nu<\kappa} A_\nu \oplus F_\nu$ and $B \oplus F = \bigcup_{\nu<\kappa} B_\nu \oplus F_\nu$ are κ-filtrations, so by assumption there is a stationary set E of ν such that $A_{\nu+1}/A_\nu \oplus F_{\nu+1}/F_\nu$ and $B_{\nu+1}/B_\nu \oplus F_{\nu+1}/F_\nu$ are not almost disjoint (cf. proof of 0.7). But then since $A_{\nu+1}/A_\nu \oplus F_{\nu+1}/F_\nu \cong A_{\nu+1}/A_\nu$ and $B_{\nu+1}/B_\nu \oplus F_{\nu+1}/F_\nu \cong B_{\nu+1}/B_\nu$, Theorem 2.1 implies that A and B are not almost disjoint, a contradiction. □

REMARK. In fact, $A \oplus F$ and $B \oplus F$ are almost disjoint for *any* free group F, since if $A \oplus F$ and $B \oplus F$ are not almost disjoint we can find a subgroup F_1 of F of rank κ such that $A \oplus F_1$ and $B \oplus F_1$ are not almost disjoint.

3. Almost disjointness for pure subgroups

If A and A' are strongly κ-free groups of cardinality κ, call them *almost disjoint for pure subgroups* if whenever there are *pure* embeddings $\theta : H \to A$, $\theta' : H \to A'$ [so $\theta(H)$ (resp. $\theta'(H)$) is a pure subgroup of A (resp. A')] then H is free.

Although quotient-equivalent groups cannot be almost disjoint, they can be almost disjoint for pure subgroups. In fact, we shall construct large families of quotient-equivalent groups of cardinality $\aleph_1$ which are pairwise almost disjoint for pure subgroups and also mutually non-embeddable.

For simplicity we begin with the following special case. We consider strongly ω_1-free groups A with an ω_1-filtration $A = \bigcup_{\nu<\omega_1} A_\nu$ such that for some fixed stationary $E \subseteq \lim(\omega_1)$, we have for all $\nu < \omega_1$:

(*) if $\nu \notin E$, A/A_ν is ω_1-free, and if $\nu \in E$, $A_{\nu+1}/A_\nu \cong Q^{(2)}$

where $Q^{(2)} = \mathbf{Z}[\frac{1}{2}] = \{m/2^n \in Q : m, n \in \mathbf{Z}\}$.

DEFINITIONS. If ϕ and ψ are functions: $\omega \to \omega$ write $\phi \ll \psi$ if $\forall r \geqq 0\ \exists N_r \geqq 0$ $\forall n \geqq N_r$ $\phi(n+r) \leqq \psi(n)$.

We let $<$ denote the lexocographical ordering on elements of ${}^\omega 2$, i.e. if η, $\zeta \in {}^\omega 2$, $\eta < \zeta$ iff $\exists n$ such that $\eta \restriction n = \zeta \restriction n$ and $\eta(n) = 0$, $\zeta(n) = 1$.

Let ζ_0 denote the element of ${}^\omega 2$ given by $\zeta_0(0) = 1$, $\zeta_0(n) = 0$ for all $n > 0$. Notice that if $\eta \in {}^\omega 2$ such that $\eta(0) = 1$, then $\zeta_0 \leqq \eta$.

3.1 LEMMA. *For each $\eta \in {}^\omega 2$ we can define a non-decreasing unbounded function $\phi_\eta : \omega \to \omega$ such that for all η, $\zeta \in {}^\omega 2$, if $\eta < \zeta$ then $\phi_\eta \ll \phi_\zeta$. Moreover given any non-decreasing unbounded function $\theta : \omega \to \omega$ we can choose the family so that $\phi_{\zeta_0} = \theta$.*

PROOF. Choose a family of non-decreasing unbounded functions $g_\eta : \omega \to \omega$ ($\eta \in {}^\omega 2$) with the property that if $\eta < \zeta$ then $\exists N$ such that for all $n \geqq N$,

