UNIVERSAL CLASSES

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Introduction

We have been interested in classifying first order theories, not in the sense of finite group theory, i.e. explicit list of families but like biology - find main taxonomies, dividing lines. See [Sh 200].

Some years ago (1982) we found what we wanted (for countable theories). We try here to develop the case of a universal class (see below). In fact we develop it less concretely, abstractly, both per se and as we shall need eventually to define inductively a sequence of such frameworks. For technical reasons only the first four chapters appear here.

Definition:

i) Let τ be a vocabulary (= signature). K will denote a class of $\tau(K)$-models.

ii) K is *universal* if K is closed under submodels and increasing chains and isomorphisms.

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Note: i) Not every elementary class is universal but many universal classes are not elementary, e.g. the locally finite groups.

ii) If K is universal, $\tau(M) = \tau(K)$ then $M \in K$ if and only if every finitely generated submodel of M belongs to K (see II 2.2B).

iii) Remember in this context the following theorem of Tarski: for a finite relational vocabulary, K is universal *if and only if* K is the class of models of a universal first order theory.

General Strategy

We shall consider various dividing lines, i.e. properties. On one side we shall prove a "non-structure results." Typically we can interpret (essentially) arbitrary linear orders I or $I = (S, \lessdot)$ with ${}^{\omega>}\lambda \subseteq S \subseteq {}^{\omega\geq}\lambda$ inside models in K. The models which we exhibit are essentially generated by such I (e.g. Ehrenfeucht Mostowski models).

In this case we get non-structure results, then we assume the negation and continue our search. The point is that the negation says much, it is a property which implies at least some structure. Sometimes this knowledge is instrumental in proving non-structure results for properties which are "buried deeper". Later we shall have cases where we get weak non structure results; seemingly as for universal class there are more possibilities. This was a successful strategy for countable first order T (see [Sh 200], [Sh A,3]) and is being written for classification over a predicate. (See [Sh 321], partial results appear in [Pi Sh 130], [Sh 234]). On non elementary classes see [Sh 48], [Sh 87], and in an abstract setting [Sh 88]. Those papers deal with categoricity. From the other end, some papers deal with properties which are sufficient for non-structure results (and hopefully their complement will be helpful). See on infinitary order property [Sh 16]. For much better results, see Grossberg and Shelah: two papers on order property [GrSh 222], [GrSh 259], one paper on unsuperstability $(({}^{\omega>}\lambda, \lessdot))$ [GrSh 238]. (On the more general situation $\{M : M \models \psi\}$, $\psi \in L_{\lambda^+,\omega}$ see [Sh 285]).

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Notation:

Set Theory

$\lambda, \mu, \chi, \kappa$ denote cardinals (usually in finite).
$\alpha, \beta, \gamma, i, j, \zeta, \xi$ denote ordinals.
δ denotes a limit ordinal.
$H(\lambda)$ denote the family of sets with transitive closure of cardinality $< \lambda$.

Model Theory

τ denotes a vocabulary, i.e. set of predicates and function symbols, each with a designated fixed (finite) arity.

M a model, $\tau(M)$ its vocabulary, for $\tau = \tau(M)$ we say M is a τ-model, $|M|$ the universe of M.

K a class of models all with the same vocabulary $\tau(K)$, for $\tau = \tau(K)$ we say K is a τ-class.

$\bar{a}, \bar{b}, \bar{c}$ denote sequences of elements from a model, not necessarily finite. The length of a sequence $\bar{a}$ is denoted by $\ell g(\bar{a})$.

$\mathcal{L}$ a logic, i.e. for every vocabulary τ, $\mathcal{L}(\tau)$ is a set of formulas $\varphi(\bar{x})$ not necessarily first order. ($\bar{x}$ is a possibly infinite sequence of variables including all free variables of φ) and we assume always $[\tau_1 \subseteq \tau_2 \Rightarrow \mathcal{L}(\tau_1) \subseteq \mathcal{L}(\tau_2)]$, $[\varphi \in \mathcal{L}(\tau_1)$ and $\varphi \in \mathcal{L}(\tau_2)$ implies $\varphi \in \mathcal{L}(\tau_1 \cap \tau_2)]$; if M is a τ-model, $\bar{c} \in {}^{\ell g(\bar{x})}|M|$, the truth value of "$M \models \varphi[\bar{c}]$" is defined, and depends only on $M \restriction \tau$ if $\varphi \in \mathcal{L}(\tau)$.

φ,ψ,θ denote formulas, on $\varphi(\bar{x})$ see above; $\varphi,\varphi(\bar{x}),\varphi(\bar{x};\bar{y})$ may be treated as objects of a different kind (see below). We sometimes separate "type", "free" variables from "parameter variables". $L_{\lambda,\kappa}$ is the set of formulas we get from the atomic formulas by closing under $\neg\varphi$ (negation) $\bigwedge_{j<\alpha} \varphi_i$ (where $\alpha < \lambda$, conjunction) and $(\exists x_0, \ldots, x_i, \cdots)_{j<\alpha}\varphi$ (where $\alpha < \kappa$, existential quantification), but for $\varphi(\bar{x}) \in L_{\lambda,\kappa}(\tau)$ we demand $\ell g(\bar{x}) < \lambda$. (So $L_{\lambda,\kappa}$ is a logic, $L_{\omega,\omega}$ first order logic).

A class K of τ-models is a $PC_{\lambda,\mu}$ if for some vocabulary τ_1, $\tau \subseteq \tau_1$, $\tau_1| \leq \lambda$ and $\psi \in L_{\lambda,\mu}(\tau_1)$ we have $K = \{M \restriction \tau : M \models \psi\}$. $PC(T_1,T)$ is the class of $\tau(T)$-reducts of models of T_1.

Lastly a class K of models is $PCT_{\lambda,\kappa}$ if for some τ_1, $\tau \subseteq \tau_1$, $|\tau_1| = \kappa$, first theory $T_1 \subseteq L_{\omega,\omega}(\tau_1)$ and set Γ of λ ($<\omega$)-types in $L_{\omega,\omega}(\tau_1)$, $K = \{M \restriction \tau : M$ a model of T_1 omitting every $p \in \Gamma\}$.

Note: for $\lambda > \kappa$ a formula of $L_{\lambda,\kappa}(\tau)$ has $< \lambda$ free variables.

Φ,Ψ,Θ will denote sets of formulas of the form $\varphi(\bar{x},\bar{y})$ or $\varphi(\bar{x})$. If $\varphi(\bar{x}) \in \Phi$ this means $\varphi(\bar{x}^1,\bar{x}^2) \in \Phi$ when $\bar{x} = \bar{x}^1 \,{}^\wedge \bar{x}^2$. These formulas may have parameters.
$tp_{\varphi(\bar{x};\bar{y})}(\bar{a},A,M) = \{\varphi(\bar{x};\bar{b}) : \bar{b} \in {}^{\ell g(\bar{y})}A,\ M \models \varphi[\bar{a};\bar{b}]\}$ where $\varphi(\bar{x};\bar{y}) \in \mathcal{L}(\tau(M))$ for some $\mathcal{L}$, and $A \subseteq |M|$.

Notation for such types is needed when a monster model ($\mathfrak{C}$) is absent (or still absent) (otherwise we can omit M). We say $p = tp_{\varphi(\bar{x};\bar{y})}(\bar{a},A,M)$ is a type (or $\varphi(\bar{x};\bar{y})$-type) inside M. Similarly for the following variants.

$$tp_{\varphi(\bar{x})}(\bar{a},\ A,M) = tp_{\varphi(\bar{x}_1;\bar{x}_2)}(\bar{a},A,M) \text{ where } \bar{x} = \bar{x}_1 \hat{}\ \bar{x}_2,\ \ell g(\bar{a}) = \ell g(\bar{x}_1)$$

$$tp_{\{\varphi\}}(\bar{a},\ A,M) = tp_{\varphi}(\bar{a},\ A,M)$$

$$tp_{\Phi}(\bar{a},\ A,M) = \bigcup_{\varphi \in \Phi} tp_{\varphi}(\bar{a},\ A,M)$$

$$S^{\alpha}_{\Phi}(A,M) = \{tp_{\Phi}(\bar{a},A,M) : \bar{a} \in {}^{\alpha}|M|\}$$

we can replace A by $\mathbf{J}$, a family of sequences, e.g.

$tp_{\varphi(\bar{x};\bar{y})}(\bar{a},\mathbf{J},M) = \{\varphi(\bar{x};\bar{b}_1,\ldots,\bar{b}_n): n < \omega,\ M \models \varphi[a,\bar{b}_1,\ldots,\bar{b}_n],\ \bar{b}_\ell \in \mathbf{J} \text{ for } \ell = 1,n\}$ or by a set of formulas with parameters e.g.

$$tp_{\varphi(\bar{x};\bar{y})}(\bar{a},\Theta,M) = \{\varphi(\bar{x};\bar{c}) : M \models \varphi(\bar{a};\bar{c}),\varphi(\bar{x};\bar{c}) \in \Theta\}$$

We then say "type over Θ" or "type over $\mathbf{J}$".

$M <_\Phi N$ *means* that for $\varphi(\bar{x}) \in \Phi$ and $\bar{a} \in {}^{\ell g(\bar{x})}|M|$:

$$[M \models \varphi[\bar{a}] \text{ if and only if } N \models \varphi[\bar{a}]\ ]$$

$\Sigma_{\lambda,\kappa}(\Phi)$ is the set of formulas of the form

$$\psi(\bar{y}) \stackrel{def}{=} (\exists x_0,\ldots,x_i,\cdots)_{i<\kappa} \bigwedge_{\alpha<\lambda} \Phi_\alpha(\bar{y}_\alpha)$$

where $\bar{y}_i \subseteq \bar{y}\ \hat{}\ \langle x_0,\ldots,x_i,\cdots\rangle_{i<\alpha}$, $|\ell g(\bar{y})| < \kappa$, $\varphi_\alpha(\bar{y}_\alpha) \in \Phi$ or $\neg\varphi_\alpha(\bar{y}_\alpha) \in \Phi$ for each α. So $\Sigma_{\lambda,\kappa}(\Phi)$ includes every $\varphi_\alpha(\bar{y}) \in \Phi$, for which $|\ell g(\bar{y})| < \kappa$.

p is a type inside M if p is a set of $\tau(M)$-formulas with parameters from M.

Chapter I: Stability Theory for a Model

§0 Introduction

In [Sh A1, Ch. I, §2] little stability theory was developed for an arbitrary model; quite naturally as this was peripheral there. More attention was given to non-structure theorems for infinitary logics (see [Sh 16, §2] and Grossberg and Shelah [GrSh 222]. [GrSh 238], [GrSh 259] and applications, see Macintyre and Shelah [MaSh 55], Grossberg and Shelah [GrSh 174]).

However, in our present framework it is important to get results on infinitary languages. As we have fewer transfer theorems, it is natural to concentrate on one model.

Surprisingly we have something to say, some of it was explicit or implicit in [Sh A1, ch. I, §2]: the theorems that non stability implies order (i.e. existence of quite long set of sequences, linearly order by a formula), that non order implies the existence of indiscernibles and (the main novelty) that we can average types, all have reasonable analogs.

Lastly we prove (in section 5) that in order to get just indiscernible sets, less "non-order" is needed, and this gives new information even on first order theories. E.g. *if* T is first order, there is no formula $\varphi(x,y,\bar{z})$ such that some model M of T has $(\varphi(x,y,\bar{z}), \aleph_0)$-order property (note x,y are not sequences), M a model of T, $a_i \in M$ for $i < (2^\lambda)^+$, $\lambda \geq |T|$, *then* for some $w \subsetneq (2^\lambda)^+$, $|w| > \lambda$, $\{a_i : i \in w\}$ is an indiscernible set in M.

§1 The order property revisited

The main results of this section are Theorem 1.2 and 1.10. We begin by recounting the appropriate definition of the order property in this context. We note in Theorem 1.2 (proved in Chapter III.3) that this relevant order property implies the existence of many non isomorphic models.

These notions have two parameters: a formula and a cardinal. As we no longer are attached to first order logic, the formula (or set of formulas) as a parameter is even more important than in [Sh A1]. As we assume generally no closure properties for the set of formulas, we have to be more explicit in asserting "there is a formula" (Note that we may have to consider several logics, simultaneously, as in [Sh 285], and that usually non-first order logics have weaker closure properties).

A new parameter is a cardinal (the length of the order). Its presence is desirable as we no longer assume compactness, so not all infinite cardinals give equivalent definitions.

Then we describe the notions of "indiscernible" and "splitting" appropriate for this context. In Theorem 1.7 we show that either for each type we can find a "base" over which it does not split or the order property holds. In Theorem 1.11 we show that for appropriate μ if the number of Φ types over a set of power μ which are realized in M is not bounded by μ then there is a Φ^* (closely associated with Φ) such that M has the (Φ^*, κ^+)-order property.

1.1 Definition:

1) M has the $(\varphi(\bar{x};\bar{y};\bar{z}),\mu)$-order property *if* there are sequences $\bar{c}$, $\bar{a}_\alpha, \bar{b}_\alpha$ from M, such that for $\alpha < \mu$:

$$M \models \varphi[\bar{a}_\alpha, \bar{b}_\beta, \bar{c}] \text{ if and only if } \alpha < \beta.$$

We extend this notion to sets (or classes) of formulas and classes of models as follows:

2) M has the (Φ,μ)-order property *if* for some $\varphi(\bar{x};\bar{y};\bar{z}) \in \Phi$, M has the $(\varphi(\bar{x};\bar{y};\bar{z}),\mu)$-order property.

3) K has the (Φ,μ)-order property if for some $M \in K$, M has the (Φ,μ)-order property.

4) M [or K] has the $(\Phi,<\mu)$-order property *if* M [or K] has the (Φ,μ_1)-order property for every $\mu_1 < \mu$.

5) replacing "order" by "nonorder" is just the negation.

6) M has the $(\pm\varphi,\mu)$-order property if it has the (φ,μ)-order property or the $(\neg\varphi,\mu)$-order property; similarly for the other definitions.

7) Let "$(\varphi(\bar{x},\bar{y}),\mu)$-order" means "$(\varphi(\bar{x};\bar{y};\bar{z}),\mu)$-order for $\bar{z}$ the empty sequence, and $(\varphi(\bar{x}),\mu)$-order means $(\varphi(\bar{x}_1;\bar{x}_2;\bar{x}_3),\mu)$-order, $\bar{x} = \bar{x}_1 {}^\wedge \bar{x}_2 {}^\wedge \bar{x}_3$ for some $\bar{x}_1,\bar{x}_2,\bar{x}_3$.

1.1A Remark: Usually $\Phi \subseteq L_{\infty,\omega}$, but sometimes $\Phi \subseteq \Delta(L_{\infty,\omega})$ (i.e. every formula and its negation is a pseudo elementary class).

On the other hand for universal K (see §2) we may well use Φ = set of quantifier free finite formulas.

Note that if M has the $(\varphi(\bar{x};\bar{y};\bar{z}),\mu)$-order property, then it has the $(\varphi(\bar{x};\bar{y}{}^\wedge\bar{z}),\mu)$- order property.

We shall prove in Chapter III (and in [Sh 220]) that order implies complexity:

1.2 Theorem:

1) If K is definable by a sentence in $\Delta(\mathcal{L}_{\lambda^+,\omega})$, and it has the $(\varphi(\bar{x};\bar{y}),<\infty)$-order property, $\varphi(\bar{x};\bar{y}) \in \Delta(\mathcal{L}_{\lambda^+,\omega})$ *then:*

(a) for every $\mu > \lambda + |\ell g(\bar{x}{}^\frown\bar{y})|^+$ the class K has 2^μ non isomorphic members of power μ. [see III 3.4 using III 1.11(3)]

(b) if $cf(\mu) > \lambda, \mu$ is regular or strong limit, then K has 2^μ nonisomorphic members of power μ which are $\mathcal{L}_{\infty,\mu}$-equivalent. [by [Sh 220], §2 (for μ regular), §3 (for μ strong limit) using III 1.11(3)].

(c) if $\mu > \lambda$ is regular, $\mu = \mu^{\ell g(\bar{x}{}^\frown\bar{y})}$ *then* K has 2^μ members of power μ, no one embeddable into another by an embedding preserving $\pm\,\varphi(\bar{x},\bar{y})$.

2) If K is definable by a sentence from $\Delta(\mathcal{L}_{\kappa^+,\omega})$ and it has the $(\varphi(\bar{x};\bar{y}),\lambda)$-order property, $\varphi(\bar{x},\bar{y}) \in \mathcal{L}_{\mu^+,\omega}$ then : [by [GrSh 222, 254])

a) if $\lambda \geq \beth_{\delta(\mu+\kappa)}$ *then* K has the $(\varphi(\bar{x};\bar{y}),<\infty)$-order property.

b) if $\lambda \geq \beth_{\delta(\mu+\kappa)}$ *then* for some $\varphi'(\bar{x}';\bar{y}') \in \mathcal{L}_{\kappa^+,\omega}$, K has the $(\varphi'(\bar{x}',\bar{y}'),<\infty)$-order property (and φ' inherits all relevant properties of φ. More exactly, [letting $H(\lambda)$ denote the family of sets hereditarily of cardinality $<\lambda$] for some $\lambda, \varphi \in H(\lambda)$, and for some elementary submodel N of $H(\lambda)$ of cardinality κ, φ' is the image of φ under the Mostowski Collapse of N).

c) if $\lambda \geq \beth_{\delta_{wo}}(\mu,\kappa)$ *then* (see definition in [GrSh222]) b)'s conclusion holds.

