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BOUSFIELD LOCALIZATION AS AN ALGEBRAIC CLOSURE OF GROUPS

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ABSTRACT
We show that the notion of HZ-local groups due to A. Bousfield which is based on considerations from algebraic topology can be defined and understood in terms of solubility of certain systems of equations over G.

In the present note we recast the basic properties of HZ-local groups in the sense of Bousfield [B] in terms of solubility of certain systems of equations over the given group. In fact we prove the following theorem (definitions are given in §1 below, proof in §2).

0.1. THEOREM. *A group G is HZ-local if and only if any Γ-system of equations over G has a unique solution.*

Systems of equations over a given group G have been discussed extensively by many authors. See a survey of Howie in [H]. Our present notion of a Γ-system is closely related to the usual concept of non-singular systems [H, p. 59]. We require uniqueness of solutions, not just existence, and that makes the arguments and characterizations easier.

For any group G Bousfield defines a functorial map $G \to LG$ which is initial among all maps of G to a local group. The group LG is the HZ-localization of G. Many properties of LG are given in [B]. We show the following consequences of 0.1:

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0.2. COROLLARY. *For any group G the cardinality* $|LG|$ *of LG is not bigger than* $|G|^{\aleph_0}$.

0.3. COROLLARY. *The functor LG commutes with linear direct limits of length* ω_1, *the first uncountable ordinal.*

PROOFS. See §3 below.

EXAMPLE. If G is finite, then $LG = G/\Gamma G$ where ΓG is the 'stable' part of the lower central series of G which is of finite length in this case, thus here the group LG is a nilpotent group.

Recall that R-nilpotent group for $R \subseteq Q$ a subring of the rationals can be defined in terms of existence and uniqueness of solutions to equations of the form $X^p = g$ for p a prime in a given set J of primes: [B–K, V, 2.6]. Namely, if J is any set of primes $R = \mathscr{Z}[J^{-1}]$, then a nilpotent group N is R-nilpotent if and only if it is uniquely J-divisible (i.e. for each $p \in J$ the p-power map $(\ )^p : G \to G$ is a bijection).

Here a characterization of HZ-local groups, and more generally HR-local groups for $R \subseteq Q$, is given in similar terms (see 1.3). The same line of thoughts can be carried also for the field $R = \mathscr{Z}/p\mathscr{Z} = \mathscr{F}_p$ but the details are more complicated and will not be given here (compare [B–K, VI, 3.5(ii)]).

Furthermore, a similar statement can be made about localization of modules over certain rings. This generalizes Bousfield's localization since one is not restricted to work with modules over group rings and can consider more general rings. An interesting corollary is an independent totally algebraic construction of the Bousfield localization of a group as an obvious "algebraic closure" construction that adjoins unique solutions to all the appropriate equations. Another consequence that will not be proven here is that the length of the lower central series of LG for any G is bounded by the first limit ordinal with cardinality $G^{\aleph_0}$.

KERVAIRE PROBLEM. It is an open question whether for any group G and for any non-singular system of equations over it [Howie] there is an overgroup H with $G \subseteq H$ in which the system has a solution. It clearly follows from (0.1) that:

0.4. COROLLARY. *If the canonical map* $G \to LG$ *is injective then the Kervaire conjecture holds for systems of equations with determinant* ± 1 *over G* (see 1.2 below).

1. Systems of equations and local groups

Our main result 0.1 identifies the class of $U\Gamma S$-groups with that of Bousfield local groups; let us define these terms:

Recall that a group L is HZ-local in the sense of Bousfield (or just *local*) if it behaves as follows: For a map of groups $f: G \to H$, *if* f induces an isomorphism on the abelianization $abf: abG \xrightarrow{\cong} abH$, where $abG = G/[G, G]$, and induces an onto map on the second homology $H_2(f): H_2(G, \mathscr{Z}) \to H_2(H, \mathscr{Z})$, *then* f induces a set isomorphism f^* between the sets of all maps $f^*: \mathrm{map}(H, L) \xrightarrow{\cong} \mathrm{map}(G, L)$. A similar definition is given by Bousfield for any commutative ring R where instead of abG, abH etc. one takes the first homology with coefficients in R and similarly one considers the map induced by f on the second homology with coefficient in R.

