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PCF ARITHMETIC WITHOUT AND WITH CHOICE

BY

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ABSTRACT

We deal with relatives of GCH which are provable. In particular, we deal with rank version of the revised GCH. Our motivation was to find such results when only weak versions of the axiom of choice are assumed, but some of the results give us additional information even in ZFC. We also start to deal with pcf for pseudo-cofinality (in ZFC with little choice).

Annotated Contents

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0. Introduction

In [Sh:460] and [Sh:513], [Sh:829] we prove in ZFC = ZF + AC relatives of GCH. Here we are mainly interested in relatives assuming only weak forms of choice, but some results add information even working in ZFC, in particular a generalization of [Sh:460] for ranks. Always we can assume ZF + DC.

Our original motivation was

Conjecture 0.1: Assume ZF + DC and μ a limit cardinal such that $\mathrm{AC}_{<\mu}$ and μ is strong limit. For every ordinal γ, for some $\kappa < \mu$, for any $\alpha < \mu$ and κ-complete filter D on α we have $\mathrm{rk}_D(\gamma) = \gamma$.

Here we get an approximation to it, i.e., for μ a limit of measurables restricting ourselves to ultrafilters; this is Conclusion 4.4 deduced by applying Theorem 3.10 to Claim 4.3. Can we do it with $\mu = \beth_\omega$?

Also, we would like to weaken $\mathrm{AC}_{<\mu}$; this is interesting per se and, as then, we will be able to combine [Sh:835] + [Sh:513] — see below. We intend to try in [Sh:955], starting with $\bar{J} = \langle J_n : n < \omega\rangle$ such that $\mathrm{IND}(\bar{J})$ or something similar.

It may be illuminating to compare the present result with (see [Sh:g, V]).

Claim 0.2: *If $\kappa \geq \theta > \aleph_0, \lambda \geq 2^{2^\kappa}$ then the following conditions are equivalent:*

- $(*)_1$ *for every θ-complete filter D on κ, we have $\mathrm{rk}_D(\lambda^+) = \lambda^+$,*
- $(*)_2$ $\alpha < \lambda^+ \Rightarrow \mathrm{rk}_D(\alpha) < \lambda^+$ *for every θ-complete filter D on κ,*
- $(*)_3$ *there is no $\mathcal{F} \subseteq {}^\kappa\lambda$ of cardinality $\geq \lambda^+$ and θ-complete filter D in κ such that $f_1 \neq f_2 \in \mathcal{F} \Rightarrow f_1 \neq_D f_2$.*

Also, we can in 0.2 replace λ^+ by a cardinal of cofinality $> 2^{2^\kappa}$. So the result in [Sh:460] implies a weak version of the conjecture above, say on $|\mathrm{rk}_D(\alpha)|$, but the present one gives more precise information. On the other hand, the present conjecture is not proved for $\mu = \beth_\omega$; also, it seems less accommodating to the possible results with $\aleph_\omega$ instead of $\beth_\omega$ in [Sh:513] below $2^{2^{\aleph_\omega}}$.

Question 0.3: In [Sh:908] can we prove that the rank is small?

Discussion 0.4: In 0.5 below we present examples showing some limitations.

In 0.5 below, part (1) of the example shows that Claim 2.3 cannot be improved too much and part (2) shows that Conclusion 4.4 cannot be improved too much. In fact, in Conjecture 0.1, if we demand only "μ is a limit cardinal" then it consistently fails. This implies that we cannot improve too much other results in §3 and §4.

We may wonder how to compare the result in [Sh:460] and Conjecture 0.1 even in ZFC.

Example 0.5: 1) If $D_\ell = \mathrm{dual}([\kappa_\ell]^{<\kappa_\ell})$ for $\ell = 1, 2$ (so if κ_ℓ is regular then $D_\ell = \mathrm{dual}(J^{\mathrm{bd}}_{\kappa_\ell})$) and $\kappa_2 < \kappa_1$ then D_2 does not 2-commute with D_1, i.e., $\boxplus^2_{D_1,D_2}$ from Definition 2.1 fails.

2) Consistently with ZFC, for every n, $\mathrm{rk}_{J^{\mathrm{bd}}_{\aleph_n}}(\aleph_\omega) > \aleph_\omega$.

Proof. 1) Let $A = \kappa_1$ and let $f_2 \in {}^{\kappa_2}\mathrm{Ord}$ be constantly 1, hence by Definition 1.10 and Claim 1.11(3) the ideal $J_2 = J[f_2, D_2]$ is $[\kappa_2]^{<\kappa_2}$. Choose a function $h : \kappa_1 \to \kappa_2$ such that $(\forall\beta < \kappa_2)(\exists^{\kappa_1}\alpha < \kappa_1)(h(\alpha) = \beta)$ and let $\langle B_\alpha : \alpha \in A\rangle$ be such that we have $B_\alpha := \kappa_2 \backslash h(\alpha)$.

So if $A_* \in D_1, B_* \in J_2^+$, then for some $\alpha_* < \kappa_1$ we have $A_* \supseteq \kappa_1 \backslash \alpha_*$ and $B_* \subseteq \kappa_2, |B_*| = \kappa_2$, and choose $t \in B_*$ and then choose $s \in A_*$ such that $h(s) = t+1$. Such s exists by the choice of h, so $(s,t) \in A_* \times B_*$ but $(s,t) \notin \{s\} \times B_s$. So $A, \langle B_\alpha : \alpha \in A \rangle, J_2 = J[f_2, D_2]$ show that $\boxplus_2$ for D_1, D_2 from Def. 2.1 fails, as promised.

2) Assume that the sequence $\langle 2^{\aleph_n} : n < \omega \rangle$ is increasing with supremum $> \aleph_\omega$, and in $\mathrm{cf}({}^{(\aleph_n)}(\aleph_n), <_{J^{\rm bd}_{\aleph_n}})$ there is an increasing sequence of length $\aleph_{\omega+1}$ for each $n \in [1,\omega)$. Hence it follows that $\mathrm{rk}_{J^{\rm bd}_{\aleph_n}}(\aleph_\omega) > \mathrm{rk}_{J^{\rm bd}_{\aleph_n}}(\aleph_n) \geq \aleph_n$ for $n \in [1,\omega)$. $\blacksquare_{0.5}$

We may hope to prove interesting things in ZF + DC by division into cases: if [Sh:835] applies, fine; if not, then we have a strict $\mathbf{p}$. But we need $\mathrm{AC}_{<\mu}$ to prove even clause (f) in 3.1; see [Sh:955]. We may consider that even in ZFC, probably [Sh:908] indicates that we can use weaker assumptions.

Let us say something about our program on set theory with little choice of which this work is a part. We always "know" that the axiom of choice is true. In addition, we had thought that there is no interesting general combinatorial set theory without AC (though equivalence of version of choice, inner model theory and some other exist). Concerning the second point, since [Sh:497] our opinion changed and we thought that there is an interesting such set theory, with "bounded choice" related to pcf. More specifically, [Sh:497] seems to prove that such theory is not empty. Then [Sh:835] suggests we look at axioms of choice "orthogonal" to "$\mathbf{V} = \mathbf{L}[\mathbb{R}]$", e.g., demand that ${}^{\omega\geq}\alpha$ can be well ordered (and weaker relatives). The results say that the universe is somewhat similar to universes obtained by Easton like forcing, blowing up 2^λ for every regular λ without well ordering the new $\mathcal{P}(\lambda)$. Continuing this Larson–Shelah [LrSh:925] generalize the classical theorem on splitting a stationary subset of a regular λ consisting of ordinals of cofinality κ.

In [Sh:955] we shall continue this work. In particular, we continue §5 to get a parallel of the pcf theorem and more, and in [Sh:1005] continue [Sh:835]. Recall that in [Sh:513] in ZFC we get connections between the existence of independent sets and a strong form of [Sh:460]. We prove related theorems on rank.

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1. Preliminaries

Context 1.1: 1) We work in ZF throughout this paper.

2) We try to say when we use DC, but assuming it always causes no great harm.

3) We shall certainly mention the use of any additional form of choice, mainly AC_A.

4) In 1.2–1.11 we quote definitions and claims to be used; see [Sh:835]. The rest of §1 is used only in §5.

Definition 1.2: 1) A filter D on Y is $(\leq B)$-complete when: if $\langle A_t : t \in B\rangle \in {}^B D$ then $A := \bigcap\{A_t : t \in B\} \in D$. We can instead say "$|B|^+$-complete" even if $|B|^+$ is not well defined.

1A) A filter D on Y is pseudo $(\leq B)$-complete when: if $\langle A_t : t \in B\rangle \in {}^B D$ then $\bigcap\{A_t : t \in B\}$ is not empty (so adopt the conventions of part (1)).

2) For an ideal J on a set Y let $\mathrm{dual}(J) = \{Y \backslash X : X \in J\}$, the dual ideal and $\mathrm{Dom}(J) = Y$; abusing notation we assume J determines Y.

3) For a filter D on a set Y let $\mathrm{dual}(D) = \{Y \backslash X : X \in D\}$, $\mathrm{Dom}(D) = Y$. We may use properties defined for filter D for the dual ideal (and vice versa).

4) For a filter D on Y let $D^+ = \{A \subseteq Y : Y \backslash A \notin D\}$, and for an ideal J on Y let $J^+ = (\mathrm{dual}(J))^+$.

Remark 1.3: It may be interesting to try to assume that relevant filters are just pseudo $(\leq B)$-complete instead of $(\leq B)$-complete. Now 1.14 clarifies the connection to some extent, but presently we do not pursue this direction.

Definition 1.4: $\mathbf{C}$ is the class of sets A such that AC_A, the axiom of choice for A non-empty sets, holds.

Definition 1.5: 1) $\theta(A) = \mathrm{Min}\{\alpha$: there is no function from A onto $\alpha\}$.

2) $\Upsilon(A) = \mathrm{Min}\{\alpha$: there is no one-to-one function from α into $A\}$ so $\Upsilon(A) \leq \theta(A)$.

Definition 1.6: 1) For D a filter on Y and $f, g \in {}^Y\mathrm{Ord}$, let $f <_D g$ or $f < g$ mod D mean that $\{s \in Y : f(s) < g(s)\} \in D$; similarly for $\leq, =, \neq$.

2) For D a filter on Y and $f \in {}^Y\mathrm{Ord}$ and $\alpha \in \mathrm{Ord} \cup \{\infty\}$, we define when $\mathrm{rk}_D(f) = \alpha$ by induction on α:

⊛ For $\alpha < \infty$, $\mathrm{rk}_D(f) = \alpha$ iff $\beta < \alpha \Rightarrow \mathrm{rk}_D(f) \neq \beta$ and for every $g \in {}^Y\mathrm{Ord}$ satisfying $g <_D f$ there is $\beta < \alpha$ such that $\mathrm{rk}_D(g) = \beta$.

3) We can replace D by the dual ideal.

Observation 1.7: 1) Let D be a pseudo $\aleph_1$-complete filter on Y. If $f, g \in {}^Y\mathrm{Ord}$ and $f \leq_D g$ then $\mathrm{rk}_D(f) \leq \mathrm{rk}_D(g)$, and so if $f =_D g$ then $\mathrm{rk}_D(f) = \mathrm{rk}_D(g)$.

2) If D_ℓ is a pseudo $\aleph_1$-complete filter on Y for $\ell = 1, 2$ then $D_1 \subseteq D_2 \wedge f \in {}^Y\mathrm{Ord} \Rightarrow \mathrm{rk}_{D_1}(f) \leq \mathrm{rk}_{D_2}(f)$.

Proof. Easy. ∎

CLAIM 1.8: *Assume D is a filter on Y such that D is $\aleph_1$-complete or just pseudo $\aleph_1$-complete (see Definition 1.2(1A)).*

1) [DC] *For $f \in {}^Y\mathrm{Ord}$, in 1.6, $\mathrm{rk}_D(f)$ is always an ordinal, i.e. $< \infty$.*

2) [DC] *If $\alpha \leq \mathrm{rk}_D(f)$ then for some $g \in \prod_{t \in Y}(f(t)+1)$ we have $\alpha = \mathrm{rk}_D(g)$. If $\alpha < \mathrm{rk}_D(f)$ we can add $g <_D f$ and we can demand $(\forall y \in Y)(g(y) < f(g) \vee g(y) = 0 = f(y))$.*

2A) *If $\mathrm{rk}_D(f) < \infty$ then part (2) holds for f (without assuming DC).*

3) *If $f, g \in {}^Y\mathrm{Ord}$ and $f <_D g$ and $\mathrm{rk}_D(g) < \infty$ then $\mathrm{rk}_D(f) < \mathrm{rk}_D(g)$.*

4) *For $f \in {}^Y\mathrm{Ord}$ we have $\mathrm{rk}_D(f) > 0$ iff $\{t \in Y : f(t) > 0\} \in D$.*

5) *If $f, g \in {}^Y\mathrm{Ord}$ and $f = g + 1$ then $\mathrm{rk}_D(f) = \mathrm{rk}_D(g) + 1$.*

Proof. Straightforward. For example:

2A) We prove this by induction on $\beta = \mathrm{rk}_D(f)$. If $\beta \leq \alpha$ there is nothing to prove. If $\beta = \alpha+1$, by the definition, there is $g <_D f$ such that $\mathrm{rk}_D(g) \geq \alpha$. Now by part (3) we have $\mathrm{rk}_D(g) < \mathrm{rk}_D(f)$, which means $\mathrm{rk}_D(g) < \alpha+1$, so together $\mathrm{rk}_D(g) = \alpha$ and let $g' \in {}^Y\mathrm{Ord}$ defined by $g'(s)$ be $g(s)$ if $g(s) < f(s)$ and 0 if $g(s) \geq f(s)$, so $g' <_D f$ and $g' \leq_D g \leq_D g'$, hence $\mathrm{rk}_D(g') = \mathrm{rk}_D(g) = \alpha$ is as required.

Lastly, if $\beta > \alpha + 1$, by the definition there is $f' <_D f$ such that $\mathrm{rk}_D(f') \geq \alpha + 1$ and by 1.7(1), without loss of generality, $t \in Y \Rightarrow f'(t) \leq f(t)$ and by part (3) $\mathrm{rk}_D(f') < \mathrm{rk}_D(f)$, so we can apply the induction hypothesis to f'. $\blacksquare_{1.8}$

CLAIM 1.9: 1) *$[\mathrm{AC}_{\aleph_0}]$ If D is an $\aleph_1$-complete filter on Y and $f \in {}^Y\mathrm{Ord}$ and $Y = \bigcup\{Y_n : n < \omega\}$ then $\mathrm{rk}_D(f) = \min\{\mathrm{rk}_{D+Y_n}(f) : n < \omega$ and $Y_n \in D^+\}$.*

2) *$[\mathrm{AC}_{\mathcal{W}}]$ If D is a $|\mathcal{W}|^+$-complete filter on $Y, \mathcal{W}$ infinite and $f \in {}^Y\mathrm{Ord}$ and $\bigcup\{Y_t : t \in \mathcal{W}\} \in D$ then $\mathrm{rk}_D(f) = \min\{\mathrm{rk}_{D+Y_t}(f) : t \in \mathcal{W}$ and $Y_t \in D^+\}$.*

Proof. Like [Sh:71].

1) By part (2).

2) Note that by $\mathrm{AC}_{\mathcal{W}}$ necessarily $\{t : Y_t \in D^+\}$ is non-empty. The inequality $\leq$ is obvious (i.e. by 1.7(2)). We prove by induction on the ordinal α that $(\forall v \in \mathcal{W})[Y_v \in D^+ \Rightarrow \mathrm{rk}_{D+Y_v}(f) \geq \alpha] \Rightarrow \mathrm{rk}_D(f) \geq \alpha$.

For $\alpha = 0$ and α is limit, this is trivial.

For $\alpha = \beta + 1$, we assume $(\forall v \in \mathcal{W})[Y_v \in D^+ \Rightarrow \mathrm{rk}_{D+Y_v}(f) \geq \alpha > \beta]$, so by Definition 1.6 it follows that

$$[v \in \mathcal{W} \wedge Y_v \in D^+ \Rightarrow (\exists g)(g \in {}^Y\mathrm{Ord} \wedge g <_{D+Y_v} f \wedge \mathrm{rk}_{D+Y_v}(g) \geq \beta],$$

hence, if $v \in \mathcal{W} \wedge Y_v \in D^+$ then $\{t \in Y : f(t) = 0\} = \emptyset \bmod (D + Y_v)$, i.e., $\{v : f(v) = 0\} \cap Y_v = \emptyset \bmod D$. As this holds for every $v \in \mathcal{W}$ and D is $|\mathcal{W}|^+$-complete, clearly we have $\{t \in Y : f(t) = 0\} = \emptyset \bmod D$. We can, by 1.7(1), replace f by $f' \in {}^Y\mathrm{Ord}$ when $\{v \in Y : f(v) = f'(v)\} \in D$, so without loss of generality $t \in Y \Rightarrow f(t) > 0$.

But $\mathcal{W} \in \mathbf{C}$, hence by 1.8(2A) there is a sequence $\langle g_v : v \in \mathcal{W}_* \rangle$ such that $\mathcal{W}_* := \{v \in \mathcal{W} : Y_v \in D^+\}$ and $g_v \in {}^Y\mathrm{Ord}$, $g_v <_{D+Y_v} f$, $\mathrm{rk}_{D+Y_v}(g_v) \geq \beta$ and $t \in Y \Rightarrow g_v(t) < f(t)$, so $g_v < f$.

As D is $|\mathcal{W}|^+$-complete, necessarily $Y_* := \bigcup\{Y_v : v \in \mathcal{W} \backslash \mathcal{W}_*\} = \emptyset \bmod D$, but $\bigcup\{Y_v : v \in \mathcal{W}\} \in D$, hence $Y_* = \bigcup\{Y_v : v \in \mathcal{W}_*\}$ belongs to D. Define $g \in {}^Y\mathrm{Ord}$ by $g(s) = \min\{g_u(s) : u \in \mathcal{W}_*$ satisfies $s \in Y_u\}$ if $s \in Y_*$ and 0 if $s \in Y \backslash Y_*$.

Hence $(\bigcup\{Y_v : v \in \mathcal{W}_*\}) \in D$ and $g \in {}^Y\mathrm{Ord}$ and $g <_D f$ (and even $g < f$), so by the induction hypothesis

- $\odot$ it suffices to prove $v \in \mathcal{W}_* \Rightarrow \mathrm{rk}_{D+Y_v}(g) \geq \beta$.

Fix $v \in \mathcal{W}_*$, and for each $u \in \mathcal{W}_*$ let $Y_{v,u} := \{t \in Y_u \cap Y_v : g(t) = g_u(t)\}$, so by the choice of $g(t)$ we have

- $\boxplus_1$ if $v \in \mathcal{W}_*, t \in Y_v$ then for some $u \in \mathcal{W}_*$ we have $t \in Y_{v,u} \subseteq Y_u$ and $g(t) = g_u(t)$.

Hence

- $\boxplus_2$ $\langle Y_{v,u} : u \in \mathcal{W}_* \rangle$ exists and $\bigcup\{Y_{v,u} : u \in \mathcal{W}_*\} = Y_v \in (D + Y_v)$.

Now

- $\boxplus_3$ if $u \in \mathcal{W}_* \wedge Y_{v,u} \in (D + Y_v)^+$ then $\mathrm{rk}_{D+Y_{v,u}}(g) \geq \beta$.

[Why? By the choice of $Y_{v,u}$ we have $g = g_u \bmod (D{+}Y_{v,u})$, hence $\mathrm{rk}_{D+Y_{v,u}}(g) = \mathrm{rk}_{D+Y_{v,u}}(g_u)$; also, $Y_{v,u} \subseteq Y_u$, hence $D{+}Y_{v,u} \supseteq D{+}Y_u$ which, by 1.7(2), implies $\mathrm{rk}_{D+Y_{v,u}}(g_u) \geq \mathrm{rk}_{D+Y_u}(g_u)$ which is $\geq \beta$. Together we are done.]