3) Similar conclusions hold for $\varphi(\bar{x};\bar{y};\bar{z})$.

Remark: 1) For a proof of more than 1.2(1) see Ch. III, §3 here.

2) On the subject and proof of 1.2(2), 1.2(3) see Shelah [Sh 16] and Grossberg and Shelah [GrSh 222,259]. Remember that $\beth_{\delta(\mu)}$ is Morley's number (See [Sh,VII,§5]). The definition of bounds on $\delta_{wo}(\mu,\kappa)$ are of Grossberg and Shelah [GrSh 222,257].

3) We do not try to get the optimal results, just previous proofs obviously give. E.g. we ignore the slightly stronger versions we can get by replacing μ by a limit cardinal (regular or use sequences of A's).

1.3 Definition:

1) $\langle \bar{a}_t : t \in I \rangle$, where I is a linear order $\bar{a}_t \in M$, is a (Φ, n)-*indiscernible sequence inside M over A* if: for all $t_1 < \cdots < t_n \in I$, $\bar{a}_{t_n} {}^\wedge \cdots {}^\wedge \bar{a}_{t_1}$ realizes the same Φ-type inside M over A.

2) Writing Φ instead (Φ, n) means "for all $n < \omega$". If we omit A we means $A = \textit{empty}$ 11.

1.3A Note: The sequences may have infinite length but $n < \omega$. I.e. we use only finitely many sequences at a time. This should not be surprising, as $\lambda \rightarrow (\mu)^{\omega}_{\chi}$ is much more difficult to have than $\lambda \rightarrow (\mu)^{<\omega}_{\chi}$.

1.4 Definition: $\{\bar{a}_t : t \in I\}$ is a (Φ, n)-indiscernible set inside M (over A) if for all distinct $t_1, \ldots, t_n \in I$

$[\bar{a}_{t_n} {}^\wedge \ldots {}^\wedge \bar{a}_{t_1}]$ realizes the same Φ-type in M (over A).

* * *

We define here the notion "$p\,(\Phi, \Psi)$-splits over A" (inside M). This says that in some weak sense, $p \restriction \Phi$ is definable over A. More specifically the Ψ-type of the parameters over A, separate between the $\bar{b}$ such that $\varphi(\bar{x}, \bar{b}) \in p$ and the $\bar{b}$ such that $\neg\varphi(\bar{x}, \bar{b}) \in p$. In Definition 1.5(2) we replace Ψ, and A by a collection of formulas Θ.

1.5 Definition:

1) A type $p = p(\bar{x})$ inside M, (Φ, Ψ)- *splits* over A *if* there are $\bar{b}, \bar{c} \in M$, and $\varphi(\bar{x}; \bar{y}) \in \Phi$ such that:

i) $\varphi(\bar{x}, \bar{b}), \neg\varphi(\bar{x}, \bar{c}) \in p$

ii) In appropriate sense, $\bar{b}$ and $\bar{c}$ realize the same Ψ-type over A inside M; more exactly : for $\psi(\bar{x};\bar{y};\bar{z}) \in \Psi$, $(\ell g(\bar{b}) = \ell g(\bar{c}) = \ell g(\bar{y}))$ and $\bar{e} \in {}^{\ell g(\bar{z})}A$, $\bar{a}' \in {}^{\ell g(\bar{x})}A$, $M \models \psi[\bar{a}',\bar{b},\bar{e}]$ if and only if $M \models \psi[\bar{a}',\bar{c},\bar{e}]$

2) A type $p = p(\bar{x})$ inside M, (Φ,Ψ)-split over Θ (Θ consisting of Ψ-formulas with parameters from M) *if* there are $\bar{b},\bar{c} \in M$ of equal length and $\varphi(\bar{x};\bar{y}) \in \Phi$ such that:

i) $\varphi(\bar{x};\bar{b}), \neg\varphi(\bar{x},\bar{c}) \in p$,

ii) if $\psi(\bar{x};\bar{y},\bar{z}) \in \Psi$, $\ell g(\bar{y}) = \ell g(\bar{c}) = \ell g(\bar{b})$ and $\bar{a}'$, $\bar{e} \in M$, $\psi(\bar{a}',\bar{y},\bar{e}) \in \Theta$ *then* $M \models \psi[\bar{a}';\bar{b},\bar{e}]$ if and only if $M \models \psi[\bar{a}';\bar{c},\bar{e}]$.

3) We define "p Ψ-split over Θ" similarly, omitting "$\varphi(\bar{x};\bar{y}) \in \Phi$".

1.5A Remark: Clearly 1.5(1) is an instance of 1.5(2).

1.6 Fact:

1) If $p = p(\bar{x})$ is a type inside M, which (Φ,Ψ)-splits over A and $p \subseteq q(\bar{x})$, with $q(\bar{x})$ a type inside M, $\Phi \subseteq \Phi_1, \Psi_1 \subseteq \Psi$ *then* $q(\bar{x})$ *(Φ_1,Ψ_1)-split over A.*

2) Suppose for $\ell = 1,2$, $p_\ell(\bar{x})$ a type inside M, which does not (Φ,Ψ)-split over Θ,Θ a set of formulas over A, $p_\ell \in S_\Phi^{\ell g(\bar{x})}(C,M)$, and $A \subseteq B \subseteq C \subseteq |M|$, and each p_ℓ is a complete Φ-type over C. If for every $\bar{b} \in C$ there is $\bar{b}' \in B$ such that for every $\varphi(\bar{a},\bar{y},\bar{e}) \in \Theta$ $M \models \varphi[\bar{a},\bar{b},\bar{e}] \equiv \varphi[\bar{a},\bar{b}',\bar{e}]$ provided that $\ell g(\bar{b}) = \ell g(\bar{b}') = \ell g(\bar{y})$ *then* $p_1 \restriction B = p_2 \restriction B$ implies $p_1 = p_2$.

3) Suppose $p(\bar{x})$ is a type inside M, $A \subseteq M$, $\Theta = \{\psi(\bar{e};\bar{y},\bar{a}) : \bar{e},\bar{a} \in A, \psi(\bar{x};\bar{y},\bar{z}) \in \Psi\}$. *Then:* $p(\bar{x})$ (Φ,Ψ)-split over A if and only if $p(\bar{x})$ (Φ,Ψ)-split over Θ.

4) If $A \subseteq B \subseteq |M|$, then $\{p \in S_\Phi^\alpha(B,M) : p$ does not (Φ,Ψ)-split over $\Theta\}$ has cardinality $\leq 2^{(2^{|\Theta|} + |\Phi|)}$

5) If $A \subseteq M$, $|S^{\alpha}_{\Phi}(A,M)| \le \Pi\{|S^{\alpha}_{\varphi(\bar{x};\bar{y})}(A,M)| : \ell g(\bar{x}) = \alpha, \varphi(\bar{x};\bar{y}) \in \Phi\}$.

1.6A Remark: We can systematically replace sets of elements by sets of formulas.

1.7 The non-splitting/order dichotomy theorem:

Suppose $M <_{\Sigma_{\chi,\kappa}(\{\varphi(\bar{x};\bar{y})\})} N$,
$\varphi(\bar{x};\bar{y})$ a $\tau(N)$-formula, $\ell g(\bar{x}) \le \ell g(\bar{y}) \le \chi$ and $\psi(\bar{x};\bar{y}_1{}^\frown\bar{y}_2) \stackrel{def}{=} [\varphi(\bar{x};\bar{y}_1) \equiv \varphi(\bar{x};\bar{y}_2)]$. then (i) or (ii) (or both) hold where:

(i) for every $\bar{c} \in |N|$, $|\ell g(\bar{c})| \le \kappa$ for some $\Theta \subseteq \{\varphi(\bar{c};\bar{y}) : \bar{c} \in |M|\}$, $|\Theta| \le \chi$ and $tp_{\varphi(\bar{x};\bar{y})}(\bar{c}, |M|, N)$ does not $(\varphi(\bar{x};\bar{y}), \varphi(\bar{x};\bar{y}))$-split over Θ.

(ii) N has the (ψ,χ^+)-order property (in fact, exemplified by sequences from M).

1.7A Fact: Note that (just combining definitions) $M <_{\Sigma_{\chi,\kappa}(\{\varphi\})} N$ means (when for simplicity $\chi^\kappa = \chi$) : $M <_\Phi N$ and *for every* $\bar{c} \in {}^\kappa N$ and $A \subseteq |M|$, such that $|A| \le \chi$ *there is* $\bar{c}' \in {}^\kappa|M|$ realizing $tp_\Phi(\bar{c},A,M)$.

1.7B Remark: In 1.7 we contrast $(\varphi(\bar{x};\bar{y}),\varphi(\bar{x};\bar{y}))$-splitting with the $(\pm\varphi(\bar{x};\bar{y}),\chi^+)$-order property where $\chi = \ell g(\bar{y})$, (and see 1.8 below). This χ is the crucial parameter because it governs our ability to continue to choose $\bar{a}_i,\bar{b}_i$.

Proof: Assume $tp_{\varphi(\bar{x};\bar{y})}(\bar{c},M,N)$ contradicts (i). We shall prove (ii). We define by induction on i, $\bar{a}_i,\bar{b}_i,\bar{c}_i$ in M with $\ell g(\bar{c}_i) = \ell g(\bar{x})$, $\ell g(\bar{b}_i) = \ell g(\bar{c}_i) = \ell g(\bar{y})$; such that:

a) $N \models [\varphi(\bar{c};\bar{a}_i) \equiv \neg\varphi(\bar{c};\bar{b}_i)]$

b) for $j < i$, $N \models \varphi(\bar{c}_j,\bar{a}_i) \equiv \varphi(\bar{c}_j,\bar{b}_i)$

c) $\bar{c}_i$ realizes $\{\varphi(\bar{x},\bar{a}_j), \neg\varphi(\bar{x},\bar{b}_j) : j \leq i\}$ inside N.

Note: (a) and (b) say exactly: $\varphi(\bar{x},\bar{y})$, $\bar{a}_i,\bar{b}_i$ exemplify $tp_{\varphi(\bar{x};\bar{y})}(\bar{c},M)$, $(\varphi(\bar{x};\bar{y}), \varphi(\bar{x};\bar{y}))$-split over $\{\varphi(\bar{c}_j;\bar{y}) : j < i\}$. Hence for $i < \chi^+$ if $\bar{c}_j,\bar{b}_j,\bar{a}_j$ $(j < i)$ are defined, we can define $\bar{a}_i,\bar{b}_i$; then using $M <_{\Sigma_{\chi,\kappa}(\varphi)} N$ we can define $\bar{c}_i$.

Having defined all $\bar{a}_j,\bar{b}_j,\bar{c}_j$ (for $j < \chi^+$), clearly $N \models \varphi(\bar{c}_\alpha,\bar{b}_\beta) \equiv \varphi(\bar{c}_\alpha,\bar{a}_\beta)$ if and only if $\alpha < \beta$. So $\{\bar{c}_\alpha : \alpha < \chi^+\}$, $\{\bar{b}_\beta {}^\wedge \bar{a}_\beta : \beta < \chi^+\}$ exemplify (ii).

1.8 Observation: Suppose φ,ψ are as in 1.7, and N has the (ψ,μ_1)-order property, $\mu_1 \rightarrow (\mu_2)^2_2$. Then N has the $(\pm\varphi(\bar{x};\bar{y}),\mu_2)$-order property.

Proof: Immediate.

1.8A Remark: Using this, and only $(\pm\varphi(\bar{x};\bar{y}),\lambda)$-order properties, the formulation of theorems in this section becomes nicer. I.e. we lose some sharpness in cardinality bounds, but we use only $\pm\varphi$-order and φ-unstability properties.

We remarked above that for non first order logics we must be careful about closure properties of sets of formulas. The following notation permit us to take this care.

1.8B Remark: 1) Theorem 1.7 has an obvious version for (Φ,Φ)- splitting and the (Φ,χ^+)-order property. To formulate it one must consider the cardinality of Φ, (use 1.6(5)).

2) We could have replaced χ^+ by a limit cardinal (sometimes of large cofinality or regular and/or > uncountable).

1.9 Definition: $\Phi^{cn} = \{\neg\varphi : \varphi \in \Phi\}$

$$\Phi^{es} = \{\psi(\bar{x};\bar{y}_1,\bar{y}_2) : \psi(\bar{x};\bar{y}_1,\bar{y}_2) \overset{def}{=} [\varphi(\bar{x};\bar{y}_1) \equiv \varphi(\bar{x};\bar{y}_2)]$$

where $\varphi(\bar{x};\bar{y}) \in \Phi\}$

$$\Phi^r = \{\psi(\bar{y};\bar{x}) : \psi(\bar{y};\bar{x}) = \varphi(\bar{x};\bar{y}) \in \Phi\}$$

$$\Phi^{rs} = ((\Phi)^r)^{es}$$

$$\Phi^{eb} = \Phi^{es} \cup \Phi^{rs}$$

If $x_1, \ldots, x_\ell \in \{cn, es, r, rs, eb, i\}$, $\Phi^{x_1,\ldots,x_\ell} = \bigcup_{m=1}^{\ell} \Phi^{x_m}$.

The next theorem connects non order and stability.

1.10 The Stability Theorem: Suppose M has the (Φ^{es},χ^+)-nonorder property, $\mu = \mu^\chi + 2^{2^\chi}$, $|\Phi| \le \chi$, $[\varphi(\bar{x}) \in \Phi \Rightarrow |\ell g(\bar{x})| \le \chi]$. *Then* for $A \subseteq M$, $|A| \le \mu$ *implies* $S^\kappa_\Phi(A,M) = \{tp_\Phi(\bar{a},A,M) : \bar{a} \in |M|^\kappa\}$ has power $\le \mu$.

Proof: There is M_1, $A \subseteq M_1 \subseteq M$, $|M_1| \le \mu$ so that $M_1 <_{\Sigma_{\chi,\chi}(\Phi)} M$. Without loss of generality replace A by M_1 and assume Φ is $\{\varphi(\bar{x};\bar{y})\}$ (by 1.6(5)). Now (ii) (of Th. 1.7) fails hence (i) (of Th. 1.7) holds. So every $p = tp_\Phi(\bar{a},M_1,M) \in S^\kappa_\Phi(M_1,M)$ does not $(\varphi(\bar{x};\bar{y}),\varphi(\bar{x};\bar{y}))$-split over some $\Theta_p \subseteq \{\varphi(\bar{c}';\bar{y}) : \bar{c}' \in |M|, \ell g(\bar{c}') = \ell g(\bar{x})\}$ which has cardinality $\le \chi$. There are at most $\||M_1\||^\chi \le \mu$ such sets Θ_p. So if the conclusion fails for some such $\Theta; |\Theta| \le \chi$ and $|\{p \in S^\kappa_\Phi(M_1,M) : \Theta_p = \Theta\}|$ is $> \mu$. Hence $\{p \in S^\kappa_\Phi(M_1,M) : p$ does not (Φ,Φ)-split over $\Theta\}$ has power $> \mu$. But it has cardinality $\le 2^{2^\chi}$ (by 1. 6(4)) (we just have to decide for p, for each $q(\bar{y}) \in S^{\ell g(\bar{y})}_\psi(B)$ (where $\psi(\bar{y},\bar{x}) = \varphi(\bar{x},\bar{y})$) whether to decide for p, for each $q(\bar{y}) \in S^{\ell g(\bar{y})}_\psi(B)$ (where $\psi(\bar{y},\bar{x}) = \varphi(\bar{x},\bar{y})$) whether $[\models q[\bar{b}], \bar{b} \in M_1 \Rightarrow \varphi(\bar{x},\bar{b}) \in p]$). Thus, by the choice of μ we finish.

1.11 Conclusion: Suppose M has the $(\pm\varphi,\chi^+)$-nonorder property $\mu = \mu^{2^\chi} + \beth_3(\chi)$, $|\Phi| \le 2^\chi$, $[\varphi(\bar{x}) \in \Phi \Rightarrow \ell g(\bar{x}) \le \chi]$. Then for $A \subseteq M$, $|A| \le \mu$ implies $S^\kappa_\Phi(A,M)$ has cardinality

$\leq \mu$.

Proof; By 1.10 and 1.8.

1.12 Exercise: 1) $|S^{\alpha}_{\Phi^x}(A,M)| \leq |S^{\alpha}_{\Phi}(A,M)|$ for $x = cn, es, i$.

2) The $(\{\varphi(\bar{x};\bar{y})\}^r, \lambda)$-order property is equivalent to the $(\neg\varphi(\bar{x},\bar{y}), \lambda)$-order property.