We now describe the systems of equations we have in mind and then give several important cases where these systems have unique solutions.

1.1. Definition. A Γ-system of equations over a group G is a (possibly infinite) set of equations of the form

$$X_\alpha = g_\alpha \cdot C_\alpha(X_{\beta_1}, \ldots, X_{\beta_n})$$

for $\alpha \in I$, $\beta_i = \beta_i(\alpha)$, in the unknowns X_α, where $g_\alpha \in G$ and C_α is a product of (high order) commutators in the X_{β_i} and elements of G and where we have exactly one equation for each X_α ($\alpha \in I$). Equivalently we can just say that $g_\alpha \cdot C_\alpha(X_{\beta_1}, \ldots, X_{\beta_n})$ is a word in the elements of G and the symbols X_{β_i} in which the total degree of X_{β_i} is 0.

A group G is Γ-soluble if *every* Γ-system of equations has a solution $X_\alpha = a_\alpha$ for some elements $a_\alpha \in G$. It is uniquely Γ-soluble ($U\Gamma S$) if every Γ-system has a unique solution.

Similarly one defines a Γ-soluble map $f: G \to H$ to be a map of groups such that for every Γ-system S of equations over G, every solution of its image $f(S)$ over H has a lifting to a solution of S in G. Thus if $H = \{e\}$ we recover the notion of Γ-soluble group. Again if the lifting of solutions is unique we say that the map f is uniquely Γ-soluble ($U\Gamma S$).

1.2. Example. The system $x = g[x, h] = gxhx^{-1}h^{-1}$ is a Γ-system of single equation over any group G with $g, h \in G$.

Remark. If $\{E_i\}$ is a finite system of equations over a group G, then it is equivalent to a finite Γ-system if and only if the associated matrix of total exponents has ± 1 as a determinant, in other words if the system has a unique solution over the abelianization of G.

It is not hard to check that for any group G any Γ-system has a (unique) solution if and only if any Γ-system with only countable number of equations (and thus of unknowns) has a (unique) solution: Suppose the latter. Take any system S of equations. To construct a solution we exhaust this system by a transfinite induction: Pick any variable X_{00}; in its equation $X_{00} = g_{00}C(X_{11}, \ldots, X_{1n})$ there are a finite number of variables $X_{11}, \ldots, X_{1n}$; in the equation of X_{1i} there are again a finite number of new variables etc. Therefore we have a countable subset of S which is a Γ-system with the equations of X_{00} in it. By assumption it has a (unique) solution. Substitute this solution into all the equations of S whenever the unknowns X_{ij} appear in them. Notice that after this substitution the set of equations still form of Γ-system. Pick a new unknown different from X_{ij} and continue the procedure by transfinite induction to solve (uniquely) all the equations in S.

As a corollary we get a construction of the Bousfield localization of a group as an algebraic closure as well as 0.2 and 0.3 (see 3.1):

1.3. Corollary. *The localization functor $G \to LG$ is identical with the obvious Γ-closure functor.*

Moreover one has (see 3.5)

1.4. Theorem. *A map of groups $G \to H$ is HZ-local in the sense of Bousfield if and only if it is UΓS.*

1.5. *HR-localizations.* Given a subring R of the rationals Q, it is determined by the set J of primes p for which $p^{-1} = 1/p \in R$. A *ΓJ-system of equations* for such a set J is a system of the form $X_\alpha^{p_\alpha} = g_\alpha \cdot C_\alpha(X_{\beta_1}, \ldots, X_{\beta_n})$ where α, β_i, g_α are as above and $p_\alpha \in J$.

