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COLOURING WITHOUT TRIANGLES AND PARTITION RELATION†

BY

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ABSTRACT

We solved questions of Erdös and Hajnal and deal with related topics. In particular, we show (CH) that in an example of $\aleph_1 \rightarrow [\aleph_1]^2_3$, there need not be a triangle with three colours.

Introduction

We say the *colouring* h of λ proves $\lambda \not\rightarrow [\mu]^2_\chi$ if h is a function from the edges of the complete graph on λ into χ such that whenever $S \subseteq \lambda, |S| = \mu$, then the range of h restricted to the edges in S is the whole of χ.

Erdös and Hajnal (Problem 68 in [3]) asked if it is true (assuming CH) that for every colouring which proves $\aleph_1 \not\rightarrow [\aleph_1]^2_3$ there is a triangle whose edges have three colours. We answer this negatively and prove the stronger result (Theorem 1.1) that, if $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, then there is a colouring which proves $\aleph_{\alpha+1} \not\rightarrow [\aleph_{\alpha+1}]^2_{\aleph_\alpha}$ and for which there is no triangle with three colours. We also show that, if $V = L$, and $\aleph_\alpha$ is regular then there is even such a colouring which proves $\aleph_{\alpha+1} \not\rightarrow [\aleph_{\alpha+1}]^2_{\aleph_{\alpha+1}}$.

In Section 2, we deal with the corresponding finite problem and confirm a conjecture of Erdös. We prove, for example, that if the edges of the complete graph on n points are colored with three colours so that no triangle has three colours, then there is a set $A \subseteq n$, such that $|A| \geqq n^c$ (c some constant) and which contains only two colours. This should be compared with the result $n \not\rightarrow [cln\ n]^2_3$ proved by Erdös [2] by a probabilistic method.

In Section three, we show that examples for 1.1 should contain quite homogeneous sets; and then introduce new partition relations, (of interest in the finite and infinite case), give some results and suggest some problems.

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It was proved by Erdös and Hajnal that any example of $\aleph_1 \not\to [[\aleph_1, \aleph_0]]^2_3$ is $\aleph_0$-universal. Erdös and Hajnal asked whether every example of $\aleph_2 \not\to [[\aleph_2, \aleph_1]]^2_3$ is $\aleph_1$-universal. We prove the consistency of the negative answer and the method is applicable in many other cases.

Colouring with no triangle of three colours for infinite cardinalities

DEFINITION 1.1. We say the colouring h of λ proves $\lambda \not\to [\mu]^2_\chi$ if h is a function from the edges of the complete graph (over) λ into χ, and if $S \subseteq \lambda$, $|S| = \mu$, then the range of the restriction of h to S is the whole χ. (Of course we can replace λ by any set S, $|S| = \lambda$. Here λ and μ denote cardinals, usually infinite.)

Problem 68 of Erdös and Hajnal [3] was whether in every colouring h of $\aleph_1$ proving $\aleph_1 \not\to [\aleph_1]^2_3$, there is a triangle with three colours. We give a negative answer.

THEOREM 1.1. *If* $2^\lambda = \lambda^+$, *and* $\lambda = \Sigma_{0 \leqq \mu < \lambda} 2^\mu$ *then there is a colouring* h *proving* $\lambda^+ \not\to [\lambda^+]^2_\lambda$, *in which there is no triangle with three colours.*

PROOF. Let ${}^\alpha 2$ be the set of sequences of length α of ones and zeros and for $\eta \in {}^\alpha 2$, $\nu \in {}^\beta 2$, let $\eta < \nu$ mean that η is an initial segment of ν. Let $l(\eta)$ be the length of η, and let $cn(\eta, \nu)$ denote the longest common initial segment of η and ν and let $l(\eta, \nu)$ denote its length. Let G_1 be the complete graph over ${}^\lambda 2$ (so it has $2^\lambda = \lambda^+$ vertices). Let f be a function from λ into λ, such that $|\{j < \lambda : f(j) = i\}| = \lambda$, $(i < \lambda)$ and if $i < \mu < \lambda$, $|\{j < \mu : f(j) = i\}| = \mu$. Let h_1 be the colouring of G_1, defined by $h_1(\{\eta, \nu\}) = f(l(\eta, \nu))$. Clearly there is no triangle with three colours.

Let $\{A_i : i < \lambda^+\}$ be a list of all subsets of ${}^\lambda 2$ of cardinality λ. We say $\eta \in \mathrm{cl} A_i$ if for every $\alpha < \lambda$ for some $\nu \in A_i$, $l(\eta, \nu) \geqq \alpha$. Now we define by induction on $j < \lambda^+$ $a_j \in G_1$ such that

(1) $a_j \notin \{a_i : i < j\}$.