§2 Convergent Indiscernible Sets

2.1 Definition: $\{\bar{a}_t : t \in \mathbf{I}\}$ is (Φ,χ)-convergent inside M if for every $\bar{c} \in M$ (of suitable length), for all but $< \chi$ members $t \in I$ $tp_{\Phi}(\bar{a}_t \wedge \bar{c}, \Phi, M)$ (Φ-type of $\bar{c} \wedge \bar{a}_t$ inside M) is constant (in particular, all $\bar{a}_t$ have the same length). We also demand, of course $|\mathbf{I}| \geq \chi$.

2.1A Remark: In the first order case we were able to show that if T is stable and $\mathbf{I}$ is an infinite set of indiscernible then $\mathbf{I}$ admits an average. Here, we do not know this. Fortunately we have a reasonable replacement: we show that if M does not have the (Φ^{bs}, χ^+)-order property then each sufficiently long indiscernible sequence from M contains a (Φ, χ^+)- convergent subsequence. Originally in the first order case we were interested in existence of indiscernible sets, but in fact we use quite extensively their being convergent. So we will be more interested in convergent sets here.

2.1B Remark: If Φ is closed enough for every (Φ,χ)- convergent $\mathbf{I}$, $\mathbf{I} > \chi$, χ regular, $|\Phi| < \chi$, $A \subseteq M$, $|A| < \chi$ there is $\mathbf{J} \subseteq \mathbf{I}$, $|\mathbf{J}| = |\mathbf{I}|$, $\mathbf{J}$ is Φ-indiscernible set over A. (Choose members of $\mathbf{J}$ one by one, see 3.5(2) below).

2.1C Remark: If $\mathbf{I}$ is (Φ_i, χ)-convergent inside M for $i < \alpha$, and $cf\, \chi > |\alpha|$ then $\mathbf{I}$ is $(\bigcup_{i<\alpha} \Phi_i, \chi)$-convergent inside M. Also obvious monotonicity holds, and (Φ,χ)-convergence implies $(\Phi^{i,es,cn},\chi)$- convergence.

2.1D Remark: We can define something similar to 2.1 for sequences (so we have that $tp_\Phi(\bar{a}_t{}^\wedge\bar{c})$, divide **I** into $<\chi$ convex subsets); but no need arises.

2.2 Definition: For **I**, (Φ,χ)-convergent inside M, and $A \subseteq |M|$, define $Av_\Phi(\mathbf{I},A,M) = \{\varphi(\bar{x},\bar{c}) : \bar{c} \in A,\ \ \varphi(\bar{x},\bar{y}) \in \Phi$ such that for at least $|\mathbf{I}|$ sequences $\bar{a} \in \mathbf{I}, M \models \varphi(\bar{a},\bar{c})\}$. Of course all members of **I** have the same length.

Note that the definition of the average does not depend on χ.

2.2A Fact; If **I** is (Φ,χ)-convergent inside M, $A \subseteq M$, $[\bar{a} \in \mathbf{I} \Rightarrow \ell g(\bar{a}) = \alpha]$ *then* $Av(\mathbf{I},A,M) \in S^\alpha_\Phi(A,M)$.

Proof: By the assumption on **I**, if $\varphi(\bar{x};\bar{y}) \in \Phi$, $\bar{c} \in A$, exactly one of $\varphi(\bar{x};\bar{c})$, $\neg\varphi(\bar{x};\bar{c})$ belongs to $Av_\Phi(\mathbf{I},A,M)$.

2.3 The set existence theorem:

Suppose M has the (Φ^{bs},χ^+)-nonorder property, $\mu = \mu^\chi + 2^{2^\chi}$, $|\Phi| \leq \chi$.

1) Let **I** be a family of α-sequences from M, $\alpha \leq \kappa$ $(\leq \chi)$ and $|\mathbf{I}| = \mu^+$; *then* there is $\mathbf{J} \subseteq \mathbf{I}$ such that;

i) $|\mathbf{J}| = \mu^+$

ii) **J** is (Φ,χ^+)-convergent.

2) If $\mathbf{I} = \{\bar{a}_\alpha : \alpha < \mu^+\}$, then there is a closed unbounded $C \subseteq \mu^+$, and a function h on μ^+ which is regressive (i.e. $h(\alpha) < 1+\alpha$) such that for every $i < \mu^+$, $\mathbf{J}_i = \{\bar{a}_\alpha : \alpha \in C, h(\alpha) = i, cf(\alpha) > \chi\}$, if not empty1, is (Φ,χ^+)-convergent.

3) If we replace "$|\Phi| \leq \chi$" by "$\mu^{|\Phi|} = \mu$", we still get a $(\Phi,\chi^+ + |\Phi|^+)$-convergent $\mathbf{J}$.

Proof: Let $\mathbf{I} = \{\bar{a}_\alpha : \alpha < \mu^+\}$. Clearly it suffices to prove (2), hence (by Fodor lemma) w.l.o.g. $\Phi = \{\varphi(\bar{x};\bar{y})\}$. Let $\psi = \psi(\bar{y},\bar{x}) \stackrel{def}{=} \varphi(\bar{x};\bar{y})$. We define by induction on $\alpha < \mu^+$ a submodel M_α of M such that:

(a) M_α is increasing continuously (in α), $\bar{a}_\alpha \in M_{\alpha+1}$.

(b) Every $p \in S_\varphi^{\ell g(\bar{x})}(M_\alpha,M) \cup S_\psi^{\ell g(\bar{y})}(M_\alpha,M)$ is realized in $M_{\alpha+1}$.

This is possible - for (b) use 1.10. Now for every $\alpha < \mu^+$, if $cf\,\alpha > \chi$ then (by (a), (b) and 1.7A,) $M_\alpha <_{\Sigma_{\chi,\chi}(\varphi)} N$. So by 1.7 there is $\Theta_\alpha \subseteq \{\varphi(\bar{a},\bar{x}) : \bar{a} \in |M_\alpha|, \ell g(\bar{x}) = \ell g(\bar{a})\}$ of cardinality $\leq \chi$ such that $tp_\varphi(\bar{a}_\alpha,M_\alpha,M)$ does not $(\varphi(\bar{x};\bar{y}),\varphi(\bar{x};\bar{y}))$- split over Θ_α. As $cf\,(\alpha) > \chi$, there is $h_0(\alpha) < \alpha$ such that $\Theta_\alpha \subseteq \{\varphi(\bar{c},\bar{y}) : \bar{c} \in M_{h_0(\alpha)}\}$. Now (by straightforward coding) for some closed unbounded subset C of μ^+ and regressive h_1, for $\alpha \in C$, $cf\,\alpha > \chi$, $tp_\varphi(\bar{a}_\alpha,M_{h_0(\alpha)+1},M)$ is determined by $h_1(\alpha)$, and also $h_0(\alpha)$ is determined by $h_1(\alpha)$. W.l.o.g. for $\alpha \in C$, if $cf(\alpha) > \chi$ then $\{\delta : h_1(\delta) = h_1(\alpha), cf\,\delta > \chi\}$ is a stationary subset of μ^+.

Now suppose $S \subseteq \{\delta \in C : cf(\delta) > \chi\}, S \neq empty11$ and h_1 is constant on S. We shall prove

(*) $\{\bar{a}_\alpha : \alpha \in S\}$ is $\varphi(\bar{x},\bar{y})$-convergent.

It is enough for the theorem to prove the claim 2.4 before [just define by induction on $i < \mu^+$, $\alpha_0 = 0$, $\beta_i = Min(S-\alpha_i)$, $\alpha_{i+1} = Min(S-(\beta_i+1))$ (so $\alpha_{i+1} = \beta_{i+1}$), $\alpha_\delta = \bigcup_{i<\delta}\alpha_i$, $M'_i = M_{\alpha_i}$, $\bar{a}'_i = \bar{a}_{\beta_i}$, and apply 2.4 to $M'_i,\bar{a}'_i(i < \mu^+)$)]

2.4 Claim: Suppose

a) $\mu = \mu^{\chi} + 2^{2^{\chi}}$, $\ell g(\bar{x}{}^\frown\bar{y}) < \chi$,

b) M has $(\{\varphi(\bar{x},\bar{y})\}^{es},\chi^{+})$-non-order property

c) $M_i, i < \mu^{+}$ is increasing $M_i \subseteq M$.

d) $\bar{a}_i \in M_{i+1}, M_{i+1} <_{\Sigma_{\chi,\chi}(\varphi)} M$

e) $\psi(\bar{y},\bar{x}) = \varphi(\bar{x},\bar{y})$

f) every $p \in S^{\ell g(\bar{y})}_{\{\psi(\bar{x},\bar{y})\}}(M_i,M)$ is realized in M_{i+1} and does not (ψ,ψ)-split over some $\Theta \subseteq \{\varphi(\bar{x};\bar{b}) : \bar{b} \in M_i\}$ of cardinality $\leq \chi$.

g) $|||M_i||| \leq \mu$

h) $tp_{\varphi}(\bar{a}_i,M_i)$ does not (φ,φ)-split over Θ where

$$\Theta \subseteq \{\varphi(\bar{c},\bar{y}) : \bar{c} \in M_0\}$$

i) $tp_{\varphi}(\bar{a}_i,M_0)$ is constant

and

j) every $p \in S^{\ell g(\bar{y})}_{\psi}(B,M) \cup S^{\ell g(\bar{x})}_{\varphi}(B,M)$ is realized in M_0.

Then $\{\bar{a}_i : i < \mu^{+}\}$ is $(\{\varphi\},\chi^{+})$-convergent.

Proof of 2.4: Let $\bar{c} \in M$, $\ell g(\bar{c}) = \ell g(\bar{y})$. We want to prove that

$$|\{i < \mu^{+} : M \models \varphi[\bar{a}_i,\bar{c}]\}| \leq \chi$$

or

$$|\{i < \mu^{+} : M \models \neg\varphi[\bar{a}_i,\bar{c}]\}| \leq \chi$$

Let $M_{\mu^{+}} = \bigcup_{i<\mu^{+}} M_i$

Now

2.4A Fact: There are sets of elements A and sets of ordinals E such that

i) $A \subseteq M_{\mu^+}$, $E \subseteq \mu^+ + 1$, and $|A| < \chi$, $|E| \leq \chi$

ii) (a) $i + 1 \in E \Rightarrow i \in E$

(b) if $\delta \in E$ and $cf\,\delta \leq \chi$ then $\delta = \sup(E \cap \delta)$

iii) if $\delta \in E$ and $cf\,\delta > \chi$ then $tp_{\varphi}(\bar{c}, M_{\delta}, M)$ does not (φ, φ)-split over $A \cap M_{\delta}$ and $A \cap M_{\delta} \subseteq M_{\sup(E \cap \delta)}$

(iv) $\mu^+ \in E$.

Proof of 2.4A: To see this, define by induction E_n, A_n for $n < \omega$, increasing as follows:

1) $E_0 = \{\mu^+\}$

2) $i + 1 \in E_n \rightarrow i \in E_{n+1}$

3) $\delta \in E_n$ and $cf\,\delta \leq \chi \Rightarrow \delta = \sup(E_{n+1} \cap \delta)$

4) $\delta \in E_n$ and $cf\,\delta > \chi \Rightarrow tp_{\psi}(\bar{c}, M_{\delta}, M)$ does not (ψ, ψ)-split over $A_{n+1} \cap M_{\delta}$

5) $A_n \cap (M_{i+1} - M_i) \neq \emptyset \Rightarrow i, i+1 \in A_{n+1}$.

5) $A_n \subseteq A_{n+1}$

7) $E_n \subseteq E_{n+1}$

8) $|E_n| + |A_n| \leq \chi$

For $n = 0$ use 1). For $n+1$, 1)-7) tell you to throw in χ sets, each of power $\leq \chi$. Take the union; for 4) use theorem 1.7. Now $\bigcup_{n<\omega} E_n$, $\bigcup_{n<\omega} A_n$ are as required in Fact 2.4A.

Continuation of the Proof of 2.4: Let $\bar{c}_i \in M_{i+1}$ realize

$$tp_\psi(\bar{c}, M_i \cup \bar{a}_i, M)$$

Now E divides $(\mu^+ + 1) - E$ naturally into $\leq \chi$ intervals. (For $\alpha \in E$, $I_\alpha \overset{def}{=} \{i < \mu^+ : \alpha = Min\{j : i < j \in E\}\}$). We first show that "$M \models \varphi(\bar{a}_i, \bar{c})$" has truth value constant on each interval, then that all intervals give the same answer. Note that $I_\alpha \neq \emptyset$ implies that α is a limit ordinal of cofinality greater than χ.

First Part:

Let $\delta_1 \in E$ and $cf\,\delta_1 > \chi$, $\delta_0 = \sup(E \cap \delta_1)$. So $I_{\delta_1} = \{i : \delta_0 \leq i < \delta_1\}$

Remember

(A) $tp_\psi(\bar{c}, M_{\delta_1}, M)$ does not (ψ,ψ)- split over $A \cap M_{\delta_1}$.

(B) $A \cap M_{\delta_1} \subseteq M_{\delta_0}$

(C) $tp_\varphi(\bar{a}_i, M_i, M)$ is increasing in i, hence

(D) $\delta_0 \leq i, j < \delta_1 \Rightarrow tp_\varphi(\bar{a}_i, M_{\delta_0}, M) = tp_\varphi(\bar{a}_j, M_{\delta_0}, M)$

Together $\varphi(\bar{a}_i, \bar{c}) \equiv \varphi(\bar{a}_j, \bar{c})$.

Second Part

Let $\delta_0 < \delta_1 < \delta_2 < \delta_3$ where $\delta_1, \delta_3 \in E$, $cf\,\delta_1, cf\,\delta_3 > \chi$, $\delta_0 = \sup(\delta_1 \cap E)$ and $\delta_2 = \sup(\delta_3 \cap E)$. We want to prove $\varphi(\bar{a}_{\delta_0}, \bar{c}) \equiv \varphi(\bar{a}_{\delta_2}, \bar{c})$. Suppose not and for example

(1) $\varphi(\bar{a}_{\delta_0}, \bar{c}) \wedge \neg\varphi(\bar{a}_{\delta_2}, \bar{c})$

Then

(2) $i \in [\delta_0,\delta_1) \Rightarrow \varphi(\bar{a}_i,\bar{c})$

[by first part]

(3) If $i < j$ are both in $[\delta_0,\delta_1)$ then $\varphi(\bar{a}_i,\bar{c}_j)$

[by choice of $\bar{c}_j$ and (2)].

(4) $j < \alpha, \beta < \mu^+ \Rightarrow \varphi(\bar{a}_\alpha,\bar{c}_j) \equiv \varphi(\bar{a}_\beta,\bar{c}_j)$

[As $tp_\varphi(\bar{a}_\alpha,M_\alpha,M)$ is increasing in α].

(5) $j_1,j_2 < \alpha < \mu^+ \Rightarrow \varphi(\bar{a}_\alpha,\bar{c}_{j_1}) \equiv \varphi(\bar{a}_\alpha,\bar{c}_{j_2})$

[As $tp_\varphi(\bar{a}_\alpha,M_\alpha,M)$ does not (φ,φ)-split over M_0 and $\bar{c}_{j_1},\bar{c}_{j_2}$ realize $tp_\psi(\bar{c},M_0)$]

(6) if $j_1 < \alpha_1 < \mu^+$, $j_2 < \alpha_2 < \mu^+$ then $\varphi(\bar{a}_{\alpha_1},\bar{c}_{j_1}) \equiv \varphi(\bar{a}_{\alpha_2},\bar{c}_{j_2})$

[combine (4) and (5) using $\varphi(\bar{a}_{\max(\alpha_1,\alpha_2)},\bar{c}_{j_\ell})$ $\ell = 1,2$ as intermediates]

(7) $j \in [\delta_2,\delta_3) \Rightarrow \neg\varphi(\bar{a}_j,\bar{c})$

[By first part and the assumption (see (1)) that $\neg\varphi(\bar{a}_{\delta_2},\bar{c})$]

(8) If $j < \alpha$ and both are in $[\delta_2,\delta_3)$ then $\neg\varphi(\bar{a}_j,\bar{c}_\alpha)$

[by combining (7) and "$\bar{c}_j$ realizes $tp_\psi(\bar{c},M_j,M)$"]

Now if $\models\varphi[\bar{a}_1,\bar{c}_0]$ then by (6) and (8) we get a φ-linear order on $\langle \bar{a}_j{}^\frown\bar{c}_j : \delta_2 < j < \delta_3\rangle$; and if $\models\neg\varphi[\bar{a}_1,\bar{c}_0]$ then by (6) and (3) we get a φ-linear order on

$$\langle \bar{a}_j{}^\frown\bar{c}_j : \delta_0 < j < \delta_1\rangle$$

as both intervals has cardinality $> \chi$ we get a contradiction.

This completes the proof of the second part. So $\varphi(\bar{a}_j,\bar{c})$ has the same truth value for all $j \in \mu^+ - E$, but $|E| \leq \chi$ so we have finished.

