

A SPACE WITH ONLY BOREL SUBSETS

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Miklós Laczkovich (Budapest) asked if there exists a Hausdorff (or even normal) space in which every subset is Borel yet it is not meager. The motivation of the last condition is that under MA_κ every subspace of the reals of cardinality κ has the property that all subsets are F_σ however Martin's axiom also implies that these subsets are meager. Here we answer Laczkovich' question. I thank Péter Komjáth – the existence of this paper owes much to him.

THEOREM. *The following are equiconsistent.*

1. *There exists a measurable cardinal.*
2. *There is a non-meager T_1 space with no isolated points in which every subset is Borel.*
3. *There is a non-meager T_4 space with no isolated points in which every subset is the union of an open and a closed set.*

PROOF. Assume first that κ is measurable in the model V . Add κ Cohen reals, that is, force with the partial ordering $\text{Add}(\omega, \kappa)$. Our model will be $V[G]$ where $G \subseteq \text{Add}(\omega, \kappa)$ is generic. We first observe that in $V[G]$ there is a κ -complete ideal on κ such that the complete Boolean algebra $P(\kappa)/I$ is isomorphic to the Boolean algebra of the complete closure of $\text{Add}(\omega, j(\kappa))$ where $j : V \rightarrow M$ is the corresponding elementary embedding. Indeed we let $X \in I$ if and only if $1 \Vdash \kappa \notin j(\tau)$ for some τ satisfying $X = \tau^G$, that is, τ is a name for $X \subseteq \kappa$. Moreover, the mapping $X \mapsto \llbracket \kappa \in j(\tau) \rrbracket$ is an isomorphism between $P(\kappa)/I$ and the regular Boolean algebra of $\text{Add}(\omega, j(\kappa) \setminus \kappa)$ (where τ is a name for X). Notice that $|j(\kappa)| = 2^\kappa$.

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We observe that this Boolean algebra has the following properties. There are 2^κ subsets $\{A_\alpha : \alpha < 2^\kappa\}$ which are independent mod I , that is, if s is a function from a finite subset of κ into $\{0, 1\}$ then the intersection

$$B_s \stackrel{\text{def}}{=} \bigcap_{\alpha \in \text{Dom}(s)} A_\alpha^{s(\alpha)}$$

is not in I (here $A^1 = A$ and $A^0 = \kappa \setminus A$). Moreover, if $A \subseteq \kappa$ then there are countably many pairwise contradictory functions s_0, s_1, \dots as above, such that

$$A/I = B_{s_0}/I \vee B_{s_1}/I \vee \dots,$$

that is, A can be written as $B_{s_0} \cup B_{s_1} \cup \dots$ add-and-take-away a set in I .

By cardinality assumptions we can assume that for every pair (X, Y) of disjoint members of I there is some $\alpha < 2^\kappa$ with $X \subseteq A_\alpha, Y \subseteq \kappa \setminus A_\alpha$.

We define a topology on κ by declaring the system

$$\{A_\alpha \setminus Z, A^1 \setminus Z : \alpha < 2^\kappa, Z \in I\}$$

a subbasis, or, what is the same, the collection of all sets of the form $B_s \setminus Z$ (where $Z \in I$) a basis.

We prove the following statements on the space.

CLAIM. *The space has the following properties.*

1. *Every set of the form B_s is clopen, every set in I is closed.*
2. *Every meager set is in I .*
3. *Every set is the union of an open and a closed set.*
4. *The closure of $B_s \setminus Z$ is B_s .*
5. *The space is T_4 .*

PROOF. 1. Straightforward.

2. Every set not in I contains a subset of the form $B_s \setminus Z$ (by one of the properties of the Boolean algebra mentioned above), which is open, so every nowhere dense, therefore every meager set is in I .

3. If $A \subseteq \kappa$ then A/I can be written as $A/I = B_{s_0}/I \vee B_{s_1}/I \vee \dots$ and then clearly

$$A = \left((B_{s_0} \setminus Z_0) \cup (B_{s_1} \setminus Z_1) \cup \dots \right) \cup Z$$

for some sets Z_0, Z_1, \dots, Z in I . But this is a decomposition into the union of an open and a closed set.

4. Clear.

5. Assume we are given the disjoint closed sets F and F' . They can be written as

$$F = (B_{s_0} \setminus Z_0) \cup (B_{s_1} \setminus Z_1) \cup \cdots \cup Z$$

and

$$F' = (B_{s'_0} \setminus Z'_0) \cup (B_{s'_1} \setminus Z'_1) \cup \cdots \cup Z'.$$

As F and F' are closed, using 4., we can assume that

$$Z_0 = Z_1 = \cdots = Z'_0 = Z'_1 = \cdots = \emptyset.$$

Set $G = B_{s_0} \cup B_{s_1} \cup \cdots$, $G' = B_{s'_0} \cup B_{s'_1} \cup \cdots$, then $F = G \cup Z$, $F' = G' \cup Z'$ and these four sets are pairwise disjoint. It suffices to separate each of the pairs (G, G') , (G, Z') , (G', Z) , and (Z, Z') . There is no problem with the first case, as G, G' are open. For the last case we use our assumption that some A_α separates Z and Z' . For the second, we can assume that G is non empty hence B_{s_0} is well defined and disjoint to Z' , now choose $\alpha < \kappa$ such that Z' is a subset of A_α , and so $G, A_\alpha \setminus B_{s_0}$ is a pair of disjoint open sets as required. Lastly the third case is similar to the second.

We have proved (1) \longrightarrow (3), and (3) \longrightarrow (2) is trivial; lastly for (2) \longrightarrow (1) assume that (X, \mathcal{T}) is a non-meager T_1 space with no isolated points in which every subset is Borel. Let $\{G_\alpha : \alpha < \tau\}$ be a maximal system of disjoint, nonempty, meager open sets. Such a system exists by Zorn's lemma. Set $Y = \bigcup\{G_\alpha : \alpha < \tau\}$. Clearly, Y is meager. As the boundary of the open Y is nowhere dense, we get that even the closure of Y is meager. Then the nonempty subspace $Z = X - \bar{Y}$ has the property that no nonempty open set is meager and every subset is Borel. If I is the meager ideal on Z then every subset is equal to some open set mod I . We claim that I is precipitous on Z which implies that in some inner model there is a measurable cardinal (see [1], [2]).

For this, assume that $\mathcal{W}^0, \mathcal{W}^1, \dots$ is a refining sequence of mod I partitions. That is, every \mathcal{W}^n is a maximal system of I -almost disjoint open sets, and if A is a member of some \mathcal{W}^{n+1} then there is some member of \mathcal{W}^n which includes A mod I . We try to find a member $A_n \in \mathcal{W}^n$ such that $\bigcap\{A_n : n < \omega\}$ is nonempty. To this, observe that the intersection of two members in \mathcal{W}^n is a meagre open set, hence is the empty set. Therefore, \mathcal{W}^n is actually a decomposition of $Z \setminus Z_n$ into the union of disjoint open sets where Z_n is a meager set. Pick an element in $Z \setminus \bigcup\{Z_n : n < \omega\}$ then it is in some member of \mathcal{W}^n for every n and we are done.

REFERENCES

- [1] T. JECH and K. PRIKRY, Ideals over uncountable sets: Application of almost disjoint functions and generic ultrapowers, *Memoirs of the A.M.S.*, **214**, 1979.
- [2] T. JECH, M. MAGIDOR, W. MITCHELL and K. PRIKRY, On precipitous ideals, *Journal of Symbolic Logic* **45** (1980), 1–8.

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