(2) if $i < j$, $a_j \in \mathrm{cl} A_i$, then $\{h_1(\{a_j, \eta\}) : \eta \in A_i\} = \lambda$.

If we succeed, the restriction of our colouring h_1 to $\{a_i : i < \lambda^+\}$ will give the required counterexample. (The requirement on triangles is already satisfied in G_1, and if $A \subseteq \{a_i : i < \lambda^+\}$, $|A| = \lambda^+$, then for some i, A_i is a dense subset of A and we can choose $j > i$ so that $a_j \in A$ and then all colours appear by (2).)

Choose $a_0 \in {}^\lambda 2$ arbitrarily. Now let $j > 0$ and assume a_i has been chosen for $i < j$. We now define a_j as follows.

First we define by induction on $\rho < \lambda$ sequences $\eta(\rho) \in \cup \{{}^{\alpha}2 : \rho \leqq \alpha < |\rho|^+ + \aleph_0\}$ such that $\xi < \rho \Rightarrow \eta(\xi) < \eta(\rho)$.

Let $\{a_i : i < j\}$ be renamed $\{b_\rho : \rho < \lambda\}$ (the b_ρ's are not neccessarily distinct).

Let $\{(A_i, \beta) : i < j, \beta < \lambda\}$ be renamed $\{(B_\rho, \beta_\rho) : \rho < \lambda\}$, such that $\beta_\rho \leqq \rho$. For $\rho = 0$ let $\eta(\rho)$ be empty, for ρ a limit ordinal, the limit of $\eta(\xi)$ $\xi < \rho$. Suppose $\eta(\rho)$ is defined. If for some $\nu \in B_\rho$, $\eta(\rho) < \nu$ choose η', $\eta(\rho) < \eta' < \nu$, $l(\eta') < |\rho|^+ + \aleph_0$, $f(l(\eta')) = \beta_\rho$ (possible by the requirement on f and the fact that $\beta_\rho \leqq \rho$), and let $\eta'' = \eta' \wedge \langle k \rangle$, $k \in \{0,1\}$, $\eta'' \nless \nu$. If there is no such ν let $\eta'' = \eta(\rho)$. Now let $\eta(\rho+1)$ be $\eta'' \wedge \langle k \rangle$, $k \in \{0,1\}$ so that $\eta(\rho+1) \nless b_\rho$. Finally put $a_j = \eta(\lambda) = \bigcup_{\rho<\lambda} \eta(\rho)$. It is easy to check that a_j satisfies (1) and (2).

THEOREM 1.2. $(V = L)$. *For λ regular there is a colouring h proving $\lambda^+ \not\to [\lambda^+]^2_{\lambda^+}$, in which there is no triangle with three colours.*

REMARK. Our proof is similar to a proof of the existence of Suslin trees of Jensen.

PROOF. As Jensen [6] proved there are sets $B_\alpha \subseteq \alpha$ $(\alpha < \lambda^+, \mathrm{cf}\,\alpha = \lambda)$ such that for every $B \subseteq \lambda^+$, $\{\alpha : \alpha < \lambda^+,\ \mathrm{cf}\,\alpha = \lambda,\ B \cap \alpha = B_\alpha\}$ is stationary. Let $f : \lambda^+ \to \lambda^+$ be such that for every $\alpha < \beta < \lambda^+$, $|\{j : \beta \leqq j < \beta + \lambda, f(j) = \alpha\}| = \lambda$. Now we define a function g such that

(1) g is from λ^+ into $\cup\{{}^{\alpha}2 : \alpha < \lambda^+\}$.

(2) g maps $\{\alpha : \alpha < \lambda\}$ onto $\cup\{{}^{\alpha}2 : \alpha < \lambda\}$.

(3) if $\lambda i \leqq \alpha < \lambda(i+1)$, $(i > 0)$, then $l(g(\alpha)) = \lambda + i$ (except for finite i, in which case $l(g(\alpha)) = \lambda + (i-1)$.)

(4) If η is in the range of g, and $\nu < \eta$ then ν is in the range of g.

(5) If $\delta < \lambda^+$, $\mathrm{cf}\,\delta < \lambda$, δ limit ordinal, $\eta \in {}^{\delta}2$, and every $\nu < \eta$ is in the range of g then η is in the range of g.

(6) If $\eta \in \mathrm{Range}(g)$, then $\eta^\wedge\langle i \rangle \in \mathrm{Range}(g)$ for $i = 0, 1$.

(7) If $\eta \in \mathrm{Range}(g)$, $l(\eta) < \alpha$, then there is $\nu \in {}^{\alpha}2 \cap \mathrm{Range}(g)$, $\eta < \nu$.