2.5 Exercise: In Theorem 2.3, replace μ^+ by a (possibly weakly) inaccessible cardinal μ.

§3 Symmetry and indiscernibility

3.1 The Symmetry Lemma:

Assume M has (φ,μ)-nonorder, $\ell = 1,2$, $\mu \le \mu_1,\mu_2$, all regular cardinals. Suppose $\mathbf{I}_\ell = \{\bar{a}^\ell_\alpha : \alpha < \mu_\ell\}$ is (φ_ℓ,μ_ℓ)-convergent inside M and

$$\varphi = \varphi(\bar{x};\bar{y};\bar{z})$$

$$\varphi_1(\overline{x;\bar{y};\bar{z}}) = \varphi(\bar{x};\bar{y};\bar{z})$$

$$\varphi_2(\bar{y};\bar{x};\bar{z}) = \varphi(\bar{x};\bar{y};\bar{z})$$

$$\ell g(\bar{a}^1_\alpha) = \ell g(\bar{x}), \ell g(\bar{a}^2_\alpha) = \ell g(\bar{y})$$

then for $\bar{c} \in M$

$$(\exists^{\ge\mu_1} \alpha < \mu_1)(\exists^{\ge\mu_2} \beta < \mu_2)\varphi(\bar{a}^1_\alpha,\bar{a}^2_\beta,\bar{c}) \text{ if and only if}$$
$$(\exists^{\ge\mu_2} \beta < \mu_2)(\exists^{\ge\mu_1} \alpha < \mu_1)\varphi(\bar{a}^1_\alpha,\bar{a}^2_\beta,\bar{c})$$

Proof: Easy.

3.2 The indiscernibility/non-splitting lemma

Let for $i < i(*)$, $\varphi_i(\bar{x}_1, \ldots ,\bar{x}_{n_i},\bar{y}_i)$ be a $\tau(M)$-formula, $\alpha = \ell g(\bar{x}^\ell)$, $\Phi_n = \{\varphi_i(\bar{x}_1, \ldots ,\bar{x}_{n_i},\bar{y}_i) : i < i(*), n_i = n\}$, and $\Phi = \bigcup_{n<\omega} \Phi_n$.

Suppose $A \subseteq |M|$, $a_i \in {}^\alpha|M|$ for $i < i(*)$ and $p^n_i \stackrel{def}{=} tp_{\Phi_n}(\bar{a}_i, A \cup \bigcup_{j<i} \bar{a}_j, M)$ does not split over Θ, where $\Theta = \bar{c}\{\varphi_i(\bar{x}_1, \ldots ,\bar{x}_{n_i},\bar{c})\colon i < i(*),\ n_i < \omega,\ \bar{c} \in A\}$ and $i < j \Rightarrow p_i \subseteq p_j$. *Then* $\langle \bar{a}_i : i < i(*)\rangle$ is a Φ-indiscernible sequence.

Proof: See [Sh A1] Lemma 2.5. p.11.

3.3 Conclusion: Suppose $\varphi(\bar{x}_n, \ldots ,\bar{x}_1,\bar{y})$ is a $\tau(M)$-formula and for $\ell = 0, \ldots , n-1$

$$\varphi_\ell(\bar{x}_n,\ldots,\bar{x}_1,\bar{y}) \stackrel{def}{=} \varphi(\bar{x}_{n-\ell},\bar{x}_{n-\ell-1},\ldots,\bar{x}_1,\bar{x}_n,\ldots,\bar{x}_{n-\ell+1},\bar{y})$$

and $\alpha = \ell g(\bar{x}_\ell)$ and let $\Phi = \{\varphi_\ell : \ell = 0,\ldots,n-1\}$.

1) If $\bar{a}_i \in {}^\alpha M$ for $i < i(*)$, $p_i = tp_\Phi(\bar{a}_i, A \cup \bigcup_{j<i} \bar{a}_j, M)$ increases with i, and is finitely satisfiable in A *then* $\langle \bar{a}_i : i < i(*) \rangle$ is a Φ-discernible sequence over A.

2) Suppose $\mathbf{J}$ is a family of sequences, $\bar{a}_i \in {}^\alpha|M|$, for $i < i(*)$ and letting $\mathbf{J}_i = \mathbf{J} \cup \{\bar{a}_j : j < i\}$

$$p_i = tp_\Phi(\bar{a}_i,\mathbf{J}_i,M) \stackrel{def}{=} \{\varphi(\bar{x},\bar{c}_1,\ldots,\bar{c}_k) : c_\ell \in \mathbf{J}_i$$

and

$$M \models \varphi[\bar{a}_i,\bar{c}_1,\ldots,\bar{c}_k]\}$$

is increasing with i and is finite satisfiable in $\mathbf{J}$. *Then* $\langle \bar{a}_i : i < i(*) \rangle$ *is a* Φ*-indiscernible (set)* over $\mathbf{J}$.

Remark: Of course we can restrict p_i to the set of formulas used.

Proof: Easy.

3.4 Lemma: Suppose $\langle \bar{a}_i : i < i(*) \rangle$ is a $(\varphi(\bar{x}_n,\ldots,\bar{x}_1,\bar{c}),n)$-indiscernible sequence but not $(\varphi(\bar{x}_n,\ldots,\bar{x}_1,\bar{c}),n)$- indiscernible set.

Let for any permutation π of $\{1,\ldots,n\}$, $\varphi_\pi(\bar{x}_n,\ldots,\bar{x}_1,y) \stackrel{def}{=} \varphi(\bar{x}_{\pi(n)},\ldots,\bar{x}_{\pi(1)},\bar{y})$ then for some permutation π and $m < n$, M has the $(\varphi_\pi(\bar{x}_n;\bar{x}_{n-1},\ldots,\bar{x}_m;\bar{a}_{m-1},\ldots,\bar{a}_0,\bar{c}),$ $(i(*)-m)/(n-m))$-order property.

3.4A Remark:

1) If $\langle \bar{a}_i : i < i(*) \rangle$ is a Φ-indiscernible sequence over A but not a Φ-indiscernible set over A, then for some $\varphi(\bar{x}_n,\ldots,\bar{x}_1,\bar{y}) \in \Phi$ $(\ell g(\bar{x}_\ell) = \ell g(\bar{a}_i))$ and $\bar{c} \in {}^{\ell g(\bar{y})}A$ the assumption of 3.4 holds.

2) In (1) we can find n and use (Φ,n)-indiscernibility.

Proof: Left to the reader (really by Morley [Mo 1], or see [Sh, AP 3.9]).

3.5 Lemma: Suppose $\mathbf{I} = \{\bar{a}_i : i < \lambda\}$ is $(\Phi,<\chi)$-convergent, $\ell g(\bar{a}_i) = \alpha$ for $i < \lambda$. Suppose further that Φ satisfies

(*) if $\varphi(\bar{x}_n, \ldots, \bar{x}_1, \bar{y}) \in \Phi$, $\ell g(\bar{x}_\ell) = \alpha$, π a permutation of $\{1, \ldots, n\}$ then $\varphi_\pi(\bar{x}_n, \ldots, \bar{x}_1, \bar{y}) \overset{def}{=} \varphi(\bar{x}_{\pi(n)}, \ldots, \bar{x}_{\pi(1)}, \bar{y})$ belongs to Φ.

Then 1) there is $\mathbf{I}' \subseteq \mathbf{I}$, $|\mathbf{I}'| = \lambda$, $\mathbf{I}$ a Φ-indiscernible set over $\mathbf{J}$.

2) In fact there is an algebra N with universe λ and $\leq |\mathbf{J}| + \chi + |\Phi|$ functions such that if for $\zeta < \lambda$, $i_\zeta < \lambda$, i_ζ not in the N-closure of $\{i_\xi : \xi < \zeta\}$ *then* $\{\bar{a}_\zeta < \lambda\}$ is an Φ-indiscernible set over $\mathbf{J}$.

Remark: If we just want "$\{a_{i_\zeta} : \zeta < \lambda\}$ is a Φ-indiscernible sequence over J" we can weaken (*) to $[\varphi \in \Phi \Rightarrow \varphi_\ell \in \Phi]$ for φ_ℓ as in 3.3.

Proof: 1) by 2)

2) By 3.4 it is enough to prove that $\langle \bar{a}_{i_\zeta} : \zeta < \lambda \rangle$ is a Φ-indiscernible sequence over $\mathbf{J}$.

We define for

$$\psi = \psi(\bar{x}_n, \ldots, \bar{x}_1, \bar{z}_m, \ldots, \bar{z}_1) \in \Phi, \ \bar{c}_i \in \mathbf{J} \ (\ell = 1,m),$$

$$\ell g(\bar{c}_\ell) = \ell g(\bar{c}_\ell)$$

and $\gamma < \chi$ a function $F^\gamma = F^{\psi,\alpha}_{\bar{c}_m,\ldots,\bar{c}_1}$ such that

(*) for $i_1, \ldots, i_{n-1} < \lambda$ the set $D_{i_1,\ldots,i_{n-1}} = \{F^\gamma(i_1, \ldots, i_n - 1) : \gamma < \chi\}$ satisfies

(a) it includes $\{i : i < \chi\}$

(b) for any $j_1, j_2 \in \lambda - D_{i_1,\ldots,i_{n-1}}$,

$$\models\psi[\bar{a}_{j_1},\bar{a}_{i_{m-1}},\ldots,\bar{a}_{i_1},\bar{c}_m,\ldots,\bar{c}_1]\equiv$$

$$\psi[\bar{a}_{j_2},\bar{a}_{i_{m-1}},\ldots,\bar{a}_{i_1},\bar{c}_m,\ldots,\bar{c}_1]$$

[this is possible as **I** is $(\Phi,<\chi)$-convergent].

Now if $\langle i_\zeta:\zeta<\lambda\rangle$ are as in 3.5(2), by 3.3(2) (with $\mathbf{J}\cup\{\bar{a}_i:i<\lambda\}$ here standing for **J** there $\langle \bar{a}_{i_\zeta}:\zeta<\lambda\rangle$ is a Φ-indiscernible sequence over **J** which suffices.

§4 What is the appropriate notion of a submodel

We want a context for non forking theory, and existence of amalgamation preferably with non-forking. For this we need a suitable notion of elementary submodel. Using $M<_{\mathcal{L}}N$, $\mathcal{L}$ strong logic, is not good enough. For example, $M_\alpha<_{\mathcal{L}}M_\beta<_{\mathcal{L}}M$ for $\alpha<\beta<\delta$ does not necessarily imply $\bigcup_{\alpha<\delta}M_\alpha<_{\mathcal{L}}M$. For δ of large cofinality this holds, but remember that if we can quantify over countable sets concepts become very dependent on the exact set theoretic hypothesis. Our problem is: Find a good notion of an elementary submodel.

We use the following relation $M<^{\kappa}_{\Phi,\mu,\chi}N$ saying mainly that types in $S^\alpha(M)$ realized in N are averages of convergent sets. (See 4.1). In lemma 4.3 we show that in the absence of ordering we are dealing with $<_{\Sigma_{\mu,(<\kappa)}}$.

4.1 Definition: $M<^{\kappa}_{\Phi,\mu,\chi}N$ if:

1) $M\subseteq N$

2) $M<_\Phi N$, that is for $\varphi(\bar{x})\in\Phi$, $\bar{c}\in M$, $M\models\varphi[\bar{c}]$ if and only if $N\models\varphi[\bar{c}]$

3) for $\bar{c}\in N$, $\ell g(\bar{c})<\kappa$ there is $\mathbf{I}=\{\bar{c}_i:i<\mu^+\}$, which is (Φ,χ^+)-convergent inside M such that $tp_\Phi(\bar{c},M,N)=Av(\mathbf{I},M,N)$

4.2 Remark:

1) Our main case:

$\Phi =$ finite quantifier free formulas, $\kappa = \aleph_0$ and μ,χ are related as in Theorem 2.3 and then we omit them and write just $<$.

2) We could separate the two roles of Φ, but we have already enough parameters.

3) Similarly we could use μ,χ instead μ^+,χ^+ gaining a little in generality.

4) Many of the "obvious" properties of a candidate for "elementary submodel" here are not so obvious. Some are proved, the failure of some is used in non structure theorems.

4.3 Lemma: Suppose $\mu = \mu^\chi + 2^{2^\chi}$, $|\Phi| \leq \chi$, $[\varphi(\bar{x}) \in \Phi \Rightarrow \ell g(\bar{x}) \leq \chi]$, and M has (Φ^{eb},χ^+)-non order *then:* $M <^\kappa_{\Phi,\mu,\chi} N$ *if and only if* $M <_{\Sigma_{\mu,(<\kappa)}(\Phi)} N$.

Proof: The direction $\Rightarrow$ is trivial. For the other direction, let $\bar{c} \in {}^\alpha N$, $\alpha < \kappa$. For notational simplicity assume (noting 2.1c, 2.3) $\Phi = \{\varphi(\bar{x};\bar{y})\}$, let $\psi(\bar{y},\bar{x}) = \varphi(\bar{x},\bar{y})$. By 1.7 for some $\Theta \subseteq \{\varphi(\bar{a},\bar{y}): a \in N\}$, $|\Theta| \leq \chi$ and $tp_{\varphi(\bar{x};\bar{y})}(\bar{c},M,N)$ does not $(\{\varphi(\bar{x},\bar{y}),\varphi(\bar{x},\bar{y})\})$-split over Θ. Choose by induction on $i < \mu^+$, $M_i,\bar{c}_i$ such that Θ is over M_0, $\|M_i\| \leq \mu$ every q such that $q \in S^{\ell g(\bar{x})}_{\varphi(\bar{x},\bar{y})}(M_i,N)$ or $q \in S^{\ell g(\bar{y})}_{\psi(\bar{y},\bar{x})}(M_i,M)$ is realized in M_{i+1} and $\bar{c}_i \in M_{i+1}$ realize $tp_{\varphi(\bar{x},\bar{y})}(\bar{c},M_i,M_i)$. This is clearly possible by 1.10. Now by 2.3, 2.4 for some $S \subseteq \mu^+$, $|S| = \mu^+$, and $\mathbf{I} \overset{def}{=} \{\bar{c}_\alpha : \alpha \in S\}$ is $\{\varphi(\bar{x},\bar{y})\}$-convergent. Hence $q = Av_\varphi(\mathbf{I},M,N)$, is well defined as is equal to $Av_{\varphi(\bar{x},\bar{y})}(\mathbf{I},M,N)$ which belongs to $S^{\ell g(\bar{x})}_\varphi(M,N)$.

Now the types q and $tp_{\varphi(\bar{x},\bar{y})}(\bar{c},M,N)$ are both in $S^{\ell g(\bar{x})}_{\varphi(\bar{x},\bar{y})}(M,N)$, does not (φ,φ)-split over M_{χ^+}, and have the same restriction to M_{χ^++1}. Hence by 1.6(2) they are equal. So we finish the second direction.

4.4 Conclusion: For κ,Φ,μ,χ as in 4.3 and models with (Φ^{eb},χ^+)-non order $<^\kappa_{\Phi,\mu,\chi}$ is transitive.

Proof: Because $<_{\Sigma_{\mu,(<\kappa)}(\Phi)}$ is transitive.

4.5 Claim:

1) If $M_1 <^{\kappa}_{\Phi,\mu,\chi} M_2$, $\mathbf{I}$ is (Φ,χ^+)-convergent inside M_1, $|\mathbf{I}| = \mu^+$ and $[\bar{a} \in \mathbf{I} \Rightarrow \ell g(\bar{a}) < \kappa]$ *then* it is (Φ,χ^+)-convergent inside M_2.

2) If $\mathbf{I}_1,\mathbf{I}_2$ are (Φ,χ^+)-convergent inside M_1, $M_1 <^{\kappa}_{\Phi,\mu,\chi} M_2$, $|\mathbf{I}_\ell| = \mu^+$, and $Av_\Phi(\mathbf{I}_1,M_1) = Av_\Phi(\mathbf{I}_2,M_1)$ *then* $Av_\Phi(\mathbf{I}_1,M_2) = Av_\Phi(\mathbf{I}_2,M_2)$.

Proof:

1) Let $\bar{c} \in M_2$. Let $\mathbf{J} \subseteq M_1$ be (Φ,χ^+).

$Av_\Phi(\mathbf{J},M_1,M_2) = tp_\Phi(\bar{c},M_1,M_{2})$ so if $\varphi(\bar{x},\bar{c})$ divides $\mathbf{J}$ into two sets $> \chi$ then so does some $\bar{c}' \in \mathbf{J}$.

2) Similar; alteratively use 4.3 (easy direction).

4.6 Union existence lemma: Let $\Phi,, \mu,\chi,\kappa$ be as in 4.3, each M_i with (Φ^{eb},χ^+)-non order. If M_i is $<^{\kappa}_{\Phi,\mu,\chi}$ increasing for $i < \delta$, $cf\,\delta > \kappa$ *then* $M_i <^{\kappa}_{\Phi,\mu,\chi} \bigcup_{j<\delta} M_j$ provided $<_\Phi$ is O.K. (i.e. $M_i <_\Phi \bigcup_{j<\delta} M_j$) which for our main case (quantifier free formulas) is O.K.

