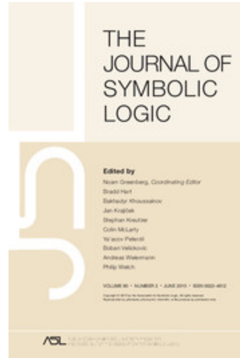

There are reasonably nice logics

Wilfrid Hodges and Saharon Shelah

The Journal of Symbolic Logic / Volume 56 / Issue 01 / March 1991, pp 300 - 322
DOI: 10.2307/2274921, Published online: 12 March 2014

THE JOURNAL OF SYMBOLIC LOGIC
Volume 56, Number 1, March 1991

THERE ARE REASONABLY NICE LOGICS

WILFRID HODGES AND SAHARON SHELAH

A well-known question of Feferman asks whether there is a logic which extends the logic $L(\exists^{\geq\aleph_1})$ $(=L(Q_1))$, is $\aleph_0$-compact and satisfies the interpolation theorem. (Cf. Makowsky [M] for background and terminology.)

The same question was open when $\aleph_1$ in $\exists^{\geq\aleph_1}$ is replaced by any other uncountable cardinal κ. We shall show that when κ is an uncountable strongly compact cardinal and there is a strongly compact cardinal $>\kappa$, then there is such a logic. It is impossible to prove the existence of uncountable strongly compact cardinals in ZFC. However, the logic that we describe has a simple and natural definition, together with several other pleasant properties. For example it satisfies Robinson's lemma, PPP (pair preservation property, viz. the theory of the sum of two models is the sum of their theories), versions of the elementary chain lemma for chains of length $<\lambda$, and isomorphism of (suitable) ultralimits.

This logic is described in §2 below; we call it $\mathscr{L}^1$. It is not a new logic—it was introduced in [Sh, Part II, §3] as an example of a logic which has the amalgamation and joint embedding properties. See the transparent presentation in [M]. But we shall repeat all the definitions. In [HS] we presented a logic with some of the same properties as $\mathscr{L}^1$, also based on a strongly compact cardinal λ; but unlike $\mathscr{L}^1$, it was not a sublogic of $\mathscr{L}_{\lambda,\lambda}$.

In §1 we describe a logic $\mathscr{L}^0$ which extends $L(\exists^{\geq\kappa})$ (and even Magidor-Malitz logic in the κ-interpretation) when κ is a weakly compact cardinal. It is also $(\aleph_0, <\kappa)$-compact. This logic seems not to satisfy the Robinson lemma or the interpolation theorem. But it does have two interesting and unusual features.

First, it satisfies the natural analogue of the Łoś-Tarski theorem (characterizing the first-order sentences preserved in submodels). This was a property that people were interested in finding in generalised logics—see for example Kaufmann's article [K]. And second, it has reasonable axioms. Very few generalised logics have axiom schemas that one can reasonably write down. (Keisler gave excellent axioms for $L(Q_1)$, and Barwise [B] and Schmerl [Sc] proved some positive results. On the other hand, Shelah and Steinhorn [SS1], [SS2] showed that there are no simple axiomatizations of $L(\exists^{\geq\beth_\omega})$ or $L(Q_2^{MM})$. See also the problem list in [CK].)

A logic along the same lines as $\mathscr{L}^0$ but extending $L(\exists^{\geq\beth_\omega})$ will be presented elsewhere.

Received September 22, 1987; revised February 6, 1990.

The logics $\mathscr{L}^0$ and $\mathscr{L}^1$ and the theorems about them are due to the second author, who wrote a draft of the whole paper and asked the first author to work out the details of Axiom K. The first author did this, and at the same time he rewrote §1, adding a number of proofs. We thank Yacov Kupelevich for checking the paper, Ian Hodkinson for many improvements, and the referee for his careful reading and corrections.

§1. A compact logic at weakly compact cardinals. Let κ be a fixed uncountable weakly compact cardinal. We shall describe a logic called $\mathscr{L}^0$, which depends on κ. The symbols of $\mathscr{L}^0$ are those of first-order logic together with a new quantifier $(\bigoplus\bar{x})$, or more precisely a family of new quantifiers $(\bigoplus\bar{x})$ depending on the variables $\bar{x}$. At a first approximation, $(\bigoplus\bar{x})\phi(\bar{x})$ means "There is a sequence of (not necessarily distinct) elements, of length κ, in which all increasing tuples satisfy ϕ."

1.1. The language $\mathscr{L}^0$ and its semantics. Let τ be a vocabulary and let M be a τ-structure. If $(X, <)$ is a linearly ordered set and $n < \omega$, we define $[X]^n$ (or more pedantically $[(X, <)]^n$) to be the set of all strictly increasing n-tuples of elements of X. $[X]^0$ is a singleton set. By a *framework* for M we mean the following: an ordered set $(X, <)$ of order-type κ, and a sequence $(f^i: i < \gamma)$ of functions, where $\gamma \leq \kappa$ and for each $i < \gamma$ there is $n(i) < \omega$ such that f^i maps $[X]^{n(i)}$ to the domain of M. When $n(i) = 0$, f^i is a constant element of M. We say that this framework is *on* $(X, <)$. Note that the images of the functions f^i need not cover the whole of M.

Let F be a framework for M, as above. If Y is a subset of X which also has order-type κ—or equivalently, a cofinal subset of X—then we write $F \mid Y$ for the framework consisting of $(Y, <)$ and the sequence of restrictions of the functions f^i to the sets $[Y]^{n(i)}$.

We write $I(F)$, the set of *F-index pairs*, for the set of all pairs $(i, \bar{\alpha})$ where $i < \gamma$ and $\bar{\alpha}$ is an element of $[\kappa]^{n(i)}$. We write ρ for an element of $I(F)$, and we introduce a variable x_ρ for each element ρ of $I(F)$. Suppose $\phi(x_1, \ldots, x_m)$ is a formula which makes sense for τ-structures, and $\rho_1, \ldots, \rho_m$ are elements of $I(F)$. Then we have a formula $\phi(x_{\rho_1}, \ldots, x_{\rho_m})$ which is interpreted as follows:

(1) *F satisfies* $\phi(x_{\rho_1}, \ldots, x_{\rho_m})$ in M iff
$M \models \phi(f^{i_1}(\bar{a}_1), \ldots, f^{i_m}(\bar{a}_m))$, where for each k $(1 \leq k \leq m)$, ρ_k is some pair $(i_k, \bar{\alpha}_k)$ and $\bar{a}_k$ is the image of $\bar{\alpha}_k$ under the order-isomorphism from κ to X.

We say that

(2) *F indiscernibly satisfies* $\phi(x_{\rho_1}, \ldots, x_{\rho_m})$ in M iff
for every subset Y of X of order-type κ, $F \mid Y$ satisfies $\phi(x_{\rho_1}, \ldots, x_{\rho_m})$.

(Note that $I(F \mid Y) = I(F)$.) Finally we define

(3) $M \models (\bigoplus x_{\rho_1} \cdots x_{\rho_m})\phi(x_{\rho_1}, \ldots, x_{\rho_m})$ iff
there is a framework for M which indiscernibly satisfies $\phi(x_{\rho_1}, \ldots, x_{\rho_m})$.

In these definitions we allow $\phi(x_{\rho_1}, \ldots, x_{\rho_m})$ to contain some elements of M as parameters. So (3) serves as a clause in an inductive truth definition for $\mathscr{L}^0$.

The set $I(F)$ depends on F, because it depends on the number γ of functions and the arity $n(i)$ of the ith function. We can remove this dependence as follows. We assume that in every framework the functions are $(f^i: i < \kappa)$, and we make each f^i a map defined on the union $[X]^0 \cup [X]^1 \cup \cdots$. So henceforth I is the set of all pairs $(i, \bar{\alpha})$ where $i < \kappa$ and $\bar{\alpha}$ is an increasing tuple of ordinals $< \kappa$.

The elements of the set I will be called simply *index pairs*. We write $\bar{\rho}$ for a tuple $(\rho_1, \ldots, \rho_m)$ of distinct index pairs. If ρ_k is the index pair $(i_k, \bar{\alpha}_k)$, then the *support* of ρ_k, in symbols $\operatorname{supp}(\rho_k)$, is the set of all ordinals in $\bar{\alpha}_k$. The *support* of $\bar{\rho}$, $\operatorname{supp}(\bar{\rho})$, is the union of the supports of the index pairs ρ_k $(1 \leq k \leq m)$.

We shall say that a sequence $\bar{\rho} = ((i_1, \bar{\alpha}_1), \ldots, (i_m, \bar{\alpha}_m))$ of index pairs is *separated by length* if $i_j = i_k$ implies that $\bar{\alpha}_j$ and $\bar{\alpha}_k$ have the same length, whenever $1 \leq j < k \leq m$. Axiom C below will tell us that every quantifier $(\bigoplus x_{\bar{\rho}})$ is equivalent to a quantifier $(\bigoplus x_{\bar{\sigma}})$ in which $\bar{\sigma}$ is separated by length.

The following paraphrase of definition (3) is often useful. For each k $(1 \leq k \leq m)$ let ρ_k be the index pair $(i_k, \bar{\alpha}_k)$. Let $\alpha_1, \ldots, \alpha_n$ be the support of $(\rho_1, \ldots, \rho_m)$, listed in increasing order. Let F be a framework for M, consisting of an ordered set $(X, <)$ and a sequence $(f^i: i < \gamma)$ of functions. Let $\bar{b}$ be an increasing n-tuple $(b_1, \ldots, b_n)$ from $(X, <)$. Then we shall say that

(4) $\bar{b}$ *satisfies* the formula $\phi(x_{\rho_1}, \ldots, x_{\rho_m})$ in F iff
$M \models \phi(f^{i_1}(\bar{b}_1), \ldots, f^{i_m}(\bar{b}_m))$, where each $\bar{b}_k$ comes from $\bar{\alpha}_k$ by replacing each α_κ by b_κ $(1 \leq \kappa \leq n)$.

LEMMA 1. *$M \models (\bigoplus x_{\rho_1} \cdots x_{\rho_m})\phi(x_{\rho_1}, \ldots, x_{\rho_m})$ if and only if there is a framework F for M, with ordered set $(X, <)$, such that (in the notation of (4) above) each n-tuple $\bar{b} \in [X]^n$ satisfies $\phi(x_{\rho_1}, \ldots, x_{\rho_m})$ in F.*

PROOF. Put $\bar{\rho} = (\rho_1, \ldots, \rho_m)$. The condition is certainly sufficient. To show that it is necessary, suppose $M \models (\bigoplus x_{\rho_1} \cdots x_{\rho_m})\phi(x_{\rho_1}, \ldots, x_{\rho_m})$, and let F be a framework for M as in (3). In the notation of (4), it follows at once that *some* n-tuples $\bar{b}$ in $[X]^n$ satisfy $\phi(x_{\rho_1}, \ldots, x_{\rho_m})$ in F; but not necessarily all n-tuples $\bar{b}$ in $[X]^n$, since there may be gaps between the ordinals in the support of $\bar{\rho}$. Let β be the smallest ordinal such that for any two distinct ordinals $\alpha_1 < \alpha_2$ occurring in the support of $\bar{\rho}$ we have $\beta \geq \alpha_1$ and $\alpha_1 + \beta \geq \alpha_2$. Let Y be the set of those elements of X which correspond in the ordering of X to the ordinals $\gamma \cdot \beta$ with $\gamma < \kappa$. Then every tuple $\bar{b}$ in $[Y]^n$ satisfies $\phi(x_{\rho_1}, \ldots, x_{\rho_m})$ in M. So the framework $F \mid Y$ satisfies the condition of the lemma. $\square$

DEFINITION 2. For a vocabulary τ, we define the language $\mathscr{L}^0(\tau)$ to be the first-order language of τ with the following changes: first, the variables are the symbols x_ρ with $\rho \in I$; and second, for all distinct $\rho_1, \ldots, \rho_m \in I$ with $m \geq 0$, there is a new quantifier $(\bigoplus x_{\rho_1} \cdots x_{\rho_m})$ in addition to the usual quantifiers $\forall x_\rho$ and $\exists x_\rho$. This describes the syntax of $\mathscr{L}^0$. The semantics is as for the usual first-order logic, with clause (3) added. We abbreviate $(\bigoplus x_{\rho_1} \cdots x_{\rho_m})\phi(x_{\rho_1}, \ldots, x_{\rho_m})$ to $(\bigoplus x_{\bar{\rho}})\phi(x_{\bar{\rho}})$, where $\bar{\rho}$ is the tuple $(\rho_1, \ldots, \rho_m)$ of index pairs.

EXAMPLE 3. The quantifier $\exists^{\geq\kappa}$ is expressible in $\mathscr{L}^0$. For this, let ρ and ρ' be the two index pairs $(0, \langle 0 \rangle)$ and $(0, \langle 1 \rangle)$. Then for any formula $\phi(x, \bar{y})$, the formula

$$(\bigoplus x_\rho x_{\rho'})(\phi(x_\rho, \bar{y}) \wedge x_\rho \neq x_{\rho'}) \tag{5}$$

expresses that there is a family $(a_i: i < \kappa)$ of elements such that $\phi(a_i, \bar{y})$ holds for each $i < \kappa$, and $a_i \neq a_j$ whenever $i < j < \kappa$. Clearly this says that at least κ elements x satisfy $\phi(x, \bar{y})$.

Sh:271

Extending this example, let $\phi(x_1,\dots,x_n)$ be a formula, and let $\psi(x_1,\dots,x_n)$ be the conjunction of the formulas $\phi(x_{\pi(1)},\dots,x_{\pi(n)})$ as π ranges over all permutations of $\{1,\dots,n\}$. Then the formula

$$(6)\quad (\bigoplus x_{(0,\langle 1\rangle)}\cdots x_{(0,\langle n\rangle)})\Big(\psi(x_{(0,\langle 1\rangle)},\dots,x_{(0,\langle n\rangle)}) \wedge \bigwedge_{1\le i<j\le n} x_{(0,\langle i\rangle)} \neq x_{(0,\langle j\rangle)}\Big)$$

is equivalent to the Magidor-Malitz formula $(Q^n x_1\cdots x_n)\phi(x_1,\dots,x_n)$ in the κ-interpretation (cf. [MM]). Thus $\mathscr{L}^0$ has at least the strength of Magidor-Malitz logic in the κ-interpretation—for example, it can say that an equivalence relation $E(x,y)$ has at least κ equivalence classes.

The quantifier $(\bigoplus x_\rho)$, with a single variable x_ρ, is equivalent to the quantifier $\exists x_\rho$. So there would be no loss of expressive power if we left $\forall$ and $\exists$ out of the language. But for the sake of familiarity we keep them in.