(8) If $\delta < \lambda^+$, $\mathrm{cf}\,\delta = \lambda$, $\eta \in {}^{\delta}2 \cap \mathrm{Range}(g)$, then either (i) there is $\nu < \eta$, such that for no $\alpha \in B_\delta$, $\nu < g(\alpha)$, or (ii) for every $i < \delta$ there is $\alpha \in B_\delta$, such that $f(l(g(\alpha), \nu)) = i$. This is easily done; and we define h on $G = \mathrm{Range}\ (g)$ by putting $h(\{\eta, \nu\}) = f(l(\eta, \nu))$. It is easy to check that this is a right example.

2. Colourings of finite sets in which there are no triangles with three colours

CLAIM 2.1. Suppose k is a natural number, h a colouring of ${}^{k}2$ with r colours, and g a function with domain $\bigcup_{l<k} {}^{l}2$ such that $h(\eta, \nu) = g(cn(\eta, \nu))$, then there is a set $A \subseteq {}^{k}2$ over which h is constant, $|A| \geqq 2^{k/r}$.

THEOREM 2.2. *Suppose h is a colouring of n by p colours (i.e. h is a function from the set of unordered pairs from* $\{0,\cdots,n-1\}$ *into some* T, $|T|=p$).

If in n there is no triangle with three colours, then there is a set $A\subseteq n$, $|A|\geqq n^{2/c}$, $[c=p(p-1)\log_2 pq$ where $q=\frac{1}{2}(p-1)^2+1]$ *such that in A only two colours appear.*

REMARK. For e.g. $p=3$, $|A|\geqq n^{1/12}$.

PROOF OF 2.2. First we observe that for any $A\subseteq n$, $|A|>1$, there are disjoint non-empty A_1, $A_2\subseteq A$, and colours t_1, t_2 such that $|A_1|\geqq|A|/p$, $|A-A_1-A_2|\leqq(p-2)|A_2|$, and for $a_l\in A_l$ $(l=1,2)$ $h(\{a_1,a_2\})$ is t_1 or t_2. This holds because if $a\in A$ and $A_t=\{b\in A: h(\{a,b\})=t\}$, then $\Sigma_{t\in T}|A_t|=|A|-1$ hence for some $t_1,t_2\in T, t_1\neq t_2$ we have $|A_{t_1}|\geqq(|A|-1)/p$, and $|A_t|\leqq|A_{t_2}|$ for $t\neq t_1$. As there is no triangle with three colours, if $a_l\in A_{t_l}$, $h(\{a_1,a_2\})$ is t_1 or t_2. Let $A_1=A_{t_1}\cup\{a\}$, $A_2=A_{t_2}$, then $|A_1|\geqq|A_{t_1}|+1\geqq(|A|-1)/p+1\geqq|A|/p$, $|A-A_1-A_2|=|\bigcup\{A_t: t\in T-\{t_1,t_2\}\}\leqq\Sigma\{|A_t|: t\in T-\{t_1,t_2\}\}\leqq(p-2)|A_{t_2}|=(p-2)|A_2|$. Clearly if $a_l\in A_l$, $h(\{a_1,a_2\}\}\in\{t_1,t_2\}$ (if $a_1\in A_{t_1}$ we have shown it, and if $a_1=a$ this holds by the definition of $A_2=A_{t_2}$). This proves the observation except when A_2 is empty. If this holds for any $a\in A$, h is constant on A, and then the proof is trivial.

Secondly we observe that if $A\subseteq n$, $|A|>1$ then there are disjoint sets $C_1,C_2\subseteq A$ and colours $t_1,t_2,\in T$ such that $|C_l|>|A|/pq$ $(l=1,2)$ and $h(\{a_1,a_2\})\in\{t_1,t_2\}$ for $a_l\in C_l$.

If $|A|<pq$ we can choose C_1, C_2 as singletons. Now assume $|A|\geqq pq$. Put $A_0=A$. We define by induction on i sets A_i, B_i, T_i, for $i\geqq 1$ so that $T_i\subseteq T$, $|T_i|=2$, A_i and B_i are disjoint subsets of A_{i-1}, $|A_i|\geqq|A_{i-1}|/p$, $|A_{i-1}-A_i-B_i|\leqq(p-2)\ |B_i|$ and $h(\{a,b\})\in T_i$ for $a\in A_i$ and $b\in B_i$. If $|A_{i-1}|>1$ the existence of A_i, B_i, T_i follows from the first observation and moreover, in this case, $B_i\neq\emptyset$. If $|A_{i-1}|=1$, we put $A_i=A_{i-1}$, $B_i=\emptyset$, $T_i=T_{i-1}$. Since $|A_{i-1}|>1$ implies that $|A_i|<|A_{i-1}|$, it follows that there is a least integer $k\geqq 1$ such that $|A_k|\leqq|A|/q$. Then $|A_{k-1}|>|A|/q$ and so $|A_k|>|A|/pq$.