4.7 The Lowenheim-Skolem Lemma: If Φ,μ,χ,κ are as in 4.3, M with (Φ^{eb},χ^+)-non order property $A \subseteq M$, $|A| \le \mu^+$ *then* there is $M', A \subseteq M' <^{\kappa}_{\Phi,\mu,\chi} M$, $\|M'\| \le \mu^+$.

Proof: Trivial for $<^{\kappa}_{\Sigma_{\mu,(<\kappa)}}$, and use 4.3.

4.8 Definition: M_0,M_1,M_2 are in (Φ,μ,χ,κ)-stable amalgamation inside M if: (for $< = <^{\kappa}_{\Phi,\mu,\chi}$, each M_ℓ has (Φ,χ^+)- non order)

1) $M_\ell < M$.

2) for every $\bar{c} \in M_2$ for some Φ-convergent $\mathbf{I} \subseteq M_0$, $|\mathbf{I}| = \mu^+$ $Av_\Phi(I,M_1,M) = tp_\Phi(\bar{c},M_1,M)$ (really every (Φ,χ)-convergent $\mathbf{I} \subseteq M_0$, if $Av_\Phi(I,M_0,M_1) = tp_\Phi(\bar{c},M_0,M_1)$ then $Av_\Phi(M_1,M) = tp_\Phi(\bar{c},M_1,M)$, (see 4.5).

§5 On the non order implies the existence of indiscernibles

5.1 Theorem: Suppose μ is a regular uncountable cardinal, M a L-model, Δ a set of $< \mu$ quantifier free L-formulas, $\varphi = \varphi(\bar{x})$ closed under negation and permuting the variables.

If $\|M\| > 2^{<\mu}$ then at least one of the following possibilities holds.

Possibility A: There is an Δ-indiscernible set $I \subseteq M$ of cardinality μ.

Possibility B: There are distinct $a_i \in M$ for $i \leq \mu$ and n, $2 \leq n < \omega$ and $\varphi = \varphi(\bar{z}, x_1, \ldots, x_n) \in \Delta$, $\bar{c} \in {}^{y(\bar{z})}M$ such that

(a) if $m < n$, $k < \omega$, $\alpha_1 < \cdots < \alpha_k$, $\alpha_k < \beta_1 < \cdots < \beta_m \leq \mu$, $\alpha_k < \gamma_1 < \cdots < \gamma_m < \mu$ and $\psi(\bar{z}, y_1, \ldots, y_k, x_1, \ldots, x_m) \in \Delta$ then:

$$M \models \psi[\bar{c}, a_{\alpha_1}, \ldots, \alpha_{\alpha_k}, \alpha_{\beta_1}, \ldots, \alpha_{\beta_m}]$$

$$M \models \psi[\bar{c}, a_{\alpha_1}, \ldots, a_{\alpha_k}, a_{\gamma_1}, \ldots, a_{\gamma_m}]$$

(b) if $\beta_1 < \cdots < \beta_m \leq \mu$, $\bar{d} = \langle a_{\beta_3}, a_{\beta_4}, \ldots, a_{\beta_n} \rangle$

$$M \models \varphi[\bar{c}, a_{\beta_1}, a_{\beta_2}, \bar{d}]$$

$$M \models \neg\varphi[\bar{c}, a_{\beta_2}, a_{\beta_1}, \bar{d}]$$

Possibility C: There are distinct $a_i \in M$ for $i \leq \mu$ and n, $\alpha \leq n < \omega$ and $\varphi = \varphi(\bar{y}, x_1, \ldots, x_n) \in \Delta$, $\bar{c} \in {}^{\ell g(\bar{y})}M$, such that:

(a) As in Possibility B.

(b) if $\alpha, \beta < \gamma_3 < \cdots < \gamma_n \leq \mu$, $\alpha \neq \beta$, $\bar{d} = \langle a_{\gamma_3}, \ldots, \alpha_{\gamma_n} \rangle$ then $M \models \varphi[\bar{c}, a_\alpha, a_\beta, \bar{d}]$ *if and only if* $Min\{\alpha, \beta\}$ is even.

Remark: We can do everything over a set of $< \mu$ parameters and find the $a_i, \bar{c}$ in some pregiven set I_0 of cardinality $2^{<\mu}$ - just expand M by individual constants or restrict its universe to I_0.

We can deal instead of elements with m-tuples (or α tuples) - replace M by an appropriate model with universe ${}^m|M|$.

5.2 Conclusion: Suppose T is first order and for no model M of T and formula $\varphi(x,y,\bar{z})$ does M have $(\varphi(x,y\,;\bar{z}),\aleph_0)$-order [i.e. for no $\bar{c}$, a_n, b_n $(n<\omega)$ from M, $M \models \varphi[a_\ell, a_k, \bar{c}]$ if and only if $\ell < k$].

If N is a model of T, $\lambda \geq |T|^+$, A, B subsets of N, $|A| < \lambda$, $|B| > 2^{<\lambda}$ *then* B has a subset of cardinality λ which is an indiscernible set over A inside N.

Proof of 5.1: Let $A^* \subseteq M$, $|A^*| = 2^{<\mu}$ be such that :

(*) if $A \subseteq A^*$, $|A| < \mu$, $a \in M$ then some $a' \in A^* - A$ realize $tp_\Delta(a,A)$.

Now for every $\bar{c} \in A^*$ and formula $\varphi = \varphi(\bar{c},\bar{x}) = \varphi(\bar{c}, x_1,\ldots,x_m)$, $(n = n(\varphi), \bar{c} = c_\varphi)$ we define a game $G_\varphi = GM_{\varphi(\bar{c},\bar{x})}$:

It lasts $n+1$ moves $(0,1,2,\ldots,n)$; in the ℓ-th move: *player I* chooses a set A_ℓ, $A_\ell \subseteq A^*$, $[m<\ell \Rightarrow A_m \cup \{a_m\} \subseteq A_\ell]$, $|A_\ell| < \mu$ *player II* choose an element a_ℓ, $a_\ell \in A^* - A_\ell$ which realize $tp_\Delta(a^*,A_\ell)$.

In the end player I wins if

$$M \models \varphi[\bar{c},a_1,a_2,a_3,\ldots,a_n] \Leftrightarrow \varphi[\bar{c},a_0,a_2,a_3,\ldots,a_n]$$

This game is clearly determined. So one of the player has a winning strategy $\bar{F}_\varphi = \langle F^\varphi_\ell : \ell \leq n\rangle$, F^φ_ℓ compute his ℓ-th move from the previous moves of his opponent. W.l.o.g. if player I wins then for every $a_0,\ldots,a_{\ell-1} \in A^*$, $F^\varphi_\ell(a_0,\ldots,a_{\ell-1})$ is a subset of A^* of cardinality $<\mu$, extending $F^\varphi_m(a_0,\ldots,a_{m-1}) \cup \{a_0,\ldots,a_m\}$ for each $m<\ell$. (So F^φ_ℓ depends on $\bar{c}$).

Case I: For every $\varphi(\bar{c},\bar{x})$ as above, player I wins the game.

We define by induction on $\alpha < \mu$, a_α, A_α such that:

(i) $\{\alpha_\beta\} \cup A_\beta \subseteq A_\alpha \subseteq A^*$ for $\beta < \alpha$ and $|A_\alpha| < \mu$

(ii) $a_\alpha \in A^* - A_\alpha$ realizes $tp_\Delta(a^*, A_\alpha)$

(iii) if $\beta < \alpha$, $\bar{c} \in A_\beta \cup \{a_\beta\}$, $\varphi(\bar{y},\bar{x}) \in \Delta$, $\ell g(\bar{y}) = \ell g(\bar{c})$, $x = \langle x_1, \ldots, x_m \rangle$, $\ell \leq n$, and $b_0, \ldots, b_{\ell-1} \in A_\beta \cup \{a_\beta\}$, *then* $F_\ell^{\varphi(\bar{c},\bar{x})}(b_0, \ldots, b_{\ell-1}) \subseteq A_\alpha$ (we can restrict further $\bar{c}, b_0, \ldots, b_{\ell-1}$)

There is no problem to do it. (in stage α, first choose A_α to satisfy (i) + (iii), [exists as the value of $F_\ell^{\varphi(\bar{c},\bar{x})}$ is always a subset of A^* of cardinality $< \mu$, μ regular $> \aleph_0$]. Then choose a_α to satisfy (ii). [exist by the choice of A^*, a^*].

Now we can prove

$(*)_a$ *if* $\alpha_1 < \cdots < \alpha_k < \beta_0 < \beta_1 < \cdots < \beta_n < \mu$, $(k < \omega)$,

$$\varphi(y_1, \ldots, y_k, x_1, \ldots, x_k) \in \Delta$$

then

$$M \models \varphi[a_{\alpha_1}, \ldots, a_{\alpha_k}, a_{\beta_1}, a_{\beta_2}, \ldots, a_{\beta_n}] \Leftrightarrow$$

$$M \models \varphi[a_{\alpha_1}, \ldots, a_{\alpha_k}, a_{\beta_0}, a_{\beta_2}, \ldots, a_{\beta_n}]$$

$(*)_b$ if $\alpha_1 < \cdots < \alpha_k < \mu$, $\alpha_k < \beta_1 < \cdots < \beta_n < \mu$, $\alpha_k < \gamma_1 < \cdots < \gamma_n < \mu$

$$\varphi(y_1, \ldots, t_k, x_1, \ldots, x_k) \in \Delta$$

then $M \models \varphi[a_{\alpha_1}, \ldots, a_{\alpha_k}, a_{\beta_1}, \ldots, a_{\beta_k}] \Rightarrow$

$$M \models \varphi[a_{\alpha_1}, \ldots, a_{\alpha_k}, a_{\gamma_1}, \ldots, a_{\gamma_k}]$$

Why this holds? As for $(*)_a$, let $\bar{c} = \langle a_{\alpha_1}, \ldots, a_{\alpha_k} \rangle$, remember that player I wins the game $GM_{\varphi(\bar{c},\bar{x})}$ and that $\langle F_\ell^{\varphi(\bar{c},\bar{x})} : \ell \leq n \rangle$ is a winning strategy for him. Let $A^\ell = F_\ell^{\varphi(\bar{c},\bar{x})}(a_{\beta_0}, a_{\beta_{\ell-1}})$. By (iii) above $A^\ell \subseteq A_{\beta_\ell}$ hence a_{β_ℓ} realize $tp(a^*, A^\ell)$, $a_{\beta_\ell} \in A^* - A^\ell$. So A^0, a_{β_0}, $A^1, a_{\beta_1}, \ldots, A^n, a_{\beta_n}$ is a play of the game $GM_{\varphi(\bar{c},\bar{x})}$ in which player I uses his winning strategy

$\langle F_\ell^{\varphi(\bar{c},\bar{x})} : \ell \le n \rangle$, so he wins the play, i.e. the conclusion of $(*)_a$ holds.

By the transitivity of equivalence we can deduce $(*)_b$.

So $\langle a_\alpha : \alpha < \mu \rangle$ is a Δ- indiscernible sequence.

If it is a Δ-indiscernible set, possibility (A) of the theorem holds. If it is not, then (by Morley's work, see e.g. [Sh, AP.3.9]) for some n, (B) of the theorem holds (i.e. use again transitivity of equivalence to get the "good form") [we have to check that address $a_\mu \stackrel{def}{=} a^*$ is O.K., but this is easy].

Case II: For some $\varphi(\bar{c}, \bar{x})$, player II wins $GM_{\varphi(\bar{c},\bar{x})}$. Choose such $\varphi_0 = \varphi_0(\bar{c}_0, x_1, \ldots, x_{n(0)})$ with minimal $n(0)$. Necessarily $n(0) \ge 2$.

We now define by induction on $\zeta < \mu$, for every $\alpha < \zeta(n(0)+1)$, A_α, a_α such that:

(i) $\bar{c} \cup \{a_\beta\} \cup A_\beta \subseteq A_\alpha \subseteq A^*$ for $\beta < \alpha$ and $|A_\alpha| < \mu$

(ii) $a_\alpha \in A^* - A_\alpha$ realizes $tp_\Delta(a^*, A_\alpha)$

(iii) if $\beta < \alpha$, $\bar{c} \in A_\beta \cup \{a_\beta\}$, $\varphi(\bar{y}, \bar{x}) \in \Delta$, $\ell g(\bar{y}) = \ell g(\bar{c})$, $\bar{x} = \langle x_1, \ldots, x_n \rangle$, $n < n(0)$, $\ell \le n$ and $b_0, \ldots, b_{\ell-1} \in A_\beta \cup \{a_\beta\}$ then

$$F_\ell^{\varphi(\bar{c},\bar{x})}(b_0, \ldots, b_{\ell-1}) \subseteq A_\alpha$$

(iv) if $\alpha = \zeta(n(0)+1)$, $\ell \le n$ then

$$a_{\alpha+\ell} = F_\ell^{\varphi_0}(A_\alpha, A_{\alpha+1}, \ldots, A_{\alpha+\ell})$$

There are no problems in carrying this out.

As in case I we can prove

$(*)_c$ *if* $n < n(0)$, $k < \omega$, $\alpha_1 < \cdots < \alpha_k$,

$$\alpha_k < \beta_1 < \cdots < \beta_n < \mu, \quad \alpha_k < \gamma_1 < \cdots < \gamma_n < \mu$$

$$\varphi(y_1, \ldots, y_k, x_1, \ldots, x_n) \in \Delta$$

then

$$M \models \varphi[a_{\alpha_1}, \ldots, a_{\alpha_k}, a_{\beta_1}, \ldots, a_{\beta_n}] \text{ if } \textit{and only} \text{ if}$$

$$M \models \varphi[a_{\alpha_1}, \ldots, a_{\alpha_k}, a_{\gamma_1}, \ldots, a_{\gamma_n}].$$

Using determinacy and possibility replacing φ_0 by $\neg\varphi_0$, w.l.o.g. $\langle F_\ell^{\varphi_0} : \ell \leq n(0) \rangle$ guarantees for every $\alpha = \zeta(n(0)+1)$

$$M \models \varphi_0[\bar{c}, a_{\alpha+1}, a_{\alpha+2}, \ldots, a_{\alpha+n}] \Leftrightarrow \neg\, \varphi_0[\bar{c}, a_\alpha, a_{\alpha+2}, \ldots, a_{\alpha+m}].$$

Let $\mathbf{t}_\zeta$ be the truth value of $M \models \varphi_0[\bar{c}, a_\alpha, a_{\alpha+2}, a_{\alpha+3}, \ldots, a_{\alpha+n}]$ where $\alpha = \zeta(n(0)+1)$.

Let $\mathbf{s}_\zeta$ be the truth value of $M \models \varphi_0[\bar{c}, a_{\alpha+1}, a_{\alpha+3}, a_{\alpha+3}, \ldots, a_{\alpha+n}]$.

There are truth value $\mathbf{t}, \mathbf{s}$ such that $S = \{\zeta < \mu : \mathbf{t}_\zeta = \mathbf{t}, \mathbf{s}_\zeta = \mathbf{s}\}$ is an unbounded subset of μ.

The rest should be clear.

Chapter II: Axiomatic Framework

§0. Introduction

We give here (§1) an axiomatic framework for dealing with classes of models which have something like "free amalgamations". We give several versions, but we shall deal here mainly with the strongest one. [Somewhere else we shall concentrate on the "prime" framework for which we can repeat the development]. We show that it holds for two main examples: stable first order T (here the models are algebraically closed subsets of $\mathbf{C}^{eq}$) and a universal class (with a special order as developed in I §4 assuming some non-order property) . So the main applications are the result for universal classes, whereas our guiding line is to make the theory similar to the one of stable first order T.

In the third section we deal with a weaker framework, but with smoothness (just as the "abstract elementary classes" of Shelah [Sh 88]). A simple observation, but with important consequences is the "model homogeneity-saturation" lemma, saying that for a model to be (D,λ)-model homogeneous, it is enough that all relevant 1-types are realized. This makes dealing with model-homogeneous models similar to saturated ones. Still, $tp(a, M, N)$ $(M \leq N,\ a \in N)$ may not be determined by the collection of $tp(a,M',N)$ for all small $M' \leq M$.

In the main framework, if M_1, M_2 are in stable amalgamation over M_0 in M , $M_1 \cup M_2$ generate a "good" submodel of M_3; in a weaker variant there is over $M_1 \cup M_2$ a prime model, and similarly for union of increasing chains. This is suitable for dealing with (D,λ)-homogeneous models (from [Sh 3] and generally continue [Sh 54] on exist existentially closed models). We can also consider Banach structures (see Stern [St 1]). Since Banach space theorists are not normally interested in the questions answered here, this is not an application to Banach space theory, and I have not developed it per se (see [Sh 54, p. 241], but it seems worthwhile to consider the example. We even consider the problem of whether any two amalgamations are necessarily compatible.

For $T \subseteq L_{\kappa,\omega}$ where κ is a compact cardinal see [Sh 285] . If we omit NF (but have smoothness and amalgamation) we can do much toward defining NF (assuming various properties hold, where their negations imply non-structure for large enough power). The results are not sufficiently cardinality free to start the theory reasonably, but we can get, e.g., universal homogeneous models in λ when $\lambda = \aleph_\lambda$.