1.2. Axioms for $\mathscr{L}^0$. We turn to some logical properties of the quantifier $(\bigoplus x_{\bar\rho})$. We write them as axiom schemas that can be added to any standard set of axioms for first-order logic. For brevity we say "axiom" rather than "axiom schema". The first two are self-evident from the definitions.

AXIOM A (Redundant variables). Let each of $\bar\rho$ and $\bar\sigma$ be a finite sequence of distinct index pairs, and let ϕ be a formula such that for every free variable x_π of ϕ, π occurs in $\bar\rho$ if and only if it occurs in $\bar\sigma$. Then $\vdash(\bigoplus x_{\bar\rho})\phi \leftrightarrow (\bigoplus x_{\bar\sigma})\phi$.

AXIOM B (Redundant quantifiers). If ϕ is a formula in which none of the variables $x_{\bar\rho}$ occur free, then $\vdash(\bigoplus x_{\bar\rho})\phi \leftrightarrow \phi$.

The next two axioms describe the effect of changing bound variables.

AXIOM C (Renaming). Suppose $i, j < \kappa$ and $n < \omega$. Suppose that $\bar\sigma$ comes from $\bar\rho$ by replacing each index pair of form $(i,\bar\alpha)$ (where $\bar\alpha$ has length n) by the pair $(j,\bar\alpha)$, and ψ comes from ϕ by replacing all free occurrences of $x_{(i,\bar\alpha)}$ (for index pairs $(i,\bar\alpha)$ as above) in ϕ by $x_{(j,\bar\alpha)}$, and that the variables $x_{(j,\bar\alpha)}$ occur nowhere in ϕ. Then $\vdash(\bigoplus x_{\bar\rho})\phi \leftrightarrow (\bigoplus x_{\bar\sigma})\psi$.

The next axiom will say that we can slide the indices of variables up and down κ, provided that their relative order stays the same; and at the same time we can add redundant indices. We have to take care that when redundant indices are added to an n-tuple $\bar\alpha$ in an index pair $(i,\bar\alpha)$, then corresponding indices are added to any index pair of form $(i,\bar\beta)$ where $\bar\beta$ has the same length as $\bar\alpha$. The following rather complex definition takes care of all this.

Let $\bar\rho$ and $\bar\sigma$ each be a sequence of distinct index pairs, $\bar\rho = ((i_1,\bar\alpha_1),\dots,(i_m,\bar\alpha_m))$ and $\bar\sigma = ((i_1,\bar\beta_1),\dots,(i_m,\bar\beta_m))$. Suppose that both $\bar\rho$ and $\bar\sigma$ are separated by length. We shall say that $\bar\rho$ is *embeddable in* $\bar\sigma$ iff there is an order-preserving map h: $\mathrm{supp}(\bar\rho) \to \mathrm{supp}(\bar\sigma)$ with the following properties:

(7) If $i_j = i_k$ then for every q, the qth ordinal in $\bar\beta_j$ is in the image of $h \mid \bar\alpha_j$ if and only if the qth ordinal in $\bar\beta_k$ is in the image of $h \mid \bar\alpha_k$;

(8) if we form $\bar\beta'_j$ by removing from $\bar\beta_j$ those ordinals which are not in the image of $h \mid \bar\alpha_j$, then the sequence $(\bar\beta'_1,\dots,\bar\beta'_m)$ is exactly the result of applying h to all the ordinals in $(\bar\alpha_1,\dots,\bar\alpha_m)$.

We call the map h an *order-embedding* of $\bar\rho$ into $\bar\sigma$. We call h an *order-isomorphism* from $\bar\rho$ to $\bar\sigma$ if for each j, $\bar\beta_j$ is exactly the image of $\bar\alpha_j$ under h.

AXIOM D (Order-embedding). Let $\bar{\rho}$ and $\bar{\sigma}$ be finite sequences of index pairs. Suppose $\bar{\rho}$ and $\bar{\sigma}$ are both separated by length, and suppose that $\bar{\rho}$ is embeddable in $\bar{\sigma}$. Suppose also $\phi(x_{\bar{\rho}})$ is a formula, and that the variables in $x_{\bar{\sigma}}$ are distinct from all variables in ϕ. Then we have $\vdash (\bigoplus x_{\bar{\rho}})\phi(x_{\bar{\rho}}) \to (\bigoplus x_{\bar{\sigma}})\phi(x_{\bar{\sigma}})$.

To prove this, suppose $M \models (\bigoplus x_{\bar{\rho}})\phi(x_{\bar{\rho}})$, so that some framework F for M indiscernibly satisfies $\phi(x_{\bar{\rho}})$. Let $(f^i: i < \kappa)$ be the functions of F. We define a framework E for M on $(\kappa, <)$ with functions $(e^i: i < \kappa)$ as follows. Let h be an order-embedding from $\bar{\rho}$ to $\bar{\sigma}$. We use the notation of (7) and (8) above.

Suppose $i = i_j$ for some j $(1 \leq j \leq m)$. Then $\bar{\beta}_j$ comes from $\bar{\alpha}_j$ by applying h and possibly adding some other ordinals. An example will show what we do. Suppose $\bar{\alpha}_j$ is $(0, 5, 8)$ and $\bar{\beta}$ is $(3, 4, 8, 11, 14)$, and h takes 0, 5, 8 to 4, 8, 14 respectively. Then for any increasing 5-tuple $\bar{\gamma} = (\gamma_1, \ldots, \gamma_5)$ of ordinals $< \kappa$, we define $e^i(\bar{\gamma})$ to be $f^i(\gamma_2, \gamma_3, \gamma_5)$. The condition (7) means that this definition is consistent when i is also i_k for some $k \neq j$. Elsewhere we define e^i in any way.

Then it is easily checked that E indiscernibly satisfies $\phi(x_{\bar{\sigma}})$, so that $M \models (\bigoplus x_{\bar{\sigma}})\phi(x_{\bar{\sigma}})$.

LEMMA 4. *Every theory in $\mathscr{L}^0$ is equivalent to a theory in which all variables are of form $x_{(i,\bar{\alpha})}$ with i finite and $\bar{\alpha}$ a sequence of finite ordinals.*

PROOF. Every sentence involves just a finite number of variables. So by Axiom C we can make each ordinal i finite. Then by Axiom D (using an order-isomorphism in both directions) we can slide the variables down into ω. □

The next three axioms describe how $\bigoplus$ interacts with other logical symbols.

AXIOM E (Monotonicity).

$$\vdash (\forall \bar{z})(\phi(\bar{z}) \to \psi(\bar{z})) \to ((\bigoplus x_{\bar{\rho}})\phi(x_{\bar{\rho}}) \to (\bigoplus x_{\bar{\rho}})\psi(x_{\bar{\rho}})).$$

This axiom implies a number of important logical properties that we shall use:

LEMMA 5. *The following hold (with the obvious conditions on the variables):*

(a) $\vdash \forall x_{\bar{\rho}}\psi(x_{\bar{\rho}}) \to (\bigoplus x_{\bar{\rho}})\psi(x_{\bar{\rho}})$.

(b) *If $x_{\bar{\rho}}$ are not free in θ then $(\bigoplus x_{\bar{\rho}})(\theta \wedge \psi(x_{\bar{\rho}}))$ is equivalent to $\theta \wedge (\bigoplus x_{\bar{\rho}})\psi(x_{\bar{\rho}})$.*

PROOF. For (a), take ϕ in Axiom E to be $\forall z\ z = z$ and use Axiom B. □

The next axiom is not quite so immediate: it uses the fact that κ is weakly compact.

AXIOM F (Disjunction). $\vdash (\bigoplus x_{\bar{\rho}})(\phi \vee \psi) \to (\bigoplus x_{\bar{\rho}})\phi \vee (\bigoplus x_{\bar{\rho}})\psi$.

For this, suppose $M \models (\bigoplus x_{\bar{\rho}})(\phi \vee \psi)$. By Lemma 1 there is a framework F for M, on an ordered set $(X, <)$, such that, in the notation of (4), every n-tuple in $[X]^n$ satisfies $\phi \vee \psi$ in F. Define a map g: $[X]^n \to 2$ by putting $g(\bar{b}) = 0 \Leftrightarrow \bar{b}$ satisfies ϕ. Since κ is weakly compact, there is a subset Y of X of order-type κ such that g is constant on $[Y]^n$. If g takes the value 0 (resp. 1) on $[Y]^n$, then every n-tuple in $[Y]^n$ satisfies ϕ (resp. ψ). So the framework $F \mid Y$ shows that $M \models (\bigoplus x_{\bar{\rho}})\phi$ or $M \models (\bigoplus x_{\bar{\rho}})\psi$ accordingly.

AXIOM G (Existential quantifiers). Let $\bar{\sigma}$ be a subsequence of $\bar{\rho}$ and let $\bar{\alpha}$ list the support of $\bar{\sigma}$. Let π be an index pair $(i, \bar{\alpha})$ such that no index pair of the form $(i, \bar{\beta})$ occurs anywhere in $\theta(x_{\bar{\rho}})$ or $\phi(y, x_{\bar{\sigma}})$. Then

$$\vdash (\bigoplus x_{\bar{\rho}})(\theta(x_{\bar{\rho}}) \wedge (\exists y)\phi(y, x_{\bar{\sigma}})) \to (\bigoplus x_{\bar{\rho}} x_{\pi})(\theta(x_{\bar{\rho}}) \wedge \phi(x_{\pi}, x_{\bar{\sigma}})).$$

Sh:271

For this, suppose $M \models (\bigoplus x_{\bar\rho})(\theta(x_{\bar\rho}) \wedge (\exists y)\phi(y, x_{\bar\sigma}))$. Then there is a framework F for M which indiscernibly satisfies $\theta(x_{\bar\rho}) \wedge (\exists y)\phi(y, x_{\bar\sigma})$. Suppose F consists of the ordered set $(X, <)$ and the functions $(f^i: i < \kappa)$. Suppose $\bar\alpha$ has length n, and let $(b_1, \ldots, b_n)$ be an increasing sequence of elements of X. We shall redefine the function f^i as follows. For each index pair $(i_k, \bar\alpha_k)$ in $\bar\sigma$, let c_k be the element $f^{i_k}(\bar b_k)$, where $\bar b_k$ is to $(b_1, \ldots, b_n)$ as $\bar\alpha_k$ is to $\bar\alpha$. Let $\bar c$ be the tuple got from $x_{\bar\sigma}$ by replacing each variable $x_{(i_k, \bar\alpha_k)}$ by the element c_k. If there is an element d such that $M \models \phi(d, \bar c)$, then choose such an element d and put $f^i(b_1, \ldots, b_n) = d$. If there is no such element d, then choose $f^i(b_1, \ldots, b_n)$ arbitrarily. Let F' be the framework F with f^i redefined in this way. Then F' satisfies $\theta(x_{\bar\rho}) \wedge \phi(x_\pi, x_{\bar\sigma})$ indiscernibly, and hence $M \models (\bigoplus x_{\bar\rho} x_\pi)(\theta(x_{\bar\rho}) \wedge \phi(x_\pi, x_{\bar\sigma}))$, as claimed.

The next three axioms describe how the quantifier $\bigoplus$ interacts with itself.

AXIOM H (Attaching tails). Let $\bar\sigma$ be a subsequence of $\bar\rho$ and let S be the support of $\bar\rho$. Let $\bar\pi$ be a tuple of index pairs $(i, \bar\alpha)$ such that (1) no index pair of form $(i, \bar\beta)$ occurs anywhere in $\theta(x_{\bar\rho})$ or in $\bar\sigma$, (2) $\bar\alpha$ contains the support of $\bar\sigma$, and (3) every ordinal in $\bar\alpha$ but not in the support of $\bar\sigma$ is greater than every ordinal in S. Let $\bar\pi'$ come from $\bar\pi$ by replacing each pair $(i, \bar\alpha)$ in $\bar\pi$ by $(i, \bar\alpha')$, where $\bar\alpha'$ lists the elements of $\bar\alpha$ which are not in S. Then

$$\vdash (\bigoplus x_{\bar\rho})(\theta(x_{\bar\rho}) \wedge (\bigoplus x_{\bar\pi'})\phi(x_{\bar\pi'}, x_{\bar\sigma})) \rightarrow (\bigoplus x_{\bar\rho} x_{\bar\pi})(\theta(x_{\bar\rho}) \wedge \phi(x_{\bar\pi}, x_{\bar\sigma})).$$

This axiom is true for reasons similar to those for Axiom G.

AXIOM I (Detaching tails). Let $\bar\rho$ and $\bar\sigma$ be sequences of index pairs such that $\bar\sigma$ is separated by length, and let δ be the greatest ordinal in the support of $\bar\rho$. Write the kth index pair in $\bar\sigma$ as $(i_k, \bar\alpha_k {}^\wedge \bar\beta_k)$, where $\bar\alpha_k$ lists those ordinals which are $\leq \delta$. Suppose that whenever $i_k = i_{k'}$ we have $\bar\alpha_k = \bar\alpha_{k'}$. For each i_k let j_k be an ordinal chosen so that no pair $(j_k, \bar\gamma)$ occurs anywhere in $\phi(x_{\bar\rho}, x_{\bar\sigma})$, and so that $i_k = i_{k'}$ if and only if $j_k = j_{k'}$. Let $\bar\pi$ be the sequence of index pairs that comes from $\bar\sigma$ by replacing each index pair $(i_k, \bar\alpha_k {}^\wedge \bar\beta_k)$ in $\bar\sigma$ by $(j_k, \bar\beta_k)$. Then we have

$$\vdash \neg(\bigoplus x_{\bar\rho {}^\wedge \bar\sigma})(\phi(x_{\bar\rho}, x_{\bar\sigma}) \wedge \neg(\bigoplus x_{\bar\pi})\phi(x_{\bar\rho}, x_{\bar\pi})).$$

To prove this, suppose to the contrary that there are a framework F (consisting of $(\kappa, <)$ and $(f^i: i < \kappa)$) for M and an integer n such that each n-tuple $\bar b \in [\kappa]^n$ satisfies $(\phi(x_{\bar\rho}, x_{\bar\sigma}) \wedge \neg(\bigoplus x_{\bar\pi})\phi(x_{\bar\rho}, x_{\bar\pi}))$ in F. (Cf. Lemma 1.) There will be some $m \leq n$ such that the first m elements in each $\bar b$ answer to the ordinals $\leq \delta$ in the support of $\bar\rho\bar\sigma$. Let $\bar c$ be the elements put for $x_{\bar\rho}$ when $\bar b$ is $(0, \ldots, n-1)$. Then we have $M \models \neg(\bigoplus x_{\bar\pi})\phi(\bar c, x_{\bar\pi})$. We contradict this by defining a new framework E on the set $Y = \{\alpha: \delta < \alpha < \kappa\}$ with functions e^i, so that each tuple $\bar d \in [Y]^{n-m}$ satisfies $\phi(\bar c, x_{\bar\pi})$ in E. Suppose i is i_k and $\bar\beta$ is an increasing sequence $(\beta_0, \beta_1, \ldots)$ of ordinals of the same length as $\bar\beta_k$ in the statement of the axiom; then put $e^i(\bar\beta) = f^i(\bar\alpha_k {}^\wedge \bar\beta)$. Elsewhere define e^i arbitrarily.