Let $B^*=\bigcup_{0<i\leqq k}B_i$, then

$$|A-A_k-B^*|=\left|\bigcup_{i<k}(A_i-A_{i+1}-B_{i+1})\right|=\sum_{i<k}|A_i-A_{i+1}-B_{i+1}|$$

$$\leqq\sum_{i<k}(p-2)|B_{i+1}|=(p-2)|B^*|.$$

As A_k, B^* are disjoint, $|B^*| \geqq |A - A_k|/(p-1) \geqq |A|(1-1/q)/(p-1)$. For each $S \subseteq T$, $|S| = 2$ let $B_S = \cup\{B_i : T_i = S,\ 0 < i \leqq k\}$, so B^* is the disjoint union of the B_S's. Hence for some S,

$$|B_S| \geqq |B^*| \Big/ \binom{p}{2} = |B^*| \frac{2}{p(p-1)} \geqq \frac{2}{p(p-1)} \cdot \frac{1}{(p-1)} \cdot \frac{q-1}{q} |A|$$

$$= |A|/pq.$$

Clearly for every $a \in A_k$ $b \in B_S$, $h(\{a,b\}) \in S$, so letting $S = \{t_1, t_2\}$, $C_1 = A_k$, $C_2 = B_S$, we prove our second observation.

Now define by induction on $l \leqq k = \log_{pq} n$ sets $B_\eta \subseteq n$, and $T_\eta \subseteq T$ for $\eta \in {}^l 2$ such that $B_{\langle\rangle} = n$, $|B_\eta| \geqq |A|/(pq)^l$ where $l = l(\eta)$, and $B_{\eta \wedge \langle 0 \rangle}$, $B_{\eta \wedge \langle 1 \rangle}$ are disjoint subsets of B_η, $|T_\eta| = 2$ and $a_r \in B_{\eta \wedge \langle r \rangle} \Rightarrow h(\{a_1, a_2\}) \in T_\eta$. This is easily done by our second observation. Choose $a_\eta \in B_\eta$ for $\eta \in {}^k 2$ and define the colouring h' of ${}^k 2$ by $h'(\{\eta, \nu\}) = T_{\cap(\eta,\nu)}$. So the theorem follows from Claim 2.1 (i.e. choose for 2.1 h' as h, k as k, T_η as $g(\eta)$, $\binom{p}{2}$ as r); and if A satisfies the conclusion of 2.1, $\{a_\eta : \eta \in A\}$ satisfies the conclusion of 2.1, its cardinality being $\geqq 2^{2k/p(p-1)} = n^{2/c}$, $c = (\log_2 pq)p(p-1)$.

Proof of 2.1. For $\eta \in {}^l 2$, $l \leqq k$, let $A_\eta = \{\nu : \nu \in {}^k 2,\ \eta \leqq \nu\}$ (so $|A_\eta| = 2^{k-l}$) and for a colour s, $f_s(\eta)$, denote the cardinality of the largest subset of A_η over which h has the constant value s. Let $r = \{0, \cdots, r-1\}$ be the set of colours. We now prove by downward induction on $l \leqq k$

$$(*) \qquad \prod_{i<r} f_i(\eta) \geqq 2^{k-l} \quad \text{for any} \quad \eta \in {}^l 2.$$

For $l = k$ this is trivial. Suppose we have proved it for $l+1$, and let us prove it for l. Assume $\eta \in {}^l 2$, then trivially

(1) for $i < r$, $f_i(\eta) \geqq \max\{f_i(\eta\hat{\ }\langle 0 \rangle),\ f_i(\eta\hat{\ }\langle 1 \rangle)\}$

(2) for $i = g(\eta) < r$, $f_i(\eta) \geqq f_i(\eta\hat{\ }\langle 0 \rangle) + f_i(\eta\hat{\ }\langle 1 \rangle)$.

Hence

$$\prod_{i<r} f_i(\eta) = \Big(\prod_{\substack{i<r \\ i \neq g(\eta)}} f_i(\eta) \Big) f_{g(\eta)}(\eta) =$$

by (2)

$$\left[\prod_{\substack{i<r\\ i\neq g(\eta)}} f_i(\eta)\right][f_{g(\eta)}(\eta\,\hat{}\,\langle 0\rangle)+f_{g(\eta)}(\eta\,\hat{}\,\langle 1\rangle]$$

$$=\left[\prod_{\substack{i<r\\ i\neq g(\eta)}} f_1(\eta)\right]f_{g(\eta)}(\eta\,\hat{}\,\langle 0\rangle+\left[\prod_{\substack{i<r\\ i\neq g(\eta)}} f_i(\eta)\right]f_{g(\eta)}(\eta\,\hat{}\,\langle 1\rangle)\geqq;$$