The corresponding statement to Bousfield's [B–K, V, 2.6] is (see 3.5):

1.6. Theorem. *A group G is HR-local if and only if any ΓJ-system has a unique solution. similarly, a map $G \to H$ is HR-local if and only if it is UΓS with respect to any Γ-systems of equations over G.*

1.7. Examples. (i) Every abelian group is clearly uniquely Γ-soluble since in such a group $C(X_\beta) = 1$, so the above system has a unique solution $X_\alpha = g_\alpha$.

(ii) Any product of $U\Gamma S$-groups is $U\Gamma S$: The direct product of solutions is a solution.

(iii) If E is the equalizer of two maps $f, g: A \rightrightarrows B$ between two $U\Gamma S$-groups A and B, then E is also $U\Gamma S$. This follows easily from uniqueness: A Γ-system over E has a solution over A. The maps f, g must carry this solution to the same unique solution over B so that solution comes from E. Uniqueness follows from uniqueness in A.

(iv) It follows from (ii), (iii) that any inverse limit over an arbitrary diagram of $U\Gamma S$-groups is $U\Gamma S$.

(v) Let $C \hookrightarrow E \twoheadrightarrow B$ be a central extension of groups. Then E is $U\Gamma S$ if and only if B is $U\Gamma S$. If a set of primes J is involved (1.5) then we must assume that C is uniquely p-divisible for any $p \in J$, to assure that any ΓJ-system has a unique solution in E if that is the case for B.

Proof of (v). Assume B is $U\Gamma S$. Let $X_\alpha = e_\alpha \omega_\alpha (X_\beta)$ be a Γ-system over E. Reducing mode C we get a system $X_\alpha = \bar{e}_\alpha \bar{\omega}_\alpha (X_\beta)$ over B that has a unique solution b_α. Each of these b_α comes from some $\tilde{b}_\alpha \in E$ and for each such choice of $\tilde{b}_\alpha$ for b_α there is an element $c_\alpha \in C$ such that $c_\alpha \cdot \tilde{b}_\alpha = e_\alpha \omega_\alpha (\tilde{b}_\beta)$. Since ω_α is a commutator in X_β and c_α is a central element we get $\omega_\alpha(\tilde{b}_\alpha) = \omega_\alpha(c_\beta \tilde{b}_\beta)$. Therefore $(c_\alpha \tilde{b}_\alpha : \alpha)$ is a solution in E to the given Γ-system. Uniqueness follows similarly.

Assume now that E is $U\Gamma S$. Then it is clear that any Γ-system over a quotient of E has a solution. To prove uniqueness of such solutions over $E/C = B$ let $X_\alpha = g_\alpha \omega_\alpha (X_\beta)$ be a Γ-system, $\alpha, \beta \in I$, with two solutions in B, namely $\{b_\alpha^1\}$ and $\{b_\alpha^2\}_\alpha$, so that $b_\alpha^i = g_\alpha \omega_\alpha (b_\beta^i)$ for $i = 1, 2$ are equalities in B. Each one of these solutions thus lifts to a solution of a lifting $x_\alpha = g_\alpha^i \omega_\alpha^i (x_\beta)$ of the system to E. But since the extension is central, $\omega_\alpha^1 = \omega_\alpha^2$ for all $\alpha \in I$. Furthermore $g_\alpha^1 = c_\alpha g_\alpha^2$ for some c_α in C. But this implies that the solutions in E of these two systems differ by elements in C so that their images in B are equal, thus $b_\alpha^1 = b_\alpha^2$ for all $\alpha \in I$.

(vi) It follows from (v) that nilpotent groups are $U\Gamma S$ and that 1.6 holds for nilpotent groups.

(vi) A slightly different example will be needed in what follows: The class of groups for which every Γ-system has *at most* one solution (but may have none!) is the class of groups G for which the transfinite intersection ΓG of the lower central series $\Gamma_\alpha G$ of G is the trivial subgroup: $\Gamma G = \{e\}$.