Our next axiom is mercifully a clear consequence of the definitions. If h is a partial isomorphism on κ (i.e. an order-preserving map $h: X \rightarrow X$ for some $X \subseteq \kappa$), and $\bar\rho$ is $((i_1, \bar\alpha_1), \ldots, (i_m, \bar\alpha_m))$, we write $h\bar\rho$ for $((i_1, h\bar\alpha_1), \ldots, (i_m, h\bar\alpha_m))$.

AXIOM J (Sliding). Let $\phi(x_{\bar\rho})$ be a formula and h a partial isomorphism on κ such that $h\bar\rho$ is defined and no variable of $x_{h\bar\rho}$ is bound in ϕ. Let $\bar\sigma$ be a tuple consisting of the index pairs which are in at least one of $\bar{\rho}$ and $h\bar{\rho}$. Then $\vdash \neg ((\bigoplus x_{\bar{\sigma}})(\phi(x_{\bar{\rho}}) \wedge \neg \phi(x_{h\bar{\rho}}))$.

There remains one axiom, the axiom of reduction.

1.3. The axiom of reduction. Broadly speaking, the larger the supports of the index pairs in $\bar{\rho}$, the more expressive is the quantifier $(\bigoplus x_{\bar{\rho}})$. But there are cases where some of the ordinals in these supports can be left out without weakening the quantifier. The axiom of reduction (Axiom K) describes some cases where this happens. We have put it into a separate subsection because it is more complicated than the other axiom schemas.

Before we can state Axiom K, we need some preliminary notions. By an *ordinal function* we shall mean a function β: $[\kappa]^r \to \kappa$ for some natural number $r = r(\beta)$. By an *indiscernible family* of ordinal functions we shall mean a set B of ordinal functions such that if β, $\gamma \in B$ and $\bar{a} = (a_0, \ldots, a_{r(\beta)-1})$, $\bar{b} = (b_0, \ldots, b_{r(\beta)-1})$, $\bar{c} = (c_0, \ldots, c_{r(\gamma)-1})$, $\bar{d} = (d_0, \ldots, d_{r(\gamma)-1})$ are increasing sequences of ordinals such that

$$\text{(9)} \qquad \text{for all } i < r(\beta) \text{ and } j < r(\gamma), \qquad a_i \gtreqless c_j \Leftrightarrow b_i \gtreqless d_j,$$

then $\beta(\bar{a}) \gtreqless \gamma(\bar{c}) \Leftrightarrow \beta(\bar{b}) \gtreqless \gamma(\bar{d})$. This condition says that the relative order of $\beta(\bar{a})$ and $\gamma(\bar{c})$ in κ depends only on the relative orders of the ordinals in $\bar{a}$ and $\bar{c}$. We call β an *indiscernible ordinal function* if β is in some indiscernible family of ordinal functions, or, equivalently, if $\{\beta\}$ is an indiscernible family.

Let $(\bigoplus x_{\bar{\rho}})$ be a quantifier; let $\bar{\rho}$ consist of the index pairs $(i_1, \bar{\alpha}_1), \ldots, (i_m, \bar{\alpha}_m)$. A *docket* $\mathscr{D}$ for the quantifier $(\bigoplus x_{\bar{\rho}})$ consists of (i) an indiscernible family of ordinal functions $\beta_{1,1}, \ldots, \beta_{1,n_1}, \beta_{2,1}, \ldots, \beta_{m,n_m}$ and (ii) a sequence $\bar{a}_1, \ldots, \bar{a}_m$ of elements of $[\kappa]^{r(1)}, \ldots, [\kappa]^{r(m)}$ respectively (where each $\beta_{k,j}$ has arity $r(k)$), such that the following hold:

$$\text{(10)} \qquad \text{if } i_k = i_{k'}, \text{ then } \beta_{k,j} = \beta_{k',j} \text{ for each } j;$$

$$\text{(11)} \qquad \text{each } \bar{\alpha}_k \text{ is } (\beta_{k,1}(\bar{a}_k), \ldots, \beta_{k,n_k}(\bar{a}_k)).$$

Write $\bar{\sigma}$ for the sequence of index pairs $((i_1, \bar{a}_1), \ldots, (i_m, \bar{a}_m))$. Then $\bar{\sigma}$ is determined by $\bar{\rho}$ and the docket $\mathscr{D}$. We shall say that $\mathscr{D}$ *reduces* $\bar{\rho}$ to $\bar{\sigma}$ (or that $\mathscr{D}$ *reduces* the quantifier $(\bigoplus x_{\bar{\rho}})$ to the quantifier $(\bigoplus x_{\bar{\sigma}})$).

AXIOM K (Reduction). Let $\phi(x_{\bar{\rho}}, \bar{z})$ be a formula and $(\bigoplus x_{\bar{\rho}})$ a quantifier in which $\bar{\rho}$ is separated by length. Suppose some docket for $(\bigoplus x_{\bar{\rho}})$ reduces it to the quantifier $(\bigoplus x_{\bar{\sigma}})$, and suppose no variable in $x_{\bar{\sigma}}$ occurs in ϕ. Then

$$\vdash (\bigoplus x_{\bar{\rho}})\phi \to (\bigoplus x_{\bar{\sigma}})\phi(x_{\bar{\sigma}}, \bar{z}).$$

We can show at once that Axiom K is always true. Suppose $M \models (\bigoplus x_{\bar{\rho}})\phi(x_{\bar{\rho}})$, where $\bar{\rho}$ is separated by length, and suppose some docket $\mathscr{D}$ for $(\bigoplus x_{\bar{\rho}})$ reduces it to the quantifier $(\bigoplus x_{\bar{\sigma}})$, where the variables $x_{\bar{\sigma}}$ occur nowhere in ϕ. Then there is a framework F for M, with functions (f^i), which indiscernibly satisfies $\phi(x_{\bar{\rho}})$. We have to define a framework E for M with functions (e^i), so that F indiscernibly satisfies $\phi(x_{\bar{\sigma}})$.

Suppose $\mathscr{D}$ consists of the indiscernible family of functions $\beta_{1,1}, \ldots, \beta_{m,n_m}$ and the sequences $\bar{a}_1, \ldots, \bar{a}_m$. Then we define each function e^k by:

$$\text{(12)} \qquad e^k(\bar{b}_1, \ldots, \bar{b}_m) = f^k(\beta_{k,1}(\bar{b}_1), \ldots, \beta_{k,n_k}(\bar{b}_k)).$$

It is straightforward to check that E satisfies $\phi(x_{\bar{a}})$ indiscernibly. This proves Axiom K.

In the next few pages we shall give a more explicit version of Axiom K, which will show that the axiom schema is primitive recursive (in the obvious sense). Our problem is to find a characterisation of finite indiscernible families of ordinal functions.

Everything rests on Lemma 7 below, which is essentially due to Kenneth Arrow [A]. This application of Arrow's result was given in [H]; it was found independently by Charretton and Pouzet [CP]. (Apart from clause (c), the lemma holds for any indiscernible function β: $[\xi]^r \to \zeta$ where ξ and ζ are infinite linear orderings, except that in clause (a) we may have $\beta(\bar{b}) > \beta(\bar{c})$ for certain values of i.) We introduce the lemma with a preliminary result:

LEMMA 6. *Suppose* β: $[\kappa]^r \to \kappa$ *is an indiscernible ordinal function. Then, writing* $\bar{a} = (a_0, \ldots, a_{r-1})$ *etc.,*

(a) *Suppose* $\bar{a}, \bar{b}, \bar{c}, \bar{d} \in [\kappa]^r$ *are such that, for each* $i < r$, $a_i \gtrless c_i \Leftrightarrow b_i \gtrless d_i$. *Then* $\beta(\bar{a}) \gtrless \beta(\bar{c}) \Leftrightarrow \beta(\bar{b}) \gtrless \beta(\bar{d})$. *(In other words, there is no need to compare* a_i *with* c_j *when* $i \neq j$.)

(b) *If* $a_i = c_i$ *for all* $i \neq j$, *then it is not possible that* $a_j < c_j$ *and* $\beta(\bar{a}) > \beta(\bar{c})$.

PROOF. (a) Consider any tuples $\bar{a}$ and $\bar{c}$ in $[\kappa]^r$. After sliding the ordinals in $\bar{a}$ and $\bar{c}$ further apart if necessary, we can find $\bar{a} = \bar{a}_0, \bar{a}_1, \ldots, \bar{a}_m = \bar{c}$ in $[\kappa]^r$ so that, for all $i < r$ and all $k < m$, $a_{k,i} \gtrless a_{k+1,i} \Leftrightarrow a_i \gtrless c_i$, and moreover if $i < j$ then $a_{k,i} < a_{k+1,j}$ and $a_{k+1,i} < a_{k,j}$. Likewise we can find $\bar{b} = \bar{b}_0, \bar{b}_1, \ldots, \bar{b}_n = \bar{d}$ with the analogous properties; using the assumption of (a), it follows that, for all $i < r$, $a_{0,i} \gtrless a_{1,i} \Leftrightarrow b_{0,i} \gtrless b_{1,i}$. Now $\beta(\bar{a}_0) \gtrless \beta(\bar{a}_1) \Leftrightarrow \beta(\bar{a}_1) \gtrless \beta(\bar{a}_2) \Leftrightarrow \cdots \Leftrightarrow \beta(\bar{a}_{m-1}) \gtrless \beta(\bar{a}_m)$, and similarly with the $\bar{b}_j$'s. Hence $\beta(\bar{a}) \gtrless \beta(\bar{c}) \Leftrightarrow \beta(\bar{a}_0) \gtrless \beta(\bar{a}_1) \Leftrightarrow \beta(\bar{b}_0) \gtrless \beta(\bar{b}_1) \Leftrightarrow \beta(\bar{b}) \gtrless \beta(\bar{d})$.

(b) Otherwise we could choose $\bar{a}$ so that there are infinitely many ordinals between a_{i-1} and a_{i+1}. Then we would have an infinite descending chain of values $\beta(\bar{c}_0) > \beta(\bar{c}_1) > \cdots$, which is impossible since κ is well-ordered. □

LEMMA 7. *Let* β: $[\kappa]^r \to \kappa$ *be an indiscernible ordinal function. Then there is an injective map* t: $s \to r$, *for some ordinal* $s \leq r$, *such that*

(a) *for all* $\bar{b} = (b_0, \ldots, b_{r-1})$ *and* $\bar{c} = (c_0, \ldots, c_{r-1})$ *in* $[\kappa]^r$ *and all* $i < s$, *if* $b_{t(0)} = c_{t(0)}, \ldots, b_{t(i-1)} = c_{t(i-1)}$ *and* $b_{t(i)} < c_{t(i)}$, *then* $\beta(\bar{b}) < \beta(\bar{c})$;

(b) *if* $b_{t(i)} = c_{t(i)}$ *for all* $i < s$, *then* $\beta(\bar{b}) = \beta(\bar{c})$; *and*

(c) *if* $s > 0$, *then* $t(i) < t(0)$ *whenever* $0 < i < s$.

PROOF. We go by induction on k. For each k, either we fix a value $\leq k$ for s, or we determine that $s > k$; at the same time we define $t(i)$ for all $i < \min(s, k)$. The induction hypothesis is that (a) holds when $i < k$, and that s is the least number $\leq k$ for which (b) holds (if there is such a number).

Put $s = 0$ iff β is a constant function. Then (a) holds trivially when $i < 0$, and (b) holds when $s = 0$. This covers the case $k = 0$.

Suppose now that the result has been proved for k; we prove it for $k + 1$. Suppose we have not chosen s to be an ordinal $\leq k$. Let $d_0, \ldots, d_{k-1}$ be distinct limit ordinals whose relative order in κ is the same as that of $t(0), \ldots, t(k-1)$, and let D be the set of all r-tuples $\bar{b} \in [\kappa]^r$ such that $b_{t(i)} = d_i$ for each $i < k$. By the choice of s as minimal for (b), $\beta(\bar{b})$ is not constant as $\bar{b}$ ranges over D.

Call a subset I of $\{0,\ldots,r-1\}\setminus\{t(0),\ldots,t(k-1)\}$ a *majority* iff, for all $\bar{b}$ and $\bar{c}$ in D,

$$(13) \qquad b_i < c_i \text{ for all } i \in I \Rightarrow \beta(\bar{b}) < \beta(\bar{c}).$$

Using Lemma 6(b) on one index at a time, we deduce that $\{0,\ldots,r-1\}\setminus\{t(0),\ldots,t(k-1)\}$ is a majority. Now we claim that if I is a majority and I is a disjoint union $J \cup K$, then either J or K is a majority. For suppose not. Then there are $\bar{a},\bar{b},\bar{c}$ in D such that (a) $a_i < b_i$ for all $i \in J$, but $\beta_j(\bar{a}) \geq \beta_j(\bar{b})$, and (b) $b_i < c_i$ for all $i \in K$, but $\beta_j(\bar{b}) \geq \beta_j(\bar{c})$. By setting off big jumps against little ones (there is plenty of room between one limit ordinal and the next), we can choose $\bar{a}$, $\bar{b}$ and $\bar{c}$ in such a way that $a_i < c_i$ for all $i \in I$, so that $\beta_j(\bar{a}) < \beta_j(\bar{c})$. But $\beta_j(\bar{a}) \geq \beta_j(\bar{c})$, so we have a contradiction. It follows that there is an $i < r$, $i \neq t(0),\ldots,t(k-1)$, such that $\{i\}$ is a majority. We put $t(k) = i$.

If $\beta(\bar{b}) = \beta(\bar{c})$ whenever $\bar{b},\bar{c} \in D$ and $b_{t(k)} = c_{t(k)}$, then we put $s = k+1$. Otherwise we record that $s > k+1$. This proves the induction hypothesis for $k+1$. The inductive argument goes on until s is chosen, which must happen after at most r steps.

There remains clause (c). Suppose $i < s$ but $t(i) \geq t(0)$. Since t is injective, it follows that $t(i) > t(0)$. Choose $\bar{b}_\alpha$ $(\alpha < \kappa)$ so that, for all $\alpha < \beta < \kappa$, $b_{\alpha,j} = b_{\beta,j}$ whenever $j < i$ but $b_{\alpha,i} < b_{\beta,i}$; and $\bar{c}$ so that $b_{0,0} < c_{0,0}$. Then $\beta(\bar{b}_0) < \beta(\bar{b}_1) < \cdots < \beta(\bar{c})$, so that $\beta(\bar{c})$ is an element of κ with κ predecessors. This contradiction completes the proof. □

We write s_β and t_β for the number s and the function t given by Lemma 7.