and by (1)

$$\geqq\left[\prod_{\substack{i<r\\ i\neq g(\eta)}} f_1(\eta\,\hat{}\,\langle 0\rangle)\right]f_{g(\eta)}(\eta\,\hat{}\,\langle 0\rangle$$

$$+\left[\prod_{\substack{i<r\\ i\neq g(\eta)}} f_i(\eta\,\hat{}\,\langle 1\rangle)\right]f_{g(\eta)}(\eta\,\hat{}\,\langle 1\rangle)$$

$$=\prod_{i<r} f_i(\eta\,\hat{}\,\langle 0\rangle)+\prod_{i<r} f_i(\eta\,\hat{}\,\langle 1\rangle).$$

By the induction hypothesis,

$$\geqq 2^{k-(l+1)}+2^{k-(l+1)}=2^{k-l}.$$

Taking $l=0$, $\eta=\langle\,\rangle$, we get that for some $i<r\ f_i(\langle\,\rangle)\geqq 2^{k/r}$, i.e. there is a set $A\subseteq A_{\langle\rangle}={}^k 2$ on which h is constant, $|A|\geqq 2^{k/r}$.

We have not tried to make the exponent in 2.2 the best possible. Clearly we cannot improve 2.1 (e.g. if $g(\eta)=l(\eta)[\mathrm{mod}\, r]$), and this gives an upper bound on the possible exponent in 2.2 ($\leqq 2/p$).

Open Question 1. What is the right exponent in 2.2?

3. Another kind of homogeneity

Theorem 3.1. *Suppose h is a colouring of a regular λ proving $\lambda\not\to[\lambda]^2_\mu$, which has no triangle with three colours. Then there is $j_0<\mu$ such that for every pair of functions g, f from $\bigcup_{l<\omega}{}^l 2$ into μ, $\{i: i<\mu,\ i\neq j_0\}$ respectively, there are distinct $a_\eta\in\lambda\,(\eta\in\bigcup_{l<\omega}{}^l 2)$ such that:*

(1) *if $\eta<\nu$ then $h(\{a_\eta,a_\nu\})=g(\eta)$; and*

(2) *if $\eta\,\hat{}\,\langle 0\rangle<\nu$, $\eta\,\hat{}\,\langle 1\rangle<\rho$ then $h(\{a_\nu,a_\rho\})=f(\eta)$.*

Proof. First we show that

(*) there are $A_0 \subseteq \lambda$, $|A_0| = \lambda$ and $j_0 < \mu$, such that for every $A \subseteq A_0$, $|A| = \lambda$ and $i < \mu$, $i \neq j_0$, there are disjoint $A_1, A_2 \subseteq A$, $|A_1| = |A_2| = \lambda$, and for every $a_l \in A_l$ $h(\{a_1, a_2\}) = i$.

For $A \subseteq \lambda$, $|A| = \lambda$ and $i \in \mu$, let $P(A,i)$ be the assertion: whenever A_1, A_2 are disjoint subsets of A, $|A_1| = |A_2| = \lambda$, then there are $a_l \in A_l$, $(l = 1,2)$ such that $h(\{a_1, a_2\}) \neq i$. If $P(A,i)$ is false for all A and i, then (*)holds with $A_0 = \lambda$ and $j_0 = 0$. Now suppose that there are $A_0 \subseteq \lambda$ and $j_0 < \mu$ such that $|A_0| = \lambda$ and $P(A_0, j_0)$ holds. We will deduce (*).

Let $A \subseteq A_0$, $|A| = \lambda$. For $a \in A$ and $j < \mu$, let $h_j(a) = \{b \in A : h(\{a,b\}) = j\}$. For any $i \in \lambda - \{j_0\}$(**) there is a $a \in A$ such that $|h_{j_0}(a)| = |h_i(a)| = \lambda$. For, if this is false, then for each $x \in A$, there is $j(x) \in \{i, j_0\}$ such that $|h_{j(x)}(a)| < \lambda$.

Then there is $A' \subseteq A$ such that $|A'| = \lambda$ and $j(x) = j_1 (x \in A')$ and so by the regularity of λ there is $A'' \subseteq A'$ such that $|A''| = \lambda$ and $y \notin h_j(x)$ for all $x, y \in A''$. This contradicts the hypothesis that h proves that $\lambda \not\to [\lambda]^2_\mu$. Therefore we can assume that $B_1 = h_{j_0}(a)$ and $B_2 = h_i(a)$ have power λ.