$g_\eta(2n) \leqq g_\zeta(n)$; and, moreover, such that $g_{\zeta_0}(n) = n$ for all n. (If the existence of such a family is not clear to the reader, see Lemma 3.4 where we describe the construction in a more general setting.) Then define ϕ_η by: $\phi_\eta(n) = \theta(g_\eta(n))$ for all n. Given $\eta < \zeta$ and given $r \geqq 0$, choose $N \geqq r$ so that $g_\eta(2n) \leqq g_\zeta(n)$ for all $n \geqq N$. Then for $n \geqq N$, $\phi_\eta(n+r) \leqq \phi_\eta(2n) = \theta(g_\eta(2n)) \leqq \theta(g_\zeta(n)) = \phi_\zeta(n)$. □

3.2 LEMMA. *Suppose that $A = \bigcup_{\nu<\omega_1} A_\nu$, and $A' = \bigcup_{\nu<\omega_1} A'_\nu$ are strongly ω_1-free groups with filtrations satisfying* (*). *Suppose that for every $\delta \in E$ there is a strictly increasing sequence $\{\rho(n): n \in \omega\}$ approaching δ, elements $y_\delta \in A_{\delta+1} - A_\delta$, $y'_\delta \in A_{\delta+1} - A'_\delta$ and functions $\phi_\delta : \omega \to \omega$, $\phi'_\delta : \omega \to \omega$ such that: $\phi_\delta \ll \phi'_\delta$ or $\phi'_\delta \ll \phi_\delta$; and for all $\nu < \delta$,*

(†) *for all $n \in \omega$, 2^{n+1} divides y_δ (resp. y'_δ) modulo A_ν (resp. A'_ν) iff $\nu > \rho(\phi_\delta(n))$ (resp. $\nu > \rho(\phi'_\delta(n))$).*

Then A and A' are almost disjoint for pure subgroups.

PROOF. Suppose $\theta : H \to A$ and $\theta' : H \to A'$ are pure embeddings. Note that H is strongly ω_1-free; let $H = \bigcup_{\nu<\omega_1} H_\nu$ be an ω_1-filtration of H s.t. for all ν, $H/H_{\nu+1}$ is ω_1-free. There is a cub $C \subseteq \omega_1$ such that for all $\nu \in C$, $\theta(H) \cap A_\nu = \theta(H_\nu)$, $\theta'(H) \cap A'_\nu = \theta'(H_\nu)$, and $A_\nu + \theta(H)$ (resp. $A'_\nu + \theta'(H)$) is a pure subgroup of A (resp. A').

It suffices to prove that for all $\delta \in C^*$ (the set of limit points of C), $H_{\delta+1}/H_\delta$ is free. So assume, to obtain a contradiction, that for some $\delta + C^* \cap E$, $H_{\delta+1}/H_\delta$ is not free. Without loss of generality, $\phi_\delta \ll \phi'_\delta$.

Let $G = \theta'^{-1}(A'_{\delta+1})$. Since $A'/A'_{\delta+1}$ is κ-free, $H_{\delta+1}/(H_{\delta+1} \cap G)$ is free, so $(H_{\delta+1} \cap G)/H_\delta \cong (\theta'(H_{\delta+1}) \cap A'_{\delta+1})/A'_\delta$ must be non-free. Hence there exists $z \in H_{\delta+1}$ such that $\theta'(z) + A'_\delta = y'_\delta + A'_\delta$. Then there is a $t \in \mathbf{Z}$ such that $\theta(z) + A_\delta = m2^t y_\delta + A_\delta$ for some m relatively prime to 2. Let $r = \max\{-t, 0\}$, and let N be such that $n \geqq N$ implies $\phi_\delta(n+r+1) \leqq \phi'_\delta(n)$. There are arbitrarily large ν such that $\nu \in C \cap \delta$ and there is an $n \geqq N$ such that $\rho(\phi_\delta(n+r)) < \nu \leqq \rho(\phi_\delta(n+r+1))$ and hence $\nu \leqq \rho(\phi'_\delta(n))$. For such an n, ν, we have by (†) that (i) 2^{n+r+1} divides y_δ modulo A_ν: but (ii) 2^{n+1} does not divide y'_δ modulo A'_ν. We shall obtain a contradiction by showing that for arbitrarily large $\nu \in C \cap \delta$, (i) implies that 2^{n+1} divides z mod H_ν and (ii) implies 2^{n+1} does not divide z mod H_ν. Part (ii) is clear since $\nu \in C$ implies $\theta'(H_\nu) \subseteq A_\nu$. As for (i), say $\theta(z) = m2^t y_\delta + u$ where $u \in A_\delta$, and choose ν large enough so that $u \in A_\nu$. By hypothesis and since $r \geqq -t$, 2^{n+1} divides $2^t y_\delta$ mod A_ν so there exists $w \in A_{\delta+1}$, $x \in A_\nu$ such that $2^{n+1} w = m2^t y_\delta + x$. But for some $\bar{w} \in H_{\delta+1}$ and $h \in H_\delta$, $2^{n+1}\bar{w} = z + h$. Thus 2^{n+1} divides $x - u - \theta(h)$ in A. Since $A_\nu + \theta(H)$ is a pure subgroup of A, there exists