The crucial question for us is how two elements $\beta(\bar{b})$ and $\gamma(\bar{c})$ compare, when $\{\beta,\gamma\}$ is an indiscernible family.

LEMMA 8. *Let B be a finite indiscernible family of ordinal functions. To each $\beta \in B$ we can associate a function $q_\beta: s_\beta + 1 \to \omega + 1$ in such a way that the following holds. Consider $\beta(\bar{b})$, and write $K(\beta,\bar{b})$ for the sequence*

$$(14) \qquad (q_\beta(0), b_{t_\beta(0)}, q_\beta(1), b_{t_\beta(1)}, \ldots, q_\beta(s)).$$

Let $\prec$ be the lexicographic ordering of sequences by first differences. Then, for any terms $\beta(\bar{b})$ and $\gamma(\bar{c})$, $\beta(\bar{b}) < \gamma(\bar{c}) \Leftrightarrow K(\beta,\bar{b}) \prec K(\gamma,\bar{c})$. (Moreover $q_\beta(i) = \omega$ if and only if $i = 0$ and $s_\beta > 0$.)

PROOF. Since B is finite, we can choose $q_\beta(0) < \omega$ for each constant function β so that $\beta < \gamma \Leftrightarrow q_\beta(0) < q_\gamma(0)$. Suppose β is a nonconstant function in B. Then Lemma 7 gives us $t_\beta(0)$; we put $q_\beta(0) = \omega$.

To justify this definition of $q_\beta(0)$, we must show first that if γ is a constant function and β is not, then $\gamma(\bar{c}) < \beta(\bar{b})$ for all $\bar{b},\bar{c} \in [\kappa]^r$. But if $\beta(\bar{b}) \leq \gamma(\bar{c})$ for some $\bar{b}$ and $\bar{c}$, we can use indiscernibility and the fact that γ is constant to choose $\bar{b}$ and $\bar{c}$ so that all of $\bar{b} >$ all of $\bar{c}$. Then we can find $\bar{b}_\alpha$ $(\alpha < \kappa)$ so that $b_{\alpha,i} \neq b_{\alpha',i}$ for all i when $\alpha \neq \alpha'$, and thus there are κ distinct ordinals $\beta(\bar{b}_\alpha)$ which are all $\leq \gamma(\bar{c})$. This contradicts the fact that $\gamma(\bar{c}) < \kappa$.

Next we must show that if β and γ are nonconstant functions in B and $i' = t_\gamma(0)$, then $b_i < c_{i'} \Rightarrow \beta(\bar{b}) < \gamma(\bar{c})$. If not, then again we can find tuples $\bar{c}_\alpha$ $(\alpha < \kappa)$ such that $c_{\alpha,i'} > b_i$ and $c_{\alpha,i'} \neq c_{\alpha',i'}$ whenever $\alpha \neq \alpha'$. Again this gives us κ distinct ordinals $\gamma(\bar{c}_\alpha)$, all of them $\leq \beta(\bar{b})$, which is impossible. This takes care of the first two terms of $K(\beta,\bar{b})$.

Suppose now that the functions q_β have been defined up as far as $q_\beta(i)$, in such a way that

$$(15)\qquad (q_\beta(0), b_{t_\beta(0)}, \ldots, b_{t_\beta(i)}) \prec (q_\gamma(0), b_{t_\gamma(0)}, \ldots, b_{t_\gamma(i)}) \Rightarrow \beta(\bar{b}) < \gamma(\bar{c})$$

(where the terms involving $q_\beta(k)$ or $t_\beta(l)$ are missing when $k > s_\beta$ or $l \geq s_\beta$, and similarly with γ). Let F be the class of functions $\beta \in B$ such that $s_\beta > i$. We define an equivalence relation $\sim$ on F by:

$$(16)\qquad \beta \sim \gamma \quad \text{iff} \quad \text{there are } \bar{b} \text{ and } \bar{c} \text{ such that } (q_\beta(0), b_{t_\beta(0)}, \ldots, b_{t_\beta(i)}) = (q_\gamma(0), b_{t_\gamma(0)}, \ldots, b_{t_\gamma(i)}).$$

(Equivalently, $\beta \sim \gamma$ iff $q_\beta(j) = q_\gamma(j)$ for all $j \leq i$ and $t_\beta(j) < t_\beta(j') \Leftrightarrow t_\gamma(j) < t_\gamma(j')$ for all $j < j' \leq i$.)

Let C be an equivalence class of $\sim$. Choose limit ordinals $d_0, \ldots, d_i$ in the same relative order as $t_\beta(0), \ldots, t_\beta(i)$ for any $\beta \in C$. When $\beta\colon [\kappa]^r \to \kappa$ is a function in C, define $D(\beta)$ to be the set of all $\bar{b} \in [\kappa]^r$ such that, for all $j \leq i$, $b_{t_\beta(j)} = d_j$.

Let β and γ be functions in C. Define

$$(17)\qquad \beta \prec \gamma \Leftrightarrow \text{for all } \bar{b} \in D(\beta) \text{ and } \bar{c} \in D(\gamma) \text{ we have } \beta(\bar{b}) < \gamma(\bar{c}).$$

The relation $\prec$ is transitive and irreflexive on C. So we can split C into disjoint classes $C_0, \ldots, C_{h-1}$ in such a way that if $\beta \in C_j$ and $\gamma \in C_{j'}$ then $\beta < \gamma \Leftrightarrow j < j'$. Put $q_\beta(i+1) = j$, where $\beta \in C_j$. If we do this simultaneously for all equivalence classes of $\sim$, we have

$$(18)\qquad (q_\beta(0), b_{t_\beta(0)}, \ldots, b_{t_\beta(i)}, q_\beta(i+1)) < (q_\gamma(0), b_{t_\gamma(0)}, \ldots, b_{t_\gamma(i)}, q_\gamma(i+1)) \Rightarrow \beta(\bar{b}) < \gamma(\bar{c}).$$

Next let C' be one of the classes C_j. If $s_\beta = i+1$ for all β in C', then, for each $\beta \in C'$ and $\bar{b} \in D(\beta)$, $\beta(\bar{b})$ depends only on β. By the definition of C' it follows that $\beta(\bar{b}) = \gamma(\bar{c})$ for all $\beta, \gamma \in C'$ and all $\bar{b} \in D(\beta)$ and $\bar{c} \in D(\gamma)$.

Otherwise let β and γ be functions in C' with $s_\beta > i+1$. Since $\beta, \gamma \in C'$, there are $\bar{b} \in D(\beta)$ and $\bar{c} \in D(\gamma)$ such that $\beta(\bar{b}) \leq \gamma(\bar{c})$; using the fact that $s_\beta > i+1$, we can choose $\bar{b}$ and $\bar{c}$ so that $\beta(\bar{b}) < \gamma(\bar{c})$. But there are also $\bar{d} \in D(\beta)$ and $\bar{e} \in D(\gamma)$ such that $\beta(\bar{d}) \geq \gamma(\bar{e})$. By indiscernibility we can arrange that $b = d$, and it follows that $s_\gamma > i+1$ too. By Lemma 7(b) we can choose $\bar{b}$ and $\bar{c}$ as above so that $b_{t_\beta(i+1)} < c_{t_\gamma(i+1)}$.

We assert that if $\bar{b}' \in D(\beta)$, $\bar{c}' \in D(\gamma)$ and $b'_{t_\beta(i+1)} < c'_{t_\gamma(i+1)}$, then $\beta(\bar{b}') < \gamma(\bar{c}')$. For suppose not. Then there are $\bar{b}' \in D(\beta)$ and $\bar{c}' \in D(\gamma)$ such that $b'_{t_\beta(i+1)} < c'_{t_\gamma(i+1)}$ but $\beta(\bar{b}') \geq \gamma(\bar{c}')$. Now (after stretching out $\bar{b}'$ and $\bar{c}'$ if necessary) we can choose $\bar{b}'' \in D(\beta)$ and $\bar{c}'' \in D(\gamma)$ so that $b'_{t_\beta(i+1)} < b''_{t_\beta(i+1)} < c''_{t_\gamma(i+1)} < c'_{t_\gamma(i+1)}$, whence $\beta(\bar{b}'') > \beta(\bar{b}') \geq \gamma(\bar{c}') > \gamma(\bar{c}'')$. At the same time we can choose $\bar{b}''$ and $\bar{c}''$ so that $\bar{b}''$ is to $\bar{c}''$ as $\bar{b}$ is to $\bar{c}$. But $\beta(\bar{b}) < \gamma(\bar{c})$ while $\beta(\bar{b}'') > \gamma(\bar{c}'')$. This contradicts the indiscernibility.

Thus we have defined the ordinals $q_\beta(i+1)$ and proved that (14) holds with $i+1$ in place of i. $\square$

Now we can give a more explicit version of Axiom K.

THEOREM 9. *Let $\phi(x_{\bar{\rho}}, \bar{z})$ be a formula and $(\bigoplus x_{\bar{\rho}})$ a quantifier in which $\bar{\rho}$ is separated by length and consists of the index pairs $(i_1, \bar{\alpha}_1), \ldots, (i_m, \bar{\alpha}_m)$, where each $\bar{\alpha}_k$ is $(\bar{\alpha}_k(0), \ldots)$. Suppose $\bar{\sigma}$ consists of the index pairs $(i_1, \bar{a}_1), \ldots, (i_m, \bar{a}_m)$, where each $\bar{a}_k$*

is $(a_k(0),\ldots)$, *and no variable in* $x_{\bar\sigma}$ *occurs in* ϕ. *For each ordinal* k $(1 \le k \le m)$, *write* r_k *for the length of* $\bar a_k$ *and* n_k *for the length of* $\bar\alpha_k$; *let* B *be the set of pairs* (k,l), *where* $1 \le k \le m$ *and* $1 \le l \le n_k$. *Then the following are equivalent*:

(a) $\vdash (\bigoplus x_{\bar\rho})\phi \to (\bigoplus x_{\bar\sigma})\phi(x_{\bar\sigma},\bar z)$ *is an instance of Axiom* K (*modulo sliding variables up or down as in Axiom* D).

(b) *For each* $\beta = (k,l)$ *in* B *there are a number* $s_\beta \le r_k$, *an injective map* $t_\beta\colon s_\beta \to r_k$ *and a map* $q_\beta\colon s_\beta + 1 \to \omega + 1$, *such that* (i) *if* $s_\beta > 0$ *then* $t_\beta(i) < t(0)$ *whenever* $0 < i < s_\beta$; (ii) $q_\beta(i) = \omega \Leftrightarrow i = 0$ *and* $s_\beta > 0$; (iii) *if* $i_k = i_{k'}$, *then* $r_k = r_{k'}$ *and, for each* l $(1 \le l \le r_k)$, $s_{(k,l)} = s_{(k',l)}$ *and likewise with* t *and* q; *and* (iv) *for all* $\beta = (k,l)$ *and* $\beta' = (k',l')$ *in* B,

$$(19)\quad \alpha_k(l) < \alpha_{k'}(l') \Leftrightarrow (q_\beta(0), a_k(t_\beta(0)), q_\beta(1),\ldots,q_\beta(s_\beta)) \prec (q_{\beta'}(0), a_{k'}(t_{\beta'}(0)), q_{\beta'}(1),\ldots,q_{\beta'}(s_{\beta'})).$$

PROOF. Lemmas 6–8 prove that (a) implies (b). In the other direction, suppose the data $s_\beta, t_\beta, q_\beta$ are given as in (b). We claim that for each pair $\beta = (k,l)$ in B we can define a function $\beta_{k,l}\colon [\kappa]^{r_k} \to \kappa$ so that these functions together with the sequences $\bar a_1,\ldots,\bar a_m$ form a docket which reduces $\bar\rho$ to $\bar\sigma$. This is enough to prove (b) $\Rightarrow$ (a) in the theorem.

Let the number Λ be $\max\{2s_\beta + 1\colon \beta \in B\}$. Let $(Z^*,<)$ be the product κ^Λ ordered lexicographically by first differences, and let Z be the subset of Z^* consisting of those sequences $(q_0, w_0, q_1, w_1,\ldots,q_s)$ such that $q_0,\ldots,q_s$ are all $\le\omega$, $w_1,\ldots,w_{s-1}$ are all $<w_0$, and $q_0 = \omega$ unless $s = 0$. Then $(Z,<)$ has order-type κ, and so there is an order-isomorphism $\eta\colon (Z,<) \to \kappa$.

Take any pair (k,l) in B; let r be r_k and let s, t, and q be $s_{(k,l)}$, $t_{(k,l)}$, and $q_{(k,l)}$. When $\bar b = (b_0,\ldots)$ is any sequence in $[\kappa]^r$, put

$$(20)\qquad \beta_{k,l}(\bar b) = \eta(q(0), b_{t(0)}, q(1),\ldots,q(s)).$$

Note that this make sense by our conditions (i) and (ii) on t. The definition implies immediately that the family of functions $\beta_{k,l}$ is indiscernible. The relative order of the ordinals $\alpha_{k,l}$ is the same as that of the ordinals $\beta_{k,l}(\bar a_k)$ by condition (iv), which yields (11) modulo some sliding. Finally, (10) follows from condition (iii). □

Axioms A–K are the full set of axioms for the logic $\mathscr{L}^0$. It is not hard to check that when τ is a primitive recursive set of symbols, these axioms form a primitive recursive set of sentences of $\mathscr{L}^0(\tau)$.

1.4. The completeness theorem. Now we shall show that these Axioms A–K, added to standard axioms for first-order logic, give us a complete axiomatisation for the logic $\mathscr{L}^0$.

THEOREM 10 (Completeness theorem). *Let* τ *be a vocabulary of cardinality* $<\kappa$, *and let* T *be a theory in* $\mathscr{L}^0 = \mathscr{L}^0(\tau)$. *Then* T *has a model if and only if* T *together with Axioms* A–K *is consistent* (*under the rules of first-order logic*).

PROOF. If T has a model M, then T and all the Axioms A–K are true in M, which proves left to right. For the rest of the proof, assume that T together with the axioms is consistent. By first-order compactness we can suppose that T is maximal consistent and hence complete (i.e. that for each sentence ϕ of $\mathscr{L}^0$, T contains just one of ϕ and $\neg\phi$).

We shall use a Henkin construction to find a model of T. Let λ be $\omega +$ (cardinality of τ). For each index pair $\rho = (i, \bar{\alpha})$ with $i < \lambda$, introduce a new constant c_ρ. If $x_{\bar{\rho}}$ is a tuple of variables, write $c_{\bar{\rho}}$ for the corresponding tuple of constants. Write $L_{(\lambda,\delta)}$ for the language got from $\mathscr{L}^0$ by adding all constants $c_{(i,\bar{\alpha})}$ where $i < \lambda$ and all the ordinals in $\bar{\alpha}$ are $< \delta$, and using only variables $x_{(i,\bar{\alpha})}$ with the same restrictions on i and $\bar{\alpha}$. Thanks to Lemma 4, we can suppose that T is a maximal first-order consistent theory in $\mathscr{L}^0 \cap L_{(\lambda,\omega)}$. (This revises an assumption from the previous paragraph.)