Clearly $h(\{b_1, b_2\}) \in \{i, j_0\}$ for every $b_l \in B_l$ (otherwise a, b_1, b_2 form a triangle with three colours). Let E be the following equivalence relation on $B_1 : b_1 E b_2$ iff for every $c \in B_2$ $h(\{b_1, c\}) = h(\{b_2, c\})$. If E has λ equivalence classes, let $\{b_i : i < \lambda\}$ be a set of representatives. As h proves $\lambda \not\to [\lambda]^2_\mu$, there are $i(1) < i(2) < \lambda$ such that $h(\{b_{i(1)}, b_{i(2)}\}) \neq i, j_0$. As $b_{i(1)}$, $b_{i(2)}$ are not E-equivalent, for some $c \in B_2$, $h(\{b_{i(1)}, c\}) \neq h(\{b_{i(2)}, c\})$ but both belong to $\{i, j_0\}$. Hence c, $b_{i(1)}$, $b_{i(2)}$ form a triangle with three colours, contradiction. So E has $< \lambda$ equivalence classes, hence by the regularity of λ, some E-equivalence class $B'_1 \subseteq B_1$ has cardinality λ. Repeating this argument, interchanging B_1 and B_2, we get $B'_2 \subseteq B_2$, $|B'_2| = \lambda$ such that for all $b_1 \in B'_1$, $b_2 \in B'_2$, $h(\{b_1 b_2\})$ has the same value, which belongs to $\{i, j_0\}$. By the choice of A_0, j_0 it cannot be j_0 so it is i. So we have proved that there are A_0, j_0 such that (*) holds.

Now we define by induction on l, for every $\nu \in \bigcup_{k<l} {}^k 2$, $\eta \in {}^l 2$, elements a_ν and sets A_η such that

(1) $a_\nu \in A_\nu$.

(2) $A_{\nu^\frown\langle 0\rangle}$, $A_{\nu^\frown\langle 1\rangle}$ are disjoint subsets of A_η.

(3) $|A_\eta| = \lambda$.

(4) if $b \in A_{\eta^\frown\langle 0\rangle} \cup A_{\eta^\frown\langle 1\rangle}$, then $h(\{a_\eta, b\}) = g(\eta)$.

(5) if $a_l \in A_{\nu^\frown\langle l\rangle}$ then $h(\{a_0, a_1\}) = f(\nu)$

The induction step is done by using (**) and then (*); and $A_{\langle\rangle} = A_0$ from (*). The a_ν's we get satisfy the requirements of the theorem.

OPEN QUESTION 2. Can we in 3.1 remove the j_0?

At this stage the following partition relations seem natural.

DEFINITION 3.1. For (not necessarily infinite) cardinals λ, μ, and for $r < \omega$ let

$$\lambda \xrightarrow{wt} (n)^r_\mu$$

means that if h is an r-colouring of λ with μ colours (i.e. a function from $[\lambda]^r = \{A : A \subset \lambda, |A| = r\}$, then there are distinct $a_\eta \in \lambda$ ($\eta \in {}^n 2$)) such that:

if $\eta(1), \cdots, \eta(r) \in {}^n 2$, $\tau(1), \cdots, \tau(r) \in {}^n 2$, and for some m $l(\eta(i), \tau(i)) \geqq m$, but $l(\eta(i), \eta(j)) < m$ for $1 \leqq i \leqq r$, $1 \leqq i < j \leqq r$ *then* $h(\{\eta(1), \cdots\}) = h(\{\tau(1), \cdots\})$.

DEFINITION 3.2. Similarly

$$\lambda \xrightarrow{st} (n)^r_\mu$$

means that if h is an r-colouring of λ with μ colours then there are distinct $a_\eta \in \lambda (\eta \in {}^n 2)$ such that:

if $\eta(1) \cdots \eta(r) \in {}^n 2$; $\tau(1), \cdots, \tau(r) \in {}^n 2$, and for all $l \leqq i < j \leqq r, 1 \leqq k \leqq r$

$$l_{i,j} \overset{df}{=\!=} l(\eta(i), \eta(j)) = l(\tau(i), \tau(j)), \text{ and the values of}$$

$\eta(i), \tau(i)$ in the $l_{k,j}$-th place are equal *then* $h(\{a_{\eta(1)}, \cdots\}) = h(\{a_{\tau(1)} \cdots\})$.

REMARK. The letters *wt*, *st* stands for weak tree, strong tree. Few facts are known.

CLAIM 3.2.

(1) If $\lambda \to (2^n)^r_\mu$, then $\lambda \xrightarrow{st} (n)^r_\mu$.

(2) If $\lambda \xrightarrow{st} (n)^r_\mu$ then $\lambda \xrightarrow{wt} (n)^r_\mu$.