Proof. Assume there exists an element $\gamma_{01} \neq e$ in ΓG. This element is then a product of commutators of elements of G with other elements

$\gamma_{1i} \in \Gamma G$, namely $\gamma_{01} = C_{01}(\gamma_1, \dots, \gamma_{1n})$. Each of γ_{1i} in turn is again $C_{1i}(\gamma_{21}, \dots, \gamma_{2n})$ where n may depend on i. One gets a system of equations $X_{ij} = C_{ij}(X_{i1}, \dots, X_{in(i,j)})$. This (possibly infinite) system has two different solutions: The first is $X_{ij} = \gamma_{ij}$, the second $X_{ij} = e$ for all i and j since $C_{ij}(e, e, \dots, e) = e$. Since $\gamma_{01} \neq e$ the solutions are different as required.

Assume now that $\Gamma G = \{e\}$; we show that if a Γ-system has a solution it is unique. Let $X_\alpha = g_\alpha C_\alpha(X_{\beta_1}, \dots, X_{\beta_n})$; $\beta_i = \beta_i(\alpha)$; $h = n(\alpha)$ is a Γ-system. Let (a_α), (b_α) be two solutions. We show that $a_\alpha^{-1} b_\alpha \in \Gamma G$ for all α. Clearly

$$a_\alpha^{-1} b_\alpha = [C_\alpha(a_1 \cdots a_n)]^{-1} [C_\alpha(b_1 \cdots b_n)]$$

for some $a_1 \cdots a_n$, $b_1 \cdots b_n$ depending on α. But the word on the right can be written as a commutator in $(a_i^{-1} b_i)$ with some elements in G; in other words, after a change of variables to $d_\alpha = a_\alpha^{-1} b_\alpha$ we can write $d_\alpha = C'_\alpha(d_i \cdots d_m)$ for each α. This means in particular that $d_\alpha \in \Gamma$. Now if for some β, $a_\beta \neq b_\beta$ or $d_\beta \neq e$, we get a contradiction to $\Gamma G = \{e\}$.

The final example of $U\Gamma S$-group is given by:

1.8. **Proposition.** *Any Bousfield local group is $U\Gamma S$.*

Proof. This is immediate from a theorem of Bousfield [B, 3.10], that every Bousfield local group can be obtained from the trivial group $\{e\}$ by repeated applications of inverse limits and central extension, possibly transfinitely many times; now apply to 1.7(iv), (v).

1.9. *$U\Gamma S$-maps.* Neither injections or surjections of groups are, in general, $U\Gamma S$-maps for obvious reasons. An interesting example of a $U\Gamma S$-inclusion is the inclusion $PG \hookrightarrow G$ of the maximal perfect subgroup of any group G into G. Any solution to a Γ-system which does not come from PG contradicts the maximality of PG. Notice also that the kernel of any $U\Gamma S$-map $G \to H$ is a $U\Gamma S$-group, since the image of a Γ-system over the kernel will always have the trivial solution over $\{e\} \hookrightarrow H$.

Similarly, any map between two $U\Gamma S$-groups is of course a $U\Gamma S$-map. In fact all the properties given in [B] of HZ-local maps are easily proved to hold for $U\Gamma S$-maps and therefore they are immediate from (1.4) above. In general, we shall see that any inclusion $G \hookrightarrow H$ can be factored to $G \hookrightarrow \bar{G} \hookrightarrow H$ where the second map is $U\Gamma S$ and the first induces a surjection on the abelianizations $abG \to ab\bar{G}$.

2.

Proof of 1.3. We now give a proof of 1.3. It relays heavily on [B] and therefore on certain topological arguments used there.