Just as in the Henkin construction, we shall build up inductively a set Γ of sentences of $L_{(\lambda,\kappa)}$ which is "consistent". The right notion of consistency for our purposes is as follows. Let U be a theory in $L_{(\lambda,\kappa)}$. Then we say that U is *$\bigoplus$-consistent* if for every conjunction $\phi(c_{\bar{\rho}})$ of sentences in U, with $\phi(x_{\bar{\rho}})$ a formula of $\mathscr{L}^0$, the sentence $(\bigoplus x_{\bar{\rho}})\phi(x_{\bar{\rho}})$ is first-order consistent with Axioms A–K.

LEMMA 11. *Let δ be any infinite ordinal $\leq \kappa$. If U is an $\bigoplus$-consistent theory in $L_{(\lambda,\delta)}$, then U can be extended to a maximal $\bigoplus$-consistent theory in $L_{(\lambda,\delta)}$.*

PROOF. $\bigoplus$-consistency is of finite character. □

The next lemma says that $\bigoplus$-consistency is preserved under certain kinds of extension.

LEMMA 12. (a) *If U is $\bigoplus$-consistent and $\phi(c_{\bar{\rho}})$ is a sentence, then at least one of $U \cup \{\phi(c_{\bar{\rho}})\}$ and $U \cup \{\neg\phi(c_{\bar{\rho}})\}$ is $\bigoplus$-consistent.*

(b) *Suppose U is $\bigoplus$-consistent, the sentence $(\exists y)\phi(c_{\bar{\rho}}, y)$ is in U, and $\sigma = (i, \bar{\alpha})$ is an index pair such that $\bar{\alpha}$ contains the support of $\bar{\rho}$ and no index pair $(i, \bar{\beta})$ occurs anywhere in U or in the sentence $(\exists y)\phi(c_{\bar{\rho}}, y)$. Then the theory $U \cup \{\phi(c_{\bar{\rho}}, c_\sigma)\}$ is $\bigoplus$-consistent.*

(c) *Suppose U is $\bigoplus$-consistent, the sentence $(\bigoplus x_{\bar{\nu}})\phi(c_{\bar{\rho}}, x_{\bar{\nu}})$ is in U, $\bar{\alpha}$ is an increasing sequence of ordinals listing the support of $\bar{\rho}$, and $\bar{\beta}$ is an increasing sequence of ordinals which are all greater than those in $\bar{\alpha}$. Suppose also that each index pair in $\bar{\nu}$ is of form $(i, \bar{\beta})$, where $(i, \bar{\alpha}^\wedge\bar{\beta})$ does not occur in $\phi(c_{\bar{\rho}}, c_{\bar{\nu}})$ or in U. Write $\bar{\pi}$ for the sequence that comes from $\bar{\nu}$ by writing $(i, \bar{\alpha}^\wedge\bar{\beta})$ for each index pair $(i, \bar{\beta})$. Then the theory $U \cup \{\phi(c_{\bar{\rho}}, c_{\bar{\pi}})\}$ is $\bigoplus$-consistent.*

PROOF. (a) Suppose not. Then there is some conjunction $\chi(c_{\bar{\sigma}})$ of sentences in U such that neither of the sentences $(\bigoplus x_{\bar{\rho}\bar{\sigma}})(\chi(x_{\bar{\sigma}}) \wedge \phi(x_{\bar{\rho}}))$ or $(\bigoplus x_{\bar{\rho}\bar{\sigma}})(\chi(x_{\bar{\sigma}}) \wedge \neg\phi(x_{\bar{\rho}}))$ is first-order consistent with the axioms. By Axioms A, E and F it follows that $(\bigoplus x_{\bar{\sigma}})\chi(x_{\bar{\sigma}})$ is not consistent with these axioms either, contradicting the $\bigoplus$-consistency of U.

(b) Suppose the conclusion fails. Then there is a conjunction $\theta(c_{\bar{\tau}})$ of sentences in U such that $\bar{\rho}$ is a subsequence of $\bar{\tau}$ and the sentence $(\bigoplus x_\sigma x_{\bar{\tau}})(\theta(x_{\bar{\tau}}) \wedge \phi(x_{\bar{\rho}}, x_\sigma))$ is not consistent with the axioms. By Axiom G it follows that the sentence $(\bigoplus x_{\bar{\tau}})(\theta(x_{\bar{\tau}}) \wedge (\exists y)\phi(x_{\bar{\rho}}, y))$ is not consistent with the axioms either. This contradicts the assumption that $(\exists y)\phi(c_{\bar{\rho}}, y)$ is in U and U is $\bigoplus$-consistent.

(c) As (b) but using Axiom H. □

The next lemma is more technical; it describes our situation in the completeness proof.

LEMMA 13. *Let δ be any infinite ordinal $\leq \kappa$. Suppose T is a maximal first-order consistent theory in $\mathscr{L}^0 \cap L_{(\lambda,\omega)}$ containing all the instances of Axioms A–K which lie in $\mathscr{L}^0 \cap L_{(\lambda,\omega)}$. Let U be a theory in $L_{(\lambda,\delta)}$ which contains T. Then:*

(a) *T is $\bigoplus$-consistent.*

Sh:271

(b) *U is $\oplus$-consistent if and only if for every conjunction $\phi(c_{\bar\rho})$ of sentences in U, with $\phi(x_{\bar\rho})$ a formula of $\mathscr{L}^0$, T contains some sentence $(\bigoplus x_{\bar\sigma})\phi(x_{\bar\sigma})$ with $\bar\sigma$ order-isomorphic to $\bar\rho$.*

(c) *Let U be maximal $\oplus$-consistent and let $\phi(x_{\bar\rho}, x_{\bar\sigma})$ be a formula of $\mathscr{L}^0 \cap L_{(\lambda,\delta)}$ meeting the conditions on $\phi(x_{\bar\rho}, x_{\bar\sigma})$ in Axiom I. Let $\bar\pi$ be as in that axiom. If the sentence $\phi(c_{\bar\rho}, c_{\bar\sigma})$ is in U then the sentence $(\bigoplus x_{\bar\pi})\phi(c_{\bar\rho}, x_{\bar\pi})$ is in U.*

(d) *Let U be maximal $\oplus$-consistent and let $\phi(x_{\bar\rho}, x_{\bar\sigma})$ be a formula of $\mathscr{L}^0 \cap L_{(\lambda,\delta)}$. If the sentence $\phi(c_{\bar\rho}, c_{\bar\sigma})$ is in U, then, for every partial order-isomorphism h such that $h\bar\rho = \bar\rho$ and $h\bar\sigma$ is defined, the sentence $\phi(c_{\bar\rho}, c_{h\bar\sigma})$ is in U.*

PROOF. For (a), suppose ϕ is a conjunction of finitely many sentences in T, such that the sentence $(\bigoplus)\phi$ is not first-order consistent with the axioms. Then since T contains these axioms and is maximal first-order consistent, it contains $\neg(\bigoplus)\phi$. This contradicts Axiom B.

(b) Right to left is immediate from Axiom D, since T is first-order consistent. In the other direction, suppose U is $\oplus$-consistent but the right-hand side fails. Thus there is $\bar\sigma$, order-isomorphic to $\bar\rho$, such that $\phi(c_{\bar\rho})$ is in U but the sentence $(\bigoplus x_{\bar\sigma})\phi(x_{\bar\sigma})$ is in $L_{(\lambda,\omega)}$ but not in T. Then, by the maximal first-order consistency of T, T contains the sentence $\neg(\bigoplus x_{\bar\sigma})\phi(x_{\bar\sigma})$. So this sentence is in U, and hence $(\bigoplus x_{\bar\rho})(\phi(x_{\bar\rho}) \wedge \neg(\bigoplus x_{\bar\sigma})\phi(x_{\bar\sigma}))$ is consistent with the axioms. But clearly it is not, by Lemma 5(b).

(c) Suppose (c) fails. Then U contains both $\phi(c_{\bar\rho}, c_{\bar\sigma})$ and $\neg(\bigoplus x_{\bar\pi})\phi(c_{\bar\rho}, x_{\bar\pi})$, by Lemma 12(a). So by (b) the theory T contains a sentence of the form $(\bigoplus x_{\bar\rho\bar\sigma})(\phi(x_{\bar\rho}, x_{\bar\sigma}) \wedge \neg(\bigoplus x_{\bar\pi})\phi(x_{\bar\rho}, x_{\bar\pi}))$. This contradicts Axiom I.

Likewise we prove (d) using Axiom J. □

Returning to the proof of the completeness theorem, we find Γ as follows. We shall inductively construct $\oplus$-consistent theories T^γ in $L_{(\lambda,\omega)}$ ($\gamma \le \lambda$) so that

(21) each T^γ is $\oplus$-consistent and contains T,

and

(22) each T^γ ($\gamma < \lambda$) contains fewer than λ of the constants $c_{(i,\bar\alpha)}$.

We take T^0 to be T; this is an $\oplus$-consistent theory by Lemma 13(a). At limit ordinals δ we take T^δ to be $\bigcup_{\alpha<\delta} T^\alpha$. Then T^δ is $\oplus$-consistent by the induction hypothesis and the fact that $\oplus$-consistency is of finite character.

There are at most λ sentences of $L_{(\lambda,\omega)}$. So we can assign each of these sentences ϕ to some $i < \lambda$. By Lemma 12(a), if T^γ is $\oplus$-consistent then at least one of $T^\gamma \cup \{\phi\}$ and $T^\gamma \cup \{\neg\phi\}$ is $\oplus$-consistent, and so we can choose $T^{\gamma+1}$ to be one of these two sets. If we do this, T^λ will automatically be maximal $\oplus$-consistent. It will follow that

(23) T^λ is a maximal $\oplus$-consistent theory in $L_{(\lambda,\omega)}$.

We also want to ensure that T^λ has witnesses for existential formulas, as in the Henkin construction. To be precise, we want the following to hold:

(24) If a sentence $(\exists y)\phi(c_{\bar\rho}, y)$ is in T^λ, then for some index pair σ the sentence $\phi(c_{\bar\rho}, c_\sigma)$ is in T^λ.

(25) Suppose a sentence $(\bigoplus x_{\bar\nu})\phi(c_{\bar\rho}, x_{\bar\nu})$ is in T^λ; suppose also that each index pair in $\bar\nu$ is of the form $(i, \bar\beta)$, where (i) no index pair $(i, \bar\gamma)$ occurs in $\bar\rho$, and (ii) every

ordinal in $\bar{\beta}$ is greater than every ordinal in the support of $\bar{\rho}$. Then T^λ contains a sentence $\phi(c_{\bar{\rho}}, c_{\bar{\pi}})$, where $\bar{\pi}$ has the following form. Let $\bar{\alpha}$ list the support of $\bar{\rho}$ in increasing order. Then $\bar{\pi}$ comes from $\bar{\nu}$ by replacing each index pair $(i, \bar{\beta})$ by the pair $(i, \bar{\alpha}{}^\wedge\bar{\beta})$.

Again this will involve at most λ steps in the construction of T^λ, and we can interlace these steps with the ones just described. Lemma 12(b) takes care of (24), and Lemma 12(c) deals with (25). We preserve (22) because we add just a finite number of witnesses at each step. Finally, as a technical convenience we can arrange (with a small amount of care) that

(26) For each $i < \lambda$ there are infinitely many $j < \lambda$ such that T^λ contains all the equations of form $c_{(i,\bar{\alpha})} = c_{(j,\bar{\alpha})}$.

This completes the construction of T^λ.

LEMMA 14. *If T^λ is as above, $\phi(x_{\bar{\rho}}, x_{\bar{\sigma}})$ is a formula of $\mathscr{L}^0 \cap L_{(\lambda,\omega)}$ and h is a partial isomorphism on ω such that $h\bar{\rho} = \bar{\rho}$ and $h\bar{\sigma}$ is defined, then $\phi(c_{\bar{\rho}}, c_{\bar{\sigma}})$ is in T^λ if and only if $\phi(c_{\bar{\rho}}, c_{h\bar{\sigma}})$ is in T^λ.*

PROOF. This follows at once from Lemma 13(d). □

It follows from Lemma 14 (with $\bar{\rho}$ empty) that we can extend T^λ uniquely to a theory Γ in $L_{(\lambda,\kappa)}$ in such a way that

(27) for every formula $\phi(x_{\bar{\sigma}})$ of $\mathscr{L}^0$ and any partial order-isomorphism h on κ such that $h\bar{\sigma}$ is defined and $\phi(x_{h\bar{\sigma}})$ is in $L_{(\lambda,\omega)}$, we have

$$\phi(c_{\bar{\sigma}}) \text{ is in } \Gamma \quad \text{iff} \quad \phi(c_{h\bar{\sigma}}) \text{ is in } T^\lambda.$$

Now (27) and Lemma 14 give us

LEMMA 15. *Suppose $\phi(x_{\bar{\rho}}, x_{\bar{\sigma}})$ is a formula of $\mathscr{L}^0$. Then for any partial order-isomorphism h on κ such that $h\bar{\rho} = \bar{\rho}$ and $h\bar{\sigma}$ is defined, the sentence $\phi(c_{\bar{\rho}}, c_{\bar{\sigma}})$ is in Γ if and only if $\phi(c_{\bar{\rho}}, c_{h\bar{\sigma}})$ is in Γ.*

PROOF. By sliding up and down κ. □

Just as in the usual Henkin argument, we readily turn Γ into an $L_{(\lambda,\kappa)}$-structure M so that the elements of M are equivalence classes of constants c_ρ (with c_ρ naming its equivalence class), and atomic sentences hold in M if and only if they are in Γ. The rest of the proof is devoted to showing that, for every sentence $\phi(c_{\bar{\rho}})$ of $L_{(\lambda,\kappa)}$,

(28) $$M \models \phi(c_{\bar{\rho}}) \Leftrightarrow \phi(c_{\bar{\rho}}) \in \Gamma.$$

When (28) has been proved, the theorem follows at once, since $T \subseteq \Gamma$ by (21).

Note that, by its construction, M has a framework F on the cardinal κ, where for each $i < \lambda$ the function f^i is defined by $f^i(\bar{\alpha}) =$ (the element named by) $c_{(i,\bar{\alpha})}$. Note also that every element of M is of form $f^i(\bar{\alpha})$ for some suitable i and $\bar{\alpha}$.