Now 1.1 through 1.4 describe the context for the entire paper. We then discuss three parallel sets of axioms in decreasing order of strength. These are $AxFr_1$ (1.4) the main framework, $AxFr_2$ (1.6) the primal framework, and $AxFr_3$ (1.5) the existential framework. The difference between these frameworks is the way in which a "cover" of a pair of models (neither contained in the other) or of an increasing sequence of models is described. In the main framework the axiom group C_{gn} express the idea that the "cover" is generated from the given models by functions. The existential framework simply demands the existence of a "cover". The primal framework express the idea that the "cover" is prime in the sense of first order model theory.

These three frameworks all avoid the introduction of element-types and deal only with models. In 1.7 we move in an orthogonal direction and describe axioms which generalize the idea of a non-forking type of element.

§1. The Framework

1.0 Notation: As we introduce axioms we give their names in round brackets, e.g. ($AxFr_2$). Later we write an axiom in square brackets to indicate in the case of a theorem that the axiom is needed to prove it and in the case of a definition that we only use the defined concept when the indicated axiom holds.

We may feel it reasonable to demand **K**, $(\mathbf{K}, \leq_{\mathbf{K}})$ (etc) are defined reasonably. Note however that by 3.8 (really by [Sh 88]), under enough (but not many) assumptions, **K** and $(\mathbf{K}, \leq_{\mathbf{K}})$ (i.e. $\{(M, N) : N \leq_{\mathbf{K}} M\}$) are $PC_{(2^\chi)^+, \aleph_0}$- classes.

1.1 Context: In all the frameworks, **K** denotes a tuple consisting of classes and relations whose properties we axiomatize. E.g. $\mathbf{K} = \langle K, \leq, NF \rangle$. For our **K**'s K will be a class of models of a fixed vocabulary $\tau(K)$, $\leq = \leq_{\mathbf{K}}$ a two-place relation on K (a generalization of being elementary submodel) and usually a four-place relation $NF = NF_K$ ($NF(M_0,M_1,M_2,M_3)$ means M_1,M_2 are in stable amalgamation over M_0 inside M_3) . [In AxFr_4 we use $NF^e = N_K^e$ ($NF^e(M_0,M_1,a,M_3)$ which means $tp(a,M_1,M_3)$ does not fork over $M_0, a \in M_3$)] . We may want to say in the former case that M_3 is generated by $M_1 \cup M_2$ ($M_3 = \langle M_1 \cup M_2 \rangle^{gn}_{M_3}$) or at least is prime over $M_1 \cup M_2$ ($Pr(M_0,M_1,M_2,M_3)$) or just any two possible M_3's are compatible. Also sometimes an increasing union is not by itself a member of K but we can close it or take over it a prime model or just any two possible bounds are compatible. Naturally,

1.2 Meta Axiom: K, and all relations on it, are closed under isomorphism.

1.3 Group A: The following axioms always will be assumed on $(K, \leq_K)$

(A0) $M \leq M$ for $M \in K$

(A1) $M \leq N$ implies $M \subseteq N$ (M a submodel of N)

(A2) $\leq$ is transitive

(A3) if $M_0 \subseteq M_1 \subseteq N$, $M_0 \leq N$ and $M_1 \leq N$ *then* $M_0 \leq M_1$

1.3A **Definition:** We say $f : M \to N$ is a $\leq$ - embedding if f is an isomorphism from M onto some $M' \leq N$.

1.4 **The Main Framework** (AxFr_1):

Here $\mathbf{K} = (K, \leq, NF, \langle\ \rangle^{gn})$ where "gn" stands for "generated".

AxFr_1 consists of (1.2, and (A0) - (A3) of 1.3 and):

(A4)Existence of General Union: If $M_i (i < \delta)$ is $\leq$ - increasing, then

$$M \overset{def}{=} \bigcup_{j<\delta} M_j \in K \quad \text{and} \quad M_j \leq M \quad \text{for } j < \delta.$$

The second group deals with the "algebraic closure."

Group B

(B0) If $B = \langle A \rangle_M^{gn}$ then $A \subseteq M \in K, A \subseteq B \subseteq M$

(B1) If $B = \langle A \rangle_M^{gn}$ then $\langle B \rangle_M^{gn} = B$

(B2) if $A \subseteq B \subseteq M$ then $\langle A \rangle_M^{gn} \subseteq \langle B \rangle_M^{gn}$

(B3) if $A \subseteq M \leq N$ then $\langle A \rangle_M^{gn} = \langle A \rangle_N^{gn}$.

The third group of axioms deals with stable amalgamation.

Group C_{gn}

(C1) If $NF(M_0,M_1,M_2,M)$ then $M_0 \leq M_1 \leq M_3$, and $M_0 \leq M_2 \leq M$

(hence $M_0,M_1,M_2,M \in K$).

(C2) **Existence:** For every M_0, M_1 and M_2 such that $M_0 \leq M_1$ and $M_0 \leq M_2$ there are M_1^*, M_2^*, M from K and f_1, f_2 such that: f_ℓ is an isomorphism from M_ℓ onto M_ℓ^* over M_0 for $\ell = 1,2$ and $NF(M_0,M_1^*,M_2^*,M)$.

(C2)$^-$ Will just state $M_0 \leq M_1^* \leq M, M_0 \leq M_2^* \leq M$ (i.e. amalgamation exists).

(C3) **Monotonicity:**

(a) $NF(M_0,M_1,M_2,M)$ implies $NF(M_0,M_1,M_2^*,M)$ when $M_0 \leq M_2^* \leq M_2$.
(b) $NF(M_0,M_1,M_2,M)$, $M \leq M^*$ implies $NF(M_0,M_1,M_2,M^*)$.
(c) $NF(M_0,M_1,M_2,M)$, $M_1 \cup M_2 \subseteq M^* \leq M$ implies $NF(M_0,M_1,M_2,M^*)$.
(a)^d $NF(M_0,M_1,M_2,M)$ implies $NF(M_0,M_1^*,M_2,M)$ when $M_0 \leq M_1^* \leq M_1$.

(C4) **Base enlargement:** $NF((M_0,M_1,M_2,M)$,
$M_0 \leq M_0' \leq M_2$ implies $NF(M_0', \langle M_1 \cup M_0' \rangle_M^{gn}, M_2, M)$.

(C5) **Uniqueness:** If for $\ell = 1,2$, $NF(M_0^\ell, M_1^\ell, M_2^\ell, M_3^\ell)$ and for $m = 0,1,2$ f_m is an isomorphism from M_m^1 onto M_n^2 and $f_0 \subseteq f_1, f_0 \subseteq f_2$ *then* for some $N \in K$, $M^2 \leq N$ there is a $\leq_K$ -embedding h of M^1 into N, which extend $f_1 \cup f_2$.

(C6) **Symmetry:** $NF(M_0,M_1,M_2,M)$ implies $NF(M_0,M_2,M_1,M)$.

(C7) **Finite Character :** If $\langle M_{1,i} : i \leq \delta \rangle$ is increasing continuous, $M_0 \leq M_{1,0}$ and

$$NF(M_0,M_{1,\delta},M_2,M) \quad then \quad \langle M_{1,\delta} \cup M_2 \rangle_M^{gn} = \bigcup_{i<\delta} \langle M_{1,i} \cup M_2 \rangle_M^{gn}.$$

1.5 THE EXISTENTIAL FRAMEWORK (AxFr_3)

Here $\mathbf{K} = (K, \leq, NF)$.

We have Axioms (A0) - (A3) from 1.3):

(A5) **Limit Existence:** If $\langle M_i : i < \delta \rangle$ is $\leq_K$ -increasing, *then* there is $M \in K$, $M_i \leq_K M$ for $i < \delta$.

(A6) **Limit Uniqueness:** If $\langle M_i : i < \delta \rangle$ is $\leq_K$ -increasing and for $\ell = 1,2$ $[i < \delta => M_i \leq_K N^\ell >]$ *then* there is N, $N^2 \leq N$ and a $\leq_K$-embedding f of N^1 into N , $f \restriction M_i = id_{M_i}$ for $i < \delta$.

Group C_{ex}: Ax(C1) (C2) , (C3) (C5) (C6) and

(C8) If $\langle M_{1,i}: i < \delta \rangle$ is increasing and $NF(M_0,M_{1,i},M_2,M)$ for each $i < \delta$ *then* for some $M_{1,\delta}$ we have $(\forall i < \delta)(M_{1,i} \leq M_{1,\delta})$ and $NF(M_0,M_{1,\delta},M_2,M)$.

(C8)$^-$ Like C8, but $M_{1,\delta}$ is found in some $\leq$-extension of M.

(C8)$_-$ *If* $\langle M_{1,i} : i \leq \delta\rangle$ *is* $\leq$*-increasing* continuous, for $i < \delta$, $NF(M_0, M_{1,i}, M_i, M)$ *then* $NF(M_0, M_{1,\delta}, M_2, M)$.

1.6 THE PRIMAL FRAMEWORK ($\mathrm{Ax}Fr_2$)

We assume the axioms of ($\mathrm{Ax}Fr_3$) the following axioms on prime models.

In the first order case one defines prime models over arbitrary subsets of members of K. Reflection shows that this cannot be expected generally, and experience has shown that it suffices to have prime models only in more specific cases: over unions of chains and over pairs of independent models. The following axioms describe the properties of such prime models.

There are (at least) three ways in which one could introduce prime models; relatively [i.e. within a specified model), compatibility (within a compatibility class cf §3) or absolutely. (The compatibility class of N : $\{N' \in K : \exists N^* \in K,\ N \leq N^* \text{ and } N' \leq N^*\}$.) Our axioms here are the compatibility version; we describe the absolute version in Definition 1.9; at present the relative version does not seem useful.

Group D: On prime models

(D1) If $\langle M_i : i < \delta\rangle$ is $\leq_K$ -increasing *then* there is a model $N^p \in K$,

$(\forall i < \delta)\ [M_i \leq N^p]$ such that

if $(\forall i < \delta) M_i \leq N \leq N^*$ *and* $N^p \leq N^*$ *then* there is a $\leq$-embedding f of N^p into N over $\bigcup_{i<\delta} M_i$.

We write in this case $\Pr(\langle M_i : i < \delta\rangle, N)$.

(D2) If $NF(M_0, M_1, M_2, M_3)$ *then* there is N prime over $M_1 \cup M_2$ inside M_3, i.e.:

(i) $M_1 \cup M_2 \subseteq N \leq M_3$ and

(ii) for every M, M_3^*, *if* M, $M_1 \cup M_2 \subseteq M_3^* \leq M$ *and* $N \leq M$ *then* there is a $\leq$-embedding f

of N into M_3^* over $M_1 \cup M_2$.
We write in this case $Pr(M_0, M_1, M_2, N)$.

(D3) **Uniqueness of the prime model over** $\langle M_i : i < \delta \rangle$:

If $Pr(\langle M_i : i < \delta \rangle, N^\ell)$, $N^\ell \leq N$ for $\ell = 1,2$ then N^1, N^2 are isomorphic over $\bigcup_{i<\delta} M_i$.

(D4) **Uniqueness of the Prime Model over** $M_1 \cup M_2$:

If $Pr(M_0, M_1, M_2, N^\ell)$, $N^\ell \leq N$ for $\ell = 1,2$ *then* N^1, N^2 are isomorphic over $M_1 \cup M_2$.

1.7 THE NF FOR ELEMENTS FRAMEWORK (Ax Fr_4)

Here $\mathbf{K} = (K, \leq, NF^e)$.

We have here Ax(A0)-(A4).

Group E :

(E1) $NF^e(M_0, M_1, a, M_3)$ implies: $M_0 \leq M_1 \leq M_3$ and $a \in M_3$

(E2) **Existence** : *For every* M_0, M_1, M_2, a such that

$a \in M_2$, $M_0 \leq M_1$, $M_0 \leq M_2$ *there are* M and f, such that

$M_1 \leq M$, f is a $\leq_{\mathbf{K}}$ -embedding of M_2 into M over M_0 , and

$NF^e(M_0, M_1, f(a), M)$.

(E3) **Monotonicity:** (a) $NF^e(M_0,M_1,a,M)$, $M_0 \le M_1^* \le M_1$ implies $NF^e(M_0,M_1^*,a,M)$

(b) $NF^e(M_0,M_1,a,M)$ and $M \le M^*$ implies $NF^e(M_0,M_1,a,M^*)$

(c) $NF^e(M_0,M_1,a,M)$, $M_1 \cup \{a\} \subseteq M^* \le M$ implies $NF^e(M_0,M_1,a,M^*)$

(E4) **Base Enlargement** : $NF^e(M_0,M_1,a,M)$ and $M_0 \le M_0^* \le M_1$ implies $NF^e(M_0^*,M_1,a,M)$

(E5) **Uniqueness**: Suppose $M_0 \le M_1 \le M$, $NF^e(M_0,M_1,a,M)$, $NF^e(M_0,M_1,b,M)$, and $M_0 \cup \{a\} \subseteq N^a \le M$, $M_0 \cup \{b\} \subseteq N^b \le M$, and there is an isomorphism from N^a onto N^b over M_0 mapping to a to b *then* there are N_a, N_b, M^* and f such that: $M \le M^*, M_1 \cup \{a\} \subseteq N_a \le M^*$, $M_1 \cup \{b\} \subseteq N_b \le M^*$ and f is an isomorphism from N_a onto N_b over M_1 mapping a to b.

(E6) **Continuity** : If $\langle M_{1,i}: i<\delta \rangle$ is $\le$-increasing, $\langle M_i: i<\delta \rangle$ is $\le$-increasing and $NF^e(M_0,M_{1,i},a,M_i)$ for every $i < \delta$, *then* we can find $M_{1,\delta}$ and M_δ such that $M_{1,i} \le M_{1,\delta}$ and $M_1 \le M_\delta$ (for $i<\delta$) and $NF^e(M_0,M_{1,\delta},a,M_\delta)$.

1.7A **Remark** : We can define variants $(AxFr_5)$, $(AxFr_6)$ of $(AxFr_2)$, $(Ax\ Fr_3)$ resp. using NF^e instead NF, i.e. we *waive* Ax(A4) replacing it by weaker axioms.

Here are some properties which do not obviously follow from the axioms we have given but are plausible additional axioms. As an example of their use note that the proof of V.1.2 (1) is carried out without recourse to (F1) but (F1) would materially simplify the proof.

1.8 **other things**

(1) (F1) **Disjointness** : $NF(M_0,M_1,M_2,M_3)$ implies $M_1 \cap M_2 = M_0$.

(F2) If $NF^e(M_0,M_1,a,M_3)$, $a \notin M_0$ then $a \notin M_1$.

(2) (G1) If $M_0 \le M_2$, $a \in M_3$, *then* there is M_2', $M_0 \cup \{a\} \subseteq M_2' \le_K M_3'$ and $NF^e(M_0,M_1,a,M_3)$, $M_2' \le_K M_3'$, implies $NF(M_0,M_1,M_2'1,M_3')$

1.9 **Definition** : Parts (1) and (2) of the following define the absolute notion of prime. As hoped for analogue of Section IV.1 would derive from (D1) a dichotomy between condition (1) and nonstructure.

(1) N is prime over $\langle M_2 : i < \delta \rangle$, ($M_i$ is $\leq$*–increasing*) if:

(a) $M_i \leq N$ for $i < \delta$ and

(b) if $(\forall i < \delta) M_i < N^*$ *then* N can be $\leq$-embedded into N^* over $\bigcup_{i<\delta} M_i$

(2) N is a prime stable amalgamation for M_0 over $M_1 \cup M_2$ if:

·(a) $NF(M_0, M_1, M_2, N)$ and

(b) if $NF(M_0, M_1^*, M_2^*, M_3^*)$,

f_1 an isomorphism from M_1 onto M_1^* over M_0

f_2 an isomorphism from M_2 onto M_2^* over M_0

then there is a $\leq$-embedding N into M^* extending $f_1 \cup f_2$.

(3) For $M \in K$ we define a relation E_M^{btp} between pairs $(\bar{a}, N)$, $\bar{a} \in N$, $M \leq N$:

$(\bar{a}_1, N_1)\ E_M^{btp}\ (\bar{a}_2, N_2)$ *if and only if* there are N_1^*, N_1^+, N_2^*, N_2^+, f such that:

$$M \leq N_1^* \leq N_1^+, \ N_1 \leq N_1^+,$$

$$M \leq N_2^* \leq N_2^+, \ N \leq N_2^+,$$

$$\bar{a}_1 \in N_1^*$$

$$\bar{a}_2 \in N_2^*$$

f is an isomorphism from N_1^* onto N_2^* over M mapping $\bar{a}_1$ to $\bar{a}_2$.

(4) E_M^{tp} will be the closure of E_M^{btp} to an equivalence relation and $tp(\bar{a},M,N)$ is $(\bar{a},N)/E_M^{btp}$ (note: if **K** has amalgamation $E_M^{tp} = E_M^{btp}$) .

Now we note some interrelations between the axioms and later define some related notions.