We begin with a reminder of HZ-closures [B]. If $G \subseteq H$ a subgroup, then one considers the largest subgroup $\bar{G}$ with $G \subseteq \bar{G} \subseteq H$ such that the inclusion $G \subseteq \bar{G}$ induces surjection $abG \to ab\bar{G}$ on the abelianizations. Such a subgroup always exists uniquely and it is called the HZ-closure of G in H.

This construction generalizes the construction of the maximal perfect subgroup $PG \subseteq G$. The HZ-closure of the trivial group $\{e\}$ inside any group G is clearly PG. We shall see immediately that in general the inclusion $\bar{G} \subseteq H$ is a $U\Gamma S$-map.

We say that a map $f: G \to H$ is HZ-closed if its image $f(G)$ is HZ-closed in H, i.e. equal to its HZ closure in H.

Given any map $f: G \to H$ we can factor it through the HZ-closure of $f(G)$ in H. This HZ-closure operation on subgroups and maps can be described in terms of solutions to Γ-systems of equations as follows:

2.2. **Proposition.** *Let $G \subseteq H$ be an inclusion of groups. Then the HZ-closure $\bar{G} \subseteq H$ is the Γ-closure of G in H, namely, it is a subgroup $\bar{\bar{G}}$ of H consisting of all solutions in H to Γ-systems of equations over G.*

Proof. The set $\bar{\bar{G}}$ of all solutions in H to all Γ-equations over G forms a subgroup of H that includes G: If $x_i = a_i$ are components in a solution to two Γ-systems E_i $(i = 1, 2)$, then we add the obvious product equation $z = x_1 \cdot x_2 = \cdots$ to the disjoint union of the two Γ-systems and get a Γ-equation with $z = a, a_2$ a component to a solution of a Γ-system. Notice that a product of two Γ-equations is a Γ-equation and so is the inverse. We first show that $\bar{\bar{G}} \subseteq \bar{G}$; for this it is enough to show that $abG \to ab\bar{\bar{G}}$ is a surjection. This is clear since, if $h \in \bar{\bar{G}} \subseteq H$, it can be written as a product of $\Pi_\alpha(g_\alpha \cdot C_\alpha(g_{\beta_1} \cdots g_{\beta_n}))$, where $g_\alpha \in G$; $g_{\beta_i} \in H$. Thus $\Pi_\alpha [g_\alpha]$ maps to $[h]$, where $[a]$ denotes the class of $a \in G$, in abG. Now we show that for every element $g \in \bar{G}$ there is a Γ-system for which $x_1 = g$ (and $h_\alpha = x_\alpha$ for some $h_\alpha \in H$) is a solution in H.

Since $abG \to ab\bar{G}$, every element $g \in \bar{G}$ can be written as $g = g \cdot C_i(g_{11} \cdots g_{1n})$ where $g_i \in \bar{G}$. Each of these g_{1i} $(1 \leqq i \leqq n)$ can be written as $g_{1i} = g_{1i} C_i(g_{21} \cdots g_{2k})$ etc., where $C_i(g_{ij})$ are commutators in $\bar{G}$. Now clearly the set of elements $\{g_{ij}\}$ obtained this way is a solution in H to a system of Γ-equations $x_{ij} = g_{ij} C_i(x_{ij})$ over G as needed.

In view of (1.8) above we must show that every $U\Gamma S$-group is Bousfield local. To show this first note that, in view of 1.7(vii), if G is $U\Gamma S$ then $\Gamma G = \{e\}$.