We prove (28) by induction on the complexity of ϕ. For ϕ atomic it holds by construction. When ϕ is built up by truth-functional operators, the argument uses (23) and Lemma 12(a) just as in the familiar first-order case. When ϕ is of form $\exists y\psi$, we use (24) as in the first-order case. We regard $\forall x$ as an abbreviation for $\neg\exists x\neg$. This leaves just the case where ϕ begins with $\bigoplus$.

Sh:271

Suppose first that Γ contains the sentence $(\bigoplus x_{\bar{v}})\phi(c_{\bar{\rho}}, x_{\bar{v}})$. By Axioms C and D, there is no loss if we suppose that the conditions on $\bar{\rho}$ and $\bar{v}$ in (25) above are satisfied, with $\bar{\alpha}$ as in (25). Then Γ contains some sentence $\phi(c_{\bar{\rho}}, c_{\bar{\pi}})$ as in (25). Hence, by Lemma 15, Γ contains $\phi(c_{\bar{\rho}}, c_{h\bar{\pi}})$ whenever h is a partial order-isomorphism for which $h\bar{\rho} = \bar{\rho}$ and $h\bar{\pi}$ is defined. By the induction hypothesis, $M \models \phi(c_{\bar{\rho}}, c_{h\bar{\pi}})$ for each such h. Let F be the framework for M which we defined above. We form F' from F by replacing the functions f^i by functions f'^i as follows: if some index pair in some tuple $h\bar{\pi}$ is $(i, \bar{\alpha}^\wedge\bar{\gamma})$, then define $f'^i(\bar{\gamma})$ to be $f^i(\bar{\alpha}^\wedge\bar{\gamma})$; otherwise define f'^i arbitrarily. Then, for some large subset Y of κ, $F' \mid Y$ satisfies $\phi(c_{\bar{\rho}}, x_{\bar{v}})$ indiscernibly. It follows that $M \models (\bigoplus x_{\bar{v}})\phi(c_{\bar{\rho}}, x_{\bar{v}})$, as required.

For the converse, suppose $M \models (\bigoplus x_{\bar{\sigma}})\phi(c_{\bar{\rho}}, x_{\bar{\sigma}})$. Then there is a framework E for M, say of the form $(X, <), (e^i\colon i < \kappa)$, which indiscernibly satisfies $\phi(c_{\bar{\rho}}, x_{\bar{\sigma}})$. Here our problem is to show that E is related to F in such a way that some sentence true in M compels us to put the sentence $(\bigoplus x_{\bar{\sigma}})\phi(c_{\bar{\rho}}, x_{\bar{\sigma}})$ into Γ.

We begin by normalising E. Let δ be the greatest ordinal in $\operatorname{supp}(\bar{\rho})$. Let $\bar{\sigma}$ be $(\sigma_1, \ldots, \sigma_m)$, where each σ_k is $(i_k, \bar{a}_k)$, and let $\bar{a}$ list the support of $\bar{\sigma}$ in increasing order. By Axiom D (order-embedding) we can suppose that all the ordinals in $\bar{a}$ are greater than δ. Using Axiom C (renaming) we can suppose that $i_k \neq i_{k'}$ when $\bar{a}_k$ and $\bar{a}_{k'}$ are of different lengths. Let n be the length of $\bar{a}$. If $1 \leq k \leq m$ and $\bar{b}$ is any increasing n-tuple from $(X, <)$, write $e_k(\bar{b})$ for the element $e^{j_k}(\bar{b}')$, where $\bar{b}'$ is the tuple related to $\bar{b}$ as $\bar{a}_k$ is related to $\bar{a}$. By our choice of E, we have

(29) $$M \models \phi(c_{\bar{\rho}}, e_1(\bar{b}), \ldots, e_m(\bar{b})) \quad \text{for every } \bar{b} \in [X]^n.$$

Now each $e_k(\bar{b})$ is an element of M. Every element of M is of form c_ρ for some index pair ρ; so we can write $e_k(\bar{b})$ as $c_{\rho(k,\bar{b})}$ for some index pair $\rho(k, \bar{b})$. Moreover our definition of $e_k(\bar{b})$ allows us to choose $\rho(k, \bar{b})$ so that

(30) $\rho(k, \bar{b})$ only depends on k and those parts of $\bar{b}$ corresponding to indices in $\bar{a}_k$,

and

(31) if $i_k = i_{k'}$, then $\rho(k, \bar{b}) = \rho(k', \bar{b}')$ whenever the part of $\bar{b}$ corresponding to $\bar{a}_k$ in $\bar{a}$ is equal to the part of $\bar{b}'$ corresponding to $\bar{a}_{k'}$ in $\bar{a}$.

The pair $\rho(k, \bar{b})$ in turn can be written as $(j(k, \bar{b}), \bar{\beta}(k, \bar{b}))$ for some $j(k, \bar{b}) < \lambda$ and some tuple $\bar{\beta}(k, \bar{b})$ in $[\kappa]^{p(k,\bar{b})}$.

None of this is altered if we replace E by some framework $E \mid Y$, where Y is a subset of X with order-type κ. So we can use the fact that Y is weakly compact in order to simplify the functions $j(k, \bar{b})$, $p(k, \bar{b})$, and $\bar{\beta}(k, \bar{b})$ as follows. Since the first two of these functions have values $< \lambda < \kappa$, we can suppose that these values depend only on k and not on $\bar{b}$. Using (26), we can arrange that

(32) $$j(k) = j(k') \quad \text{if and only if} \quad i_k = i_{k'}.$$

(Right to left in (32) holds by (31).) Hence there is no loss in supposing that $j(k) = i_k$ for each k. Also $\bar{\beta}(k, \bar{b})$ is a $p(k)$-tuple $(\beta(k, 1, \bar{b}), \ldots \beta(k, p(k), \bar{b}))$.

In this notation we have

(33) $$M \models \phi(c_{\bar{\rho}}, c_{(i_1, (\beta(1,1,\bar{b}), \beta(1,2,\bar{b}), \ldots, \beta(1,p(1),\bar{b})))}, \ldots, c_{(i_m, (\beta(m,1,\bar{b}), \beta(m,2,\bar{b}), \ldots, \beta(m,p(m),\bar{b})))})$$

Sh:271

for every increasing n-tuple $\bar{b}$ from $(X,<)$. Note that, by (32), if we write the sentence in (33) as $\phi(c_{\bar{\rho}},c_{\bar{\sigma}})$, then $\bar{\sigma}$ is separated by length.

Using weak compactness again, we can suppose that, for any k and k' between 1 and m inclusive, and any g and g' with $1 \leq g \leq p(k)$ and $1 \leq g' \leq p(k')$,

(34) the relative order of $\beta(k,g,\bar{b})$ and $\beta(k',g',\bar{b}')$ in κ depends only on the relative order of the elements of $\bar{b}$ and $\bar{b}'$ in $(X,<)$,

and moreover, for each g and k,

(35) if there is $\bar{b}$ such that $\beta(k,g,\bar{b}) \leq \delta$, then $\beta(k,g,\bar{b}') = \beta(k,g,\bar{b})$ for all $\bar{b}'$.

(Recall that δ is the greatest ordinal in $\operatorname{supp}(\bar{\rho})$.) By (31), if g and k are such that the conclusion of (35) holds, and $i_k = i_{k'}$, then $\beta(k,g,\bar{b}) = \beta(k',g,\bar{b}')$ for all $\bar{b}'$.

Now, by the induction hypothesis (28) on ϕ, the sentence (33) lies in Γ. By Lemma 13(c) and Lemma 15 it follows that Γ contains a sentence which is almost

$$(36)\qquad \begin{aligned}&(\textstyle\bigoplus x_{(i_1,(\beta(1,1,\bar{b}),\beta(1,2,\bar{b}),\ldots,\beta(1,(p(1),\bar{b})))}\\ &\quad,\ldots,x_{(i_m,(\beta(m,1,\bar{b}),\beta(m,2,\bar{b}),\ldots,\beta(m,p(m),\bar{b})))})\\ &\quad\phi(c_{\bar{\rho}},x_{(i_1,(\beta(1,1,\bar{b}),\beta(1,2,\bar{b}),\ldots,\beta(1,p(1),\bar{b})))}\\ &\quad,\ldots,x_{(i_m,(\beta(m,1,\bar{b}),\beta(m,2,\bar{b}),\ldots,\beta(m,p(m),\bar{b})))}).\end{aligned}$$

More precisely, it contains (32) with all ordinals $\beta(k,j,\bar{b}) \leq \delta$ removed. To save notation we shall assume that all the ordinals $\beta(k,j,\bar{b})$ are already greater than δ.

Write $\beta_{k,j}(\bar{b}')$ for $\beta(k,j,\bar{b})$, where $\bar{b}'$ is to $\bar{b}$ as $\bar{a}_k$ is to $\bar{a}$. In general this will only define $\beta_{k,j}(\bar{b}')$ when $\bar{b}'$ has gaps large enough to accommodate the other ordinals in $\bar{b}$. But $\beta_{k,j}(\bar{b}')$ is certainly defined when $\bar{b}'$ is a sequence of limit ordinals; so we can make the functions $\beta_{k,j}$ be defined everywhere by the simple trick of replacing κ by the set of limit ordinals $<\kappa$.

Thus the following sentence is in Γ:

$$(37)\qquad \begin{aligned}&(\textstyle\bigoplus x_{(i_1,(\beta_{1,1}(\bar{a}_1),\beta_{1,2}(\bar{a}_1),\ldots,\beta_{1,p(1)}(\bar{a}_1))}\\ &\quad,\ldots,x_{(i_m,(\beta_{m,1}(\bar{a}_m),\beta_{m,2}(\bar{a}_m),\ldots,\beta_{m,p(m)}(\bar{a}_m))})\\ &\quad\phi(c_{\bar{\rho}},x_{(i_1,(\beta_{1,1}(\bar{a}_1),\beta_{1,2}(\bar{a}_1),\ldots,\beta_{1,p(1)}(\bar{a}_1))}\\ &\quad,\ldots,x_{(i_m,(\beta_{m,1}(\bar{a}_m),\beta_{m,2}(\bar{a}_m),\ldots,\beta_{m,p(m)}(\bar{a}_m))})\end{aligned}$$

But now the functions $\beta_{1,1},\ldots,\beta_{m,p(m)}$ and the tuples $\bar{a}_1,\ldots,\bar{a}_m$ form a docket $\mathscr{D}$ for the quantifier at the beginning of (37). (The functions $\beta_{k,j}$ form an indiscernible family by (34).) This docket reduces the quantifier to $(\bigoplus x_{\bar{\sigma}})$. So by Axiom K the sentence $(\bigoplus x_{\bar{\sigma}})\phi(c_{\bar{\rho}},x_{\bar{\sigma}})$ lies in Γ. This concludes the proof of the completeness theorem. □

1.5. Compactness and other properties of $\mathscr{L}^0$.

Corollary 16. *Let T be a theory in $\mathscr{L}^0$, whose variables $x_{(i,\bar{\alpha})}$ all satisfy the restriction that the ordinals $\bar{\alpha}$ are less than ω. Then T has a model if and only if T is first-order consistent with those instances of Axioms* A–K *which satisfy this same restriction on the variables.* □

Theorem 17 (Compactness theorem). Let T be a theory in $\mathscr{L}^0$ such that T has cardinality $<\kappa$ and every finite subset of T has a model. Then T has a model. □

Sh:271

DEFINITION 18. We say that a formula of $\mathscr{L}^0$ is *existential* if it is built up from quantifier-free formulas by means of $\exists$, $\bigoplus$, $\wedge$ and $\vee$. A formula is *universal* if it is the negation of an existential formula. (Or one can define "universal" directly in the usual way.)

THEOREM 19. *Let τ be a finite vocabulary. A sentence θ in $\mathscr{L}^0(\tau)$ is preserved in submodels if and only if it is equivalent to a universal sentence.*

PROOF. Right to left is easy. In the other direction we take Θ to be the set of all universal sentences of $\mathscr{L}^0(\tau) \cap L_{(\omega,\omega)}$ which are consequences of θ in the logic of $\mathscr{L}^0$. Since we have compactness, and the conjunction of a finite number of universal sentences is clearly equivalent to a universal sentence, it suffices to show that Θ implies θ. We assume that $\Theta \cup \{\neg\theta\}$ has a model, and we aim for a contradiction.

Suppose that $\Theta \cup \{\neg\theta\}$ has a model. Construct a model M of this theory, exactly as in the proof of the completeness theorem. Let Δ be the set of all quantifier-free sentences $\phi(c_{\bar{\rho}})$ of $L_{(\omega,\omega)}$ which are true in M.

We claim that the theory $\Delta \cup \{\theta\}$ is $\bigoplus$-consistent. For suppose not. Then (since Δ is closed under conjunctions) there is a sentence $\phi(c_{\bar{\rho}})$ in Δ such that the sentence $(\bigoplus x_{\bar{\rho}})(\theta \wedge \phi(x_{\bar{\rho}}))$ is first-order inconsistent with the axioms; in other words, using Lemma 5(b), $\theta \vdash \neg(\bigoplus x_{\bar{\rho}})\phi(x_{\bar{\rho}})$ in the logic of $\mathscr{L}^0$. It follows that the sentence $\neg(\bigoplus x_{\bar{\rho}})\phi(x_{\bar{\rho}})$ is in Θ. But this contradicts Lemma 13(c) in the construction of the model M.

So the theory $\Delta \cup \{\theta\}$ is $\bigoplus$-consistent. By Lemma 11 we can extend it to a maximal $\bigoplus$-consistent theory in $L_{(\omega,\omega)}$, and then proceed to construct a model N of this theory just as in the proof of the completeness theorem. Inside N consider the substructure M' consisting of those elements named by constants c_ρ which were used to name elements of M. Clearly this substructure M' is isomorphic to M, and so it is a model of $\neg\theta$. This contradicts the assumption that θ was preserved in submodels. □

THEOREM 20. *Łoś's theorem holds for $\mathscr{L}^0$ and ultraproducts with index set of cardinality $<\kappa$.*

PROOF. Use the fact that $\kappa \to (\kappa)^n_\lambda$ for all $\lambda < \kappa$. □

§2. Extending $L_{\omega,\omega}(\exists^{\geq\kappa})$ to a κ-compact logic with INT.

2.1. THEOREM. *Suppose κ and λ are regular compact cardinals, $\aleph_0 < \kappa < \lambda$. Then there is a logic $\mathscr{L}^1 = \mathscr{L}[\kappa, \lambda]$ which is $[\aleph_0, <\kappa)$-compact, $[\lambda, <\infty)$-compact, satisfies ROB (Robinson lemma), INT (the interpolation theorem), PPP (the pair preservation theorem) and generalizations, the theorems on elementary chains shorter than $<\kappa$, and the existence of isomorphic ultralimits of $\mathscr{L}$-equivalent models.*

From now on, κ and λ will be fixed.

2.1. REMARK. We can of course prove the suitable preservation theorems, such as "a sentence is preserved under submodels iff it is equivalent to a universal sentence".