1.10 **Lemma** : 1) [Ax Fr_1 , or just (A0), (B), (C1), (C4)]

If $NF(M_0,M_1,M_2,M)$ then $M_3 \overset{def}{=} \langle M_1 \cup M_2 \rangle_M^{gn}$ (i.e. the restriction of M to this set is well defined), is a member of K and $M_1 \cup M_2 \subseteq M_3 \leq M$

2) [Ax Fr_1 or just (B),$(C2)^-$,]

Suppose that the conclusion of 1.10(1) holds, then Ax(C5) is equivalent to:

(*) if $NF((M_0^\ell,M_1^\ell,M_2^\ell,M^\ell)$ for $\ell = 1,2$

and for $m = 0,1,2$ f_m is an isomorphism from M_m^1 onto M_m^2 and $f_0 \subseteq f_1, f_0 \subseteq f_2$ *then* $f_1 \cup f_2$ can be extended to an isomorphism from $\langle M_1^1 \cup M_2^1 \rangle_{M^3}^{gn}$ onto $\langle M_1^2 \cup M_2^2 \rangle_{M^3}^{gn}$

3) AxFr_1 implies AxFr_2 which implies AxFr_3

4) $\mathrm{Ax(C8)}_-$ follows from (C2),(C5) and smoothness (see 1.12 below)

5) If $Pr(\langle M_i : i<\delta \rangle,\ M)$ and Ax(A6) then M is prime over $\langle M_i : i<\delta \rangle$

6) If $Pr(M_0,M_1,M_2,M)$ and Ax(C5) then M is a prime stable amalgam for M_0 over $M_1 \cup M_2$

7) Ax (C8) $\Rightarrow$ Ax $(C8)^-$, Ax $(C8)_-$.

8) If **K** is smooth, then Ax $(C8)_-$ implies Ax (C8).

9) If $NF(M_0, M_1, M_0, M_2)$ when $M_0 \le M_1 \le M_2$ (this follows from Ax (C2,C3) *then Ax (C8) is equivalent to Ax* $(C8)_-$ *+ smoothness.*

Proof : 1) Apply Ax(C4) with $M_0^1 = M_2$. [Note $M_0 \le M_0$ as $M_0 \le M_2$ by Ax(C1). $M^1 \le M_2$ by Ax(A,0)]. So $NF(M_2, \langle M_1 \cup M_2 \rangle_M^{gn}, M_2, M\ )$. Now by Ax(C1) this implies $M_1 \cup M_2 \subseteq \langle M_2 \cup M_2 \rangle_M^{gn} \le M$.

4) See Lemma IV 1.5.

The other proofs are left to the readers.

There are more implications

1.11 **Definition** : 1) **K** has the λ-Lowenheim-Skolem property (λ–*LSP*) if:

$[A \subseteq M$ and $|A| \le \lambda] \Rightarrow (\exists N \le M)[A \subseteq N$ and $|||N||| \le \lambda]$

2) The $(<\lambda)$-Lowenheim-Skolem property ($(<\lambda)$-LSP) means:

$$[A \subseteq M \text{ and } |A| < \lambda] \Rightarrow (\exists N \le M)[A \subseteq N \text{ and } |||N||| < \lambda]$$

3) $LS(\mathbf{K})$ is the minimal λ for which **K** has λ-LSP. We also write $\chi_{\mathbf{K}}$ for LS (**K**).

4) Instead λ-LSP we also write $LSP(\lambda)$. $LSP(\mu,\lambda)$ means in (1) $|A| \le \lambda$, $|||M||| \le \mu$. We define $LSP(<\mu, <\lambda)$ etc. similarly.

1.11A **Remark** : The statement "$\lambda < \mu$ and λ - Lowenheim-Skolem property $\Rightarrow \mu$-Lowenheim-Skolem property" will be considered.

1.12 **Definition** : 1) **κ-smoothness** means:

If $\langle M_i : i < \kappa \rangle$ is increasing, then there is N prime over $\langle M_i : i < \kappa \rangle$ (For $AxFr_1$ this means: if each $M_i \le M$ and $\langle M_i : i \le \kappa \rangle$ is $\le$-increasing, *then* $\bigcup_{i<\kappa} M_i \le M$).

2) **The weak κ- smoothness** means (for $AxFr_1$): if $\langle M_i : i < \kappa \rangle$ is $\le$-increasing continuous, $M_i \le M$ *then* $\bigcup_{i<\kappa} M_i \le M$. [This condition is weaker than 1.12 1) since we have assumed the $\langle M_i : i < \kappa \rangle$ is continuous.]

3) Let (λ, κ)-smoothness be defined as in (1) but demanding $|||M_i||| \le \lambda$, and $|||M||| \le \lambda$.

Let $(\lambda, \kappa)^+$-smoothness be defined as in (1) but demanding only $|||M_i||| \le \lambda$ for $i < \kappa$.

4) $(<\kappa)$-smoothness, etc. has the obvious meaning.

1.12A **Remark.** Smoothness and (A4) are (in this context) the Tarski-Vaught theorem.

1.13 **Claim** : (1) [weak] κ-smoothness is equal to [weak] $cf(\kappa)$-smoothness

(2) Our framework is $(<\kappa)$-smooth *if and only if* our framework in weakly $(<\kappa)$-smooth

Proof: Check.

1.14 **Definition** : "NF is κ-based" means:

if $M \le N$, $A \subseteq N$ and $|||N||| \le \kappa$ *then* there are M_0, M_1, such that $NF(M_0, M, M_1, N)$, $|||M_1||| \le \kappa$ and $A \subseteq M_1$.

1.15 **Definition** : 1) $\lambda_0(\mathbf{K}) = \lambda_0(K)$ is the first λ such that K is a $PC(L_{\lambda^+,\omega})$-class, i.e., the class of $\tau(K)$-reducts of models of some $\psi \in L_{\lambda^+,\omega}$.

2) $\lambda_1(\mathbf{K}) = \lambda(K, \le)$ is the first λ such that $\{(M, N) : M \in K, N \in K, N \le M\}$ is a $PC(L_{\lambda^+,\omega})$-class

3) $\lambda_2(\mathbf{K}) = \lambda(NF, gn)$ is the first cardinal λ such that

$\{(M_1,M_0,M_1,M_2) : NF(M_0,M_1,M_2,M),\ M = (\langle M_1 \bigcup M_2 \rangle)^{gn}_M\}$ is a $PC(L_{\lambda^+,\omega})$ class

4) $\lambda(\mathbf{K}) = \sum_{\ell<3} \lambda_\ell(\mathbf{K})$ and $\lambda_{\ell_1,\ell_2}(\mathbf{K}) = \lambda_{\ell_1}(\mathbf{K}) + \lambda_{\ell_2}(\mathbf{K}) + \cdots$.

1.16 **Definition** : λ is **K**-inaccessible if:

1) for $M_0 \le M_1,\ M_2$ (in K) each of cardinality $<\lambda$, there is $M \in K$, $|||M||| < \lambda$, and for $\ell = 1,2$ $\le$-embeddings f_ℓ of M_ℓ into M over M_0 such that $NF(M_0, f(M_1), f(M_2), M)$

2) If $\delta < \lambda$, $|||\bigcup_{i<\delta} M_i||| < \lambda, \langle M_i : i < \delta\rangle$ is $\le$-increasing, then for some $M \in K$ of cardinality $< \lambda$, $M_i \le M$ for $i < \lambda$.

The following definition of pseudo cardinality is an attempt to axiomatize the idea of a structure being generated by χ elements.

1.17 **Definition** : [AxFr$_2$]

We define pscard $^\chi_{\mathbf{K}}(M)$ as follows:

(I) for $M \in K$, pscard$(M) = \chi$ if $|||M||| \le \chi$

(II) for $M \in K, \lambda \ge \chi$: pscard$^\chi_{\mathbf{K}}(M) = \lambda$ *iff*

(i) for some $\le$ increasing sequence $\langle M_i : i < \delta\rangle$:

(a) $\delta \le \lambda$

(b) Pr($\langle M_i : i < \delta\rangle$, M_δ)

(c) pscard$^\chi_{\mathbf{K}}(M) < \lambda$

(ii) for no $\mu < \lambda$, pscard$^\chi_{\mathbf{K}}(M) = \mu$

1.7A Remark: Rather than defining pscard, we can use it as a basic function and put on it an axiom.

§2. The Main Examples

2.1 First Order Theories

Let T be a *stable* first order theory. Assume that T^{eq} has elimination of quantifiers. Let

(i) $K = \{M : M$ is a submodel of some $N \models T^{eq}$ and $|M| = ac\ell_N(M)\}$

(If you want--omit the unnecessary elements of N)

(ii) $\leq_{\mathbf{K}}$ is being a submodel

(iii) Let for some N, $M \subseteq N \models T^{eq}$ *then:* $B = \langle A \rangle_M^{gn}$ if and only if $A \subseteq M$, $B = ac\ell_N A$ (i.e. is B the algebraic closure of A inside N)

iv) $NF(A_0, A_1, A_2, A)$. Let $A_\ell \subseteq N$ for $\ell < 4$, $N \models T^{eq}$, $NF(A_0, A_1, A_2, A_3)$ holds if and only if:

$A_\ell = ac\ell_N A_\ell$ for $\ell = 0,1,2,3$ $A_0 \subseteq A_1 \subseteq A_3$ and $A_0 \subseteq A_2 \subseteq A_3$ and $tp_*(A_2, A_1)$ does not fork over A_0.

Remark : In this context "models" disappear. I.e. "model" in our context, is just an algebraically closed set. Later "λ-saturated model, $\lambda > |T|$" are defined. But "models of T" are not naturally defined in this context. As we prefer to have theorems which say something when specialized to this case, we will try to have non-structure saying not only

"there are many $M \in K$" but

"there are many quite homogeneous ($\equiv$ quite saturated) models"
or at least
"there are many models in $\mathbf{K}_\mu^{us}$" (see Definition 3.12 below).

2.1A **Fact** : All axioms from §1 hold under those circumstances.

So, most of [Sh] can be done in this framework, and many of the proofs here are adaptations of proofs from [Sh] to our context under this translation.

2.2 Universal Classes

2.2A **Definition** : A class K of $\tau(K)$-models is called *universal* if it is closed under submodels and under unions of increasing chain.

2.2B **Claim** : The following are equivalent for a class K of $\tau(K)$- models

(i) K is a universal class

(ii) a $\tau(K)$ -model M belongs to K *iff* every finitely generated submodel of M belongs to K.

Proof . Now (ii) $\Rightarrow$ (i) should be clear.

So assume (i). Let M be a $\tau(K)$-model.

(a) If $M \in K$ then every finitely generated submodel of K belongs to N.

It is true as "membership in K" is hereditary.

(b) If every finitely generated submodel of K belongs to K *then* $M \in K$.

We prove by induction on κ that if $M = \langle A \rangle_M^{gn}$, $|A| \leq \kappa$ and every finitely generated $N \subseteq M$ belongs to K , *then* $M \in K$.

For κ finite ($< \aleph_0$) it is trivial.

For $\kappa \geq \aleph_0$ let $A = \{a_i : i < |A|\}$

$$M_i = \langle a_j : j < i \rangle^{gn}.$$

So M_i ($i < |A|$) is increasing and $M = \bigcup_{i<\kappa} M_i$. Every finitely generated submodel of M_i belongs to K hence by the inductive hypothesis (as $|\{a_j: j < i\}| \leq |i| < \kappa$) $M_i \in K$. But K is closed under unions of increasing chains, hence

$$M = \bigcup_{i<\kappa} M_i \in K\$.$$

2.2.C **Hypothesis** : K has $(\chi^+, q.f.)$ - nonorder, $\chi \geq |\tau(K)|$.

2.2D **Convention** : Let $\mu = 2^{2^\chi}$, $\leq = \leq^{\aleph_0}_{qf, \mu^+, \chi^+}$

$\langle A \rangle^{gn}_N$ be the closure of A under the functions of N and $NF(M_0, M_1, M_2, M_3)$ *iff* M_0, M_1, M_2 are in (qf, $\mu, \chi, \aleph_0$)- stable amalgamation inside M_3 (see 14.8) and $\langle M_1 \bigcup M_2 \rangle^{gn}_{M_3} \leq M_3$.

2.2E **Lemma** : From the axioms from §1 AxFr $_1$ + (E 1) holds

Proof: Most are totally routine (using Lemma I2.3).

2.2E1 **Sublemma** : Ax C2 (Existence) holds

Proof : So suppose $M_\ell \in K$ for $\ell < 3$, $M_0 \leq M_1$ and $M_0 \leq M_2$.

We shall find $M, M_0 < M$ and $\leq$-embeddings $f_\ell: M_\ell \rightarrow M$ over $M_\ell, \ell = 1,2$ (i.e. f_ℓ is an isomorphism from M_ℓ onto $M'_\ell, M'_\ell < M$, $f_\ell \restriction M_0 = identity$) , such that $M = \langle f_1(M_1) \bigcup f_2(M_2) \rangle^{gn}_M$ and M_0, M_1, M_2 are in stable amalgamation inside M.

We let $M_\ell = \{c^\ell_i : i < \|M_\ell\|\}$. The universe of M will be the set $\{\sigma(\bar{c}^1, \bar{c}^2): \bar{c}^\ell \in \|M_\ell\|, \sigma$ a τ(**K**)-term$\}$ ($\ell = 1,2$) divided by an equivalence relation E defined below. The operations are defined in the obvious way.

Let $\Gamma = \{\varphi(\sigma_1(\bar{c}^{1,1}, \bar{c}^{2,1}), \ldots, \sigma_m(\bar{c}^{1,m}, \bar{c}^{2,m}))$: for some $q.f.$ formula φ and $(q.f., \mu^+, \chi^+)$-convergent family $\mathbf{J} \subseteq M_0$ of sequences of length $\lg(c^{2,1} \wedge \cdots \wedge \bar{c}^{2,m})$, $\varphi(\sigma_1(\bar{c}^{1,1}, \bar{x}_1), \ldots, \sigma_m(\bar{c}^{1,m}, \bar{x}_m)) \in Av(\mathbf{J}, M_1)$ where

$\bar{x} = \bar{x}_1 \hat{} \cdots \hat{} \bar{x}_m,\ \ell g(\bar{x}_\ell) = \ell g(\bar{c}^{2,\ell})\}$.

The average is well defined as **J** is convergent. Note that the definition of Γ does not depend on the choice of **J** by (2) of the claim I4.5. So Γ is complete ($\varphi \in \Gamma$ or $\neg\phi \in \Gamma$) as there are such **J** with the convergent property because $M_0 \le M_2$. Also every finite subset of Γ is realized in M_0.

Next E is defined by:

$$\sigma_1(\bar{c}^1, \bar{c}^2) E \sigma_2(\bar{d}^1, \bar{d}^2)$$

if and only if:

$$[\sigma_1(\bar{c}^1, \bar{c}^2) = \sigma_2(\bar{d}^1, \bar{d}^2)] \in \Gamma.$$

As Γ is finitely satisfiable in M_0, E is a congruence relation (and of course an equivalence relation). So M is well defined, f_ℓ are defined naturally and they are embeddings.

Now, why is $M \in K$? It is enough that every finitely generated submodel is in K. Say such a submodel is generated by $\bar{c}^\ell \in M_\ell$ (really $\bar{c}^\ell / E$). But if $\mathrm{Av}_{qf}(\mathbf{J}, M_0, M_0) = tp_{qf}(\bar{c}^2, M_0, M_2)$ and **J** is (qf, μ^+, χ^+) -convergent *then:* for all but $< \chi$ of the sequence $\bar{d}^2 \in \mathbf{J}$ the quantifier free type of $\bar{c}^1 \hat{} \bar{c}^2$ in M_1 is equal to the quantifier free type of $\bar{c}^1 \hat{} \bar{d}^2$ in M. The models they generate are isomorphic but the first being a submodel of M_1 is in K so also the second one is in K. Now $M_0 \le M_1$ is quite easy, thus we finish proving 2.2E1.

2.2E2 **Sublemma** : Ax(C5) (symmetry) holds, i.e.

M_0, M_1, M_2 is in stable amalgamation inside M if and only if

M_0, M_2, M_1 is in stable amalgamation inside M.

Proof : Assume the former. We prove the latter.

Let $\bar{a} \in M_1$, $\mathbf{J} \subseteq M_0$, $|\mathbf{J}| = \mu^+$, **J** (qf, μ^+, χ^+) - convergent, $\mathrm{Av}_{qf}(\mathbf{J}, M_0, M_0) = tp_{qf}(\bar{a}, M_0, M_1)$; hence $q \overset{def}{=} Av_{qf.}(\mathbf{J}, M_2, M_2)$ is well defined. We should show it is equal to $tp_{qf}(\bar{a}, M_2, M)$. So assume $\bar{b} \in M_2$, φ quantifier free, and $M \models \varphi[\bar{a}, \bar{b}]$ and it is enough to show $\varphi(\bar{x},\bar{b})\in q$.

Let $\mathbf{I}\subseteq M_0$, $|\mathbf{I}| = \mu^+$ be (qf,μ^+,χ^+) - convergent and $Av(\mathbf{I},M_0,M_0) = tp(\bar{b},M_0,M_1)$

Picture :

$\bar{a}\in M_1$ $\bar{b}\in M_2$

M_0

I

Now $\mathrm{Av}_{qf}(\mathbf{I},M,M) = tp_{qf}(\bar{b},M_1,M)$.