We proceed by induction on the length λ of the transfinite lower central series of G. If $\lambda = \beta + 1$ for some ordinal β, then G is a central extension (see 1.3(v))

$$\{e\} \to \Gamma_\lambda G \to G \to G/\Gamma_\lambda G \to \{e\}$$

where the length of $G/\Gamma_\lambda G$ is $\beta < \lambda$ and so it is Bousfield local if it is $U\Gamma S$. By [B, 3.10] and 1.7(v) G is also Bousfield local. Thus the interesting case is when λ is a limit ordinal. In that case, since $\Gamma G = \{e\}$ we have an injection

$$G \to \varprojlim_{\beta<\lambda} G/\Gamma_\beta G$$

into the inverse limit of all the lower central series quotients. Now by the inductive assumption $G/\Gamma_\beta G$ is Bousfield local for all $\beta < \lambda$, since its lower series length is smaller than λ, and it is $U\Gamma S$, by the following lemma:

Lemma. *If G is $U\Gamma S$, then so is $G/\Gamma_\beta G$ for all β.*

Proof. By transfinite induction on β. If $\beta = \alpha + 1$ then it is clear from the central extension $\Gamma_\alpha G/\Gamma_\beta G \to G/\Gamma_\beta G$. If β is a limit ordinal we have a factorization

$$G \to G/\Gamma_\beta G \to \varprojlim_{\alpha<\beta} G/\Gamma_\alpha G.$$

Now by induction and 1.7(iv) both

$$G/\Gamma_\alpha G \quad \text{for } \alpha < \beta \quad \text{and} \quad \varprojlim_{\alpha<\beta} G/\Gamma_\alpha G$$

are $U\Gamma S$. But since G is $U\Gamma S$ this means that G is HZ-closed in the limit.

Since the image of G is $G/\Gamma_\beta G$, the latter is also HZ-closed. Since the limit is $U\Gamma S$, any HZ-closed subgroup is, by Proposition 2.2, also $U\Gamma S$, as needed.

To proceed with the proof, we note that by the above lemma $G/\Gamma_\beta G$ is $U\Gamma S$ and thus, by induction, Bousfield local for every β smaller than the limit ordinal λ. Again we have an imbedding

$$G/\Gamma_\lambda G = G \hookrightarrow \lim_{\beta<\lambda} G/\Gamma_\beta G.$$

Now G is $U\Gamma S$; by the above lemma so is $G/\Gamma_\beta G$ and now, by 1.7(iv), so is the limit. Thus G is HZ-closed by 1.6 above. But by [B] an HZ-closed subgroup in a Bousfield local group is also Bousfield local, hence $G$is local as needed.

3. A few consequences

We now draw some consequences from Theorem 0.1 and, in particular, prove 0.2 and 0.3.

3.1. *Construction of Bousfield's HZ-localization as an algebraic closure*
Given a group G one can consider the set $\mathcal{S}$ of all countable (see Remark 1.1) Γ-systems $\{E_\alpha\}_{\alpha\in I}$ of equations over G. A system E_α is a countable set of equations of the form $X_{\alpha j} = g_{\alpha j} C_{\alpha j}(X_{\alpha i})$. Clearly there are altogether in $\mathcal{S}$ at most $G^{\aleph_0}$ equations with altogether at most the same number of unknown $\{X_{\alpha j}\}_{\alpha,j}$. We now construct an algebraic closure CG with respect to the set $\mathcal{S}$, in which any Γ-system has at least one solution set.

To get CG from G, one simply adds $\{X_{\alpha j}\}$ as new generators and all the equations in all the Γ-systems E_α in $\mathcal{S}$ as additional relations. Thus every Γ-system $E_\alpha \in \mathcal{S}$ has a solution $\{a_{\alpha_j} = |X_{\alpha_j}|\}$ in CG. Moreover, we claim that any Γ-system of equations over CG has a solution in CG: Take $\{y_j = h_j C(y_{j1}\cdots y_{jn})\}$ a Γ-system over CG. We associate with it a Γ-system over G in the following way: Let h_j be a constant in some equation of y_j that does not belong to G. Then h_j can be written (not uniquely) in CG as a product of some new generators $X_{\alpha j}$ with possibly some elements in G. In each equation one replaces h_j by the product of the expressions $g_{\alpha j} C_{\alpha j}$ that correspond to $X_{\alpha j}$ in the system E_α from $\mathcal{S}$. After adding the corresponding equation form $X_{\alpha j}$ from E_α, one gets a system of equations over G which is still a Γ-system with the unknowns y_j, $X_{\alpha j}$. This system has a solution in CG. By going back to the original equation using the relations in CG we see that the solution of the Γ-system over G gives us a solution of the original Γ-system over CG.