2.2. DEFINITION. We define the logic $\mathscr{L}^1 = \mathscr{L}^1[\kappa, \lambda)$ (which is a sublogic of $\mathscr{L}_{\lambda,\lambda}$). For a vocabulary τ we define the *set of formulas* of $\mathscr{L}^1(\tau)$ just as those of first-order logic (if τ has a member with infinite arity $<\lambda$, there is no problem). The set of formulas of $\mathscr{L}^1(\tau)$ is the closure of the set of atomic formulas under negation $(\neg\psi)$, conjunction $(\psi \wedge \phi)$, disjunction $(\psi \vee \phi)$, quantifications $((\exists x_0, \ldots, x_i, \ldots)_{i<\mu}\psi$, where $\mu < \lambda$), and the Boolean operation ($\bigwedge^{D}_{i\in\mu}\phi_i$, *where* $\mu < \lambda$ and D is a κ-complete ultrafilter over μ).

The set of free variables, sentences, and $\phi(\bar{x})$ are defined as usual.

Satisfaction is defined as usual (including $M \models \phi[\bar{a}]$), and we have to say only that

$$M \models \bigwedge^{D}_{i<\mu} \phi_i[\bar{a}] \text{ iff } \{i < \mu\colon M \models \phi_i[\bar{a}]\} \in D.$$

2.2A. *Fact.* (1) *We can identify* $\bigwedge^{D}_{i<\mu}$ *with* $\bigvee_{A\in D}(\bigwedge_{i\in A}\phi_i)$ *or equivalently*

$$\bigvee_{A\in D}\Big(\bigwedge_{i\in A}\phi_i \wedge \bigwedge_{\substack{i<\mu\\ i\notin A}} \neg\phi_i\Big)$$

and get that $\mathscr{L}^1$ *is a sublogic of* $\mathscr{L}_{\lambda,\lambda}$.

(2) $\mathscr{L}^1$ *satisfies the strong Löwenheim-Skolem theorem*: *if* $\chi = \chi^{<\lambda}$, $A \subseteq M$, *and* $|A| + |\tau(M)| \leq \chi$, *then there is* $N <_{\mathscr{L}} M$, *such that* $A \subseteq |N|$ *and* $\|N\| \leq \chi$.

2.3. Definition. (1) For $E \subseteq \mathscr{P}(\mu)$, define

$$\bigwedge^{E}_{i<\mu}\phi_i = \bigvee_{A\in E}\Big(\bigwedge_{i\in A}\phi_i \wedge \bigwedge_{\substack{i\in\mu\\ i\notin A}}\neg\phi_i\Big).$$

(2) If $A_i \subseteq \mu$ for $i < \theta$ and D is an ultrafilter on θ, then

$$\operatorname*{Lim}_D A_i = \{\alpha < \mu\colon \{i < \theta\colon \alpha \in A_i\} \in D\}.$$

(3) We call $E \subseteq \mathscr{P}(\mu)$ $(<\kappa)$-*closed* if, for every cardinal $\theta < \kappa$ and ultrafilter D on θ and subsets A_i of μ (for $i < \theta$), $\{i\colon A_i \in E\} \in D$ implies $\mathrm{Lim}_D A_i \in E$.

(4) We call $E \subseteq \mathscr{P}(\mu)$ $(<\kappa)$-*biclosed* if E and $\mathscr{P}(\mu) - E$ are $(<\kappa)$-closed.

(5) We call $E \subseteq \mathscr{P}(\mu)$ *upward-closed* if $A \subseteq B \subseteq \mu \wedge A \in E \Rightarrow B \in E$. *Downward closed* is defined similarly.

2.4. *Fact. If* D *is a* κ-*complete ultrafilter on* μ, *then* D (*as a subset of* $\mathscr{P}(\mu)$) *is* $(<\kappa)$-*biclosed.*

2.5. *Fact. The set of formulas of* $\mathscr{L}$ *is closed under* $\bigwedge^{E}_{i<\mu}\phi_i$, *where* $\mu < \lambda$ *and* $E \subseteq \mathscr{P}(\mu)$ *is* $(<\kappa)$-*biclosed* (*ignoring trivialities*).

Proof. See [Sh, Part II, 3.3]. □

2.6. *Fact.* (1) *If* μ *is a cardinal* $<\lambda$, $E \subseteq \mathscr{P}(\mu)$ *is* $(<\kappa)$-*closed, and* $\mu \notin E$, *then there is a* $(<\kappa)$-*biclosed set* $E_1 \subseteq \mathscr{P}(\mu) - E$ *to which* μ *belongs.*

(2) *We can demand in addition that* E_1 *is upward-closed.*

Proof. (1) is 3.2 from [Sh, Part II]; and the same proof works for (2). □

2.7. *Fact.* $\mathscr{L}$ *satisfies Łoś's theorem for ultraproducts with index-set* $<\kappa$.

Proof. This is Fact B of [Sh, Part II, §3]. □

2.8. *Fact. If* T *and* Γ *are sets of formulas in* $\mathscr{L}^1$, *and* $T \cup \Gamma$ *does not have a model, then, for some* $\mu < \lambda$, $\phi_i \in \Gamma$ $(i < \mu)$, *and upward-closed nonempty* $(<\kappa)$-*biclosed* $E \subseteq \mathscr{P}(\mu)$, *the set* $T \cup \{\bigwedge^{E}_{i<\mu}\phi_i\}$ *does not have a model and* $\mu \in E$.

Proof. This proof is like that of [Sh, Part II, §3, Fact C]. As $\mathscr{L}^1$ is $[\lambda, <\infty]$-compact, for some $\Gamma_1 \subseteq \Gamma$ of cardinality $<\lambda$, $T \cup \Gamma_1$ has no model. Let $\Gamma_1 = \{\phi_i\colon i < \mu\}$ and $E = \{A \subseteq \mu\colon T \cup \{\phi_i\colon i \in A\}$ has a model$\}$. So $\mu \notin E$.

Sh:271

Now E is $(<\kappa)$-closed (by Fact 2.7); hence by Fact 2.6(2) there is an $E_1 \subseteq \mathscr{P}(\mu) - E$, upward-closed, $(<\kappa)$-biclosed, and with $\mu \in E_1$. Now E_1 is as required. □

2.9. *Fact. If* $\phi(\ldots,\bar{x}_\gamma,\ldots)_{\gamma<\gamma_0}$ *belongs to* $\mathscr{L}^1$, $\gamma_0 < \kappa$, *then for some finite* $w \subseteq \gamma_0$, ϕ *depends on* $\bar{x}_\gamma$ $(\gamma \in w)$ *only, i.e., for any model M and sequences* $\bar{a}_\gamma$ *and* $\bar{b}_\gamma$ *from the model M of length* $l(\bar{x}_\gamma)$ *for* $\gamma < \gamma_0$,

$$\bigwedge_{\gamma\in w} \bar{a}_\gamma = \bar{b}_\gamma \Rightarrow M \models \phi(\ldots,\bar{a}_\gamma,\ldots) \equiv \phi(\ldots,\bar{b}_\gamma,\ldots).$$

Proof. We use 2.7. Let I be the family of finite subsets of γ_0. It is known that there is an ultrafilter D on I such that, for every $\gamma < \gamma_0$, $\{w \in I: \gamma \in w\} \in D$. Assume that the conclusion of 2.9 fails, so for every $w \in I$ there is a model M_w and sequences $\bar{a}^w_\gamma$, $\bar{b}^w_\gamma$ of length $l(\bar{x}_\gamma)$ for $\gamma < \gamma_0$ and, for $\gamma \in w$, $\bar{a}^w_\gamma = \bar{b}^w_\gamma$ but $M_w \models \phi(\ldots,\bar{a}^w_\gamma,\ldots) \cong \neg\phi(\ldots,\bar{b}^w_\gamma,\ldots)$. We can assume that the vocabulary of M_w is that of ϕ, and in $M = \prod_{w\in I} M_w/D$ let (essentially) $\bar{a}^w_\gamma = \langle\ldots,a^w_\gamma,\ldots\rangle_{w\in I}/D$ and $\bar{b}_\gamma = \langle\ldots,\bar{b}^w_\gamma,\ldots\rangle_{w\in I}/D$ for $\gamma < \gamma_0$. So by Łoś's theorem, $M \models \bar{a}_\gamma = \bar{b}_\gamma$ for $\gamma < \gamma_0$, but $M \models \phi(\ldots,\bar{a}_\gamma,\ldots) \cong \neg\phi(\ldots,b_\gamma,\ldots)$. Contradiction. □

We recall a definition.

2.10. Definition. $M <_{\mathscr{L}^1} N$ (for τ-models M and N) means that $(M,c)_{c\in M}$ and $(N,c)_{c\in M}$ have the same $\mathscr{L}^1$-theory.

2.11. *Fact.* (1) *If* $\delta < \kappa$ *and* $M_i <_{\mathscr{L}^1} M_j$ *for* $i < j < \delta$, *then* $M_i <_{\mathscr{L}^1} \bigcup_{j<\delta} M_j$.

(2) *If in addition* $M_i <_{\mathscr{L}^1} N$ *for* $i < \delta$, *then* $\bigcup_{i<\delta} M_i <_{\mathscr{L}^1} N$.

Proof. (1) Let $M = \bigcup_{i<\delta} M_i$. We prove by induction on $\phi(\bar{x})$ that, for every $i < \delta_i$ and $\bar{a} \subseteq M$,

$$M_i \models \phi[\bar{a}] \quad \text{iff} \quad M \models \phi[\bar{a}].$$

For ϕ atomic there is no problem. Also for $\phi = \neg\psi$ or $\phi = \bigwedge^E_{i<\mu} \phi_i$ we have no problem. So let $\phi(\bar{x}) = (\exists\bar{y})\psi(\bar{y},\bar{x})$. If $M_i \models \phi[\bar{a}]$, there is again no problem.

So suppose $M \models \phi[\bar{a}]$, and hence, for some $\bar{b} \subseteq M$, $M \models \psi[\bar{b},\bar{a}]$. By renaming, assume $\bar{b} = (\bar{b}_0{}^\wedge \cdots \bar{b}_\gamma{}^\wedge \cdots)_{\gamma<\delta}$, $\bar{b}_\gamma \subseteq M_\gamma$. By Fact 2.9 there is a finite $w \subseteq \delta$ as described there. Choose an ordinal $j > i$ with $\max(w) < j < \delta$, and choose sequences $\bar{b}'_\gamma$ $(\gamma < \delta)$ as follows: for $\gamma < j$ we take $\bar{b}'_\gamma = \bar{b}_\gamma$, and for $j \leq \gamma < \kappa$ we choose $\bar{b}'_\gamma$ to be any sequence from M_0 of the same length as $\bar{b}_\gamma$. By Fact 2.9,

$$M \models \psi[\bar{b}'_0,\ldots,\bar{b}'_\gamma,\ldots,\bar{a}] \equiv \psi[\bar{b}_0,\ldots,\bar{b}_\gamma,\ldots,\bar{a}];$$

but by assumption $M \models \psi[\bar{b}_0,\ldots,\bar{b}_\gamma,\ldots,\bar{a}]$, and hence $M \models \psi[\bar{b}'_0,\ldots,\bar{b}'_\gamma,\ldots,\bar{a}]$. By the induction hypothesis

$$M_j \models \psi[\bar{b}'_0,\ldots,\bar{b}'_\gamma,\ldots,\bar{a}],$$

so $M_j \models (\exists\bar{y})\psi(\bar{y},\bar{a})$, and hence $M_j \models \phi[\bar{a}]$; but $M_i <_{\mathscr{L}^1} M_j$, whence $M_i \models \phi[\bar{a}]$ as required.

(2) is also easy. □

2.11A. Remark. By 2.11(2) we can strengthen 2.2A(2) (χ may be e.g. any strong limit cardinal of cofinality $<\kappa$).

2.12. *Fact. Let* $\tau \subseteq \tau_1$, *let* M_1 *be a* τ_1-*model and* M *a* τ-*model, and let* $M_1 \restriction \tau <_{\mathscr{L}^1} M$. *Then there is a* τ_1-*model* N *such that* $M_1 <_{\mathscr{L}^1} N$ *and* $M <_{\mathscr{L}^1} N \restriction \tau$.

Proof. Let $M'_1 = (M_1, c)_{c \in M_1}$ and $M' = (M, c)_{c \in M}$. It is enough to prove that $\mathrm{Th}_{\mathscr{L}^1}(M'_1) \cup \mathrm{Th}_{\mathscr{L}^1}(M')$ has a model. If not, then by Fact 2.8, for some $\mu < \lambda$, $\phi_i \in \mathrm{Th}_{\mathscr{L}^1}(M')$ (for $i < \mu$) and upward-closed nonempty $(<\kappa)$-biclosed $E \subseteq \mathscr{P}(\mu)$,

$$\mathrm{Th}_{\mathscr{L}^1}(M'_1) \cup \left\{ \bigwedge_{i<\mu}^{E} \phi_i \right\} \text{ has no model and } \mu \in E.$$

Let $\bar{a}$ be a list of the $c \in M$ appearing in some ϕ_i; so $\bar{a}$ has length $< \lambda$ and without loss of generality $\bar{a} = \bar{b}^\frown \bar{c}$, $\bar{b} \subseteq |M| - |M_1|$, $\bar{c} \subseteq |M_1|$ and $\phi_i = \phi_i(\bar{b}, \bar{c})$, where $\phi_i(\bar{y}, \bar{z}) \in \mathscr{L}^1(\tau)$. Now $M \models \phi_i[\bar{b}, \bar{c}]$ for $i < \mu$, and $\mu \in E$; hence $M \models \bigwedge_{i<\mu}^{E} \phi_i[\bar{b}, \bar{c}]$, and so $M \models (\exists \bar{y}) \bigwedge_{i<\mu}^{E} \phi_i(\bar{y}, \bar{c})$. As the formula $(\exists \bar{y}) \bigwedge_{i<\mu}^{E} \phi_i(\bar{y}, \bar{z})$ is in $\mathscr{L}^1$ (by the choice of E) and as $M_1 \restriction \tau <_{\mathscr{L}^1} M$ and $\bar{c} \subseteq M_1$, clearly $M_1 \models (\exists \bar{y}) \bigwedge_{i<\mu}^{E} \phi_i(\bar{y}, \bar{c})$. Hence, for some $\bar{b}' \subseteq M_1$, $M_1 \models \bigwedge_{i<\mu}^{E} \phi_i(\bar{b}', \bar{c})$.