By $\boxplus_2{+}\boxplus_3$ and the induction hypothesis it follows that $v{\in}\mathcal{W}_* \Rightarrow \mathrm{rk}_{D+Y_v}(g){\geq}\beta$, so by $\odot$ we are done. $\blacksquare_{1.9}$

Definition 1.10: For Y, D, f in 1.6 let $J[f,D] =: \{Z \subseteq Y : Y \backslash Z \in D$ or $(Y\backslash Z) \in D^+ \wedge \mathrm{rk}_{D+Z}(f) > \mathrm{rk}_D(f)\}$.

Claim 1.11 (DC+AC$_{\mathcal{Y}}$): *Assume D is an $\aleph_1$-complete $|\mathcal{Y}|^+$-complete filter on Y.*

1) *If $f \in {}^Y\mathrm{Ord}$ then $J[f,D]$ is an $\aleph_1$-complete and $|\mathcal{Y}|^+$-complete ideal on Y.*

2) *If $f_1, f_2 \in {}^Y\mathrm{Ord}$ and $J = J[f_1,D] = J[f_2,D]$ then $\mathrm{rk}_D(f_1) < \mathrm{rk}_D(f_2) \Rightarrow f_1 < f_2 \bmod J$ and $\mathrm{rk}_D(f_1) = \mathrm{rk}_D(f_2) \Rightarrow f_1 = f_2 \bmod J$.*

3) *If $f \in {}^Y\mathrm{Ord}$ is, e.g., constantly 1 then $J[f,D] = \mathrm{dual}(D)$.*

4) *If $f \in {}^Y\mathrm{Ord}$ and $A \in (J[f,D])^+$ then ($A \in D^+$ and) $\mathrm{rk}_{D+A}(f) = \mathrm{rk}_D(f)$.*

Proof. 1) By 1.9.

2) As J is an ideal on Y (by part (1)) this should be clear by the definitions; that is, let $A_0 := \{t \in Y : f_1(t) < f_2(t)\}, A_1 := \{t \in Y : f_1(t) = f_2(t)\}$ and $A_2 := \{t \in Y : f_1(t) > f_2(t)\}$. Now $\langle A_0, A_1, A_0\rangle$ is a partition of Y.

First, assume $A_0 \in J^+$, then by the definition of $J[f_1,D]$ we have $\neg(\mathrm{rk}_D(f_1) < \mathrm{rk}_{D+A_0}(f_1))$; i.e., $\mathrm{rk}_{D+A_0}(f_1) \leq \mathrm{rk}_D(f_1)$ and so by 1.7(2) we have $\mathrm{rk}_D(f_1) = \mathrm{rk}_{D+A_0}(f_1)$. Now as $A_0 \in J^+$, by the choice of $A_0, f_1 <_{D+A_0} f_2$, hence $\mathrm{rk}_D(f_1) = \mathrm{rk}_{D+A_0}(f_1) < \mathrm{rk}_{D+A_0}(f_2) = \mathrm{rk}_D(f_2)$.
[Why? By the previous sentence, by 1.8(3), by the previous sentence, respectively.]

Second, similarly if $A_2 \in J^+$ then $f_2 < f_1 \bmod (D + A_2)$ and $\mathrm{rk}_D(f_1) > \mathrm{rk}_D(f_2)$.

Lastly, if $A_1 \in J^+$ then, by 1.7(1), $f_1 = f_2 \bmod (D{+}A_1)$, hence $\mathrm{rk}_{D+A_1}(f_1) = \mathrm{rk}_{D+A_1}(f_2)$ and $\mathrm{rk}_D(f_1) = \mathrm{rk}_{D+A_1}(f_2) = \mathrm{rk}_{D+A_1}(f_2) = \mathrm{rk}_D(f_2)$.

By the last three paragraphs at most one of A_0, A_1, A_2 belongs to J^+ and, as $A_0 \cup A_1 \cup A_2 = Y$, at least one of A_0, A_1, A_2 belongs to J^+, so easily we are done.

3) Obvious.

4) Proved within the proof of part (2). $\blacksquare_{1.11}$

Definition 1.12: 1) Let $\mathrm{FIL}^{\mathrm{cc}}_S(Y)$ or $\mathrm{FIL}^{\mathrm{pcc}}_S(Y)$ be the set of D such that:

D is a filter on the set Y which is $|S|^+$-complete and is $\aleph_1$-complete or is pseudo $|S|^+$-complete and pseudo $\aleph_1$-complete.

2) Let $\mathrm{FIL}_{\mathrm{cc}}(Y)$ or $\mathrm{FIL}_{\mathrm{pcc}}(Y)$ be $\mathrm{FIL}^{\mathrm{cc}}_{\emptyset}$ or $\mathrm{FIL}^{\mathrm{pcc}}_{\emptyset}$.

3) Omitting Y means for some Y and then we let $Y = \mathrm{Dom}(D)$.

Without enough choice, the minimal $(\leq S)$-complete filter extending a filter D is obtained in stages.

Definition 1.13: 1) For a filter D on Y and set S we define $\mathrm{comp}_{S,\gamma}(D)$ by induction on $\gamma \in \mathrm{Ord} \cup \{\infty\}$.

$\underline{\gamma = 0}$: $\mathrm{comp}_{S,\gamma}(D) = D$.

$\underline{\gamma = \text{limit}}$: $\mathrm{comp}_{S,\gamma}(D) = \bigcup\{\mathrm{comp}_{S,\beta}(D) : \beta < \gamma\}$.

$\underline{\gamma = \beta + 1}$: $\mathrm{comp}_{S,\gamma}(D) = \{A \subseteq Y : A$ belongs to $\mathrm{comp}_{S,\beta}(D)$ or includes the intersection of some S-sequence of members of $\mathrm{comp}_{S,\beta}(D)$, i.e., $\bigcap\{A_s : s \in S\}$, where $\langle A_s : s \in S\rangle$ is a sequence of members of $\mathrm{comp}_{S,\beta}(D)\}$.

2) Similarly for a family $\mathscr{S}$ of sets replacing S by "some member of $\mathscr{S}$", e.g., we define $\mathrm{com}_{\in \mathscr{S},\gamma}(D)$ by induction on γ using $(\in \mathscr{S})$-sequences, i.e., S-sequence for some $S \in \mathscr{S}$.

3) If $\gamma = \infty$ we may omit it. We say that D is pseudo $(\leq S, \gamma)$-complete when $\emptyset \notin \mathrm{comp}_{S,\gamma}(D)$.

Observation 1.14: 1) If D is a filter on Y and S is a set, then:

(a) $\langle \mathrm{comp}_{S,\gamma}(D) : \gamma \in \mathrm{Ord} \cup \{\infty\}\rangle$ is an $\subseteq$-increasing sequence of filters of Y (starting with D);
(b) if $\mathrm{comp}_{S,\gamma+1}(D) = \mathrm{comp}_{S,\gamma}(D)$ then for every $\beta \geq \gamma$ we have $\mathrm{comp}_{S,\beta}(D) = \mathrm{comp}_{S,\gamma}(D)$;
(c) there is an ordinal $\gamma = \gamma_S(D) < \theta(\mathcal{P}(Y))$ such that $\mathrm{comp}_{S,\gamma}(D) = \mathrm{comp}_{S,\gamma+1}(D)$ and $\langle \mathrm{comp}_{S,\beta}(D) : \beta \leq \gamma\rangle$ is strictly $\subset$-increasing.

2) Assume AC_S. Then for any filter D on Y we have $\gamma_S(D) \leq \theta$ when $\theta := \min\{\lambda : \lambda$ a cardinal such that $\mathrm{cf}(\lambda) \geq \theta(S)\}$.

3) Assume $\mathrm{DC} + \mathrm{AC}_S + |S \times S| = |S|$. Then for any filter D on Y we have $\gamma_S(D) \leq 1$ and $\mathrm{comp}_{S,1}(D)$ is an $(\leq S)$-complete filter or is $\mathcal{P}(Y)$; the latter holds iff D is not pseudo $(\leq S)$-complete.

4) Similarly to part (2) for "$\in \mathscr{S}$" but AC_S is replaced by $S \in \mathscr{S} \Rightarrow \mathrm{AC}_S$ and $\theta = \min\{\kappa : \kappa$ regular and $S \in \mathscr{S} \Rightarrow \kappa \geq \theta(\mathscr{S})\}$.

Remark 1.15: Note that in part (2) of 1.14, θ is regular and $\theta \leq \theta({}^{\omega>}S)$ but the inverse is not true if $\theta(S) = \aleph_0$, but holds if $\theta(S) > \aleph_0$.

Proof. We prove the versions with $\mathscr{S}$, i.e., for (4). Let $D_\gamma = \text{comp}_{\in \mathscr{S},\gamma}(D)$ for $\gamma \in$ Ord.

1) Clause (a) is by the definition; clause (b) is proved by induction on $\beta \geq \gamma$: for $\beta = \gamma$ this is trivial, for $\beta = \gamma + 1$ use the assumption and for $\beta > \gamma + 1$ use the definition and the induction hypothesis. As for clause (c) let

$$\gamma_* = \min\{\gamma \in \text{Ord} \cup \{\infty\};\ \text{if } \gamma < \infty \text{ then } D_\gamma = D_{\gamma+1}\},$$

so $\langle D_\gamma : \gamma \leq \gamma_*\rangle$ is $\subset$-increasing continuous by clause (a), and by clause (b), $\langle D_\gamma : \gamma \geq \gamma_*\rangle$ is constant. Now define $h : \mathcal{P}(Y) \to \gamma_*$ by:

$$A \in D_{\gamma+1} \backslash D_\gamma \Rightarrow h(A) = \gamma \quad \text{and } h(A) = 0$$

when there is no such γ. So h is onto γ_*, hence $\gamma_* < \theta(\mathcal{P}(A))$, so γ_* is as required on $\gamma_S(D)$.

2) We prove also the relevant statement in part (4), so $S \in \mathscr{S} \Rightarrow \text{AC}_S \wedge \text{cf}(\theta) \geq \theta(S)$. Let γ be an ordinal.

Let

$$\begin{aligned} \mathcal{T}^1_n = \{\Lambda : &\Lambda \text{ is a set of sequences of length } \leq n, \\ &\text{closed under initial segments such that for every} \\ &\text{non-maximal } \eta \in \Lambda \text{ for some } S \in \mathscr{S} \text{ we have} \\ &\eta\hat{\ }\langle s\rangle \in \Lambda \Leftrightarrow s \in S\}; \end{aligned}$$

$$\begin{aligned} \mathcal{T}^2_{\gamma,n} = \{\mathbf{x} : &\text{(a)} \quad \mathbf{x} \text{ has the form } \langle Y_\eta, \gamma_\eta : \eta \in \Lambda\rangle, \\ &\text{(b)} \quad \Lambda \in \mathcal{T}^1_n \text{ and } Y_\eta \subseteq Y \text{ for } \eta \in \Lambda, \\ &\text{(c)} \quad Y_\eta = \bigcap\{Y_{\eta\hat{\ }\langle s\rangle} : s \text{ satisfies } \eta\hat{\ }\langle s\rangle \in \Lambda\} \text{ if } \eta \in \Lambda \\ &\qquad \text{but } \eta \text{ is not } \triangleleft\text{-maximal in } \Lambda, \\ &\text{(d)} \quad \eta \triangleleft \nu \in \Lambda \Rightarrow \gamma_\nu < \gamma_\eta < 1 + \gamma, \\ &\text{(e)} \quad Y_\eta \in D \text{ if } \eta \in \Lambda \text{ is } \triangleleft\text{-maximal in } \Lambda \\ &\qquad \text{but } \ell g(\eta) < n\}; \end{aligned}$$

$$\mathcal{T}^2_n = \bigcup\{\mathcal{T}^2_{\gamma,n} : \gamma \text{ is an ordinal}\}.$$

Let $\mathbf{n}(\mathbf{x}) = n$ for the minimal possible n such that $\mathbf{x} \in \mathcal{T}^2_n$ and let $\mathbf{x} = \langle Y^{\mathbf{x}}_\eta, \gamma^{\mathbf{x}}_\eta : \eta \in \Lambda_{\mathbf{x}}\rangle$.

Let $\mathcal{T}^3_\gamma = \bigcup\{\mathcal{T}^2_{\gamma,n} : n < \omega\}$ and let $<_*$ be the natural order on $\mathcal{T}^3_\gamma : \mathbf{x} <_* \mathbf{y}$ iff $n(\mathbf{x}) < n(\mathbf{y}), \Lambda_{\mathbf{x}} = \Lambda_{\mathbf{y}} \cap {}^{n(\mathbf{x})\geq}(\bigcup\{S : S \in \mathscr{S}\})$ and $(Y^{\mathbf{x}}_\eta, \gamma^{\mathbf{x}}_\eta) = (Y^{\mathbf{y}}_\eta, \gamma^{\mathbf{y}}_\eta)$ for $\eta \in \Lambda_{\mathbf{x}}$.

Now

⊛ $A \in D_\gamma$ iff there is an ω-branch $\langle \mathbf{x}_n : n < \omega\rangle$ of $(\mathcal{T}^3_\gamma, <_*)$ such that $Y^{\mathbf{x}_n}_{\langle\rangle} = A$.

[Why? We prove it by induction on the ordinal γ. For $\gamma = 0$ and γ limit this is obvious, so assume we have it for γ and we shall prove it for $\gamma + 1$.

First assume $A \in D_{\gamma+1}$ and we shall find such an ω-branch; if $A \in D_\gamma$ this is obvious, otherwise there are $S \in \mathscr{S}$ and a sequence $\langle A_s : s \in S\rangle$ of members of D_γ such that $A = \bigcap\{A_s : s \in S\}$. So $X_s := \{\bar{\mathbf{x}} : \bar{\mathbf{x}}$ witness $A_s \in D_\gamma\}$ is well defined and non-empty by the induction hypothesis; clearly the sequence $\langle X_s : s \in S\rangle$ exists, hence we can use AC_S to choose $\langle \bar{\mathbf{x}}_s : s \in S\rangle$ satisfying $\bar{\mathbf{x}}_s \in X_s$.

Now define $\bar{\mathbf{x}} = \langle \mathbf{x}_n : n < \omega\rangle$ as follows:

$$\Lambda_{\mathbf{x}_n} = \{\langle\rangle\} \cup \{\langle s\rangle\hat{\ }\eta : \eta \in \Lambda_{\mathbf{x}_{s,n-1}} \text{ and } s \in S\}, \quad \gamma^{\mathbf{x}_n}_{\langle\rangle} = \bigcup\{\gamma^{\mathbf{x}_{s,n}}_{\langle\rangle} + 1 : s \in S\}$$

and $Y^{\mathbf{x}}_{\langle\rangle} = A$ and $Y^{\mathbf{x}_n}_{\langle s\rangle\hat{\ }\eta} = Y^{\mathbf{x}_{s,n-1}}_\eta$. Now check.

Second, assume that there is such an ω-branch $\langle \mathbf{x}_n : n < \omega\rangle$ of $(\mathcal{T}^3_\gamma, <_*)$ such that $Y^{\mathbf{x}_n}_{\langle\rangle} = A$. Let $S = \{\eta(0) : \eta \in \Lambda_{\mathbf{x}_1}\}$ so necessarily $S \in \mathscr{S}$. For each $n < \omega$ and $s \in S$ we define $\mathbf{y}_{n,s}$ as follows: $\Lambda^{\mathbf{y}_{n,s}}_s = \{\nu : \langle s\rangle\hat{\ }\nu \in \Lambda_{\mathbf{x}_{n+1}}\}$, and for $\nu \in \Lambda^n_s$, let $\gamma^{\mathbf{y}_{n,s}}_\nu = \nu^{\mathbf{x}_{n+1}}_{\langle s\rangle\hat{\ }\nu}$ and $Y^{\mathbf{y}_{n,s}}_\nu = Y^{\mathbf{x}_{n+1}}_{\langle s\rangle\hat{\ }\nu}$. Now clearly $\langle \mathbf{y}_{n,s} : n < \omega\rangle$ is an ω-branch of $(\mathcal{T}^3_\gamma, \leq_*)$, so by the induction hypothesis $A^{\mathbf{x}_1}_{\langle s\rangle} \in D$, $\mathrm{comp}_{S,\gamma}(D)$ and $Y^{\mathbf{x}_0}_{\langle\rangle} = A = \bigcap\{Y^{\mathbf{x}}_{\langle s\rangle} : \langle\,\rangle \in \Lambda_{\mathbf{x}_1}\} \in \ \mathrm{comp}_{S,\gamma+1}(D)$. So we are done.]

Now toward a contradiction assume that $\gamma_S(D) > \theta$, so there is $A \in D_{\theta+1}\backslash D_\theta$, hence there is an ω-branch $\langle \mathbf{x}_n : n < \omega\rangle$ of $\mathcal{T}^3_\gamma$ witnessing that $A \in D_{\theta+1}$; let $\Lambda = \bigcup\{\Lambda_{\mathbf{x}_n} : n < \omega\}$ and $\gamma_\eta = \gamma^{\mathbf{x}_n}_\eta$ for every $n < \omega$ large enough. So Λ is well founded (recalling $\eta \triangleleft \nu \in \Lambda \Rightarrow \gamma_\eta > \gamma_\nu$) and we can choose $\langle \gamma'_\eta : \eta \in \Lambda\rangle$ such that $\gamma'_\eta = \sup\{\gamma_\nu + 1 : \eta \triangleleft \nu \in \Lambda$ and $\ell g(\nu) = \ell g(\eta) + 1\}$. If $\gamma_{\langle\rangle} < \theta$ we are done, otherwise let $\eta \in \Lambda$ be $\triangleleft$-maximal such that $\gamma'_\eta \geq \theta$ hence $\eta \triangleleft \nu \Rightarrow \gamma'_\nu < \theta$, so necessarily $\gamma'_\eta = \theta = \bigcup\{\gamma'_\nu + 1 : \eta \triangleleft \nu \in \Lambda, \ell g(\nu) = \ell g(\eta) + 1\}$. Let $S \in \mathscr{S}$ be such that $\eta\hat{\ }\langle s\rangle \in \Lambda \Leftrightarrow s \in S$, so $\{\gamma'_{\eta\hat{\ }\langle s\rangle} : s \in S\}$ is an unbounded subset of

θ, hence $\mathrm{cf}(\theta) \le \theta(S) < \theta$. This takes care of the first possibility for θ, so the second case is easier.

3) It suffices to show that we can replace $\mathbf{x} \in \mathcal{T}_2^2$ by $\mathbf{x} \in \mathcal{T}_1^2$. $\blacksquare_{1.14}$

Definition 1.16: 1) For a filter D on a set Y and a set S let $\gamma_S(D)$ be as in clause (c) of Observation 1.14(1).

1A) Similarly with "$\in \mathscr{S}$" instead S.

2) D is pseudo (S,γ)-complete if $\emptyset \notin \mathrm{comp}_{S,\gamma}(D)$.

2A) Similarly with "$\in \mathscr{S}$" instead S.

Observation 1.17: If h is a function from S_1 onto S_2, then $\theta(S_1) \ge \theta(S_2)$ and every [pseudo] $(\le S_1)$-complete filter is a [pseudo] $(\le S_2)$-complete filter.

2. Commuting ranks

The aim of this section is to sort out when two ranks, rk_{D_1} and rk_{D_2}, do so-called commute.