(3) If $\lambda(1) \geqq \lambda$, $\mu(1) \leqq \mu$, $n(1) \leqq n$ then $\lambda \xrightarrow{xt} (n)^r_\mu$ implies $\lambda(1) \xrightarrow{xt} (n(1))^r_{\mu(1)}$ for $x = w, s$. Similar monotonicity results hold for the index $r \leqq n$.

PROOF. Trivial.

THEOREM 3.3. *For every n, $r < \omega$ there is $l = l(n,r) < \omega$ such that: for every $k < \omega$, $k^l \xrightarrow{wt} (n)^l_k$.*

PROOF. In a different notation, this is proved in Shelah [7].

THEOREM 3.4. $\aleph_{\alpha+n} \xrightarrow{wt} (n)^2_{\aleph_\alpha}$ *for $n < \omega$.*

PROOF: We prove it by induction on n, for $n = 0$, $n = 1$ it is trivial; so let us prove it for $n + 1$.

Clearly there is $A_0 \subseteq \aleph_{\alpha+n+1}$, $|A_0| = \aleph_{\alpha+n}$, such that for every $m < \omega$, $b_l \in A_0 (l < m)$ and $c \in \aleph_{\alpha+n+1}$ if $\{a : a \in \aleph_{\alpha+n+1}$ and for every $l < m$, $h(\{a, b_l\}) = h(\{c, b_l\})$ has cardinality $\leqq \aleph_{\alpha+n}$, then it is a subset of A_0.

Now choose $c \in \aleph_{\alpha+n+1}$, $c \notin A_0$, and $i < \aleph_\alpha$ such that $A_1 = \{a : a \in A, h(\{c, a\}) = i\}$ has cardinality $\aleph_{\alpha+n}$. Choose $a^0_\eta \in A_1 (\eta \in {}^n 2)$ by the induction hypothesis. Now let $A_2 \subseteq \{a : a \in \aleph_{\alpha+n+1}$ and $\eta \in {}^n 2 \Rightarrow h(\{a^0_\eta, a\}) = h(\{a^0_\eta, c\})\}$, $|A_2| = \aleph_{\alpha+n}$, and choose $a^1_\eta \in A_2 (\eta \in {}^n 2)$ by the induction hypothesis. Now define $a_{\eta \wedge \langle l \rangle} = a^l_\eta$ for $l = 0, 1, \eta \in {}^n 2$, and we get the conclusion for $n + 1$.

LEMMA 3.5. *If $\lambda \leqq \mu < \operatorname{Ded} \lambda$* (i.e. *there is an ordered set I, $|I| = \mu$, with a dense subset J, $|J| = \lambda$*), *then there is not $\mu \xrightarrow{st} (2)^2_\lambda$.*

PROOF: Let, for $a \neq b \in I$, $h(\{a, b\})$ be any element of J which is between a and b.

We can prove in a way similar to [7]:

THEOREM 3.6. *If λ is real valued measurable, $\mu < \lambda$ then $\lambda \xrightarrow{wt} (n)^r_\mu$.*

OPEN QUESTION 3. Is there $l = l(n, r) < \omega$ for every n, $r < \omega$ such that $\aleph_{\alpha+l} \xrightarrow{wt} (n)^r_{\aleph_\alpha}$.

(By Claim 3.2, it is interesting only without G.C.H., see [4].)

OPEN QUESTION 4. Is it true that for every $k < \omega$ big enough (relative to n, r)

$$2^k \xrightarrow{st} (n)^r_k?$$

OPEN QUESTION 5.
(1) Find the best l in 3.3.

(2) Is there $l = l(n,r)$ such that $k^l \xrightarrow{st} (n)^r_k$ (e.g., we do not know the answer even for $n = r = 2$, where we asked for the existence of $a_l (l = 1,\cdots,4)$ such that a_1a_3, a_1a_4, a_2a_3, a_2a_4 have the same colour, and a_1a_2, a_3a_4 have the same colour)?

4. Universality and negative partition relations

DEFINITION 4.1. *A colouring h of λ is μ-universal, if for every colouring h' of μ with the same set of colours, there is a one-to-one function g from μ into λ such that $h'(\{i,j\}) = h(\{g(i),g(j)\})$ for any distinct i, $j < \mu$.*

The following theorem is included by the kind permission of Erdös and Hajnal.