$\tilde{h} \in H_\nu$ and $v \in A_\nu$ such that $2^{n+1}v = x - u - \theta(\tilde{h})$. Thus $2^{n+1}(w - v) = m2^r y_\delta + u + \theta(\tilde{h}) = \theta(z + \tilde{h})$ so by the purity of $\theta(H)$ in A, $w - v \in \theta(H)$. Thus 2^{n+1} divides z in $H_{\delta+1}$ modulo H_ν. □

3.3 THEOREM. *For every $0 \neq \tilde{E} \in D(\omega_1)$ there is a family of 2^{ω_1} ω_1-separable ω_1-free groups $\{A_i : i < 2^{\omega_1}\}$ in $\Gamma^{-1}(\tilde{E})$ such that if $i \neq j$, A_i and A_j satisfy (*) (hence are quotient-equivalent), A_i is not embeddable in A_j, and A_i and A_j are almost disjoint for pure subgroups. Moreover given any group A in $\Gamma^{-1}(\tilde{E})$ satisfying (*) we can choose the family so that $A_0 = A$.*

PROOF. Let $\{\phi_\eta : \eta \in {}^\omega 2\}$ be the family of functions constructed in Lemma 3.1 with $\theta =$ the identity. Let $\{\tilde{f}_i : \omega_1 \to Y \mid i < 2^{\omega_1}\}$ be a family of pairwise almost disjoint functions with codomain $Y =$ the set of all $\eta \in {}^\omega 2$ such that $\eta(0) = 1$; clearly $|Y| = 2^{\aleph_0}$ so the family exists by Lemma 1.3, and, moreover, we may assume that $\tilde{f}_0$ is the constant function ζ_0. Write E as a disjoint union of stationary sets: $E = \coprod_{\mu < \omega_1} E_\mu$ (cf. 0.6(i)). Let $\{S_i : i < 2^{\omega_1}\}$ be a family of subsets of ω_1 such that for all $i \neq j$, S_i is not contained in S_j. Define $\hat{\zeta}_0 = \zeta_0$, and if $\eta \in Y - \{\zeta_0\}$, let $\hat{\eta}$ be the element of ${}^\omega 2 - Y$ such that $\hat{\eta}(0) = 0$ and $\hat{\eta} \restriction \omega - \{0\} = \eta \restriction \omega - \{0\}$. Notice that if $\eta \in Y - \{\zeta_0\}$, then $\hat{\eta} < \zeta_0 < \eta$.

Now define $f_i : E \to {}^\omega 2$ as follows. Fix $\nu \in E$; say $\nu \in E_\mu$ and $\tilde{f}_i(\nu) = \eta$. Define

$$f_i(\nu) = \begin{cases} \eta & \text{if } \mu \in S_i, \\ \hat{\eta} & \text{if } \mu \notin S_i. \end{cases}$$

Clearly for $i \neq j$, f_i and f_j are almost disjoint, since $\tilde{f}_i$ and $\tilde{f}_j$ are almost disjoint. Moreover if $\mu \in S_j - S_i$, then $\phi_{f_i(\nu)} \ll \phi_{f_j(\nu)}$ for almost all $\nu \in E_\mu$ (i.e. for all $\nu \in E_\mu$ except those in the non-stationary set of ν such that $f_i(\nu) = f_j(\nu) = \zeta_0$).