So M_1 has an expansion which is a model of $\mathrm{Th}_{\mathscr{L}^1}(M'_1) \cup \{\bigwedge_{i<\mu}^{E} \phi_i(\bar{b}, \bar{c})\}$, contradicting the choice of E. □

2.13. *Fact. If T has a model, T is a complete theory in $\mathscr{L}^1(\tau_1)$, $\tau_0 \subseteq \tau_1$, and M is a τ_0-model of $T \cap \mathscr{L}^1(\tau_0)$, then, for some τ_1-model N of T, $M <_{\mathscr{L}^1} N \restriction \tau_0$.*

Proof. Exactly like that of 2.12. □

2.14. *Fact. $\mathscr{L}^1$ has the Robinson property, i.e., if $\tau_0 = \tau_1 \cap \tau_2$ are vocabularies, T_l is a complete $\mathscr{L}^1(\tau_l)$-theory ($l \le 2$) which has a model, and $T_0 = T_1 \cap T_2$, then $T_1 \cup T_2$ has a model.*

Proof. We define, by induction on n, a model M_n such that:

(a) if n is even, M_n is a τ_2-model of T_2;

(b) if n is odd, M_n is a τ_1-model of T_1;

(c) $M_n \restriction \tau_0 <_{\mathscr{L}^1} M_{n+1} \restriction \tau_0$; and

(d) $M_n <_{\mathscr{L}^1} M_{n+2}$.

For $n = 0$ use "T_2 has a model"; for $n = 1$, Fact 2.13; for $n > 1$, Fact 2.12.

Let M be the unique $(\tau_1 \cup \tau_2)$-model such that $M \restriction \tau_1 = \bigcup_{n<\omega} M_{2n+1}$ and $M \restriction \tau_2 = \bigcup_{n<\omega} M_{2n}$. By 2.11 applied to $(M_{2n}: n < \omega)$, $M \restriction \tau_2$ is a model of T_2. By 2.11 applied to $(M_{2n+1}: n < \omega)$, $M \restriction \tau_1$ is a model of T_1.

2.15. Remark. The completeness of T_1 and T_2 is not needed in 2.14. We could complete them.

2.16. *Fact. $\mathscr{L}^1$ satisfies the interpolation theorem.*

2.16A. Remark. $\psi \models \phi$ means every model of ψ is a model of ϕ (if its vocabulary is rich enough).

Proof. Suppose $\tau_0 = \tau_1 \cap \tau_2$ and $\psi_l \in \mathscr{L}^1(\tau_l)$ for $l = 1, 2$; assume $\psi_1 \models \psi_2$ but for no $\phi \in \mathscr{L}^1(\tau_0)$ is it true that $\psi_1 \models \phi$ and $\phi \models \psi_2$. Let $T_0 = \{\phi \in \mathscr{L}^1(\tau_0): \psi_1 \models \phi\}$.

Now T_0 has a model (as ψ_1 has a model—otherwise trivially there is an interpolant). Choose if possible a complete $T \subseteq \mathscr{L}^1(\tau_0)$ such that $T \cup \{\neg \psi_2\}$ has a model. If $\{\psi_1\} \cup T$ has no model, then, by 2.8, for some $\mu < \lambda$, $\phi_i \in T$ for $i < \mu$, and $(<\kappa)$-biclosed $E \subseteq \mathscr{P}(\mu)$, $\mu \in E$, and $\{\psi_1\} \cup \{\bigwedge_{i<\mu}^{E} \phi_i\}$ has no model. Hence $\neg \bigwedge_{i<\mu}^{E} \phi_i \in T_0$, and so $\neg \bigwedge_{i<\mu}^{E} \phi_i \in T$. As $\phi_i \in T$ for each i, and $\mu \in E$, T can have no model; contradiction. So $\{\psi_1\} \cup T$ has a model. By 2.14 and 2.15, $\{\psi_1\} \cup T \cup \{\neg \psi_2\}$ has a model; contradiction. Hence there is no such T. So $T_0 \cup \{\neg \psi_2\}$ has no model. So, by 2.8, for some $\mu < \lambda$ we have $\phi_i \in T_0$ (for $i < \mu$) and $(<\kappa)$-biclosed $E \subseteq \mathscr{P}(\mu)$, $\mu \in E$, and $\{\bigwedge_{i<\mu}^{E} \phi_i\} \cup \{\neg \psi_2\}$ has no model. So $\bigwedge_{i<\mu}^{E} \phi_i \models \psi_2$;

on the other hand, $\psi_1 \vDash \phi_i$ (as $\phi_i \in T_0$), and hence, as $\mu \in E$, $\psi_1 \vDash \bigwedge^E_{i<\mu} \phi_i$. As $\phi_i \in T_0$, $\bigwedge^E_{i<\mu} \phi_i \in \mathscr{L}(\tau_0)$, so it is an interpolant as required. □

2.17. *Fact.* (1) *$\mathscr{L}^1$ is stronger than $\mathscr{L}_{\omega,\omega}(\exists^{\geq\kappa})$.*

(2) *If $\kappa \leq \mathrm{cf}(\mu) \leq \mu < \lambda$ and $(\forall\theta < \mu)(\forall\sigma < \kappa)[\theta^\sigma < \mu]$, then $\mathscr{L}'$ is stronger than $\mathscr{L}_{\omega,\omega}(\exists^{\geq\mu})$.*

Proof. (1) For any formula $\phi(x, \bar{y}) \in L$, and D a κ-complete uniform ultrafilter on μ, $\kappa \leq \mu < \lambda$, we have that

$$\psi(\bar{y}) = (\exists x_0, \ldots, x_i \ldots)_{i<\mu}\left[\bigwedge^D_{i<\mu}\left(\bigwedge^D_{j<\mu}(\phi(x_j, \bar{y}) \wedge x_i \neq x_j)\right)\right] \quad \text{is in } \mathscr{L}^1$$

and

$$M \vDash \psi[\bar{b}] \quad \text{iff} \quad |\{a \in M: M \vDash \phi[a, \bar{b}]\}| \geq \kappa.$$

(2) Let D be a κ-complete ultrafilter on $\mu \times \mu$ such that, for every function f from μ to any $\alpha < \mu$,

$$\{\langle i, j\rangle: i < \theta, j < \theta, f(i) = f(j)\} \in D$$

(as κ is a compact cardinal, there is such an ultrafilter). Now let

$$\psi'(\bar{y}) = (\exists \ldots, x_{\langle i,j\rangle}, \ldots)_{\langle i,j\rangle \in \mu\times\mu}\left[\bigwedge^D_{\langle i,j\rangle \in \mu\times\mu} \phi(x_i, \bar{y}) \wedge \phi(x_j, \bar{y}) \wedge x_i \neq x_j\right].$$

Now $M \vDash \psi'[\bar{b}]$ iff $|\{a \in M: M \vDash \varphi[a, \bar{b}]\}| > \mu$. □

2.18. *Fact. If $A_i \in D$ for $i < \mu$ then $\mathrm{Lim}_{D_1} A_i \in D$.*

2.19. *Fact.* (1) *An ultrafilter D over χ is $[\kappa, \lambda)$-complete iff, for every μ, $\kappa \leq \mu < \lambda$, every κ-complete ultrafilter D_1 over μ, and every $l(i, \alpha) \in \{0, 1\}$ $(i < \mu, \alpha < \chi)$ and $l \in \{0, 1\}$:*

$$\{i < \mu: \{\alpha < \chi: l(i, \alpha) = l\} \in D\} \in D_1$$
$$\textit{iff } \{\alpha < \chi: \{i < \mu: l(i, \alpha) = l\} \in D_1\} \in D.$$

(2) *Łoś's theorem holds for formulas of $\mathscr{L}^1$ for $[\kappa, \lambda)$-complete ultrafilters.*

Proof. (1) Check.

(2) Standard (using (1)). □

2.20. *Fact.* (1) *If M and N are $\mathscr{L}^1$-equivalent τ-models, and $\chi = \chi^{<\lambda} \geq |\tau(N)| + \|N\| + \|M\|$, then for some $[\kappa, \lambda)$-complete ultrafilter D over χ, M can be $\mathscr{L}^1$-elementarily embedded into N^χ/D.*

(2) *If, in (1), $M' <_{\mathscr{L}^1} M$, $N' <_{\mathscr{L}^1} N$ and h is an isomorphism from M' onto N', we can demand that the embedding there extend h.*

(3) *If M and N are $\mathscr{L}^1$-equivalent τ-models, $\|M\| + \|N\| \leq \chi_n$, and $2^{\chi_n} \leq \chi_{n+1}$, then, for some $[\kappa, \lambda)$-complete ultrafilters D_n on χ_n, the corresponding ultralimits are isomorphic. [I.e., let $M_0 = M$, $N_0 = N$, $M_{n+1} = M_n^{\chi_n}/D_n$, $N_{n+1} = N_n^{\chi_n}/D_n$, and, by* 2.19(2) *and suitable identification, $M_n <_{\mathscr{L}^1} M_{n+1}$ and $N_n <_{\mathscr{L}^1} N_{n+1}$. Then $\bigcup_{n<\omega} M_n$ and $\bigcup_{n<\omega} N_n$ are isomorphic.]*

Proof. (1) Like the proof of 2.12.

(2) By (1).

(3) Like 2.14.

2.21. *Fact.* (1) *The* $\mathscr{L}^1$*-theory of* $M + N$ *is totally determined by* $\mathrm{Th}_{\mathscr{L}^1}(M)$ *and* $\mathrm{Th}_{\mathscr{L}^1}(N)$.

(2) *The same is true for any operation as in Feferman and Vaught* [FV], *as long as we use sequences of elements of length uniformly bounded below* κ (*and on* $<\kappa$ *models*).

(3) *We can replace* $\mathscr{L}^1$ *by the set of sentences in* $\mathscr{L}^1$ *of quantifier depth* $<\alpha$.

Proof. (1) Follows by 2.20(3), or by (3).

(2) and (3) are like the generalizations of [FV] to the set of sentences in $L_{\infty,\lambda}$ of depth $<\alpha$ (see [D, §§2 and 4]). □

2.22. *Fact (pre-nice normal form). Every formula in* $\mathscr{L}^1$ *is equivalent to a formula of the form*

$$\forall \bar{x}_0 \exists \bar{y}_1 \forall \bar{x}_1 \exists \bar{y}_2 \cdots \forall \bar{x}_n \exists \bar{y}_n \Phi,$$

where

(1) $n < \omega$, *and* $\bar{x}_l$ *and* $\bar{y}_l$ *are sequences of variables of length* $<\lambda$ *with no variable appearing twice in* $\bar{x}_0{}^\wedge \bar{y}_0{}^\wedge \bar{x}_1{}^\wedge \bar{y}_1{}^\wedge \cdots {}^\wedge \bar{y}_n$); *and*

(2) Φ *has the form* $\bigwedge^E_{i<\mu} \phi_i$, *where* $\mu < \lambda$, $E \subseteq \mathscr{P}(\mu)$ *is* ($<\kappa$)*-biclosed and each* ϕ_i *is atomic, or, equivalently,* Φ *is a finite Boolean combination of formulas of the form* $\bigwedge^D_{i<\mu} \phi_i$, *where each* ϕ_i *is atomic,* $\mu < \lambda$, *and* D *is a* κ*-complete ultrafilter.*

Proof. By induction on the depth of the formula (and see 2.5). □

2.23. *Discussion.* We may be unsatisfied that as an extension of $\mathscr{L}^1(\exists^{\geq\kappa})$ which has INT we suggest $\mathscr{L}$ with $|\mathscr{L}^1(\tau)| = \lambda$ even for finite τ. However, we can find $\mathscr{L}'$ such that $\mathscr{L}'(\tau) \subseteq \mathscr{L}^1(\tau)$ (for each τ', $\mathscr{L}'(\tau) = \bigcup\{\mathscr{L}^1(\tau'): \tau' \subseteq \tau \text{ is finite}\}$), $\mathscr{L}'$ satisfies interpolation and extends $\mathscr{L}(\exists^{\geq\kappa})$, and $|\mathscr{L}'(\tau)| \leq \kappa + |\tau|$. Clearly many of the good properties of $\mathscr{L}^1$ are inherited.

REFERENCES

[A] KENNETH J. ARROW, *A difficulty in the concept of social welfare*, ***Journal of Political Economy***, vol. 58 (1950), pp. 328–346.

[B] K. J. BARWISE, *Some applications of Henkin quantifiers*, ***Israel Journal of Mathematics***, vol. 25 (1976), pp. 47–63.

[BF] J. BARWISE and S. FEFERMAN (editors), ***Model-theoretic logics***, Springer-Verlag, New York, 1985.

[CK] C. C. CHANG and H. J. KEISLER, ***Model theory***, North-Holland, Amsterdam, 1973.

[CP] CHRISTINE CHARRETTON and MAURICE POUZET, *Comparaison des structures engendrées par des chaînes*, ***Proceedings of the Dietrichshagen Easter conference on model theory***, Seminarbericht No. 49, Sektion Mathematik, Humboldt-University, Berlin, 1983, pp. 17–27.

[D] M. A. DICKMANN, *Larger infinitary languages*, in [BF], pp. 317–363.

[FV] S. FEFERMAN and R. L. VAUGHT, *The first-order properties of algebraic systems*, ***Fundamenta Mathematicae***, vol. 47 (1959), pp. 57–103.

[H] WILFRID HODGES, ***Some questions on the structure of models***, D.Phil. Thesis, Oxford University, Oxford, 1969.

[HS] WILFRID HODGES and SAHARON SHELAH, *Infinite games and reduced products*, ***Annals of Mathematical Logic***, vol. 20 (1981), pp. 77–108.

[K] M. KAUFMANN, *The quantifier "There exist uncountably many" and some of its relatives*, in [BF], pp. 123–176.

[M] J. A. MAKOWSKY, *Compactness, embeddings and definability*, in [BF], pp. 645–716.

[MM] M. MAGIDOR and J. J. MALITZ, *Compact extensions of* $L(Q)$ (Part 1a), ***Annals of Mathematical Logic***, vol. 11 (1977), pp. 217–261.

[Sc] JAMES H. SCHMERL, *An axiomatization for a class of two-cardinal models*, this JOURNAL, vol. 42 (1977), pp. 174–178.

[Sh] SAHARON SHELAH, *Remarks on abstract model theory*, ***Annals of Pure and Applied Logic***, vol. 29 (1985), pp. 255–288.

[SS1] SAHARON SHELAH and CHARLES STEINHORN, *The non-axiomatizability of $L(Q^2_{\aleph_1})$ by finitely many schemata*, ***Notre Dame Journal of Formal Logic***, vol. 31 (1990), pp. 1–13.

[SS2] ———, *On the non-axiomatizability of some logics by finitely many schemas*, ***Notre Dame Journal of Formal Logic***, vol. 27 (1986), pp. 1–11.

SCHOOL OF MATHEMATICAL SCIENCES
QUEEN MARY AND WESTFIELD COLLEGE
LONDON E1 4NS, ENGLAND

DEPARTMENT OF MATHEMATICS
THE HEBREW UNIVERSITY
GIVAT RAM, JERUSALEM, ISRAEL