Definition 2.1: Assume that D_ℓ is an $\aleph_1$-complete filter on Y_ℓ for $\ell = 1,2$. For $\iota \in \{1,2,3,4,5\}$ we say D_2 ι-commutes with D_1 when: $\boxplus_\iota = \boxplus^\iota_{D_1,D_2}$ holds where:

$\boxplus_1$ if $A \in D_1$ and $\bar{B} = \langle B_s : s \in A\rangle \in {}^A(D_2)$ then we can find A_*, B_* such that: $A_* \in D_1, B_* \in D_2$ and $A_* \times B_* \subseteq \bigcup\{\{s\} \times B_s : s \in A\}$, so $A_* \subseteq A$;

$\boxplus_2$ if $A \in D_1$ and $\bar{B} = \langle B_s : s \in A\rangle \in {}^A(D_2)$ and $J_2 = J[f_2, D_2]$ for some $f_2 \in {}^{Y_2}\mathrm{Ord}$ then we can find A_*, B_* such that $A_* \in D_1, B_* \in J_2^+$ and $A_* \times B_* \subseteq \bigcup\{\{s\} \times B_s : s \in A\}$, so $A_* \subseteq A$;

$\boxplus_3$ if $A \in D_1$ and $\bar{B} = \langle B_s : s \in A\rangle \in {}^A(D_2)$ and $J_1 = J[f_1, D_1]$ for some $f_1 \in {}^{Y_1}\mathrm{Ord}$ then we can find A_*, B_* such that $A_* \in J_1^+, A_* \subseteq A, B_* \in D_2$ and $s \in A_* \Rightarrow B_* \subseteq B_s$;

$\boxplus_4$ if $A \in D_1$ and $\bar{B} = \langle B_s : s \in A\rangle \in {}^A(D_2)$ and $\bar{J}^1 = \langle J^1_t : t \in Y_2\rangle$ satisfies $J^1_t \in \{J[f, D_1] : f \in {}^{Y_1}\mathrm{Ord}\}$ and $J_2 \in \{J[f, D_2] : f \in {}^{Y_2}\mathrm{Ord}\}$ then we can find A_*, B_* such that $B_* \in J_2^+$ and $t \in B_* \Rightarrow A_* \in (J^1_t)^+$ and $(s,t) \in A_* \times B_* \Rightarrow s \in A \wedge t \in B_s$, hence $A_* \subseteq A, A_* \in D_1^+$;

$\boxplus_5$ like $\boxplus_4$, but we omit the sequence $\bar{J}^1$ and the demand on A_* is $A_* \in D_1^+$.

Remark 2.2: 1) These are seemingly not commutative relations.

2) We shall first give a consequence and then give sufficient conditions.

3) We intend to generalize to systems (see 3.1 and 3.8).

4) Can we use "$D_\ell \in \text{FIL}_{\text{pcc}}(Y_1)$" below; see Definition 1.12? Yes, but only when we do not use $D + A, A \in D^+$.

Claim 2.3: $\text{rk}_{D_1}(f) \le \text{rk}_{D_2}(g)$ *when:*

$\oplus$ (a) $D_\ell \in \text{FIL}_{cc}(Y_\ell)$ *for* $\ell = 1, 2$,
(b) $\bar{g} = \langle g_t : t \in Y_2 \rangle$,
(c) $g_t \in {}^{Y_1}\text{Ord}$,
(d) $g \in {}^{Y_2}\text{Ord}$ *is defined by* $g(t) = \text{rk}_{D_1}(g_t)$,
(e) $\bar{f} = \langle f_s : s \in Y_1 \rangle$,
(f) $f_s \in {}^{Y_2}\text{Ord}$ *is defined by* $f_s(t) = g_t(s)$,
(g) $f \in {}^{Y_1}\text{Ord}$ *is defined by* $f(s) = \text{rk}_{D_2}(f_s)$;

$\boxplus$ (a) D_2 *does 2-commute with* D_1, *and even* $D_2 + A$ *when* $A \in D_2^+$ *(really folow),*
(b) AC_{Y_1} *holds.*

Remark 2.4: In order not to use DC in the proof we should consider ∞ as a member of Ord in clauses (d),(g) of $\boxplus$.

Proof. We prove by induction on the ordinal ζ that

$\boxdot_\zeta$ if $\oplus + \boxplus$ above hold for $D_1, D_2, f, g, \bar{f}, \bar{g}$ and $\text{rk}_{D_1}(f) \ge \zeta$, then $\text{rk}_{D_2}(g) \ge \zeta$.

The case $\zeta = 0$ is trivial and the case ζ a limit ordinal follows by the induction hypothesis. So assume that $\zeta = \xi + 1$.

Let

$(*)_1$ $A := \{s \in Y_1 : f(s) > 0\}$.

As we are assuming $\text{rk}_{D_1}(f) > \xi \ge 0$ by 1.8(4) necessarily

$(*)_2$ $A \in D_1$.

For each $s \in A$, $f(s) > 0$, so applying clause (g) of $\oplus$ we get

$(*)_3$ $\text{rk}_{D_2}(f_s) > 0$ when $s \in A$,

hence

$(*)_4$ $B_s := \{t \in Y_2 : f_s(t) > 0\}$ belongs to D_2 when $s \in A$.

So $\langle B_s : s \in A \rangle \in {}^A(D_2)$. Recall (see $\boxplus$(a) of the assumption) that D_2 does 2-commute with D_1, apply it to $A, \langle B_s : s \in A \rangle, J_2 := J[g, D_2]$; so we can find A_*, B_* such that

$(*)_5$ (a) $A_* \in D_1$ (and $A_* \subseteq A$),

(b) $B_* \in J_2^+$ recalling $J_2 = J[g, D_2]$, so $B_* \in D_2^+$ and (by Definition 1.10) $\mathrm{rk}_{D_2+B_*}(g) = \mathrm{rk}_{D_2}(g)$,

(c) $(s,t) \in A_* \times B_* \Rightarrow s \in A \wedge t \in B_s$.

Now by the present assumption of $\boxdot_\zeta$ we have

$(*)_6$ $\mathrm{rk}_{D_1}(f) \geq \zeta = \xi + 1$.

Hence by the definition of rk and 1.8(2) we can find f' such that

$(*)_7$ (a) $f' \in {}^{Y_1}\mathrm{Ord}$ and $\mathrm{rk}_{D_1}(f') \geq \xi$,

(b) $f' <_{D_1} f$,

(c) by $(*)_1$, without loss of generality, $s \in A \Rightarrow f'(s) < f(s)$.

For each $s \in A$, clearly $f'(s) < f(s) = \mathrm{rk}_{D_2}(f_s) \leq \mathrm{rk}_{D_2+B_*}(f_s)$, by 1.8(3) clause (g) of $\oplus$ and 1.7(2) (as $D_2 \subseteq D_2 + B_*$) respectively, hence by 1.8(2) for each $s \in Y_1$ there is a function f'_s such that

$(*)_8$ (a) $f'_s \in {}^{(Y_2)}\mathrm{Ord}$,

(b) $f'_s < f_s \bmod D_2$ if $s \in A$ and $t \in Y_2 \Rightarrow f'_s(t) < f_s(t) \vee f'_s(t) = 0 = f_s(t)$,

(c) $\mathrm{rk}_{D_2+B_*}(f'_s) = f'(s)$; we may require this only for $s \in A$.

As $Y_1 \in \mathbf{C}$ by $\boxplus$(b) of the assumption, clearly

$(*)_8^+$ there is such a sequence $\bar{f}' = \langle f'_s : s \in Y_1 \rangle$.

As $s \in A_* \wedge t \in B_* \Rightarrow f_s(t) > 0$, see $(*)_4 + (*)_5$, clearly

$(*)_9$ if $s \in A_*$ and $t \in B_*$ then $f'_s(t) < f_s(t)$.

We now define $\bar{g}' = \langle g'_t : t \in Y_2 \rangle$ by

$(*)_{10}$ $g'_t(s) = f'_s(t)$ for $s \in Y_1, t \in Y_2$, so $g'_t \in {}^{Y_1}\mathrm{Ord}$.

Therefore

$(*)_{11}$ $s \in A_* \wedge t \in B_* \Rightarrow g'_t(s) = f'_s(t) < f_s(t) = g_t(s)$,

hence (recalling $A_* \in D_1$ by $(*)_5$(a) and 1.8(3))

$(*)_{12}$ if $t \in B_*$ then $g'_t <_{D_1} g_t$, hence $\mathrm{rk}_{D_1}(g'_t) < \mathrm{rk}_{D_1}(g_t)$.

Define $g' \in {}^{(Y_2)}\mathrm{Ord}$ by $g'(t) := \mathrm{rk}_{D_1}(g'_t)$, hence (recalling $\mathrm{rk}_{D_1}(g_t) = g(t)$)

$(*)_{13}$ $g' < g \bmod D_2 + B_*$.

Note that here $D_1 + A_* = D_1$ (though not so when we shall prove 2.9).

Now we apply the induction hypothesis to $g', f', \bar{g}' := \langle g'_t : t \in Y_2 \rangle$, $\bar{f}' := \langle f'_s : s \in Y_1 \rangle, D_1 + A_*, D_2 + B_*$ and ξ and get

$(*)_{14}$ $\xi \leq \mathrm{rk}_{D_2+B_*}(g')$.

[Why is this legitimate? First, obviously clauses (a),(b) of $\boxplus$ hold, second, we have to check that clauses (a)–(g) of $\oplus$ hold in this instance.

Clause (a): First "$D_1 + A_* \in \mathrm{FIL}_{\mathrm{cc}}(Y_1)$" as we assume $D_1 \in \mathrm{FIL}_{\mathrm{cc}}(Y_1)$ and $A_* \in D_1$; see $(*)_5$(a), actually $A_* \in D_1^+$ suffice (used in proving 2.9).

Second, "$D_2 + B_* \in \mathrm{FIL}_{\mathrm{cc}}(Y_2)$" as $D_2 \in \mathrm{FIL}_{\mathrm{cc}}(Y_2)$ and $B_* \in D_2^+$ by $(*)_5$(b).

Clause (b): "$\bar{g}' = \langle g'_t : t \in Y_2\rangle$" by our choice.

Clause (c): "$g'_t \in {}^{Y_1}\mathrm{Ord}$" by $(*)_{10}$.

Clause (d): "$g' \in {}^{Y_2}\mathrm{Ord}$ is defined by $g'(t) = \mathrm{rk}_{D_1}(g'_t)$" by its choice after $(*)_{12}$.

Clause (e): "$\bar{f}' = \langle f'_s : s \in Y\rangle$" by our choice in $(*)_8^+$.

Clause (f): "$f'_s \in {}^{Y_2}\mathrm{Ord}$ is defined by $f'_s(t) = g'_t(s)$" holds by $(*)_{10}$.

Clause (g): "$f' \in {}^{Y_1}\mathrm{Ord}$ is defined by $f'(s) = \mathrm{rk}_{D_2+B_*}(f'_s)$" holds by $(*)_7$(a) + $(*)_8$(c).

Now $\boxdot_\xi$, the induction hypothesis, assumes "$\mathrm{rk}_{D_1+A_*}(f') \geq \xi$", which holds by $(*)_7$(a) + $(*)_5$(a); actually, $A_* \in D_1^+$ suffices here and its conclusion is $\xi \leq \mathrm{rk}_{D_2+B_*}(g')$ as promised in $(*)_{14}$.]

Next

$(*)_{15}$ $\xi < \mathrm{rk}_{D_2}(g)$.

[Why?

- $\bullet_1$ $\xi \leq \mathrm{rk}_{D_2+B_*}(g')$ by $(*)_{14}$,
- $\bullet_2$ $\mathrm{rk}_{D_2+B_*}(g') < \mathrm{rk}_{D_2+B_*}(g)$ by $(*)_{13}$ and 1.8(3),
- $\bullet_3$ $\mathrm{rk}_{D_2+B_*}(g) = \mathrm{rk}_{D_2}(g)$ by $(*)_5(b)$.

Together $(*)_{15}$ holds.]

So

$(*)_{16}$ $\zeta = \xi + 1 \leq \mathrm{rk}_{D_2}(g)$

as promised. Together we are done. $\blacksquare_{2.3}$

Claim 2.5: *Assume $D_\ell \in \mathrm{FIL}_{cc}(Y_\ell)$ for $\ell = 1, 2$.*

If D_2 does ι_1-commute with D_1 then D_2 does ι_2-commute with D_1 when $(\iota_1, \iota_2) = (1, 2), (1, 3), (2, 4), (1, 4), (1, 5), (4, 5)$.

Proof. Obvious for (4,5); use 1.11(3). $\blacksquare_{2.5}$

Claim 2.6: *Assume $D_\ell \in \mathrm{FIL}_{cc}(Y_\ell)$ for $\ell = 1, 2$. If at least one of the following cases occurs, then D_2 does 1-commute (hence 2-commute) with D_1.*

Case 1: *D_2 is $|Y_1|^+$-complete.*
Case 2: *D_1 is an ultrafilter which is $|Y_2|^+$-complete.*
Case 3: *D_1, D_2 are ultrafilters and if $\bar{A} = \langle A_t : t \in Y_2 \rangle \in {}^{Y_2}(D_1)$ then for some $A_* \in D_1$ we have $\{t : A_t \supseteq A_*\} \in D_2$.*

Proof. So let $A \in D_1$ and $\langle B_s : s \in A \rangle \in {}^{A}(D_2)$ be given.

Case 1: Let $A_* = A$ and $B_* = \bigcap\{B_s : s \in A\}$, so $A_* \in D_1$ by an assumption, and $B_* \in D_2$ as we assume $\{B_s : s \in A\} \subseteq D_2$ and D_2 is $|Y_1|^+$-complete (and necessarily $|A| \le |Y_1|$).

Case 2: For each $t \in Y_2$ let $A'_t := \{s \in Y_1 : s \in A$ and $t \in B_s\}$ and let A''_t be the unique member of $\{A'_t, Y_1 \backslash A'_t\} \cap D_1$, recalling D_1 is an ultrafilter on Y_1. Clearly the functions $t \mapsto A'_t$ and $t \mapsto A''_t$ are well defined, hence the sequences $\langle A'_t : t \in Y_2 \rangle, \langle A''_t : t \in Y_2 \rangle$ exist and $\{A''_t : t \in Y_2\} \subseteq D_1$.

As D_1 is $|Y_2|^+$-complete, necessarily $A_* := \cap\{A''_t : t \in Y_2\} \cap A$ belongs to D_1, and clearly $A_* \subseteq A$. Let $B_* = \{t \in Y_2 : A''_t = A'_t\}$.

So now choose any $s_* \in A_*$ (possible as $A_* \in D_1$ implies $A_* \neq \emptyset$), so $B_{s_*} \in D_2$ and $t \in B_{s_*} \Rightarrow s_* \in A'_t \Rightarrow s_* \in A'_t \cap A_* \Rightarrow A'_t \cap A_* \neq \emptyset \Rightarrow A''_t = A'_t \Rightarrow t \in B_*$, so $B_{s_*} \subseteq B_*$; but $B_{s_*} \in D_2$, hence $B_* \in D_2$. So A_*, B_* are as required.

Case 3: Like Case 2. $\blacksquare_{2.6}$

Claim 2.7: *Assume* $\mathrm{AC}_{\mathcal{P}(Y_2)}$.

1) *Assume $D_1 \in \mathrm{FIL}_{cc}(Y_1)$ and $D_2 \in \mathrm{FIL}_{cc}(Y_2)$.*
Then D_2 does 3-commute with D_1 when D_1 is $(\le \mathcal{P}(Y_2))$-complete.
2) *In part (1), if $E \subseteq D_2$ is $(D_2, \subseteq)$-cofinal, it suffices to assume D_1 is $(\le E)$-complete.*

Remark 2.8: For part (1) in the definition of $(\le \mathcal{P}(Y_2))$-complete, we can use just partitions, but not so in part (2).

Proof. 1) So let $A \in D_1$ and $\bar{B} = \langle B_s : s \in A \rangle \in {}^{A}(D_2)$ and $J_1 = J[f_1, D_1]$ for some $f_1 \in {}^{Y}\mathrm{Ord}$ be given. So $s \mapsto B_s$ is a function from $A \in D_1$ to $D_2 \subseteq \mathcal{P}(Y_2)$, hence as $\mathrm{AC}_{\mathcal{P}(Y_2)}$ is assumed, recalling that by 1.11(1) the ideal J_1 on Y_1 is $(\le \mathcal{P}(Y_2))$-complete, there is $B_* \in D_2$ such that $A_* := \{s \in A : B_s = B_*\} \in J_1^+$. Clearly A_*, B_* are as required.

2) For $B \in E$ let $A^*_B = \{s \in A : B \subseteq B_s\}$, so clearly $\langle A^*_B : B \in E \rangle$ is a sequence of subsets of $A \in D_1$ with union A, so again by 1.11(1), for some, $B_* \in E$, the set $A_* := \{s \in A : B_* \subseteq B_s\}$ belongs to J_1^+, so we are done. $\blacksquare_{2.7}$

CLAIM 2.9: $\mathrm{rk}_{D_1}(f) \leq \mathrm{rk}_{D_2}(g)$ *when:*

$\oplus$ *as in 2.3,*

but we replace clause ($\boxplus$) *there by*

$\boxplus'$ (a) D_2 *does 4-commute with* D_1,
(b) AC_{Y_1} *holds.*

Proof. We repeat the proof of 2.3 but:

First change: we replace $(*)_5$ and the paragraph before it by the following: So $\bar{B} = \langle B_s : s \in A\rangle \in {}^A(D_2)$.

Recall that D_2 does 4-commute with D_1. Apply this to $A, \langle B_s : s \in A\rangle, \bar{J}^1 = \langle J^1_t : t \in Y_2\rangle$ where $J^1_t := J[g_t, D_1], J_2 := J[g, D_2]$, and we get A_*, B_* such that:

$(*)'_5$ (a) $A_* \in D_1^+$ and $A_* \subseteq A$,
(b) $B_* \in J_2^+$, hence $B_* \in D_2^+$ and $\mathrm{rk}_{D_2+B_*}(g) = \mathrm{rk}_{D_2}(g)$,
(c) $(s,t) \in A_* \times B_* \Rightarrow s \in A \wedge t \in B_s$,
(d) if $t \in B_*$, then $A_* \in (J^1_t)^+$, hence $t \in B_* \Rightarrow \mathrm{rk}_{D_1+A_*}(g_t) = \mathrm{rk}_{D_1}(g_t) = g(t)$.

Second change: we replace $(*)_{12}$ and the line before, the line after it and $(*)_{13}$ by:

Define $g' \in {}^{Y_2}\mathrm{Ord}$ by $g'(t) = \mathrm{rk}_{D_1+A_*}(g'_t)$.

Now:

$(*)'_{12}$ if $t \in B_*$ then
(a) $g'_t <_{D_1+A_*} g_t$ by $(*)_{11}$,
(b) $\mathrm{rk}_{D_1+A_*}(g'_t) < \mathrm{rk}_{D_1+A_*}(g_t)$ by (a) and 1.8(3),
(c) $\mathrm{rk}_{D_1+A_*}(g_t) = \mathrm{rk}_{D_1}(g_t)$ recalling $(*)'_5(d)$ and $J^1_t = J[g_t, D_1]$,
(d) $\mathrm{rk}_{D_1}(g_t) = g(t)$ by clause (d) of $\oplus$,
(e) $\mathrm{rk}_{D_1+A_*}(g_t) = g(t)$ by (c), (d) above, hence
(f) $g'(t) < g(t)$ by the choice of g', clause (b) and clause (e).

Hence by $(*)'_{12}(f)$ we have

$(*)'_{13}$ $g' < g \bmod D_2 + B_*$.

Concerning the rest, we quote $(*)_5$(b) twice but $(*)'_5$(b)= $(*)_5$(b), and quote $(*)_5$(a) twice but noted there that $(*)'_5$(a) suffice and g' is defined before $(*)'_{12}$ rather than apply $(*)_{12}$. $\blacksquare_{2.9}$

3. Rank systems and a relative of GCH

To phrase our theorem we need to define the framework.