Given A satisfying (*) choose for each $\delta \in E$, $y_\delta \in A_{\delta+1} - A_\delta$ and for all $n \in \omega$, let $\rho^\delta(n) =$ the least $\mu < \delta$ such that 2^{n+1} divides y_δ modulo $A_{\mu+1}$; this defines a non-decreasing sequence whose limit is δ.

For each $i \in 2^{\omega_1} - \{0\}$ we shall define A_i as a certain subgroup of

$$D = \bigoplus_\nu Qx_\nu \oplus \bigoplus_{\delta \in E} Qy_\delta$$

where ν ranges over the ordinals $< \omega_1$. Given $n \in \omega$, $i \in 2^{\omega_1} - \{0\}$, and $\delta \in E$, let $x\langle i, \delta, n\rangle$ be x_μ where $\mu = \rho^\delta(\phi_{f_i(\delta)}(n))$. Then for all $\tau \in \omega_1$, define $A_{i,\tau}$ to be the subgroup of D generated by $\{x_\nu : \nu < \tau\} \cup \{y_\delta : \delta \in E \cap \tau\}$ and

$$\frac{y_\delta - \sum_{j=0}^{n} 2^j x\langle i, \delta, j\rangle}{2^{n+1}}$$

for all $\delta \in E \cap \tau$, $n \in \omega$. Let $A_i = \bigcup_{\tau < \omega_1} A_{i,\tau}$. One may check that: A_i is ω_1-separable with $\Gamma(A_i) = \tilde{E}$; if $\tau \notin E$, $A_{i,\tau}$ is ω_1-pure in A_i; and if $\delta \in E$, $A_{i,\delta+1}/A_{i,\delta} \doteq Q^{(2)}$ (cf. [3; p. 99] and [9; pp. 1213–1215]).

Notice that if $i \in 2^{\omega_1}$ and $\delta \in E$, then for all $n \in \omega$, 2^{n+1} divides y_δ modulo $A_{i,\tau}$ in A_i iff $\tau > \rho^\delta(\phi_{f_i(\delta)}(n))$. (In particular this holds for $i = 0$, with $A_0 = A$, and $A_{0,\tau} = A_\tau$, since for all n, $f_0(\delta) = \zeta_0$, so $\phi_{f_0(\delta)}(n) = n$.) Thus Lemma 3.2 implies that for all $i \neq j$, A_i and A_j are almost disjoint for pure subgroups (since by construction there is a cub C such that for all $\delta \in E \cap C$, either $\phi_{f_i(\delta)} \ll \phi_{f_j(\delta)}$ or vice versa).

We must prove that for $i \neq j$, A_i is not embeddable in A_j. So suppose, to obtain a contradiction, that there is an embedding $\theta : A_i \to A_j$. Then there is a cub C such that for $\tau \in C$, $\theta(A_{i,\tau}) = \theta(A_i) \cap A_{j,\tau}$. Let $\mu \in S_j - S_i$ and let $\delta \in E_\mu \cap C^*$. Let $\eta = f_i(\delta)$, $\zeta = f_j(\delta)$; then $\eta < \zeta$. Now for some $t \geqq 0$, $\theta(2^s y_\delta) = 2^t m y_\delta + a$ where $(2, m) = 1$ and $a \in A_{j,\sigma}$ for some $\sigma < \delta$. Let $r = \max\{t - s, 0\}$ and choose N so that $n \geqq N$ implies $\phi_\eta(n + r + 1) \leqq \phi_\zeta(n)$. Now since $\delta \in C^*$ it is the limit of a strictly increasing sequence $\{\tau_n : n \in \omega\}$ of elements of C. Choose $n \geqq N$ so that there is an m such that

$$\rho^\delta(\phi_\eta(n + r)) < \tau_m \leqq \rho^\delta(\phi_\eta(n + r + 1))$$

and $a \in A_{j,\tau_m}$. Then by construction, 2^{n+r+1} divides y_δ modulo A_{i,τ_m}. Therefore, since $\tau_m \in C$, $2^{n+r+s+1}$ divides $\theta(2^s y_\delta)$ modulo A_{j,τ_m} so $2^{n+r+s+1}$ divides $2^t y_\delta$ modulo A_{j,τ_m}; thus 2^{n+1} divides y_δ modulo A_{j,τ_m} since $r \geqq t - s$. But $\tau_m \leqq \rho^\delta(\phi_\eta(n + r + 1)) \leqq \rho^\delta(\phi_\zeta(n))$ so by construction 2^{n+1} does not divide y_δ modulo A_{j,τ_m}, a contradiction. □

Now we want to extend Theorem 3.3 to arbitrary quotient-equivalence classes. We begin with a generalization of Lemma 3.1.