To construct the localization of G out of CG we must take care that there is always a solution to any Γ-system. This can be done either by dividing CG by ΓCG using 1.7(vii) above or by adding new relations to CG identifying all different solutions of a given system. We claim that the group

$UG = CG/\Gamma G$ is exactly the Bousfield localization LG of G. It is clear that UG is local, since any equation over UG has at most one solution and, being a quotient of CG, it has at least one solution obtained from any solution to any lifting of the Γ-system to CG itself, so UG is a $U\Gamma S$ group and by 0.1 it is an HZ-local group.

3.2. PROPOSITION. *The map $u: G \to UG$ defined above is naturally the same as the HZ-localization map from* (0.2).

PROOF. *Claim*. Given any map $f: G \to A$ to an HZ-local group A there is a unique map $g: UG \to A$ such that $gu = f$. Since $\Gamma A = \{e\}$ it is enough to get a map $\tilde{g}: CG \to A$. Now, given an unknown $X_{\alpha j}$ in CG we take $\tilde{g}(X_{\alpha_j})$ to be the j-component a_{α_j} of the unique solution in A to the system of equations $E_\alpha \in \mathcal{S}$. We can extend this map to the free product $G * F$ of G with the free group $F = F(X_{\alpha_j})$ generated by all the unknowns X_{α_j} and, since all the relations in CG as a quotient of $G * F$ are satisfied in A, there is a unique extension of $\tilde{g}$ to CG. It is clear that any homomorphism $l: CG \to H$ commuting with the maps from G must carry (X_{α_j}) to the unique solution of E_α, otherwise there would be more than one solution in A to the image of the system E_α in A, but A is $U\Gamma S$ by 0.1.

3.3. PROOF OF 0.3. Let G be an infinite group. In 3.1, for a fixed α, the cardinality of the set $\{X_{\alpha j}\}$ of variables is of course at most the cardinality of G raised to the power $\aleph_0$ since there are only $|G|$ possible *finite* words involving the elements of G and a *countable* set of variables $(X_{\alpha 1}, X_{\alpha 2}, \ldots)$ which, by Remark 1.1, is all we need. Therefore there are at most $\Pi_{i<\infty}|G|$ possible countable Γ-systems of equations over G, involving altogether at most the said number of unknowns.

Now by Proposition 3.2 we get the desired result.

3.4. PROOF OF 0.4. Let $G_0 \to G_1 \to \cdots \to G_\alpha \to \cdots$ be a direct system of groups of length ω_1. We claim that there is an isomorphism:

$$\Omega : U\left(\varinjlim_{\alpha<\omega_1} G_\alpha\right) \xrightarrow{\cong} \varinjlim_{\alpha<\omega_1} UG_\alpha .$$

To see that $\varinjlim UG_\alpha$ is $U\Gamma S$ let E_α be a Γ-system over the limit, then it is already well defined over UG_β for some $\beta < \omega_1$ since $|E| < \omega_1$, so E_α has a solution in UG_β, thus in the limit. Uniqueness follows similarly. So we have the map Ω as above.

The inverse to Ω is induced from the maps $Ui_\alpha : UG_\alpha \to U \varinjlim G_\alpha$, obtained from applying U to the natural map of G_α to the limit group. It is straightforward to check that one gets a two-sided inverse to Ω.

3.5. *HR-localizations and UΓS-map*

It is straightforward to check that all our arguments work with minimal changes in the case of localizations with respect to subrings of the rationals, and in the relative case of 1.4 and 1.6. We omit the details for the sake of brevity. □

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