Definition 3.1: Main Definition: We say that

$$\mathbf{p} = (\mathbb{D}, \mathrm{rk}, \Sigma, \mathbf{j}, \mu) = (\mathbb{D}_{\mathbf{p}}, \mathrm{rk}_{\mathbf{p}}, \Sigma_{\mathbf{p}}, \mathbf{j}_{\mathbf{p}}, \mu_{\mathbf{p}})$$

is a weak (rank) 1-system when:

(a) μ is singular;

(b) each $\mathbf{d} \in \mathbb{D}$ is (or we can just compute from it) a pair $(Y, D) = (Y_{\mathbf{d}}, D_{\mathbf{d}}) = (Y[\mathbf{d}], D_{\mathbf{d}}) = (Y_{\mathbf{p},\mathbf{d}}, D_{\mathbf{p},\mathbf{d}})$ such that:
  (α) $\theta(Y_{\mathbf{d}}) < \mu$, on $\theta(-)$, see Definition 1.5,
  (β) $D_{\mathbf{d}}$ is a filter on $Y_{\mathbf{d}}$;

(c) for each $\mathbf{d} \in \mathbb{D}$, a definition of a function $\mathrm{rk}_{\mathbf{d}}(-)$ with domain ${}^{Y[\mathbf{d}]}\mathrm{Ord}$ and range $\subseteq$ Ord, that is $\mathrm{rk}_{\mathbf{p},\mathbf{d}}(-)$ or $\mathrm{rk}^{\mathbf{p}}_{\mathbf{d}}(-)$;

(d) (α) Σ is a function with domain $\mathbb{D}$ such that $\Sigma(\mathbf{d}) \subseteq \mathbb{D}$,
  (β) if $\mathbf{d} \in \mathbb{D}$ and $\mathbf{e} \in \Sigma(\mathbf{d})$ then $Y_{\mathbf{e}} = Y_{\mathbf{d}}$ [natural to add $D_{\mathbf{d}} \subseteq D_{\mathbf{e}}$, this is not demanded but see 3.8(2)];

(e) (α) $\mathbf{j}$ is a function from $\mathbb{D}$ onto $\mathrm{cf}(\mu)$,
  (β) let $\mathbb{D}_{\geq i} = \{\mathbf{d} \in \mathbb{D} : \mathbf{j}(\mathbf{d}) \geq i\}$ and $\mathbb{D}_i = \mathbb{D}_{\geq i} \backslash \mathbb{D}_{i+1}$,
  (γ) $\mathbf{e} \in \Sigma(\mathbf{d}) \Rightarrow \mathbf{j}(\mathbf{e}) \geq \mathbf{j}(\mathbf{d})$;

(f) for every $\sigma < \mu$ for some $i < \mathrm{cf}(\mu)$, if $\mathbf{d} \in \mathbb{D}_{\geq i}$, then $\mathbf{d}$ is $(\mathbf{p}, \leq \sigma)$-complete where:
  ($*$) we say that $\mathbf{d}$ is $(\mathbf{p}, \leq X)$-complete (or $(\leq X)$-complete for $\mathbf{p}$) when: if $f \in {}^{Y[\mathbf{d}]}\mathrm{Ord}$ and $\zeta = \mathrm{rk}_{\mathbf{d}}(f)$ and $\langle A_j : j \in X\rangle$ a partition[1] of $Y_{\mathbf{d}}$, then for some $\mathbf{e} \in \Sigma(\mathbf{d})$ and $j \in X$ we have $A_j \in D_{\mathbf{e}}$ and $\zeta = \mathrm{rk}_{\mathbf{e}}(f)$; so this is not the same as "$D_{\mathbf{d}}$ is $(\leq X)$-complete"; we define $(\mathbf{p}, |X|^+)$-complete, i.e., $(\mathbf{p}, < |X|^+)$-complete, similarly;

(g) no hole[2]: if $\mathrm{rk}_{\mathbf{d}}(f) > \zeta$ then for some pair $(\mathbf{e}, g)$ we have: $\mathbf{e} \in \Sigma(\mathbf{d})$ and $g <_{D[\mathbf{e}]} f$ and $\mathrm{rk}_{\mathbf{e}}(g) = \zeta$;

(h) if $f = g + 1 \bmod D_{\mathbf{d}}$ then $\mathrm{rk}_{\mathbf{d}}(f) = \mathrm{rk}_{\mathbf{d}}(g) + 1$;

(i) if $f \leq g \bmod D_{\mathbf{d}}$ then $\mathrm{rk}_{\mathbf{d}}(f) \leq \mathrm{rk}_{\mathbf{d}}(g)$.

---

[1] As long as X is a well-ordered set, it does not matter whether we use a partition or just a covering, i.e., $\cup\{A_j : j \in X\} = Y_{\mathbf{d}}$.

[2] We may use another function Σ here, as in natural examples here we use $\Sigma(\mathbf{d}) = \{\mathbf{d}\}$ and not so in clause (f).

Definition 3.2: 1) We say $\mathbf{p} = (\mathbb{D}, \mathrm{rk}, \Sigma, \mathbf{j}, \mu)$ is a weak (rank) 2-system (if we write system we mean 2-system) when in 3.1 we replace clauses (d),(f),(g) by:

(d)′ (α) Σ is a function with domain $\mathbb{D}$,

(β) for $\mathbf{d} \in \mathbb{D}$ we have $\Sigma(\mathbf{d}) \subseteq \{(\mathbf{e}, h) : \mathbf{e} \in \mathbb{D}_{\geq \mathbf{j}(\mathbf{d})}$ and $h : Y_{\mathbf{e}} \to Y_{\mathbf{d}}\}$; writing $\mathbf{e} \in \Sigma(\mathbf{d})$ then means $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$ for some function h;

(f)′ for every $\sigma < \mu$ for some $i <$ cf(μ), if $\mathbf{d} \in \mathbb{D}_{\geq i}$, then $\mathbf{d}$ is $(\mathbf{p}, \leq \sigma)$-complete where:

($*$) we say that $\mathbf{d}$ is $(\mathbf{p}, \leq X)$-complete (for $\mathbf{p}$) when: if $f \in {}^{Y[\mathbf{d}]}\mathrm{Ord}$ and $\zeta = \mathrm{rk}_{\mathbf{d}}(f)$ and $\langle A_j : j \in X \rangle$ a partition[3] of $Y_{\mathbf{d}}$, then for some $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$ and $j \in X$ we have $h^{-1}(A_j) \in D_{\mathbf{e}}$ and $\zeta = \mathrm{rk}_{\mathbf{e}}(f \circ h)$; we define "$(\mathbf{p}, |X|^+)$-complete" similarly;

(g)′ no hole: if $\mathrm{rk}_{\mathbf{d}}(f) > \zeta$, then for some $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$ and $g \in {}^{Y[\mathbf{e}]}\mathrm{Ord}$ we have $g < f \circ h \bmod D_{\mathbf{e}}$ and $\mathrm{rk}_{\mathbf{e}}(g) = \zeta$.

Definition/Claim 3.3: Let $\mathbf{p}$ be a weak rank 1-system; we can define $\mathbf{q}$ and prove it is a weak rank 2-system by $\mathbb{D}_{\mathbf{q}} = \mathbb{D}_{\mathbf{p}}$, $\mathrm{rk}_{\mathbf{q}} = \mathrm{rk}_{\mathbf{p}}$, $\Sigma_{\mathbf{q}}(\mathbf{d}) = \{(\mathbf{e}, \mathrm{id}_{Y[\mathbf{d}]}) : \mathbf{e} \in \Sigma_{\mathbf{p}}(\mathbf{d})\}$, $\mathbf{j}_{\mathbf{q}} = \mathbf{j}_{\mathbf{p}}, \mu_{\mathbf{q}} = \mu_{\mathbf{p}}$.

Convention 3.4: 1) We use $\mathbf{p}$ only for systems as in Definition 3.1 or 3.2.

2) We may not distinguish $\mathbf{p}$ and $\mathbf{q}$ in 3.3 so deal only with 2-systems.

Remark 3.5: The following is an alternative to Definition 3.2. As in 3.1 we can demand $\mathbf{e} \in \Sigma(\mathbf{d}) \Rightarrow Y_{\mathbf{e}} = Y_{\mathbf{d}}$ but for every $\mathbf{d}$ we have a family $\mathcal{E}_{\mathbf{d}}$, i.e., the function $\mathbf{d} \mapsto \mathcal{E}_{\mathbf{d}}$ is part of $\mathbf{p}$ and make the following additions and changes:

(α) $\mathcal{E}_{\mathbf{d}}$ is a family of equivalence relations on $Y_{\mathbf{d}}$,

(β) we replace ${}^{Y[\mathbf{d}]}\mathrm{Ord}$ by $\{f \in {}^{Y[\mathbf{d}]}\mathrm{Ord}$: $\mathrm{eq}(f) := \{(s,t) : s,t \in Y_{\mathbf{d}}$ and $f(s) = f(t)\} \in \mathcal{E}_{\mathbf{d}}\}$,

(γ) if E_1, E_2 are equivalence relations on $Y_{\mathbf{d}}$ such that E_2 refines E_1 then $E_2 \in \mathcal{E}_{\mathbf{d}} \Rightarrow E_1 \in \mathcal{E}_{\mathbf{d}}$,

(δ) if $\mathbf{e} \in \Sigma(\mathbf{d})$ then $Y_{\mathbf{e}} = Y_{\mathbf{d}}$ and $\mathcal{E}_{\mathbf{d}} \subseteq \mathcal{E}_{\mathbf{e}}$.

Definition 3.6: For $\iota = 1, 2$ we say that $\mathbf{p} = (\mathbb{D}, \mathrm{rk}, \Sigma, \mathbf{j}, \mu)$ is a strict ι-system when it satisfies clauses (a)–(i) from 3.1 or from 3.2 and

[3] As long as X is a well-ordered set, it does not matter whether we use a partition or just a covering, i.e., $\cup\{A_j : j \in X\} = Y_{\mathbf{d}}$.

(j) for every $\mathbf{d} \in \mathbb{D}$ and ζ, ξ, f, j_0 satisfying $\boxplus$ below, there[4] is $j < \operatorname{cf}(\mu)$ such that: there are no $\mathbf{e}, g$ satisfying $\oplus$ below, where:

$\oplus$ $\bullet_1$ $\mathbf{e} \in \mathbb{D}_{\geq j}$,

$\bullet_2$ $g \in {}^{Y[\mathbf{e}]}\zeta$,

$\bullet_3$ $\{g(t) : t \in Y_{\mathbf{e}}\} \subseteq [\xi, \zeta_*)$ for some $\zeta_* < \zeta$,

$\bullet_4$ $j \geq j_0$,

$\bullet_5$ $\operatorname{rk}_{\mathbf{e}}(g) \geq \zeta$;

$\boxplus$ $\bullet_1$ $f \in {}^{Y[\mathbf{d}]}\xi$,

$\bullet_2$ $\operatorname{rk}_{\mathbf{d}}(f) = \zeta$,

$\bullet_3$ $\xi < \zeta$,

$\bullet_4$ $\operatorname{cf}(\mu) = \operatorname{cf}(\zeta)$,

$\bullet_5$ $j_0 < \operatorname{cf}(\mu)$,

$\bullet_6$ $s \in Y_{\mathbf{d}} \wedge \mathbf{e}' \in \mathbb{D}_{\geq j_0} \Rightarrow \operatorname{rk}_{\mathbf{e}'}(f(s)) = f(s)$.

Observation 3.7: 1) If $\mathbf{p}$ is a strict ι-system then $\mathbf{p}$ is a weak ι-system.

2) In Definition 3.6, from $(j) \boxplus \bullet_6$ recalling $(j) \oplus \bullet_1 + \bullet_4$ we can deduce: $\operatorname{rk}_{\mathbf{e}}(f(s)) = f(s)$ for $s \in Y_{\mathbf{d}}$.

3) In $\oplus \bullet_5$ of (j) of 3.6, without loss of generality, $\operatorname{rk}_{\mathbf{e}}(y) > \zeta + 7$ as we can use $g + 7$.

Definition 3.8: 1) We say that a weak ι-system $\mathbf{p}$ is weakly normal when:

- $\bullet_1$ in (d)(β) of Definition 3.1 we add $\mathbf{e} \in \Sigma(\mathbf{d}) \Rightarrow D_{\mathbf{d}} \subseteq D_{\mathbf{e}}$,
- $\bullet_2$ in (d)$'(\beta)$ of Definition 3.2 we add: if $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$ then $(\forall A \in D_{\mathbf{d}})(h^{-1}(A) \in D_{\mathbf{e}})$.

2) We say $\mathbf{p}$ is normal when it is weakly normal and

- $\bullet_3$ in Definition 3.1, if $A \in D^+_{\mathbf{d}}, \mathbf{d} \in \mathbb{D}, f \in {}^{Y[\mathbf{d}]}\mathrm{Ord}$ and $\zeta = \operatorname{rk}_{\mathbf{d}}(f)$ then for some $\mathbf{e} \in \Sigma(\mathbf{d})$ we have $A \in D_{\mathbf{e}}$ and $\operatorname{rk}_{\mathbf{e}}(f) = \zeta$,
- $\bullet_4$ in Definition 3.2, if $\mathbf{d} \in \mathbb{D}, f \in {}^{Y[\mathbf{d}]}\mathrm{Ord}$, $\operatorname{rk}_{\mathbf{d}}(f) = \zeta$ and $A \in D^+_{\mathbf{d}}$ then for some $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$ we have $\{s \in Y_{\mathbf{e}} : h(s) \in A\} \in D_{\mathbf{e}}$ and $\operatorname{rk}_{\mathbf{e}}(f \circ h) = \zeta$.

3) We say $\mathbf{p}$ is semi-normal when it is weakly normal and we have $\bullet'_3, \bullet'_4$ holds, where they are as above just ending with $\geq \zeta$.

Claim 3.9: *Assume $\mathbf{p}$ is a weak ι-system and $\mathbf{d} \in \mathbb{D}_{\mathbf{p}}$.*

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[4] Can we make j depend on f and (a partition of) $Y_{\mathbf{d}}$? Anyhow, later we use $\mathbf{d}' \in \Sigma(\mathbf{d})$, if $\{\mathbf{d}\} \neq \Sigma(\mathbf{d})$. Also, therefore $\iota = 1, 2$ may make a difference.

0) *If* $\mathbf{p}, \mathbf{q}$ *are as in 3.3, then* $\mathbf{p}$ *is [weakly] normal iff* $\mathbf{q}$ *is.*

1) *If* $f, g \in {}^{Y[\mathbf{d}]}\mathrm{Ord}$ *and* $f <_{D_\mathbf{d}} g$ *then* $\mathrm{rk}_\mathbf{d}(f) < \mathrm{rk}_\mathbf{d}(g)$.

2) *If* $f \in {}^{Y[\mathbf{d}]}\mathrm{Ord}$ *and* $\mathrm{rk}_\mathbf{d}(f) > 0$ *then* $\{s \in Y_\mathbf{d} : f(s) > 0\} \in D^+_\mathbf{d}$.

2A) *If, in addition,* $\mathbf{p}$ *is semi-normal then* $\{s \in Y_\mathbf{d} : f(s) = 0\} = \emptyset \bmod D_\mathbf{d}$.

3) $\mathrm{rk}_\mathbf{d}(f)$ *depends just on* $f/D_\mathbf{d}$ *(and* $\mathbf{d}$ *and, of course,* $\mathbf{p}$*).*

Proof. 0) Easy; note that by this part, below without loss of generality $\iota_\mathbf{p} = 2$.

1) Let $f_1 \in {}^{Y[\mathbf{d}]}\mathrm{Ord}$ be defined by $f_1(s) = f(s) + 1$. So clearly $f_1 \leq_{D_\mathbf{d}} g$, hence by clause (i) of 3.1 (equivalently 3.2) we have $\mathrm{rk}_\mathbf{d}(f_1) \leq \mathrm{rk}_\mathbf{d}(g)$. Also $f_1 = f + 1 \bmod D_\mathbf{d}$, hence by clause (h) of 3.1 (equivalently 3.2) we have $\mathrm{rk}_\mathbf{d}(f_1) = \mathrm{rk}_\mathbf{d}(f) + 1$. The last two sentences together give the desired conclusion.

2) Toward a contradiction assume the conclusion fails. Let $f' \in {}^{Y_\mathbf{d}}\mathrm{Ord}$ be constantly zero, so $f = f' \bmod D_\mathbf{d}$, hence by part (3) we have $\mathrm{rk}_\mathbf{d}(f') = \mathrm{rk}_\mathbf{d}(f) > 0$. By clause $(g)'$ of Definition 3.2, the "no hole" applied to $(f', \mathbf{d})$, there is a triple $(\mathbf{e}, h, g)$ as there, so $B := \{s : s \in Y_\mathbf{e} \text{ and } g(s) < f(h(s))\} \in D_\mathbf{e}$, i.e., $\{s \in Y_\mathbf{e} : g(s) < 0\} \in D_\mathbf{e}$, contradiction.

2A) Let $A = \{s \in Y_\mathbf{a} : f(s) = 0\}$, so toward a contradiction assume $A \in D^+_\mathbf{d}$. As $\mathbf{p}$ is semi-normal we can find a pair $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$ as in $\bullet'_4$ of 3.8(3), so $A_1 = \{t \in Y_\mathbf{e} : h_1(t) \in A\} \in D_\mathbf{e}$ and $\mathrm{rk}_{\mathbf{e}_1}(f \circ h) \geq \mathrm{rk}_\mathbf{d}(f) > 0$, but clearly $\{t \in Y_\mathbf{e} : (f \circ h)(t) = 0\} \in D_\mathbf{e}$, contradiction by part (2).

3) Use clause (i) of Definition 3.1 twice. $\blacksquare_{3.9}$

Theorem 3.10 (ZF): *Assume that* $\mathbf{p} = (\mathbb{D}, \mathrm{rk}, \Sigma, \mathbf{j}, \mu)$ *is a strict rank 1-system (see Main Definition 3.1) or just a strict 2-system. Then for every ordinal* ζ *there is* $i < \mathrm{cf}(\mu)$ *such that: if* $\mathbf{d} \in \mathbb{D}_{\geq i}$ *then* $\mathrm{rk}_\mathbf{d}(\zeta) = \zeta$*, i.e.,* $\mathrm{rk}_\mathbf{d}(\langle \zeta : s \in Y_\mathbf{d} \rangle) = \zeta$.

Proof. We shall use the notation:

$\odot_0$ If there is an i as required in the theorem for the ordinal ζ then let $\mathbf{i}(\zeta)$ be the minimal such i (otherwise, $\mathbf{i}(\zeta)$ is not well defined).

Without loss of generality,

$\odot_1$ every $\mathbf{d} \in \mathbb{D}_\mathbf{p}$ is $(\mathbf{p}, \leq (\mathrm{cf}(\mu)))$-complete, i.e., clause (f) of 3.1 for $\sigma_* := \mathrm{cf}(\mu)^+$ holds for every $\mathbf{d} \in \mathbb{D}$.

[Why? Let i_* be the $i <$ cf(μ) which exists by clause (f) of Definition 3.1, 3.2 for σ_*. Now we just replace $\mathbb{D}$ by $\mathbb{D}_{\geq i_*}$ (and $\mathbf{j}$ by $\mathbf{j} \restriction \mathbb{D}_{\geq i_*}$, etc).]

Clearly we have

$\odot_2$ $\text{rk}_{\mathbf{d}}(\zeta) \geq \zeta$ for ζ an ordinal and $\mathbf{d} \in \mathbb{D}$.

[Why? We can prove this by induction on ζ for all $\mathbf{d} \in \mathbb{D}$, by clauses (h) + (i) of Definition 3.1.]

As a warmup we shall note that:

$\odot_3$ if $\mathbf{d} \in \mathbb{D}$ and $\zeta < \sigma_*$ or just $\mathbf{d} \in \mathbb{D}$ and is hereditarily $(\mathbf{p}, \leq \zeta)$-complete, which means that every $\mathbf{e}$ in the Σ-closure of $\{\mathbf{d}\}$ is $(\mathbf{p}, \leq \zeta)$-complete, then:

(α) $\text{rk}_{\mathbf{d}}(\zeta) = \zeta$,

(β) $f \in {}^{Y[\mathbf{d}]}\zeta \Rightarrow \text{rk}_{\mathbf{d}}(f) < \zeta$.