3.4 LEMMA. *Given a strictly increasing function $\Phi : \omega \to \omega$, there exists a family of non-decreasing unbounded functions $\phi_\eta : \omega \to \omega$ ($\eta \in {}^\omega 2$) such that for all $\eta, \zeta \in {}^\omega 2$, if $\eta < \zeta$ then there exists N such that for all $n \geqq N$, $\phi_\eta(\Phi(n)) \leqq \phi_\zeta(n)$. Moreover, given any non-decreasing unbounded $\theta : \omega \to \omega$ we can choose the family so that $\phi_{\zeta_0} = \theta$.*

PROOF. We shall define a family of strictly increasing functions $k_\eta : \omega \to \omega$ ($\eta \in {}^\omega 2$) such that:

(a) if $\eta < \zeta$ $\exists N$ such that for all $n \geqq N$, $k_\eta(n) \geqq \Phi(k_\zeta(n))$. Suppose for the moment that we can do this. Then define $g_\eta : \omega \to \omega$ as follows. Let $g_{\zeta_0} =$ identity. For $\eta \neq \zeta_0$, for any $n \in \omega$, if $k_\eta(m) \leqq n < k_\eta(m + 1)$, then

$$g_\eta(n) = \begin{cases} k_{\zeta_0}(m+1) & \text{if } \zeta_0 < \eta, \\ k_{\zeta_0}(m) & \text{if } \eta < \zeta_0. \end{cases}$$

Then one may easily check, using (a), that if $\eta < \zeta$ there exists N such that for $n \geqq N$, $g_\eta(\Phi(n)) \leqq g_\zeta(n)$. Hence if we let $\phi_\eta = \theta \circ g_\eta$, we have the desired family of functions.

Thus it remains only to construct the functions k_η. We shall define $k_\eta(m)$ by induction on m so that the following additional property is satisfied:

(b) $\{\eta \mid k_\eta(m) : \eta \in {}^\omega 2\}$ is a finite family of functions such that for all η, ζ, if $\eta \mid k_\eta(m-1) = \zeta \mid k_\eta(m-1)$ then $k_\eta(m+1) = k_\zeta(m+1)$.

Define $k_\eta(0) = 0$ for all η. Suppose that $k_\eta(m)$ has been defined for all m and that $\eta_r < \cdots < \eta_1$ represent all the elements of $\{\eta \mid k_\eta(m) : \eta \in {}^\omega 2\}$. Now by induction on $j \leqq r$ define $k_{\eta_j}(m+1)$: let $k_{\eta_1}(m+1) = k_{\eta_1}(m) + 1$; and for $j > 1$ let $k_{\eta_j}(m+1) = \Phi(k_{\eta_{j-1}}(m+1))$. Then for every ζ, let $k_\zeta(m+1) = k_{\eta_j}(m+1)$ if $\zeta \mid k_\zeta(m) = \eta_j \mid k_{\eta_j}(m)$. It is then easy to check that (b) holds for $m+1$. Moreover (a) holds, since if $\eta < \zeta$ and we choose N such that $\eta \mid N-1 < \zeta \mid N-1$ then for all $n \geqq N$, $\eta \mid k_\eta(n-1) < \zeta \mid k_\zeta(n-1)$, so by construction $k_\eta(n) \geqq \Phi(k_\zeta(n))$. □

3.5 Theorem. *For every non-free strongly ω_1-free A of cardinality ω_1, there is a family $\{A_i : i < 2^{\omega_1}\}$ of strongly ω_1-free groups of cardinality ω_1 such that for all i, A_i is quotient-equivalent to A and for all $i \neq j$, A_i and A_j are almost disjoint for pure subgroups and A_i cannot be embedded in A_j. Moreover we can choose the family so that $A_0 = A$.*