THEOREM 4.1. *Let h be a colouring of $\aleph_{\alpha+1}$ with range C such that $h(A \times B) = C$ whenever A, B are disjoint subsets of $\aleph_{\alpha+1}$ such that $|A| = \aleph_\alpha$ and $|B| = \aleph_{\alpha+1}$. Then h is $\aleph_0$-universal.*

PROOF. Let h' be a colouring of ω with the same set of colours, μ, so w.l.o.g. $\mu \leqq \aleph_0$. We define by induction on $n \geqq 0$, α_n and A^n_η $(\eta \in {}^n\mu)$:
(1) $A^n_\eta \subseteq \aleph_{\alpha+1}, |A^n_\eta| = \aleph_{\alpha+1}$; and $n < m, g = f \restriction n, f \in {}^m\mu$ implies $A^m_f \subseteq A^n_g$;
(2) $\alpha_n < \aleph_{\alpha+1}$, $\alpha_n < \alpha_{n+1}$; and $\alpha_n \in A^n_{\eta_n}$ where $\eta_n(i) = h'(\{n,i\})$ for $i < n$.
(3) If $i < n$, $\alpha \in A^n_\eta$, then $h(\{\alpha_i, \alpha\}) = \eta(i)$.

Put $A^0_{\langle\rangle} = \aleph_{\alpha+1}$. Now suppose that $n \geqq 0$ and that A^n_η has been defined for $\eta \in {}^n\mu$. We shall define α_n and A^{n+1}_η for $\eta \in {}^{n+1}\mu$. For any $\gamma \in A^n_{\eta_n}$ and $\eta \in {}^{n+1}\mu$, let

$$A^{n+1}_\eta(\gamma) = \{\beta : \beta \in A^n_{\eta \restriction n}, \text{and } h(\{\gamma,\beta\}) = \eta(n)\}.$$

If for some such γ, $|A^{n+1}_\eta(\gamma)| = \aleph_{\alpha+1}$ for all $\eta \in {}^{(n+1)}\mu$, then let $\alpha_n = \gamma$, $A^{n+1}_\eta = A^{n+1}_\eta(\gamma)$. Suppose on the contrary, that for every $\gamma \in A^n_{\gamma_n}$ there is some $\tau_\gamma \in {}^{n+1}\mu$ such that $|A^{n+1}_{\tau_\gamma}(\gamma)| \leqq \aleph_\alpha$. Since $\mu \leqq \aleph_0 < \aleph_{\alpha+1}$, there is $\tau \in {}^{n+1}\mu$ such that $B = \{\gamma \in A^n_{\eta_n} : \tau_\gamma = \tau\}$ has cardinality $\aleph_{\alpha+1}$. This means that $A^{n+1}_\tau(\gamma)| \leqq \aleph_\alpha$

for $\gamma \in B$. Choose $B_0 \subset B$ such that $|B_0| = \aleph_\alpha$, and let $B_1 = A^n_{\tau \restriction n} - \bigcup_{\gamma \in B_0} A^{n+1}_\tau(\gamma)$. Then $|B_1| = \aleph_{\alpha+1}$ and $h(\{\beta,\gamma\}) \neq \tau(n)$ for $\beta \in B_1$, $\gamma \in B_0$. This contradicts the hypothesis of the theorem and shows that there is $\gamma \in A^n_{\eta_m}$ such that $|A^{n+1}_\eta(\gamma)| = \aleph_{\alpha+1}$ for all $\eta \in {}^{n+1}\mu$.

THEOREM 4.1. *It is consistent with ZFC + GCH that:*

There is a colouring h of $\aleph_2$ by two colours, which contains no monochromatic $[[\aleph_1,\aleph_2]]$*-biparite graph; and is not $\aleph_1$-universal.*

PROOF. Let h be such a colouring on L, and we prove that this statement holds in $L(G) - L$ when we add to it one generic subset of ω (see P. Cohen [1]). Now we have to prove the statements:

(1) Also in $L(G)$, there is nomonochromatic $[[\aleph_1,\aleph_0]]$-biparite graph.

(2) $(\aleph_2, h)$ is not $\aleph_1$-universal.

Remember that in $L(G)$, the cofinality function and the cardinals are the same as in L.

Now (1) holds as for each set A in $L(G)$, $A \subseteq L$, $\mathrm{cf}|A| > \aleph_0$, there is $B \subseteq A$, $|B| = |A|$, $B \in L$.

As for (2), let G_1 be a graph of power $\aleph_1$. If it can be embedded, by an embedding f, some countable subset of G_1 is mapped into a subset B of $\aleph_2$ which $\in L$. Each $\alpha < \omega_2$ induces a partition of B into two sets, $((\{a \in B : h(\{a,\alpha\}) = c\}$, c a colour), but all those partition $\in L$, so not all possible partition appears. But we could have chosen G_1 so that this is impossible.

REMARK. This same method can be applied to many other negative partition relations.

It seemed likely that this statement is not only consistent with *ZFC*, but is provable in *ZFC*.

Note added in proof: (1) It seems likely that the regularity of λ can be omitted in 12 using Jensen's square (just as he proved the existence of Suslin trees).

(2) Question 3 has been answered positively in: S. Shelah, *A two-cardinal theorem and a combinatorial theorem* (submitted to Proc. Amer. Math. Soc.).

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