[Why? Note that as $\zeta < \sigma_*$, clearly $\mathbf{d}$ is $(\mathbf{p}, \leq \zeta)$-complete by $\odot_1$ and clause (f) of 3.1, so we can assume that $\mathbf{d}$ is hereditarily $(\mathbf{p}, \leq \zeta)$-complete. We prove the statement inside $\odot_3$ by induction on the ordinal ζ (for all hereditarily $(\mathbf{p}, \leq \zeta)$-complete $\mathbf{d} \in \mathbb{D}$). Note that for $\varepsilon < \zeta$, "$\mathbf{d}$ is $(\mathbf{p}, \leq \zeta)$-complete" implies "$\mathbf{d}$ is $(\mathbf{p}, \leq \varepsilon)$-complete"; we shall use this freely.

Arriving at ζ, to prove clause (β) let $f \in {}^{Y[\mathbf{d}]}\zeta$ and for $\varepsilon < \zeta$ we define $A_\varepsilon := \{t \in Y_{\mathbf{d}} : f(t) = \varepsilon\}$, so $\langle A_\varepsilon : \varepsilon < \zeta \rangle$ is a well-defined partition of $Y_{\mathbf{d}}$ so the sequence exists, hence as "$\mathbf{d}$ is hereditarily $(\mathbf{p}, \leq \zeta)$-complete" recalling $(*)$ from clause (f)$'$ of 3.2 for some triple $(\mathbf{e}, h, \varepsilon)$ we have $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$ and $\varepsilon < \zeta$ and $h^{-1}(A_\varepsilon) \in D_{\mathbf{e}}$ and $\text{rk}_{\mathbf{e}}(f \circ h) = \text{rk}_{\mathbf{d}}(f)$.

Now $f \circ h = \langle \varepsilon : t \in Y_{\mathbf{e}} \rangle \bmod D_{\mathbf{e}}$, hence by Claim 3.9(3) we have $\text{rk}_{\mathbf{e}}(f \circ h) = \text{rk}_{\mathbf{e}}(\varepsilon)$. But the assumptions on $\mathbf{d}$ hold for $\mathbf{e}$, hence by the induction hypothesis on ζ we know that $\text{rk}_{\mathbf{e}}(\varepsilon) = \varepsilon$ and $\varepsilon < \zeta$ so together $\text{rk}_{\mathbf{d}}(f \circ h) < \zeta$, therefore clause ($\beta$) of $\odot_3$ holds.

To prove clause (α) first consider $\zeta = 0$; if $\text{rk}_{\mathbf{d}}(\zeta) > 0$ by clause (g) of Definition 3.1, 3.2 there are $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$ and $g \in {}^{Y[\mathbf{e}]}\text{Ord}$ such that $g < \langle \zeta : t \in Y_{\mathbf{e}} \rangle \bmod D_{\mathbf{e}}$, so for some $t \in Y_{\mathbf{e}}$ we have $g(t) < \zeta$, but $\zeta = 0$, contradiction; this is close to 3.9(2).

Second, consider $\zeta > 0$, so by $\odot_2$ we have $\text{rk}_{\mathbf{d}}(\zeta) \geq \zeta$ and assume toward a contradiction that $\text{rk}_{\mathbf{d}}(\zeta) > \zeta$, so by clause (g) of Definition 3.1, 3.2 there is a triple $(\mathbf{e}, h, g)$ as there. Now apply clause (β) of $\odot_3$ for ζ (which we have already proved) recalling $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$, so also $\mathbf{e}$ is $(\mathbf{p}, \leq \zeta)$-complete. We get $\text{rk}_{\mathbf{e}}(g) < \zeta$, a contradiction. So $\odot_3$ indeed holds.]

Now as in the desired equality we have already proved one inequality in $\odot_2$, we need to prove only the other inequality. We do it by induction on ζ.

Case 1: $\zeta < \mu$.

By clause (f) of Def. 3.1 and Def. 3.2 for some $i < \mathrm{cf}(\mu)$ we have $\mathbf{d} \in \mathbb{D} \wedge \mathbf{j}(\mathbf{d}) \geq i \Rightarrow \mathbf{d}$ is $(\mathbf{p}, \leq \zeta)$-complete, hence by $\odot_3(\alpha)$ we have $\mathrm{rk}_{\mathbf{d}}(\zeta) = \zeta$, as required.

CASE 2: $\zeta = \xi + 1$.

By clause (h) of Definition 3.1 we have $\mathbf{d} \in \mathbb{D} \Rightarrow \mathrm{rk}_{\mathbf{d}}(\zeta) = \mathrm{rk}_{\mathbf{d}}(\xi) + 1$. Hence $\mathbf{d} \in \mathbb{D}_{\geq \mathbf{i}(\xi)} \Rightarrow \mathrm{rk}_{\mathbf{d}}(\zeta) = \mathrm{rk}_{\mathbf{d}}(\xi) + 1 = \xi + 1 = \zeta$, so $\mathbf{i}(\xi)$ exemplifies that $\mathbf{i}(\zeta)$ exists and is $\leq \mathbf{i}(\xi)$ so we are done.

CASE 3: ζ is a limit ordinal $\geq \mu$ of cofinality $\neq \mathrm{cf}(\mu)$.

So for each $\xi < \zeta$ by the induction hypothesis $\mathbf{i}(\xi) < \mathrm{cf}(\mu)$ is well defined. For $i < \mathrm{cf}(\mu)$ let $u_i := \{\xi < \zeta : \mathbf{i}(\xi) \leq i\}$, so is well defined; moreover, the sequence $\langle u_i : i < \mathrm{cf}(\mu)\rangle$ exists and is $\subseteq$-increasing. If $i < \mathrm{cf}(\mu) \Rightarrow \sup(u_i) < \zeta$ then $\langle \sup(u_i) : i < \mathrm{cf}(\mu)\rangle$ is a $\leq$-increasing sequence of ordinals $< \zeta$ with limit ζ. So as $\mathrm{cf}(\zeta) \neq \mathrm{cf}(\mu)$ necessarily for some $i_* < \mathrm{cf}(\mu)$ the set $S := \{\xi : \xi < \zeta$ and $\mathbf{i}(\xi) < i_*\}$ is an unbounded subset of ζ. We shall prove that $\mathbf{i}(\zeta)$ is well defined and $\leq i_*$.

SUBCASE 3A: $\mathrm{cf}(\zeta) \geq \mu$.

Let $\mathbf{d} \in \mathbb{D}_{\geq i_*}$ and $g \in {}^{Y[\mathbf{d}]}\zeta$ be given. Clearly $\mathrm{Rang}(g)$ is a subset of ζ of cardinality $< \theta(Y_{\mathbf{d}})$ which, by clause (b)(α) of 3.1, is $< \mu \leq \mathrm{cf}(\zeta)$, hence we can fix $\xi \in S$ such that $\mathrm{Rang}(g) \subseteq \xi$, hence by clause (i) of 3.1, $\mathrm{rk}_{\mathbf{d}}(g) \leq \mathrm{rk}_{\mathbf{d}}(\xi)$ but $\mathbf{i}(\xi) = i_*$ and $\mathbf{d} \in \mathbb{D}_{\geq i_*}$, hence $\mathrm{rk}_{\mathbf{d}}(\xi) = \xi < \zeta$, so together $\mathrm{rk}_{\mathbf{d}}(g) < \zeta$. As this holds for every $\mathbf{d} \in \mathbb{D}_{\geq i_*}$, by the no-hole clause (g)$'$ and clause (e)(γ) of 3.2 it follows that $\mathbf{d} \in \mathbb{D}_{\geq i_*} \Rightarrow \mathrm{rk}_{\mathbf{d}}(\zeta) \leq \zeta$ as required.

SUBCASE 3B: $\mathrm{cf}(\zeta) < \mu$ (but still $\mathrm{cf}(\zeta) \neq \mathrm{cf}(\mu)$).

Let $\langle \zeta_\varepsilon : \varepsilon < \mathrm{cf}(\zeta)\rangle$ be an increasing sequence of ordinals from S with limit ζ. Now let $j_* < \mathrm{cf}(\mu)$ be such that $\mathbf{d} \in \mathbb{D}_{\geq j_*} \Rightarrow \mathbf{d}$ is $(\mathbf{p}, \mathrm{cf}(\zeta)^+)$-complete; see clause (f) of Definition 3.1.

Now assume $\mathbf{d} \in \mathbb{D}_{\geq \max\{i_*, j_*\}}$ and $g \in {}^{Y[\mathbf{d}]}\zeta$. For $\varepsilon < \mathrm{cf}(\zeta)$ let $A_\varepsilon = \{t \in Y_{\mathbf{d}} : g(t) < \zeta_\varepsilon$ but $j < \varepsilon \Rightarrow g(t) \geq \zeta_j\}$, so $\langle A_\varepsilon : \varepsilon < \mathrm{cf}(\zeta)\rangle$ is well defined and is a partition of $Y_{\mathbf{d}}$. Hence by clause (f) of Definition 3.2 for some $\varepsilon < \mathrm{cf}(\zeta)$ and $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$ we have $h^{-1}(A_\varepsilon) \in D_{\mathbf{e}}$ and $\mathrm{rk}_{\mathbf{d}}(g) = \mathrm{rk}_{\mathbf{e}}(g \circ h)$; but $\mathbf{j}(\mathbf{e}) \geq \mathbf{j}(\mathbf{d}) \geq i_*, j_*$ and, by the choice of A_ε and clause (i) of 3.1, the latter is $\leq \mathrm{rk}_{\mathbf{e}}(\zeta_\varepsilon)$, hence as $\mathbf{i}(\zeta_\varepsilon) \leq i_*$ the latter is $= \zeta_\varepsilon < \zeta$. As this holds for every $\mathbf{d} \in \mathbb{D}_{\geq \max\{i_*, j_*\}}$ and $g \in {}^{Y[\mathbf{d}]}\zeta$, by the no-hole clause (g)$'$ of 3.2 necessarily $\mathrm{rk}_{\mathbf{d}}(\zeta) \leq \zeta$. Hence $\max\{i_*, j_*\} < \mathrm{cf}(\mu)$ is as required, so we are done.

CASE 4: $\zeta \geq \mu$ is a limit ordinal such that $\mathrm{cf}(\zeta) = \mathrm{cf}(\mu)$.

Let $\langle \zeta_i : i < \text{cf}(\zeta) \rangle$ be increasing with limit ζ. Assume toward a contradiction that for every $i < \text{cf}(\mu)$ there is $\mathbf{d}_i \in \mathbb{D}_{\geq i}$ such that $\text{rk}_{\mathbf{d}_i}(\zeta) > \zeta$ but we do not assume that such a sequence $\langle \mathbf{d}_i : i < \text{cf}(\mu) \rangle$ exists. Choose such $\mathbf{d}_0$; as $\text{rk}_{\mathbf{d}_0}(\zeta) > \zeta$, clearly there are $f_0 \in {}^{Y[\mathbf{d}'_0]}\zeta$ and a member $\mathbf{d}'_0$ of $\Sigma(\mathbf{d}_0)$, though not necessarily $Y_{\mathbf{d}'_0} = Y_{\mathbf{d}}$, such that

$\odot_4$ $\text{rk}_{\mathbf{d}'_0}(f_0) = \zeta$.

[Why? By using clause (g)$'$ of 3.2.]

Note

$\odot_5$ $\mathbf{i}(f_0(t))$ is well defined for every $t \in Y_{\mathbf{d}'_0}$.

[Why does it hold? Because $f_0(t) < \zeta$ and the induction hypothesis.]

For $j_1 < \text{cf}(\zeta), j_0 < \text{cf}(\mu)$ let $A_{j_1,j_0} = \{t \in Y_{\mathbf{d}'_0} : f_0(t) < \zeta_{j_1}$ and $(\forall j < j_1)(f_0(t) \geq \zeta_j)$ and $\mathbf{i}(f_0(t)) = j_0\}$. By clause (f)$'$ of 3.2 applied to the pair $(\mathbf{d}'_0, f_0)$ and the partition $\langle A_{j_1,j_0} : j_1 < \text{cf}(\zeta), j_0 < \text{cf}(\mu) \rangle$, for some $(\mathbf{d}_*, h_*) \in \Sigma(\mathbf{d}'_0)$ we have $\text{rk}_{\mathbf{d}_*}(f_0 \circ h_*) = \zeta$ and for some j_1, j_0 we have $h_*^{-1}(A_{j_1,j_0}) \in D_{\mathbf{d}_*}$. By 3.9(3) for some $f = f_0 \circ h_*$ mod $D_{\mathbf{d}_*}$ and letting $\mathbf{d} := \mathbf{d}_*$ we have

$\odot_6$ (a) $\mathbf{d} \in \mathbb{D}$,

(b) $f \in {}^{Y[\mathbf{d}]}\text{Ord}$,

(c) $\text{rk}_{\mathbf{d}}(f) = \zeta$,

(d) $t \in Y_{\mathbf{d}} \Rightarrow \mathbf{i}(f(t)) = j_0 \wedge f(t) < \zeta_{j_1} \wedge (\forall j < j_1)(f(t) \geq \zeta_j)$.

Next

$\odot_7$ letting $\xi := \zeta_{j_1}$, clause $\boxplus$ from 3.6 for $(\mathbf{d}, \zeta, \xi, f, j_0)$.

[Why? We check the six demands:

- $\bullet_1$ "$f \in {}^{Y[\mathbf{d}]}\xi$" which holds by $\odot_6$(b) + (d),
- $\bullet_2$ "$\text{rk}_{\mathbf{d}}(f) = \zeta$" which holds by $\odot_6$(c),
- $\bullet_3$ "$\xi < \zeta$" which holds as $(\forall i < \text{cf}(\mu))(\zeta_i < \zeta)$,
- $\bullet_4$ "$\text{cf}(\zeta) = \text{cf}(\mu)$" which holds by the case assumption,
- $\bullet_5$ $j_0 < \text{cf}(\mu)$ obvious,
- $\bullet_6$ $s \in Y_{\mathbf{d}} \wedge \mathbf{e}' \in \mathbb{D}_{\geq j_0} \Rightarrow \text{rk}_{\mathbf{e}'}(f(s)) = f(s)$ holds by $\odot_6$(d).

So $\odot_7$ indeed holds.]

Now by $\odot_7$, clause (j) of Definition 3.6(1) applied with $\mathbf{d}, \zeta, \xi = \zeta_{j_1}, f, j_0$ here standing for $\mathbf{d}, \zeta, \xi, f, j_0$ there, we can find j as there. Let $i_2 = \max\{j, j_1, j_0, \mathbf{i}(\zeta_{j_1})\}$ so $i_2 < \text{cf}(\mu)$, and choose $\mathbf{e}_0 \in \mathbb{D}_{\geq i_2}$ such that $\text{rk}_{\mathbf{e}_0}(\zeta) > \zeta$ as in the beginning of the case. As $\text{rk}_{\mathbf{e}_0}(\zeta) > \zeta$ by clause (g)$'$ of 3.2 there are $\mathbf{e}_1 \in \Sigma(\mathbf{e}_0)$ and

$g \in {}^{Y[\mathbf{e}_1]}\zeta$ such that $\text{rk}_{\mathbf{e}_1}(g) \geq \zeta$, so $g < \langle \zeta : t \in Y_{\mathbf{e}_1} \rangle$. Now, without loss of generality,

$\odot_8$ (a) $\text{rk}_{\mathbf{e}_1}(g) = \zeta + 1$,
(b) $\zeta_* = \sup\{g(t) + 1 : t \in Y_{\mathbf{e}_1}\} < \zeta$,
(c) $\mathbf{j}(\mathbf{e}_1) \geq i_2$.

[Why? Because we can use $g' \in {}^{Y[\mathbf{e}_1]}\zeta$ defined by $g'(t) = g(t) + 2$ for $t \in Y_{\mathbf{e}_1}$, by clause (h) of 3.1, $\text{rk}_{\mathbf{e}_1}(g') = \text{rk}_{\mathbf{e}_1}(g) + 2 > \zeta$. By clause (e)($\gamma$) we have $\mathbf{j}(\mathbf{e}_1) \geq \mathbf{j}(\mathbf{e}_0) \geq i_2$. Now we find $(\mathbf{d}''_2, h'') \in \Sigma(\mathbf{e}_1)$ and g_2 as in the proof of $\odot_6$ and rename.]

Also, without loss of generality,

$\odot_9$ $t \in Y_{\mathbf{e}_1} \Rightarrow g(t) \geq \zeta_{j_1}$.

[Why? Let $A_0 = \{t \in Y_{\mathbf{e}_1} : g(t) < \zeta_{j_1}\}$, $A_1 = \{t \in Y_{\mathbf{e}_1} : g(t) \geq \zeta_{j_1}\}$, so by clause (f)$'$ of 3.2 for some pair $(\mathbf{e}_2, h) \in \Sigma(\mathbf{e}_1)$ we have $\text{rk}_{\mathbf{e}_2}(g \circ h) = \text{rk}_{\mathbf{e}_1}(g) = \zeta + 1$ and $(h^{-1}(A_0) \in D_{\mathbf{e}_2}) \vee (h^{-1}(A_1) \in D_{\mathbf{e}_2})$. So if $h^{-1}(A_0) \in D_{\mathbf{e}_2}$ then, by clause (i) of 3.1, $\text{rk}_{\mathbf{e}_1}(g \circ h) \leq \text{rk}_{\mathbf{e}_2}(\xi)$; but $\mathbf{i}(\xi)$ is well defined $\leq i_2 \leq \mathbf{j}(\mathbf{e}_1) \leq \mathbf{j}(\mathbf{e}_2)$ so $\text{rk}_{\mathbf{e}_2}(\xi) = \xi$ together $\text{rk}_{\mathbf{e}_2}(g \circ h) \leq \xi$, contradicting the previous sentence. Hence $h^{-1}(A_0) \notin D_{\mathbf{e}_2}$ so $h^{-1}(A_1) \in D_{\mathbf{e}_2}$. Let $g' \in {}^{Y[\mathbf{e}_2]}\text{Ord}$ be defined by: $g'(t)$ is $(g \circ h)(t)$ if $t \in h^{-1}(A_1)$ and is $\zeta_{j_1} + 1$ if $t \in h^{-1}(A_0)$. By Claim 3.9(3) we have $\text{rk}_{\mathbf{e}_2}(g') = \text{rk}_{\mathbf{e}_2}(g \circ h)$, so $(\mathbf{e}_2, g')$ satisfies all requirements on the pair $(\mathbf{e}_1, g)$ and $t \in Y_{\mathbf{e}_2} \Rightarrow g'(t) \geq \zeta_{j_1} > 0$, so we have justified the non-loss of generality.]

Recall $\xi := \zeta_{j_1}$ and let $\mathbf{e} = \mathbf{e}_1$. By the choice of j after $\odot_6$, i.e., as in clause (j) of 3.6, recalling $\mathbf{e} \in \mathbb{D}_{\geq j}$ we shall get a contradiction to the choice of $(\mathbf{d}, \xi, \zeta, f, j_0, \mathbf{e}, g, j)$. To justify it we have to recall by $\odot_7$ that the quintuple $(\mathbf{d}, \zeta, \xi, f, j_0)$ satisfies $\boxplus$ of 3.6(j) and then we prove that the triple $(\mathbf{e}, g, j)$ satisfies $\oplus$ of 3.6(j).