Proof. Without loss of generality $A = \bigcup_{\nu < \omega_1} A_\nu$ where $E = \{\delta \in \omega_1 : A_\delta$ is not ω_1-pure in $A\} = \{\delta \in \omega_1 : A_{\delta+1}/A_\delta$ is not free$\} \subseteq \lim(\omega_1)$, and for $\nu \notin E$, $A_{\nu+1}/A_\nu$ has infinite rank. For each $\delta \in E$, choose $\{y_i^\delta \mid i < \alpha\} \subseteq A_{\delta+1}$ for some $\alpha = \alpha^\delta \leqq \omega$ such that the y_i^δ are independent mod A_δ and $A_{\delta+1}/A_\delta = \langle \{y_i^\delta : i < \alpha\} + A_\delta \rangle_* / A_\delta$. (Call this group G_δ; it is a countable non-free torsion-free group; by an abuse of language we write y_i^δ for $y_i^\delta + A_\delta \in G_\delta$.) Now fix $\delta \in E$ and write $\bar{y}^\delta = \{y_i^\delta : i < \alpha\}$. We claim that we can choose terms t_n and integers $d_n \geqq 2$ (depending on δ but we omit all superscript δ's) such that:

(1) $G_\delta = \langle \bar{y} \cup \{t_n(\bar{y})/d_n : n \in \omega\} \rangle$,

(2) for all n, $t_n(\bar{y})/d_n \notin \langle t_i(\bar{y})/d_i : i < n \rangle + \langle \bar{y} \rangle$,

(3) if $\{f_r : L_r \to L'_r \mid r \in \omega\}$ is an enumeration of all isomorphisms between a pure finite-rank non-free subgroup L_r of G_δ and a subgroup L'_r of G_δ, then there is a family $\{D_r \mid r \in \omega\}$ of pairwise disjoint infinite subsets of ω such that for all $r \in \omega$, $L'_r \supseteq \{t_n(\bar{y})/d_n : n \in D_r\}$.

Write $\{2k : k \in \omega\}$ as a disjoint union of infinite subsets D_r. We shall define t_n

and d_n by induction on n. If $n = 2k + 1$ we choose t_n, d_n so that, in the end, (1) will hold. Notice that $G_\delta/\langle\bar{y}\rangle$ is not finitely-generated so we can do this while satisfying (2). If $n = 2k$ and $n \in D_r$, choose t_n, d_n so that $t_n(\bar{y})/d_n \in L'_r$; we can do this while satisfying (2) because L'_r is of finite rank but not finitely generated.

For each $\delta \in E$ choose a non-decreasing sequence $\{\rho^\delta(n): n \in \omega\}$ with limit δ.

Continuing to hold δ fixed, for each $r \in \omega$ let $\sigma_r : \omega \to D_r$ be a strictly increasing enumeration of D_r. For each $r \in \omega$ let $\psi_r : D_r \to \omega$ be a strictly increasing function such that for all $n \in D_r$, $f_r^{-1}(t_n(\bar{y})/d_n) \in \langle t_i(\bar{y})/d_i : i \leqq \psi_r(n)\rangle + \langle\bar{y}\rangle$. Finally, define $\Phi: \omega \to \omega$ as follows: if $n \in D_r$, $n = \sigma_r(z)$, then

$$\Phi(n) = \max(\psi_r(\sigma_r(z+1)), \Phi(n-1)+1);$$

if $n \notin \bigcup_r D_r$, $\Phi(n) = \Phi(n-1)+1$.

Now let $\{\phi_\eta \mid \eta \in {}^\omega 2\}$ be the family of functions given by 3.4 for this Φ and for θ = identity. (In fact we have such a family for each δ; if necessary for clarity, we write ϕ^δ_η instead of ϕ_η, Φ^δ instead of Φ, etc.). Let $f_i : E \to {}^\omega 2$ ($i \in 2^{\omega_1}$) be as in the proof of 3.3.

Then for every $i < 2^{\omega_1}$ we can construct a group A_i quotient-equivalent to A such that for all $\delta \in E$ there are elements $\bar{y} = \{y^\delta_l \mid l < \alpha^\delta\} \subseteq A_{i,\delta+1}$ such that $G_\delta \cong \langle\bar{y} + A_{i,\delta}\rangle_*/A_{i,\delta} = A_{i,\delta+1}/A_{i,\delta}$, and for all n, d_n divides $t_n(\bar{y})$ mod $A_{i,\tau}$ iff $\tau > \rho^\delta(\phi^\delta_{f_i(\delta)}(n))$. (See Lemma 3.6 following.)

We claim that for $i \neq j$, A_i and A_j are almost disjoint for pure subgroups. To obtain a contradiction, suppose that θ_i (resp. θ_j) embeds a non-free H as a pure subgroup of A_i (resp. A_j). Let C be as in the proof of 3.2 and let $\delta \in C^* \cap E$ such that $H_{\delta+1}/H_\delta$ is not free. let $\phi_\delta = \phi^\delta_{f_i(\delta)}$, $\phi'_\delta = \phi^\delta_{f_j(\delta)}$. By construction and without loss of generality we may assume that there exists N such that for all $n \geqq N$, $\phi_\delta(\Phi(n)) \leqq \phi'_\delta(n)$. By means of the identification of G_δ with $\langle\bar{y} + A_{i,\delta}\rangle_*/A_{i,\delta}$ and with $\langle\bar{y} + A_{j,\delta}\rangle_*/A_{j,\delta}$, θ_i and θ_j induce an isomorphism $f_r : L_r \to L'_r$ between non-free finite rank pure subgroups of G_δ. We can pick arbitrary large $\nu \in C^* \cap \delta$ and $z \in D_r$ such that if $n = \sigma_r(z)$, we have $n \geqq N$ and

$$\rho^\delta(\phi_\delta(\psi_r(\sigma_r(z)))) < \nu \leqq \rho^\delta(\phi_\delta(\psi_r(\sigma_r(z+1)))).$$

By construction and the definition of ψ_r, d_n divides $f_r^{-1}(t_n(\bar{y}))$ in $A_{i,\delta+1}$ mod $A_{i,\nu}$. But, since $\nu \leqq \rho^\delta(\phi_\delta(\psi_r(\sigma_r(z+1)))) \leqq \rho^\delta(\phi_\delta(\Phi(n))) \leqq \rho^\delta(\phi_\delta(n))$, d_n does not divide $t_n(\bar{y})$ in $A_{j,\delta+1}$ mod $A_{j,\nu}$. By choosing a sufficiently large ν we obtain a contradiction from this as in the proof of 3.2.

The proof that A_i does not embed in A_j is similar.

Note that we can, in fact, construct the family $\{A_i : i < 2^{\omega_1}\}$ so that $A_0 = A$. To do this, we impose an additional condition on the t_n and d_n, viz.

Sh:145

(4) $\rho^\delta(n) \overset{\text{def}}{=}$ least μ such that $d_n^\delta \mid t_n(\bar{y}^\delta) \bmod A_{\mu+1}$ defines a non-decreasing function of n.

Then we use this function ρ^δ in the above construction of the A_i. □

All that remains is to sketch the proof of the following lemma which justifies the construction in the proof of 3.5.

3.6 LEMMA. *Given* $A_\delta = \bigcup_{\nu<\delta} A_\nu$, G_δ, $t_n(\bar{y})$ *and* d_n *as in* (1) *and* (2) *of* 3.5, *and given a non-decreasing function* $\psi:\omega \to \delta$ *with range cofinal in* δ, *such that for all* n, $A_{\psi(n)+1}/A_{\psi(n)}$ *is free, then there is a countable free group* $A_{\delta+1} \supseteq A_\delta$ *containing elements* $\bar{y} = \{y_l \mid l < \alpha\}$ *independent over* A_δ *such that* $A_{\delta+1}$ *is the pure closure of* $\langle \bar{y} \rangle + A_\delta$ *and*:

(i) $G_\delta \cong A_{\delta+1}/A_\delta$;

(ii) *for all* $\nu < \delta$, $A'_{\delta+1}/A_{\nu+1}$ *is free*; *and*

(iii) *for all* $n \in \omega$ *and all* $\tau < \delta$, d_n *divides* $t_n(\bar{y}) \bmod A_\tau$ *iff* $\tau > \psi(n)$.