Now $\oplus$ of 3.6 says:

$\bullet_1$ "$\mathbf{e} \in \mathbb{D}_{\geq j}$"
as $j \geq i_2, \mathbf{e}_0 \in \mathbb{D}_{\geq i_2}$ and $\mathbf{e} = \mathbf{e}_1 \in \Sigma(\mathbf{e}_0)$,

$\bullet_2$ "$g \in {}^{Y[\mathbf{e}]}\zeta$"
which holds as $g \in {}^{Y[\mathbf{e}]}\zeta$,

$\bullet_3$ "$g(t) \in [\xi, \zeta_*)$"
holds as $g(t) < \zeta$ by $\bullet_2 + \odot_8$(b) and $g(t) \geq \zeta_{j_1} = \xi$ by $\odot_9$,

$\bullet_4$ "$j \geq j_0$"
holds as $j \geq i_2 \geq j_0$,

$\bullet_5$ "$\text{rk}_{\mathbf{e}}(g) > \zeta$"

holds by $\odot_8$(a).

So we really get a contradiction. $\blacksquare_{3.10}$

Definition 3.11: 1) We say that the pair $(\mathbf{d},\mathbf{e})$ commute (or 6-commute) for $\mathbf{p}$ when $\mathbf{d},\mathbf{e} \in \mathbb{D}_{\mathbf{p}}$ and $\mathrm{rk}_{\mathbf{d}}(f) \ge \mathrm{rk}_{\mathbf{e}}(g)$ whenever $(f,g,\bar f,\bar g)$ is a $(\mathbf{p},\mathbf{d},\mathbf{e})$-rectangle, see below; fixing f,g we may say $(\mathbf{d},\mathbf{e})$ commute for f,g.

2) We say that $(\mathbf{d},\mathbf{e},f,g,\bar f,\bar g)$ is $\mathbf{p}$-rectangle or $(f,g,\bar f,\bar g)$ is a $(\mathbf{p},\mathbf{d},\mathbf{e})$-rectangle when:

$\circledast$ (a) $\mathbf{d} \in \mathbb{D}_{\mathbf{p}}$,
(b) $\mathbf{e} \in \mathbb{D}_{\mathbf{p}}$,
(c) $\bar g = \langle g_t : t \in Y_{\mathbf{e}}\rangle$ and $g_t \in {}^{Y[\mathbf{d}]}\mathrm{Ord}$ for $t \in Y_{\mathbf{e}}$,
(d) $g \in {}^{Y[\mathbf{e}]}\mathrm{Ord}$ is defined by $g(t) = \mathrm{rk}_{\mathbf{d}}(g_t)$,
(e) $f_s \in {}^{Y[\mathbf{e}]}\mathrm{Ord}$ is defined by $f_s(t) = g_t(s)$,
(f) $\bar f = \langle f_s : s \in Y[\mathbf{d}]\rangle$,
(g) $f \in {}^{Y[\mathbf{d}]}\mathrm{Ord}$ is defined by $f(s) = \mathrm{rk}_{\mathbf{e}}(f_s)$.

Claim 3.12 (Assume ZF + $\mathrm{AC}_{<\mu}$): *If $\mathbf{p} = (\mathbb{D},\mathrm{rk},\Sigma,\mathbf{i},\mu)$ is a weak rank 1-system then $\mathbf{p}$ is a strict rank 1-system when there is a function Σ_1 such that (and we may say Σ_1 witnesses it):*

$(*)_0$ *Σ_1 is a function with domain $\mathbb{D}$,*
$(*)_1$ *$\Sigma_1(\mathbf{d}) \subseteq \Sigma(\mathbf{d})$ is non-empty for $\mathbf{d} \in \mathbb{D}$,*
$(*)_2$ *for every $\mathbf{d},\zeta,\xi,f,j_0$ satisfying $\boxplus$ of 3.6, for some $j < \mathrm{cf}(\mu)$ for every $\mathbf{e} \in \mathbb{D}_{\ge j}$ we have*
(a) *$\mathbf{e}$ is $(\mathbf{p}, \le \Sigma_1(\mathbf{d}))$-complete,*
(b) *if $\mathbf{d}_* \in \Sigma_1(\mathbf{d}), \mathbf{e}_* \in \Sigma_1(\mathbf{e})$ then $(\mathbf{d}_*,\mathbf{e}_*)$ commute (for $\mathbf{p}$), see 3.11, at least for any $f' \in {}^{Y[\mathbf{d}]}\zeta$ and any $g' \in {}^{Y[\mathbf{e}]}[\xi,\zeta)$,*
$(*)_3$ *we strengthen clause* (g) *of Definition 3.1 to*
(g)$^+$ *add:* $\mathrm{rk}_{\mathbf{e}}(f) = \mathrm{rk}_{\mathbf{d}}(f)$ *and* $\mathbf{e} \in \Sigma_1(\mathbf{d})$,
$(*)_4$ $\mathrm{AC}_{Y[\mathbf{d}]}$ *and* $\mathrm{AC}_{\Sigma_1(\mathbf{d})}$ *whenever* $\mathbf{d} \in \mathbb{D}$.

Remark 3.13: 1) In $(*)_2$, can we make j depend on f and a partition of $Y_{\mathbf{d}}$? It will be somewhat better.

2) We can similarly prove this for a weak rank 2-system. It is natural, though not necessary, to add $(\mathbf{e},h) \in \Sigma_1(\mathbf{d}) \Rightarrow Y_{\mathbf{e}} = Y_{\mathbf{d}} \wedge h = \mathrm{id}_{Y_{\mathbf{d}}}$.

3) We can add $\Sigma_1(\mathbf{d}) = \{\mathbf{d}\}$, so $(g)^+$ means $\mathbf{e} = \mathbf{d}$, and $(*)_2$(a) and $AC_{\Sigma_1(\mathbf{d})}$ in $(*)_4$ are trivial.

Proof. Let $\mathbf{d}, \zeta, \xi, f, j_0$ satisfying $\boxplus$ of 3.6(j) be given, and we should find $j < \mathrm{cf}(\mu)$ such that for no pair $(\mathbf{e}, g)$ clause $\oplus$ there holds. By 3.9(2) and 3(f) without loss of generality, $s \in Y_{\mathbf{d}} \Rightarrow f(s) > 0$.

Let $j < \mathrm{cf}(\mu)$ be as in $(*)_2$ in the claim and, without loss of generality, $j > j_0$ and we shall prove that j is as required in clause (j) of Definition 3.6; this is enough. So assume $\mathbf{e} \in \mathbb{D}_{\geq j}, g \in {}^{Y[\mathbf{e}]}[\xi, \zeta_*), \xi < \zeta_* < \zeta$ and, toward a contradiction, $(j, \zeta, \xi, \mathbf{e}, g)$ satisfy $\oplus$ there. For each $t \in Y_{\mathbf{e}}$ clearly $g(t) < \zeta = \mathrm{rk}_{\mathbf{d}}(f)$, hence by clause $(\mathrm{g})^+$ of $(*)_3$, see (g) of Definition 3.1, "no hole", there are $g_t \in {}^{Y[\mathbf{d}]}\xi$ and $\mathbf{d}_t \in \Sigma_1(\mathbf{d})$ such that $g_t <_{D_{\mathbf{d}_t}} f$ and $\mathrm{rk}_{\mathbf{d}_t}(g_t) = g(t)$, without loss of generality $g_t < \max(f, 1_{Y[\mathbf{d}]}) = f$ and, by the $(*)_3$, "we add" also $\mathrm{rk}_{\mathbf{d}_t}(f) = \mathrm{rk}_{\mathbf{d}}(f)$.

As $\mathrm{AC}_{Y_{\mathbf{e}}}$ by $(*)_4$, we can choose such a sequence $\langle (g_t, \mathbf{d}_t) : t \in Y_{\mathbf{e}} \rangle$. Now $\mathbf{e}$ is $(\mathbf{p}, \leq \Sigma_1(\mathbf{d}))$-complete and $(\mathbf{d}, \mathbf{e})$ commute for $\mathbf{p}$, by clauses (a),(b) respectively of $(*)_2$ (i.e., by the choice of j and as $\mathbf{e} \in \mathbb{D}_{\geq j}$), hence we can find $\mathbf{e}_* \in \Sigma_1(\mathbf{e})$ and $\mathbf{d}_* \in \Sigma_1(\mathbf{d})$ such that $\mathrm{rk}_{\mathbf{e}_*}(g) = \mathrm{rk}_{\mathbf{e}}(g) = \zeta$ and $\{t \in Y_{\mathbf{e}} : \mathbf{d}_t = \mathbf{d}_*\}$ belongs to $D_{\mathbf{e}_*}$. For $s \in Y_{\mathbf{d}} = Y_{\mathbf{d}_*}$ let $f_s \in {}^{Y[\mathbf{e}_*]}\mathrm{Ord}$ be defined by $f_s(t) = g_t(s)$ so $f_s(t) = g_t(s) < \xi$ and let $f' \in {}^{Y[\mathbf{d}_*]}\mathrm{Ord}$ be defined by $f'(s) = \mathrm{rk}_{\mathbf{e}_*}(f_s)$ and let $\bar{f} = \langle f_s : s \in Y_{\mathbf{d}_*} \rangle$.

Fixing $s \in Y_{\mathbf{d}_*}$ we have $t \in Y_{\mathbf{e}_*} \Rightarrow f_s(t) = g_t(s) < \mathrm{Max}\{f(s), 1\} = f(s)$, i.e., $f_s < \langle f(s) : t \in Y_{\mathbf{e}_*} \rangle$ hence $\mathrm{rk}_{\mathbf{e}_*}(f_s) < \mathrm{rk}_{\mathbf{e}_*}(f(s))$. Now by $\boxplus \bullet_6$ from 3.6, as $j_0 \leq j \leq \mathbf{j}(\mathbf{e}_*)$ we have $s \in Y_{\mathbf{e}_*} \Rightarrow \mathrm{rk}_{\mathbf{e}_*}(f(s)) = f(s)$, so $s \in Y_{\mathbf{e}_*} \Rightarrow \mathrm{rk}_{\mathbf{e}_*}(f_s) < f(s)$, i.e., $f' < f$.

Clearly $(f', g, \bar{f}, \bar{g})$ is a $(\mathbf{p}, \mathbf{d}_*, \mathbf{e}_*)$-rectangle, hence by clause (b) of $(*)_2$ of the assumptions, i.e., the choice of $(\mathbf{e}, g)$ and Definition 3.11(2), we know that $\mathrm{rk}_{\mathbf{d}_*}(f') \geq \mathrm{rk}_{\mathbf{e}_*}(g)$.

But recall that $\mathrm{rk}_{\mathbf{e}_*}(g) = \mathrm{rk}_{\mathbf{e}}(g)$ by the choice of $\mathbf{e}_*$. We get a contradiction by

$$(*) \quad \zeta = \mathrm{rk}_{\mathbf{d}}(f) = \mathrm{rk}_{\mathbf{d}_*}(f) > \mathrm{rk}_{\mathbf{d}_*}(f') \geq \mathrm{rk}_{\mathbf{e}_*}(g) = \mathrm{rk}_{\mathbf{e}}(g) \geq \zeta.$$

[Why those inequalities? By $\bullet_2$ of $\boxplus$ from 3.6 we are assuming; as $\mathbf{d}_* \in \{\mathbf{d}_t : t \in Y_{\mathbf{e}}\}$ and the choice of the $\mathbf{d}_t$'s; as $f' <_{D_{\mathbf{d}_*}} f$ and 3.9(3); by an inequality above; by the choice of $\mathbf{e}_*$; by $\bullet_5$ of $\oplus$ of 3.6.] $\blacksquare_{3.12}$

4. Finding systems

4A. Building weak rank systems and measurable.

Claim 4.1 (ZF + DC and $\mathrm{AC}_{<\mu}$): *If* $\circledast_1$ *holds and* $\mathbf{p}_{\kappa^-,\theta^*} = \mathbf{p}_{\bar\kappa} = \mathbf{p} = (\mathbb{D}, \mathrm{rk}, \Sigma, \mathbf{j}, \mu)$ *is defined in* $\circledast_2$ *then* $\mathbf{p}$ *is a weak rank 1-system, even semi-normal (and (g)*$^+$ *of 3.12 holds) where:*

$\circledast_1$ (a) $\bar\kappa = \langle \kappa_i : i < \partial\rangle$ *is an increasing sequence of regular cardinals* $> \partial = \mathrm{cf}(\partial)$ *with limit* μ *such that if* $i < \partial$ *is a limit ordinal then* $\kappa_i = (\Sigma\{\kappa_j : j < i\})^+$,

(b) θ^* *is a cardinal or* ∞;

$\circledast_2$ (a) $\mathbb{D}_i = \{J : J$ *is a* κ_i*-complete ideal on some* $\kappa = \kappa_J < \mu$ *including* $[\kappa]^{<\kappa}$ *and satisfying* $\mathrm{cf}(J, \leq) < \theta^*$ *(and if* $\theta^* = \infty$ *we stipulate this as the empty demand) such that* $\beta < \kappa \Rightarrow \{\beta\} \in J\}$ *and let* $\mathbb{D} = \mathbb{D}_0$,

(b) *if* $\mathbf{d} = J \in \mathbb{D}_i, J$ *an ideal on* $\kappa_J := \cup\{A : A \in J\}$, *then we let* $Y_{\mathbf{d}} = \kappa_J$ *and* $D_{\mathbf{d}}$ *be the filter dual to the ideal* J,

(c) $\mathbf{j}(J) = \min\{i : J$ *is not* κ^+_{i+1}*-complete*$\}$,

(d) $\Sigma(J) = \{J + B : B \supseteq A$ *and* $\kappa_J \backslash B$ *is not in* $J\}$,

(e) $\mathrm{rk}_J(f)$ *is as in Definition 1.6.*

Proof. So we have to check all the clauses in Definition 3.1.

Clause (a): As $\mu = \Sigma\{\kappa_i : i < \partial\}$, the sequence $\langle\kappa_i : i < \partial\rangle$ is increasing and $\kappa_0 > \partial$ (all by $\circledast_1$); clearly μ is a singular cardinal (and $\partial = \mathrm{cf}(\mu)$).

Clause (b): Let $\mathbf{d} \in \mathbb{D}$, so $\mathbf{d} = J$.

Subclause (α): So $Y_{\mathbf{d}} = \kappa_J < \mu$, hence $\theta(Y_{\mathbf{d}}) = \theta(\kappa_J) = \kappa_J^+ < \mu$ recalling μ is a limit cardinal and the definition of $\mathbb{D} = \mathbb{D}_0$ in clause (a) of $\circledast_2$.

Subclause (β): Also obvious.

Clause (c): For $f \in {}^{(\kappa_{\mathbf{d}})}\mathrm{Ord}$, $\mathrm{rk}_{\mathbf{d}}(f)$, as defined in $\circledast_2(e)$, is an ordinal recalling Claim 1.8(1).

Clause (d)(α): Trivial.

Clause (d)(β): Trivially $\mathbf{e} \in \Sigma(\mathbf{d}) \Rightarrow Y_{\mathbf{e}} = Y_{\mathbf{d}} \wedge D_{\mathbf{e}} \supseteq D_{\mathbf{d}}$; so "$\mathbf{p}$ is weakly normal", see Definition 3.8. Moreover, "$\mathbf{p}$ is semi-normal" as $\mathrm{rk}_D(f) \leq \mathrm{rk}_{D+A}(f)$ for $A \in D^+$.

Clause (e): Obvious from the definitions.

Clause (f): Let $\sigma < \mu$ be given and choose $i < \partial$ such that $\sigma < \kappa_i$. Let $\mathbf{d} \in \mathbb{D}$ be such that $j = \mathbf{j}(\mathbf{d}) \geq i$, hence $D = D_{\mathbf{d}}$ is a filter on some κ_J, so assume $\bigcup\{A_\varepsilon : \varepsilon < \varepsilon^*\} = \kappa_J$ and $\varepsilon^* < \kappa_i$. Now D is κ_i-complete and (see 1.9(2) as

AC_{κ_J} is assumed) we have $\mathrm{rk}_{D_{\mathbf{d}}}(f) = \min\{\mathrm{rk}_{D+A_\varepsilon}(f) : \varepsilon < \varepsilon^* \text{ and } A_\varepsilon \in D^+_{\mathbf{d}}\}$, which is what is needed as $A_\varepsilon \in D^+_{\mathbf{d}} \Rightarrow \mathbf{d} + (\kappa_j \backslash A_\varepsilon) \in \Sigma(\mathbf{d})$.

Clause (g): By 1.8(2). Moreover, the stronger version with $\mathbf{e} = \mathbf{d}$ holds so, in particular, $(g)^+$ of 3.12 holds.

Clause (h): Easy. On the one hand, as $g < f$, the definition of $\mathrm{rk}_{\mathbf{d}}(f)$, we have $\mathrm{rk}_{\mathbf{d}}(f) \geq \mathrm{rk}_{\mathbf{d}}(g) + 1$. On the other hand, if $g' < f \bmod D_{\mathbf{d}}$ then $g' \leq g \bmod D$, hence by clause (i) below we have $\mathrm{rk}_{\mathbf{d}}(g') \leq \mathrm{rk}_{\mathbf{d}}(g) < \mathrm{rk}_{\mathbf{d}}(g) + 1$; as this holds for every $g' < f \bmod D_{\mathbf{d}}$ we have $\mathrm{rk}_{\mathbf{d}}(f) \leq \mathrm{rk}_{\mathbf{d}}(g) + 1$. Together we are done.

Clause (i): Obvious. $\blacksquare_{4.1}$

Discussion 4.2: Assume μ is a singular cardinal, $\mu = \sum_{i<\kappa} \mu_i, \kappa = \mathrm{cf}(\kappa) < \mu_i < \mu$ and μ_i is increasing with i. Assume that for each i there is a pair (D, Y); D is a μ_i-complete ultra-filter on $Y, \theta(Y) < \mu$. This seems to be a good case, but either we have "D is a $(\leq \theta(Y))$-complete" so ${}^Y\mathrm{Ord}/D$ is "dull", or $\theta(Y) > \kappa =$ completeness(D) and so there is a κ-complete non-principal ultrafilter on κ and on $\kappa < \mu$, so $\mu = \sup(\text{measurables} \cap \mu)$.

Claim 4.3 ($ZF + DC + AC_{<\mu}$): *Assume μ is singular and $\mu = \sup(\mu \cap$ the class of measurable cardinals) (equivalently, for every $\kappa < \mu$, there is a κ-complete non-principal ultrafilter on some $\kappa' < \mu$). Let $\bar\kappa = \langle \kappa_i : i < \mathrm{cf}(\mu) \rangle$ be increasing with limit $\mu, \kappa_i > \mathrm{cf}(\mu)$ such that, for i limit, $\kappa_i = (\Sigma\{\kappa_j : j < i\})^+$ and κ_i is measurable for i non-limit.*

Then $\mathbf{p} = \mathbf{p}^{\mathrm{uf}}_{\bar\kappa}$ is a strict rank 1-system where $\mathbf{p}$ is defined by:

⊛ (a) $\mathbb{D}_{\geq i} = \{J$: *dual* (J) *is a non-principal ultra-filter which is* κ_i-*complete on some* $\kappa = \kappa_J < \mu\}$, *so naturally* $Y_J = \kappa_J$ *and* $D_J = \mathrm{dual}(J)$,

(b) $\mathbf{j}(J) = \min\{i : J$ *is not* κ^+_{i+1}-*complete*$\}$, *well defined,*

(c) $\Sigma(J) = \{J\}$,

(d) $\mathrm{rk}_J(f) = \mathrm{rk}_{\mathrm{dual}(J)}(f)$ *as in 1.6.*

Proof. We can check clauses (a)–(i) of 3.1 as in the proof of 4.1.

We still have to prove the "strict", i.e., we should prove clause (j) from Definition 3.6. We prove this using Claim 3.12; we choose $\Sigma_1(\mathbf{d}) := \{\mathbf{d}\} \subseteq \Sigma(\mathbf{d})$ for $\mathbf{d} \in \mathbb{D}_{\mathbf{p}}$ so it suffices to prove $(*)_0 - (*)_4$ of 3.12.

Thus in Claim 3.12, we have $(*)_0, (*)_1$ hold by the choice of Σ_1, and concerning $(*)_3$ in 4.1 easily $(\mathrm{g})^+$, and $(*)_4$ holds because for each $\kappa < \mu$ we have AC_κ as $\kappa < \mu$ by an assumption and for $\mathbf{d} \in \mathbb{D}$ we have $\mathrm{AC}_{\Sigma_1(\mathbf{d})}$, as $\Sigma_1(\mathbf{d})$ is a singleton.

Note that

$\boxplus_1$ if $\kappa < \chi < \theta(\mathcal{P}(\kappa))$ then χ is not measurable.

Now we are left with proving $(*)_2$, so let $\mathbf{d} \in \mathbb{D}_{\mathbf{p}}, \zeta, \xi, f \in {}^{Y[\mathbf{d}]}\zeta$ be given as in $\boxplus$ of (j) in 3.6, and we should find j as there.

Let $j < \partial = \mathrm{cf}(\mu)$ be such that $\theta(\mathcal{P}(\kappa)) < \kappa_j$, and let $\mathbf{e} \in \mathbb{D}_{\geq j}$. Now clause (a) is trivial as $|\Sigma_1(\mathbf{d})| = 1$, and clause (b) says that "the pair $(\mathbf{d}, \mathbf{e})$ commute"; see Definition 3.11 recalling $\Sigma_1(\mathbf{d}) = \{\mathbf{d}\}$. So let $(f, g, \bar{f}, \bar{g})$ be a $(\mathbf{p}, \mathbf{d}, \mathbf{e})$-rectangle, see Definition 3.11(2), and we should prove that $\mathrm{rk}_{\mathbf{e}}(g) \leq \mathrm{rk}_{\mathbf{d}}(f)$; let $Y_1 = Y_{\mathbf{e}}, Y_2 = Y_{\mathbf{d}}$.

To prove this we apply 2.3 or 2.9, but the $f, \bar{f}$ are interchanged with $g, \bar{g}$; we check $\oplus$(a)–(g) from 2.3. They hold by $\circledast$(a)–(f) of Definition 3.11.

Concerning $\boxplus$(a),(b) from 2.3, "AC_{Y_1}" holds as $\mathrm{AC}_{<\mu}$ holds and the definition of $\mathbf{p}$. Lastly, we should prove $\boxplus$(a) there, which says "$D_{\mathbf{d}}$ does 2-commute with $D_{\mathbf{e}}$", which holds by Case 2 of Claim 2.6. $\blacksquare_{4.3}$

Conclusion 4.4 ($\mathrm{AC}_{<\mu}, \mu$ a singular cardinal): Assume $\mu = \sup\{\lambda < \mu : \lambda$ is a measurable cardinal$\}$. Then for every ordinal ζ for some $\kappa < \lambda$ we have $\mathrm{rk}_D(\zeta) = \zeta$ for every κ-complete ultrafilter on some cardinal $< \mu$.

Proof. It suffices to prove this for the case μ has cofinality $\aleph_0$. Now we can apply Claim 4.3 and Theorem 3.10. $\blacksquare_{4.4}$

5. Pseudo true cofinality, pseudo pcf

We try to develop pcf theory with little choice. We deal only with $\aleph_1$-complete filters, and replace cofinality and other basic notions by pseudo ones; see below. This is quite reasonable as with choice there is no difference.

The main results of this section are 5.9, existence of filters with pseudo true cofinality; 5.19, giving a parallel of $J_{<\lambda}[\alpha]$.

In the main case we may (in addition to ZF) assume DC + $\mathrm{AC}_{\mathcal{P}(\mathcal{P}(Y))}$; this will be continued in [Sh:955].

Hypothesis 5.1: ZF

Definition 5.2: 1) We say that a partial order P is $(<\kappa)$-directed when every subset A of P of power $<\kappa$ has a common upper bound.

1A) Similarly P is $(\leq S)$-directed.

2) We say that a partial order P is pseudo $(<\kappa)$-directed when it is $(<\kappa)$-directed and, moreover, every subset $\bigcup\{P_\alpha : \alpha < \delta\}$ has a common upper bound when:

(a) if $\delta < \kappa$ is a limit ordinal,
(b) $\bar{P} = \langle P_\alpha : \alpha < \delta \rangle$ is a sequence of non-empty subsets of P,
(c) if $\alpha_1 < \alpha_2, p_1 \in P_{\alpha_1}$ and $p_2 \in P_{\alpha_2}$ then $p_1 <_P p_2$.

2A) For a set S we say that the partial order P is pseudo $(\leq S)$-directed when $\bigcup\{P_s : s \in S\}$ has a common upper bound whenever

(a) $\langle P_s : s \in S \rangle$ is a sequence,
(b) $P_s \subseteq P$,
(c) if $s \in S$ then P_s has a common upper bound.

Definition 5.3: We say that a partial (or quasi) order P has pseudo true cofinality δ when: δ is a limit ordinal and there is a sequence $\langle P_\alpha : \alpha < \delta \rangle$ such that

(a) $P_\alpha \subseteq P$ and $\delta = \sup\{\alpha < \delta : P_\alpha \text{ non-empty}\}$,
(b) if $\alpha_1 < \alpha_2 < \delta, p_1 \in P_{\alpha_1}, p_2 \in P_{\alpha_2}$ then $p_1 <_P p_2$,
(c) if $p \in P$ then for some $\alpha < \delta$ and $q \in P_\alpha$ we have $p \leq_P q$.

Remark 5.4: 0) See 5.2(2) and 5.8(1).

1) We could replace δ by a partial order Q.

2) The most interesting case is in Definition 5.6.

3) We may in Definition 5.3 demand δ is a regular cardinal.

4) Usually in clause (a) of Definition 5.3, without loss of generality, $\bigwedge\limits_\alpha P_\alpha \neq \emptyset$ as, without loss of generality, $\delta = \mathrm{cf}(\delta)$ using $P'_\alpha = P_{f(\alpha)}$, where $f(\alpha) =$ the α-th member of C where C is an unbound subset of $\{\beta < \delta : P_\beta \neq \emptyset\}$ of order type $\mathrm{cf}(\delta)$. Why do we allow $P_\alpha = \emptyset$? Because it is more natural in 5.17(1), but we can usually ignore it.

Example 5.5: Suppose we have a limit ordinal δ and a sequence $\langle A_\alpha : \alpha < \delta \rangle$ of sets with $\prod_{\alpha<\delta} A_\alpha = \emptyset$; moreover, $u \subseteq \delta = \sup(u) \Rightarrow \prod_{\alpha \in u} A_\alpha = \emptyset$. Define a partial order P by:

(a) its set of elements is $\{(\alpha, a) : a \in A_\alpha \text{ and } \alpha < \delta\}$,

(b) the order is $(\alpha_1, a_1) <_P (\alpha_2, a_2)$ iff $\alpha_1 < \alpha_2$ (and $a_\ell \in A_{\alpha_\ell}$ for $\ell = 1, 2$).

It seems very reasonable to say that P has pseudo true cofinality but there is no increasing cofinal sequence.

Definition 5.6: 1) For a set Y and sequence $\bar\alpha = \langle \alpha_t : t \in Y\rangle$ of ordinals and cardinal κ we define

$$\text{ps-tcf-fil}_\kappa(\bar\alpha) = \{D : D \text{ a } \kappa\text{-complete filter on } Y \text{ such that } (\Pi\bar\alpha/D) \text{ has a pseudo true cofinality}\};$$

see below.

2) We say that $\Pi\bar\alpha/D$ or $(\Pi\bar\alpha, D)$ or $(\Pi\bar\alpha, <_D)$ has pseudo true cofinality γ when D is a filter on $Y = \text{Dom}(\bar\alpha)$ and γ is a limit ordinal and the partial order $(\Pi\bar\alpha, <_D)$ essentially does[5], i.e., there is a sequence $\bar{\mathcal{F}} = \langle \mathcal{F}_\beta : \beta < \gamma\rangle$ satisfying:

$\circledast_{\bar{\mathcal{F}}}$ (a) $\mathcal{F}_\beta \subseteq \{f \in {}^Y\text{Ord} : f <_D \bar\alpha\}$,

(b) $\mathcal{F}_\beta \neq 0$,

(c) if $\beta_1 < \beta_2, f_1 \in \mathcal{F}_{\beta_1}$ and $f_2 \in \mathcal{F}_{\beta_2}$ then $f_1 < f_2 \mod D$,

(d) if $f \in {}^Y\text{Ord}$ and $f < \bar\alpha \mod D$ then for some $\beta < \gamma$ we have $g \in \mathcal{F}_\beta \Rightarrow f < g \mod D$ (by clause (c) this is equivalent to: for some $\beta < \gamma$ and some $g \in \mathcal{F}_\beta$ we have $f \leq g \mod D$).

3) $\text{ps-pcf}_\kappa(\bar\alpha) = \text{ps-pcf}_{\kappa\text{-comp}}(\bar\alpha) := \{\gamma$: there is a κ-complete filter D on Y such that $\Pi\bar\alpha/D$ has pseudo true cofinality γ and γ is minimal for $D\}$.

4) $\text{pcf-fil}_{\kappa,\gamma}(\bar\alpha) = \{D : D$ a κ-complete filter on Y such that $\Pi\bar\alpha/D$ has pseudo true cofinality $\gamma\}$.

5) In part (2), if γ is minimal we call it $\text{ps-tcf}(\Pi\bar\alpha, D)$ or simply $\text{ps-tcf}(\Pi\bar\alpha, <_D)$; note that it is a well defined (regular cardinal).

Claim 5.7: 1) *If* $\lambda = \text{ps-tcf}(\Pi\bar\alpha, <_D)$, *then* $(\Pi\bar\alpha, <_D)$ *is pseudo* $(< \lambda)$*-directed.*

1A) *If* $\theta(S) < \lambda = \text{ps-tcf}(\Pi\bar\alpha, <_D)$ *then* $(\Pi\bar\alpha, <_D)$ *is pseudo* $(\leq S)$*-directed.*

2) *Similarly for any quasi order.*

3) *If* $\text{cf}(\alpha_t) \geq \lambda = \text{cf}(\lambda)$ *for* $t \in Y$ *then* $(\Pi\bar\alpha, <_D)$ *is* λ*-directed.*

4) *Assume* AC_α *for* $\alpha < \lambda$. *If* $\text{cf}(\alpha_s) \geq \lambda$ *for* $s \in Y$ *then* $(\Pi\bar\alpha, <_D)$ *is pseudo* λ*-directed.*

---

[5] So necessarily $\{s \in Y : \alpha_s > 0\}$ belongs to D but is not necessarily empty; if it is $\neq Y$ then $\Pi\bar\alpha = \emptyset$, so pedantically this is wrong, $(\Pi\bar\alpha, <_D)$ does not have any pseudo true cofinality, hence we say "essentially" but usually we shall ignore this or assume $\bigwedge_t \alpha_t \neq 0$ when not said otherwise.

Proof. 1), 1A), 2) As in 5.8(1) below.

3) So assume $\mathcal{F} \subseteq \Pi\bar{\alpha}$ satisfies $|\mathcal{F}| < \lambda$. Hence there is a sequence $\langle f_\alpha : \alpha < \mu\rangle$ listing $\mathcal{F}$ for some $\mu < \lambda$. Let $f \in \Pi\bar{\alpha}$ be defined by $f(s) = \sup\{f_\alpha(s) : \alpha < \mu\}$; now $f(s) < \alpha(s)$ as $\mathrm{cf}(\alpha_s) \geq \lambda > \mu$.

4) So assume $\bar{P} = \langle P_\alpha : \alpha < \delta\rangle, \delta$ a limit ordinal $< \lambda$ and $P_\alpha \subseteq \Pi\bar{\alpha}$ non-empty and $\alpha < \beta < \delta \wedge f \in P_\alpha \wedge g \in P_\beta \Rightarrow f <_D g$. As AC_δ holds we can find a sequence $\bar{f} = \langle f_\alpha : \alpha \in \delta\rangle \in \prod_{\alpha<\beta} P_\alpha$ and apply part (3). $\blacksquare_{5.7}$

Claim 5.8: *Let $\bar{\alpha} = \langle \alpha_s : s \in Y\rangle$ and D is a filter on Y.*

0) *If $\Pi\bar{\alpha}/D$ has pseudo true cofinality then* $\text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ *is a regular cardinal; similarly for any partial order.*
1) *If $\Pi\bar{\alpha}/D$ has pseudo true cofinality γ_1 and true cofinality γ_2 then* $\mathrm{cf}(\gamma_1) = \mathrm{cf}(\gamma_2) = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$*; similarly for any partial order.*
2) $\text{ps-pcf}_\kappa(\bar{\alpha})$ *is a set of regular cardinals, so if $\Pi\bar{\alpha}/D$ has pseudo true cofinality then* $\text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ *is γ where $\gamma = \mathrm{cf}(\gamma)$ and $\Pi\bar{\alpha}/D$ has pseudo cofinality γ.*
3) *Always* $\text{ps-pcf}_\kappa(\bar{\alpha})$ *has cardinality $< \theta(\{D : D$ a κ-complete filter on $Y\})$.*
4) *If $\bar{\beta} = \langle \beta_s : s \in Y\rangle \in {}^Y\mathrm{Ord}$ and $\{s : \beta_s = \alpha_s\} \in D$ then* $\text{ps-tcf}(\Pi\bar{\alpha}/D) = \text{ps-tcf}(\Pi\bar{\beta}/D)$*, so one is well defined iff the other is.*

Proof. 0) By the definitions.

1) Let $\langle \mathcal{F}^\ell_\beta : \beta < \gamma_\ell\rangle$ exemplify "$\Pi\bar{\alpha}/D$ has pseudo true cofinality γ_ℓ" for $\ell = 1, 2$. Now

(∗) if $\ell \in \{1, 2\}$ and $\beta_\ell < \gamma_\ell$, then for some $\beta_{3-\ell} < \gamma_{3-\ell}$ we have $g_1 \in \mathcal{F}^\ell_{\beta_\ell} \wedge g_2 \in \mathcal{F}^{3-\ell}_{\beta_{3-\ell}} \Rightarrow g_1 <_D g_2$.

[Why? Choose $g^\ell \in \mathcal{F}^\ell_{\beta_\ell+1}$, choose $\beta_{3-\ell} < \gamma_{3-\ell}$ and $g_{3-\ell} \in \mathcal{F}^{3-\ell}_{\beta_{3-\ell}}$ such that $g^\ell < g^{3-\ell} \bmod D$. Clearly $f \in \mathcal{F}^\ell_{\beta_\ell} \Rightarrow f <_D g^\ell <_D g^{3-\ell}$, so $g^{3-\ell}$ is as required.]

Hence

(∗) $h_1 : \gamma_1 \to \gamma_2$ is well defined when

$$h_1(\beta_1) = \mathrm{Min}\{\beta_2 < \gamma_2 : (\forall g_1 \in \mathcal{F}^1_{\beta_1})(\forall g_2 \in \mathcal{F}^2_{\beta_2})(g_1 < g_2 \bmod D)\}.$$

Clearly h is non-decreasing and it is not eventually constant (as $\bigcup\{\mathcal{F}^1_\beta : \beta < \gamma_1\}$ is cofinal in $\Pi\bar{\alpha}/D$) and has range unbounded in γ_2 (similarly).

The rest should be clear.

2) Follows.

3), 4) Easy. $\blacksquare_{5.8}$

Concerning [Sh:835]

Claim 5.9 (The Existence of true cofinality filter [$\kappa > \aleph_0 + \mathrm{DC} + \mathrm{AC}_{<\kappa}$]): *Suppose*

(a) *D is a κ-complete filter on Y,*
(b) $\bar{\alpha} \in {}^Y\mathrm{Ord}$,
(c) $\delta := \mathrm{rk}_D(\bar{\alpha})$ *satisfies* $\mathrm{cf}(\delta) \geq \theta(\mathrm{Fil}^1_\kappa(Y))$, *see below.*

Then for some D' we have

(α) *D' is a κ-complete filter on Y,*
(β) $D' \supseteq D$,
(γ) $\Pi\bar{\alpha}/D'$ *has pseudo true cofinality, in fact,* $\text{ps-tcf}(\Pi\bar{\alpha}, <_{D'}) = \mathrm{cf}(\mathrm{rk}_D(\bar{\alpha}))$.

Recall from [Sh:835]

Definition 5.10: 0) $\mathrm{Fil}^1_\kappa(Y) = \{D : D$ a κ-complete filter on $Y\}$, and if $D \in \mathrm{Fil}^1_\kappa(Y)$ then $\mathrm{Fil}^1_\kappa(D) = \{D' \in \mathrm{Fil}^1_\kappa(Y) : D \subseteq D'\}$.
1) $\mathrm{Fil}^4_\kappa(Y) = \{(D_1, D_2) : D_1 \subseteq D_2$ are κ-complete filters on $Y\}$.
2) $J[f, D]$, where D is a filter on Y and $f \in {}^Y\mathrm{Ord}$, is $\{A \subseteq Y : A = \emptyset \bmod D$ or $\mathrm{rk}_{D+A}(f) > \mathrm{rk}_D(f)\}$.

Remark 5.11: 1) On the definition of pseudo $(< \kappa, 1+\gamma)$-complete D, see 1.13; we may consider changing the definition of $\mathrm{Fil}^1_\kappa(Y)$ to: D is $\aleph_1$-complete and a pseudo$(< \kappa, 1+\gamma))$-complete filter on Y.

Proof of the Claim 5.9. Recall $\{y \in Y : \alpha_y = 0\} = \emptyset \bmod D$ as $\mathrm{rk}_D(\langle \alpha_y : y \in Y\rangle) = \delta > 0$, but $f_1, f_2 \in {}^Y \mathrm{Ord} \wedge (f_1 = f_2 \bmod D) \Rightarrow \mathrm{rk}_D(f_1) = \mathrm{rk}_D(f_2)$, hence without loss of generality $y \in Y \Rightarrow \alpha_y > 0$.

Let $\mathbb{D} = \{D' : D'$ is a filter on Y extending D which is κ-complete$\}$. So $\theta(\mathbb{D}) \leq \theta(\mathrm{Fil}^1_{\aleph_1}(Y)) \leq \mathrm{cf}(\delta)$. For any $\gamma < \mathrm{rk}_D(\bar{\alpha})$ and $D' \in \mathbb{D}$ let

$(*)_2$ (a) $\mathcal{F}_{\gamma,D'} = \{f \in \Pi\bar{\alpha} : \mathrm{rk}_D(f) = \gamma$ and D' is $\mathrm{dual}(J[f,D])\}$,
(b) $\mathcal{F}_{D'} = \bigcup\{\mathcal{F}_{\gamma,D'} : \gamma < \mathrm{rk}_D(\bar{\alpha})\}$,
(c) $\Xi_{\bar{\alpha},D'} = \{\gamma < \mathrm{rk}_D(\bar{\alpha}) : \mathcal{F}_{\gamma,D'} \neq \emptyset\}$,
(d) $\mathcal{F}_\gamma = \bigcup\{\mathcal{F}_{\gamma,D''} : D'' \in \mathbb{D}\}$.

Now

$(*)_3$ if $\gamma < \mathrm{rk}_D(\bar{\alpha})$ then $\mathcal{F}_\gamma \neq \emptyset$.

[Why? By 1.8(2) there is $g \in {}^Y\mathrm{Ord}$ such that $g < \bar{\alpha} \bmod D$ and $\mathrm{rk}_D(g) = \gamma$ and, without loss of generality, $g \in \Pi\bar{\alpha}$. Now let $D' = \mathrm{dual}(J[g,D])$, so

$(D, D') \in \mathrm{Fil}^4_\kappa(Y)$; by 1.11(1) (using $\mathrm{AC}_{<\kappa}$) the filter D' is κ-complete so $D' \in \mathbb{D}$ and clearly $g \in \mathcal{F}_{\gamma,D'}$, see 1.8(2), but $\mathcal{F}_{\gamma,D'} \subseteq \mathcal{F}_\gamma$, so $\mathcal{F}_\gamma \neq 0$; here we use $\mathrm{AC}_{<\kappa}$.]