SKETCH OF PROOF. We shall define $A_{\delta+1}$ as a subgroup of $D = A_\delta \oplus \bigoplus_{l<\alpha} Qy_l$ by defining by induction elements $a_n \in A_{\psi(n)+1} - A_{\psi(n)}$ and setting

$$A_{\delta+1} = \left\langle A_\delta, \left\{ \frac{t_n(\bar{y}) - a_n}{d_n} : n \in \omega \right\} \cup \{\bar{y}\} \right\rangle \subseteq D.$$

We must choose the a_n so that the rule

$$(*) \qquad \frac{t_n(\bar{y})}{d_n} \mapsto \frac{t_n(\bar{y}) - a_n}{d_n} + A_\delta;\ y_j \mapsto y_j + A_\delta$$

defines an isomorphism θ of G_δ onto $A_{\delta+1}/A_\delta$. Suppose a_i chosen for $i < n$. Choose $x_n \in A_{\psi(n)+1}$ such that $x_n + A_{\psi(n)}$ generates a direct summand of $A_{\psi(n)+1}/A_{\psi(n)}$ independent mod $A_{\psi(n)}$ from $\langle a_i : i < n \rangle$. Let $k \neq 0$ be minimal such that

$$\frac{kt_n(\bar{y})}{d_n} \in \left\langle \frac{t_i(\bar{y})}{d_i} : i < n \right\rangle + \langle \bar{y} \rangle \ (\subseteq G_\delta).$$

By hypothesis (2) of the proof of 3.5, $k > 1$ and by minimality k divides d_n.

If

$$\frac{kt_n(\bar{y})}{d_n} \equiv \sum_{i<n} c_i \frac{t_i(\bar{y})}{d_i} (\operatorname{mod} \langle \bar{y} \rangle),$$

then, by induction, $(d_n/k)\Sigma_{i<n} c_i a_i/d_i$ is a well-defined element of $A_{\psi(n)}$, so we can let

$$a_n = \frac{d_n}{k}\left(\sum_{i<n} c_i \frac{a_i}{d_i} + x_n \right).$$

Sh:145

Then it is easy to check that

$$k\left(\frac{t_n(\bar{y})-a_n}{d_n}\right) \equiv \sum_{i<n} c_i \frac{(t_i(\bar{y})-a_i)}{d_i} (\operatorname{mod}\langle\bar{y}\rangle + A_{\psi(n+1)})$$

so θ is well-defined by (*). Also, since $k > 1$ the choice of x_n insures that d_n does not divide $t_n(\bar{y}) \bmod A_{\psi(n)}$. □

References

1. A. Beller and A. Litman, *A strengthening of Jensen's □ principles*, J. Symb. Logic **45** (1980), 251–264.
2. P. Eklof, *On the existence of κ-free abelian groups*, Proc. Am. Math. Soc. **47** (1975), 65–72.
3. P. Eklof, *Set Theoretic Methods in Homological Algebra and Abelian Groups*, les Presses de l'Université de Montréal, 1980.
4. P. Eklof and A. Mekler, *Infinitary stationary logic and abelian groups*, Fund. Math. **112** (1981), 1–15.
5. L. Fuchs, *Infinite Abelian Groups*, Vol. II, Academic Press, New York, 1973.
6. P. Hill, *New criteria for freeness in abelian groups, II*, Trans. Am. Math. Soc. **196** (1974), 191–201.
7. T. Jech, *Set Theory*, Academic Press, 1978.
8. A. Mekler, *The number of κ-free abelian groups and the size of* Ext, in *Abelian Group Theory*, Springer-Verlag LNM No. 616, 1977, pp. 323–331.
9. A. Mekler, *How to construct almost free groups*, Can. J. Math. **32** (1980), 1206–1228.
10. M. Magidor and S. Shelah, to appear.
11. S. Shelah, *On successor of singular cardinals*, in *Logic Colloquium 78*, North-Holland, 1979, pp. 357–380.

Institute for Advanced Studies
The Hebrew University of Jerusalem
Jerusalem, Israel

Current address of first author
Department of Mathematics
University of California
Irvine, CA 92717 USA

Current address of second author
Department of Mathematics
Simon Fraser University
Burnaby, British Columbia V5A 1S6, Canada

Current address of third author
Institute of Mathematics
The Hebrew University of Jerusalem
Jerusalem, Israel