- $(*)_4$ $\{\sup(\Xi_{\bar{\alpha},D'}) : D' \in \mathbb{D}$ and $\Xi_{\bar{\alpha},D'}$ is bounded in $\mathrm{rk}_D(\bar{\alpha})\}$ is a subset of $\mathrm{rk}_{D'}(\bar{\alpha})$ which has cardinality $< \theta(\mathbb{D}) \leq \theta(\mathrm{Fil}^1_\kappa(Y)) \leq \mathrm{cf}(\delta)$.

[Why? The function $D' \mapsto \sup(\Xi_{\bar{\alpha},D'})$ witnesses this.]

- $(*)_5$ The set in $(*)_4$ is bounded below $\mathrm{rk}_D(\bar{\alpha})$, so let $\gamma(*) < \mathrm{rk}_D(\bar{\alpha})$ be its supremum.

[Why? By $(*)_4$.]

- $(*)_6$ There is $D' \in \mathbb{D}$ such that $\Xi_{\bar{\alpha},D'}$ is unbounded in $(\Pi\bar{\alpha}, <_{D'})$.

[Why? Choose $\gamma < \mathrm{rk}_D(\bar{\alpha})$ such that $\gamma > \gamma(*)$. By $(*)_3$ there is $f \in \mathcal{F}_{\gamma(*)}$, and by $(*)_2$(d) for some $D' \in \mathbb{D}$ we have $f \in \mathcal{F}_{\gamma(*),D'}$, so by the choice of $\gamma(*)$ the set $\Xi_{\bar{\alpha},D'}$ cannot be bounded in $\mathrm{rk}_D(\bar{\alpha})$.]

- $(*)_7$ If $\gamma_1 < \gamma_2$ are from $\Xi_{\bar{\alpha},D'}$ and $f_1 \in \mathcal{F}_{\gamma_1,D'}, f_2 \in \mathcal{F}_{\gamma_2,D'}$ then $f_1 <_{D'} f_2$.

[Why? By 1.8.]

Together we are done: by $(*)_6$ there is $D' \in \mathbb{D}$ such that $\Xi_{\bar{\alpha},D'}$ is unbounded in $\mathrm{rk}_D(\bar{\alpha})$. Hence $\bar{\mathcal{F}} = \langle \mathcal{F}_{\gamma,D'} : \gamma \in \Xi_{\bar{\alpha},D'}\rangle$ witnesses that $(\Pi\bar{\alpha}, <_{D'})$ has pseudo true cofinality by $(*)_7$, and so $\text{ps-tcf}(\Pi\bar{\alpha}, <_D) = \mathrm{cf}(\mathrm{otp}(\Xi_{\bar{\alpha},D'})) = \mathrm{cf}(\mathrm{rk}_D(\bar{\alpha}))$, hence we are done. $\blacksquare_{5.9}$

Therefore we have

Definition/Claim 5.12: 1) We say that $\delta = \text{ps-tcf}_{\bar{D}}(\bar{\alpha})$, where δ is a limit ordinal when, for some set Y:

(a) $\bar{\alpha} \in {}^Y\mathrm{Ord}$,
(b) $\bar{D} = (D_1, D_2)$,
(c) $D_1 \subseteq D_2$ are $\aleph_1$-complete filters on Y,
(d) $\mathrm{rk}_{D_1}(\bar{\alpha}) = \delta = \sup(\Xi_{\bar{D},\bar{\alpha}})$ where $\Xi_{\bar{D},\bar{\alpha}} = \{\gamma < \mathrm{rk}_{D_1}(\bar{\alpha})$: for some $f < \bar{\alpha}$ mod D_1, we have $\mathrm{rk}_{D_1}(f) = \gamma$ and $D_2 = \mathrm{dual}(J[f, D_1]\}$.

2) [DC] If D_1 is an $\aleph_1$-complete filter on $Y, \bar{\alpha} = \langle \alpha_t : t \in Y\rangle$ and $\mathrm{cf}(\mathrm{rk}_D(\overline{\alpha})) \geq \theta(\mathrm{Fil}^1_{\aleph_1}(Y))$ then for some $\aleph_1$-complete filter D_2 on Y extending D_1 we have $\text{ps-tcf}_{(D_1,D_2)}(\bar{\alpha})$ is well defined.

3) [DC] Moreover, in part (2) there is a definition giving for any $(Y, D_1, D_2, \bar{\alpha})$, as there, a sequence $\langle \mathcal{F}_\gamma : \gamma < \delta\rangle$ exemplifying the value of $\text{ps-tcf}_{\bar{D}}(\bar{\alpha})$.

Proof. 2), 3) Let $\delta := \operatorname{rk}_{D_1}(f)$, so we have $\operatorname{cf}(\delta) \geq \theta(\operatorname{Fil}^1_{\aleph_1}(Y))$, hence by Claim 5.9 above and its proof the conclusion holds: the inequality is needed for "$\delta = \sup(\Xi_{\bar{D},\alpha})$". $\blacksquare_{5.12}$

Observation 5.13: Assume D is an $\aleph_1$-complete filter on Y and $f, f_n \in {}^Y\mathrm{Ord}$ for $n < \omega$ and $f(t) = \sup\{f_n(t) : n < \omega\}$. Then $\operatorname{rk}_D(f) \geq \sup\{\operatorname{rk}_D(f_n) : n < \omega\}$.

Remark 5.14: Similarly for other amounts of completeness, see 5.18.

Proof. As $f_n \leq f$ clearly $\operatorname{rk}_D(f_n) \leq \operatorname{rk}_D(f)$ for each n hence $\operatorname{rk}_D(f) \geq \sup\{\operatorname{rk}_D(f) : n < \omega\}$. $\blacksquare_{5.13}$

Remark 5.15: Also in 1.9(2) we can use AC_Y only, i.e., omit the assumption DC, a marginal point here.

Claim 5.16 ($\mathrm{AC}_{<\theta}$): *The ordinal δ has cofinality $\geq \theta$ when:*

$\circledast$ (a) $\delta = \operatorname{rk}_D(\bar{\alpha})$,
(b) $\bar{\alpha} = \langle \alpha_y : y \in Y \rangle \in {}^Y\mathrm{Ord}$,
(c) *D is an $\aleph_1$-complete filter on Y,*
(d) $y \in Y \Rightarrow \operatorname{cf}(\alpha_y) \geq \theta$.

Proof. Note that $y \in Y \Rightarrow \alpha_y > 0$. Toward a contradiction assume $\operatorname{cf}(\delta) < \theta$, so δ has a cofinal subset C of cardinality $< \theta$. For each $\beta < \delta$ for some $f \in {}^Y\mathrm{Ord}$ we have $\operatorname{rk}_D(f) = \beta$ and $f <_D \bar{\alpha}$ and, without loss of generality, $f \in \prod_{y \in Y} \alpha_y$. By $\mathrm{AC}_{<\theta}$ there is a sequence $\langle f_\beta : \beta \in C \rangle$ such that $f_\beta \in \prod_{y \in Y} \alpha_y, f_\beta <_D \bar{\alpha}$ and $\operatorname{rk}_D(f_\beta) = \beta$. Define $g \in \prod_{y \in Y} \alpha_y$ by $g(y) = \bigcup\{f_\beta(y) : \beta \in C$ and $f_\beta(y) < \alpha_t\}$. By clause (d) we have $[y \in Y \Rightarrow g(y) < \alpha_y]$, so $g <_D \bar{\alpha}$, hence $\operatorname{rk}_D(\bar{g}) < \operatorname{rk}_D(\alpha)$; but by the choice of g we have $\beta \in C \Rightarrow f_\beta \leq_D g$, hence $\beta \in C \Rightarrow \beta = \operatorname{rk}_D(f_\beta) \leq \operatorname{rk}_D(g)$, hence $\delta = \sup(C) \leq \operatorname{rk}_D(g)$, contradiction. $\blacksquare_{5.16}$

Observation 5.17: 1) Assume $(\bar{\alpha}, D)$ satisfies

(a) D is a filter on Y and $\bar{\alpha} = \langle \alpha_t : t \in Y \rangle$, α_t an ordinal of uncountable cofinality,
(b) $\bar{\mathcal{F}} = \langle \mathcal{F}_\beta : \beta < \partial \rangle$ exemplifies $\partial = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ so we demand just $\partial = \sup\{\beta < \partial : \mathcal{F}_\beta \neq \emptyset\}$,
(c) $\mathcal{F}'_\beta = \{f \in \prod_{t \in Y} \alpha_t$: for some $g \in \mathcal{F}_\beta$ we have $f = g \bmod D\}$.

Then: $\langle \mathcal{F}'_\beta : \beta < \partial \rangle$ exemplify $\partial = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$, that is,

(α) $\bigcup_{\beta<\gamma} \mathcal{F}'_\beta$ is cofinal in $(\Pi\bar{\alpha}, <_D)$,

(β) for every $\beta_1 < \beta_2 < \partial$ and $f_1 \in \mathcal{F}'_{\beta_1}$ and $f_2 \in \mathcal{F}'_{\beta_2}$ we have $f_1 < f_2$ mod D.

2) Similarly, if $D, \bar{\mathcal{F}}$ satisfy clauses (a),(b) above and D is $\aleph_1$-complete and $\partial = \mathrm{cf}(\partial) > \aleph_0$ then we can "correct" $\bar{\mathcal{F}}$ to make it $\aleph_0$-continuous, that is, $\langle \mathcal{F}''_\beta : \beta < \partial\rangle$ defined in $(\mathrm{c})_1$ + $(\mathrm{c})_2$ below satisfies (α) + (β) above and (γ) below and so is $\aleph_0$-continuous, (see below) where

$(\mathrm{c})_1$ if $\beta < \partial$ and $\mathrm{cf}(\beta) \neq \aleph_0$ then $\mathcal{F}''_\beta = \mathcal{F}'_\beta$;

$(\mathrm{c})_2$ if $\beta < \partial$ and $\mathrm{cf}(\beta) = \aleph_0$ then $\mathcal{F}''_\beta = \{\sup\langle f_n : n < \omega\rangle$: for some increasing sequence $\langle \beta_n : n < \omega\rangle$ with limit β we have $n < \omega \Rightarrow f_n \in \mathcal{F}'_{\beta_n}\}$, see below;

(γ) if $\beta < \partial$ and $\mathrm{cf}(\beta) = \aleph_0$ and $f_1, f_2 \in \mathcal{F}''_\beta$ then $f_1 = f_2$ mod D.

3) This applies to any increasing sequence $\langle \mathcal{F}_\beta : \beta < \delta\rangle, \mathcal{F}_\beta \subseteq {}^Y\mathrm{Ord}$, δ a limit ordinal.

Proof. Straightforward. $\blacksquare_{5.17}$

Definition 5.18: 0) If $f_n \in {}^Y\mathrm{Ord}$ for $n < \omega$, then $\sup\langle f_n : n < \omega\rangle$ is defined as the function f with domain Y such that $f(t) = \bigcup\{f_n(t) : n < \omega\}$.

1) We say $\bar{\mathcal{F}} = \langle \mathcal{F}_\beta : \beta < \lambda\rangle$, exemplifying $\lambda = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$, is weakly $\aleph_0$-continuous when:

if $\beta < \partial$, $\mathrm{cf}(\beta) = \aleph_0$ and $f \in \mathcal{F}_\beta$ then for some sequence $\langle(\beta_n, f_n) : n < \omega\rangle$ we have $\beta = \bigcup\{\beta_n : n < \omega\}, \beta_n < \beta_{n+1} < \beta, f_n \in \mathcal{F}_{\beta_n}$ and $f = \sup\langle f_n : n < \omega\rangle$; so if D is $\aleph_1$-complete then $\{f/D : f \in \mathcal{F}_\beta\}$ is a singleton.

2) We say it is $\aleph_0$-continuous if we can replace the last "then" by "iff".

Theorem 5.19 (The Canonical Filter Theorem): *Assume* DC *and* $\mathrm{AC}_{\mathcal{P}(Y)}$.

Assume $\bar{\alpha} = \langle \alpha_t : t \in Y\rangle \in {}^Y\mathrm{Ord}$ *and* $t \in Y \Rightarrow \mathrm{cf}(\alpha_t) \geq \theta(\mathcal{P}(Y))$ *and* $\partial \in \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$, *hence it is a regular cardinal. Then there is* $D = D^{\bar{\alpha}}_\partial$, *an* $\aleph_1$*-complete filter on* Y *such that* $\partial = \text{ps-tcf}(\Pi\bar{\alpha}/D)$ *and* $D \subseteq D'$ *for any other such* $D' \in \mathrm{Fil}^1_{\aleph_1}(D)$.

Remark 5.20: 1) By 5.9 there are some such ∂.

2) We work to use just $\mathrm{AC}_{\mathcal{P}(Y)}$ and not more.

3) If $\kappa > \aleph_0$ we can replace "$\aleph_1$-complete" by "κ-complete".

Proof. Let

$\boxplus_1$ (a) $\mathbb{D} = \{D : D$ is an $\aleph_1$-complete filter on Y such that $(\Pi\bar{\alpha}/D)$ has pseudo true cofinality $\partial\}$,

(b) $D_* = \bigcap\{D : D \in \mathbb{D}\}$.

Now obviously

(c) D_* is an $\aleph_1$-complete filter on Y.

For $A \subseteq Y$ let $\mathbb{D}_A = \{D \in \mathbb{D} : A \notin D\}$ and let $\mathcal{P}_* = \{A \subseteq Y : \mathbb{D}_A \neq \emptyset\}$. As $\mathrm{AC}_{\mathcal{P}(Y)}$ we can find $\langle D_A : A \in \mathcal{P}_*\rangle$ such that $D_A \in \mathbb{D}_A$ for $A \in \mathcal{P}_*$. Let $\mathbb{D}_* = \{D_A : A \in \mathcal{P}_*\}$; clearly

$\boxplus_2$ $D_* = \bigcap\{D : D \in \mathbb{D}_*\}$ and $\mathbb{D}_* \subseteq \mathbb{D}$ is non-empty.

As $\mathrm{AC}_{\mathcal{P}_*}$ holds clearly

$(*)_0$ We can choose $\langle \bar{\mathcal{F}}^A : A \in \mathcal{P}_*\rangle$ such that $\bar{\mathcal{F}}^A$ exemplifies $D_A \in \mathbb{D}$ as in 5.17(1),(2), so in particular is $\aleph_0$-continuous.

For each $\beta < \partial$ let $\mathcal{F}^*_\beta = \bigcap\{\mathcal{F}^A_\beta : A \in \mathcal{P}_*\}$; now

$(*)_1$ $\mathcal{F}^*_\beta \subseteq \Pi\bar{\alpha}$.

[Why? As by 5.17(1)(c) we have $\mathcal{F}^A_\beta \subseteq \Pi\bar{\alpha}$ for each $A \in \mathcal{P}_*$.]

$(*)_2$ If $\beta_1 < \beta_2 < \partial, f_1 \in \mathcal{F}^*_{\beta_1}$ and $f_2 \in \mathcal{F}^*_{\beta_2}$ then $f_1 < f_2 \mod D_*$.

[Why? Note that $A \in \mathcal{P}_* \Rightarrow f_1 <_{D_A} f_2$ by the choice of $\langle \mathcal{F}^*_\beta : \beta < \partial\rangle$, hence the set $\{t \in Y : f_1(t) < f_2(t)\}$ belongs to D_A for every $A \in \mathcal{P}_*$, hence by $\boxplus_2$ it belongs to D_* which means that $f_1 <_{D_*} f_2$, as required.]

$(*)_3$ If $f \in \Pi\bar{\alpha}$ then for some $\beta_f < \partial$ we have

$$f' \in \cup\{\mathcal{F}^*_\beta : \beta \in [\beta_f, \partial)\} \Rightarrow f < f' \mod D_*.$$

[Why? For each $A \in \mathcal{P}_*$ there are β, g such that $\beta < \partial, g \in \mathcal{F}^A_\beta$ and $f < g$ mod D, hence $\beta' \in [\beta+1, \partial) \wedge f' \in \mathcal{F}^A_{\beta'} \Rightarrow f < g < f'$ mod D_A. Let β_A be the minimal such ordinal $\beta < \delta$. As $\mathrm{cf}(\delta) \geq \theta(\mathcal{P}(Y)) \geq \theta(\mathcal{P}_*)$, clearly $\beta_* = \sup\{\beta_A + 1 : A \in \mathcal{P}_*\}$ is $< \delta$. So $A \in \mathcal{P}_* \wedge g \in \bigcup\{\mathcal{F}^*_\beta : \beta \in [\beta_*, \delta)) \Rightarrow f <_{D_A} g$. By $\boxplus_2$ the ordinal α_* is as required on β_f.]

Moreover:

$(*)_4$ There is a function $f \mapsto \beta_f$ in $(*)_3$.

[Why? As we can (and will) choose β_f as the minimal β such that ...]

$(*)_5$ For every $\beta_* < \partial$ there is $\beta \in (\beta_*, \partial)$ such that $\mathcal{F}^*_\beta \neq \emptyset$.

[Why? We choose by induction on n a sequence $\bar{\beta}_n = \langle \beta_{n,A} : A \in \mathcal{P}_*\rangle$ and a sequence $\bar{f}_n = \langle f_{n,A} : A \in \mathcal{P}_*\rangle$ and a function f_n such that

(α) $\beta_n < \partial$ and $m < n \Rightarrow \beta_m < \beta_n$,

(β) $\beta_0 = \beta_*$ and for $n > 0$ we let $\beta_n = \sup\{\beta_{m,A} : m < n, A \in \mathcal{P}_*\}$,

(γ) $\beta_{n,A} \in (\beta_n, \partial)$ is minimal such that there is $f_{n,A} \in \mathcal{F}^A_{\beta_{n,A}}$ satisfying $n = m + 1 \Rightarrow f_m < f_{\beta_{n,A}} \bmod D_A$,

(δ) $\langle f_{n,A} : A \in \mathcal{P}_*\rangle$ is a sequence such that each $f_{n,A}$ is as in clause (γ),

(ε) $f_n \in \Pi\bar{\alpha}$ is defined by $f_n(t) = \sup\{f_{m,A}(t) + 1 : A \in \mathcal{P}_*$ and $m < n\}$.

[Why can we carry the induction? Arriving to n first, f_n is well defined $\in \Pi\bar{\alpha}$ by clause (ε) as $\mathrm{cf}(\alpha_t) \geq \theta(\mathcal{P}_*)$ for $t \in Y$. Second, by clause (γ) and the choice of $\langle\langle\bar{\mathcal{F}}^A_\beta : \beta < \partial\rangle : A \in \mathcal{P}_*\rangle$ in $(*)_0$ the sequence $\langle\beta_{n,A} : A \in \mathcal{P}_*\rangle$ is well defined. Third, by clause (δ) we can choose $\langle f_{m,A} : A \in \mathcal{P}_*\rangle$ because we have $\mathrm{AC}_{\mathcal{P}_*}$. Fourth, β_n is well defined by clause (β) as $\mathrm{cf}(\delta) \geq \theta(\mathcal{P}_*)$.

Lastly, the inductive construction is possible by DC.]

Let $\beta^* = \cup\{\beta_n : n < \omega\}$ and $f = \sup\langle f_n : n < \omega\rangle$. Easily $f \in \cap\{\mathcal{F}^A_{\beta^*} : A \in \mathcal{P}_*\}$ as each $\langle\mathcal{F}^A_\beta : \beta < \partial\rangle$ is $\aleph_0$-continuous.]

$(*)_6$ If $f \in \Pi\bar{\alpha}$ then for some $\beta < \gamma$ and $f' \in \mathcal{F}^*_\beta$ we have $f < f' \bmod D^*$.

[Why? By $(*)_3 + (*)_5$.]

So we are done. $\blacksquare_{5.19}$

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