Now as M_0,M_1,M_2 are in stable amalgamation inside M

$$\vDash\varphi(a,\bar{b})\Rightarrow(\exists^{>\chi}\bar{b}^+\in\mathbf{I})\varphi(\bar{a},\bar{b}')$$

$$\Rightarrow(\exists^{>\chi}\bar{b}'\in\mathbf{I})[\exists^{\chi}a'\in\mathbf{J}]\varphi(\bar{a}',\bar{b}')$$

by choice of **J**

But then for each $\bar{b}'$

$$(\exists^{\chi}\bar{a}'\in\mathbf{J})(\exists^{\chi}\bar{b}'\in\mathbf{I})\varphi(\overline{a}',\bar{b}')$$

by the symmetry Lemma I. 3.1

By the proof of I2.3 for some **J** (remember bs stand for "atomic and negation of atomic formulas)

$$\mathbf{J}^*\subseteq\bigcup_{\alpha}\mathbf{J}_\alpha, Av_{bs}(\mathbf{J}^*,\ N_0) = tp_{bs}(\bar{d},\ N_0)$$

hence

$$Av_{bs}(J^*,\ N_0 \cup B) = tp_{bs}(\bar{d},\ N_0 \cup B).$$

But (*) contradicts the choice of φ .

2.2F **Sublemma** Ax (C4) (base enlargement) holds.

Proof : So suppose $N_0 \leq N \in K$.

If N_0, B, C is in stable amalgamation inside N (in particular $N_0 \leq B \leq N, N_0 \leq C \leq N$) and $N_0 \subseteq C' < C$ *then* C',B,C is in stable amalgamation inside N where $B' = \langle C' \cup B \rangle_N^{gn}$

Let $\bar{d} \in C$, so $\exists \mathbf{I} \subseteq C' [\,|\mathbf{I}| = \mu^+ , Av_{bs}(\mathbf{I},C) = tp_{bs}(\bar{d},C')]$.

We want to show $Av_{bs}(\mathbf{I},C' \cup B) = tp_{bs}(\bar{d},C' \cup B)$ (or over $\langle C' \cup B \rangle_N^{gn}$ --same thing).

So suppose $\bar{e} \in C', \bar{b} \in B, \varphi$ is basic, $\models \neg\varphi[\bar{d},\bar{e},\bar{b}]$ but w.l.o.g. $(\forall \bar{d},\bar{e},\mathbf{I}) \models \varphi(\bar{d}',\ \bar{e}\ \bar{b})$. Let $\mathbf{I} = \{\bar{d}_\alpha : \alpha < \chi_1^\dagger\}$ and

$\mathbf{J}_\alpha = \{\bar{d}_{\alpha,\gamma} {}^\wedge \bar{e}_\gamma : \gamma < \mu_1^\dagger\} \subseteq N$ ind $\mathrm{Av}_{bs}(\mathbf{J}_\alpha,N_0) = tp_{bs}(\bar{d}_\alpha {}^\wedge \bar{e},N_0)$

and N_0,B,C is in stable amalgamation inside N, $N_0 \leq N$ $\mathrm{Av}_{bs}(\mathbf{J}_\alpha,N_0 \cup B) = tp_{bs}(\bar{d}_\alpha {}^\wedge \bar{e},N_0 \cup B)$. So w.l.o.g. $\models \varphi[\bar{d}_{\alpha,\gamma},\bar{e}_\gamma,\bar{b}]$ for $\alpha,\gamma < \mu_1^\dagger$.

2.3 **Sequence homogeneous models**

Let τ be a vocabulary, Δ a set of $L_{\omega,\omega}(\tau)$ - formulas, where in $\mu \geq |D|$ D a set of types, each a complete (Δ,n) - type for some n. And let $\mu \geq \aleph_0$; D is μ-good if there is a (D,μ) - homogeneous model closed under subformulas, (see [Sh 3]]. Now $K = K_D^\mu$ is the set of τ - model M which are (D,μ) - homogeneous; $M \leq N$ iff $M \leq_\Delta N$. We assume $\kappa^-(D) = \aleph_0$, i.e. if $M \leq N \in K, \bar{a} \in {}^{\omega>}N$ then $tp_\Delta(\bar{a},M,N)$ does not split strongly over some finite subset of M (by [Sh 3] $\kappa(D) > \aleph_0$ (with the additional assumption D is good), implies non structure.) Sometimes we use the stronger assumption $\kappa(D) = \aleph_0$: if $A \subseteq N \in K, \bar{a} \in {}^{\omega>}N$ then $tp_\Delta(\bar{a},A,N)$ does not split strongly over some finite subset of A (equivalent to $\kappa^-(D) = \aleph_0$ when D is good).

We let $NF(M_0,M_1,M_2,M_3)$ mean: $M_0 \leq M_1 \leq M_3$, $M_0 \leq M_2 \leq M_3$, and for $\bar{a} \in {}^{\omega>}M_2$, the type $tp_\Delta(\bar{a},M_1,M_3)$ does not split strongly over some finite subset of M_0. Clearly $NF^e(M_0,M_{1,} \ a,M\ )$ is defined similarly. Let $\lambda(D)$ (see [Sh 3]) be minimal λ such that D in λ -stable. Let us check when the axioms holds: (we use goodness and $\mu > \lambda(D)$ freely)

Ax (A0): Holds

Ax (A1) : Holds

Ax (A2) : Holds

Ax(A3): Holds

Ax(A4) The problem is whether $r \overset{def}{=} \bigcup_{i<\delta} M_i$ is (D,μ) - homogeneous. For $\mu = \aleph_0$ this is trivial. Generally it still holds if $\kappa(D) = \aleph_0$, D good

Ax(A5) Follows from $Ax\,(A\,4)$

Ax(A6) Follows from $Ax\,(A\,4)$

Ax(C1) Obvious

Ax(C2) If D is good, $\mu > \lambda(D)$, it is clear by [Sh 3]

Ax(C3) Easy

Ax(C5) Holds for good D

Ax(C6) Holds

Ax(C7) Holds

Ax(C8) Holds

Ax(D1) : Obvious

Ax(D2) : This is how $Ax(C2)$ was proved (for D good. $\mu > \lambda(D)$) .

Ax (D4)(D4): We have to generalize the theorem on the uniqueness of prime models of [Sh IV §4] (we can use induction on rank, D good, $\mu > \lambda(D)$)

Ax(E1) : clear

Ax(E2) : Holds for D good

Ax(E3) : Obvious

Ax(E4) : Obvious

Ax(E5) : True for D good

Ax(E6) : True (take unions), when $Ax(A4)$ holds

Ax(F1) : Holds

Ax(G1) : Holds

2.4 **Problem** : What if for D good, $\mu > \lambda(D)$, we assume just $\kappa(D) < \infty$, and $K = \{M : M\ (D,\mu)$ - homogeneous $\}$: We have many results, but not yet enough to prove the main gap.

§3 Existence/uniqueness of homogeneous quite universal models

Hypothesis : the axioms of group A or just $(A0)(A1)(A2)(A3)(A5)$) and existence of amalgamation $(C2)^-$) , $\chi_1 = LS(\mathbf{K})$.

3.1 **Definition** : We define a two place relation $E_{\mathbf{K}}$ on K :

$ME_{\mathbf{K}}N$ *iff* they are isomorphic to $\leq$-submodels of some common member of K. Since K has amalgamations it is straightforward to show:

3.2 **Fact:** 1) $E_{\mathbf{K}}$ is an equivalence relation with $\leq 2^{LS(\mathbf{K})+|\tau(K)|}$ equivalence classes, each having a member of power $\leq LS(\mathbf{K})$. So (see 3.3 below)

2) $K - \{N \in K : \|N\| < LS(\mathbf{K})\} = \bigcup \{K_{\mathcal{D}} \; \mathcal{D} \in \mathcal{D}'_{\mathbf{K}}\}$ (disjoint union); for each $\mathcal{D} \in \mathcal{D}'_{\mathbf{K}}$, $(K_{\mathcal{D}}, \leq)$ has the amalgamation and disjoint embedding property; and if we are in AxFr_ℓ then $\mathbf{K}_{\mathcal{D}} = \langle K_{\mathcal{D}}, \leq, \langle\,\rangle^{gn}, NF\rangle$ satisfies AxFr_ℓ with Lowenheim number $\leq LS(\mathbf{K})$.

3.3 **Definition:** 1) For $M \in K$, $\|M\| \geq \chi_1$ let

$$\mathcal{D}_M = \mathcal{D}(M) = \{N/\cong : \|N\| = LS(\mathbf{K}), N \leq M\}$$

$$2) \mathcal{D}^0_{\mathbf{K}} = \{\mathcal{D}(M) : M \in K, \|M\| \geq \chi_1\}$$

$$\mathcal{D}^M_{\mathbf{K}} = \bigcup \{\mathcal{D}(N) : M \leq N \in \mathbf{K}\}$$

$$\mathcal{D}'_{\mathbf{K}} = \{\mathcal{D}^M_{\mathbf{K}} : M \in \mathbf{K}, \; \|M\| \geq \chi_1\}$$

$$\mathcal{D}_{\mathbf{K}} = \bigcup \{\mathcal{D}(M) : M \in K\}$$

$$3) \textit{ For } \mathcal{D} \subseteq \mathcal{D}_{\mathbf{K}}, \; K_{\mathcal{D}} = \{M \in K : \mathcal{D}_M \subseteq \mathcal{D}\}$$

Translating the symobls into words we have: $\mathcal{D}_M$ is the collection of isomorphism types of models of power χ_1, which are *embeddable* in M, $\mathcal{D}^M_{\mathbf{K}}$ is the collection of isomorphism types of models of power χ_1 which are *compatible* with M.

$\mathcal{D}^0_{\mathbf{K}}$ is the collection of $\mathcal{D}_M$ for $M \in K$ with $|M| \geq \chi_1$.

$\mathcal{D}'_{\mathbf{K}}$ is the collection of $\mathcal{D}^M_{\mathbf{K}}$ for $M \in K$ with $|M| \geq \chi_1$.

$\mathcal{D}_{\mathbf{K}}$ is in fact that set of isomorphism types of members of K with power χ_1. But in the sense (not denotation see Frege) of our definition, $\mathcal{D}_{\mathbf{K}}$ is the union over all $M \in K$ of the collection $\mathcal{D}_M$ of isomorphism-types of models of power χ_1 which can be embedded in M. Thus $\mathcal{D}^0_{\mathbf{K}}$, $\mathcal{D}'_{\mathbf{K}}$ are objects of one higher type than $\mathcal{D}_{\mathbf{K}}$, $\mathcal{D}_M$ and $\mathcal{D}^M_{\mathbf{K}}$. Finally, if $\mathcal{D}$ is a collection of isomorphism types of models in K, each with power χ_1, $K_{\mathcal{D}}$ is the collection of those M such that each K-submodel of M with power χ_1 is isomorphic to a member of $\mathcal{D}$.

To clarify our notation, note that when $\mathcal{D}$ appears with a subscript $\mathbf{K}$, $\mathcal{D}$ is naming a function and $\mathcal{D}_{\mathbf{K}}$ is the value of that function at the class $\mathbf{K}$. Thus, in Convention 3.4 we write $\mathcal{D}_{\mathbf{K}}$ for $\mathcal{D}$ because we are thinking of $\mathcal{D}_{\mathbf{K}}$ as $\mathcal{D}_{\mathbf{K}_{\mathcal{D}}}$.

In the following convention we are fixing a particular compatibility class (to guarantee joint embedding) and restricting our attention to it.

3.4 **Convention:** We fix $\mathcal{D} \in \mathcal{D}'_{\mathbf{K}}$ and we replace $\mathbf{K}$ by $\mathbf{K}_{\mathcal{D}}$. We write $\mathcal{D}_{\mathbf{K}}$ for this $\mathcal{D}$ We can then have $\mathbf{C}$ ($\mathcal{D}$, $< \infty$)- homogeneous as in [Sh I. §1] (but for uniqueness we have to assume smoothness). The existence of $\mathbf{C}$ is proved in 3.1.

3.5 **Definition:** 1) $M \in K_{\mathcal{D}}$ is $(\mathcal{D}, \lambda)$ -homogeneous (where $\lambda \geq \chi_1^{+}$) *if*

(a) *for* N_0, N_1 satisfying $N_0 \leq M$, $N_0 \leq N_1 \in K_{\mathcal{D}}$, $|||N_1||| < \lambda$ *there is* a $\leq$-embedding of N_1 into M over N_0

(b) every $N_1 \in K_{\mathcal{D}}$ of cardinality $< \lambda$ can be $\leq$- embedded into M.

2) $M \in K_{\mathcal{D}}$ is strongly $(\mathcal{D},\lambda)$-homogeneous (where $\lambda \geq \chi_1^{+}$) if (b) above holds and

(a)$^+$ for $N_0 \leq M$, $N_1 \leq M$, h an isomorphism from N_0 onto N_1 if $|||N_0||| < \lambda$ *then* h can be extended to an automorphism of M.

Remark: By 3.4, part (b) is usually redundant.

3.6 **Definition:** $\mathbf{K}$ is trivial if $[M \leq N \Rightarrow M = N]$; hence K has a unique member up to isomorphism.

3.7 **Lemma:** 1) If λ is $\mathbf{K}$-inaccessible and regular, $\lambda = \lambda^{<\lambda} > |\tau(\mathbf{K})|$ *then* there is $M \in K$ of power λ which is $(\mathcal{D}_{\mathbf{K}}, \lambda)$ -homogeneous and M is smooth (i.e., $M = \bigcup_{i<\lambda} M_i$, $|||M_i||| < \lambda$, M_i-increasing continuous $M_i \leq M$ for $i < \lambda$).

2) If λ is regular, M,N are $(\mathcal{D}_{\mathbf{K}},\lambda)$ - homogeneous of power λ and are smooth, *then* $M \cong N$.

Remark We can weaken somewhat the λ - inaccessibility demands

3.8 **Claim :** 1) If $\mathbf{K}$ has smoothness, $\lambda > LS(\mathbf{K})$, *then* λ is $\mathbf{K}$ -inaccessible (and for $A \subseteq M \in K$, $|||A||| < \lambda \leq |||M|||$ there is $N < M$, $|||N||| = \lambda$, $A \subseteq N$).

2) If (in addition to axioms (A0)-(A4)), $LS(\mathbf{K}) + |\tau(\mathbf{K})| < \chi$ and $\mathbf{K}$ has smoothness, then K and $\{ (M, N): M \leq N \}$ are $PC_{(2^\chi)^+,\omega}$ -class, hence $\mathbf{K}_\mu \neq \emptyset \Rightarrow (\forall \lambda \geq \chi) K_\lambda \neq \phi$ where $\mu = (2^\chi)^+$.

Remark: Using *NF*, we can improve 3.8(2).

Proof: See [Sh 88].

3.9 **Lemma:** If K has smoothness, λ is regular, $|||M||| = \lambda > LS(\mathbf{K})$, *then* every $M \in K$ of power λ is smooth.

Remark: We can begin classification theory for a class satisfying Ax(A0)-(A4)+ smoothness + amalgamation $(+ Ax(C2)^-) + \chi = LS(\mathbf{K})$, using strong splitting. But we do not succeed to move the properties between cardinals. We can arrive, e.g. , that for a class of suitable λ either union of $(\mathcal{D}_{\mathbf{K}},\lambda)$ - homogeneous is $(\mathcal{D}_{\mathbf{K}},\lambda)$-homogeneous, or suitable non-structure results holds.

3.10 **The Model-homogeneity = Saturativity Lemma**

Let $\mu > LS(K)$, K satisfies smoothness

1) M is $(\mathcal{D}_{\mathbf{K}},\mu)$-homogeneous *if and only if* for every $N_1 \leq N_2 \in K$, $|||N_2||| < \mu$, N_1 , M , and $a \in N_2 - N_1$ there are models $N'_2, N_3 \in K$, such that $N_1 \leq N'_2 \leq N_3, N_2 \leq N_3$, $a \in N'_2$ and there is a $\leq_{\mathbf{K}}$-embedding f of N'_2 into M over N_0

2) $M \leq \underset{=}{\mathbf{C}}$ is $(\mathcal{D}_{\mathbf{K}},\mu)$-homogeneous *if and only if* for every $N \leq M$, $|||N||| < \mu$ and $a \in \underset{=}{\mathbf{C}}$, there is $a' \in M$ realizing $tp(a,N,\mathbf{C})$, i.e. there is an automorphism f of $\mathbf{C}, f \restriction N = id_N$ and $f(a) \in N$ (or use Definition 1.8(4).

Proof: 1) clearly w.l.o.g. μ is regular. The "only if" direction is trivial. Let us prove the other direction.

Let $|N_2| = \{a_i : i < \kappa\}$, and we know $\kappa < \mu$. We define by induction on $i \leq \kappa$, N_1^i, N_2^i, f_i such that:

(a) $N_1^i \leq N_2^i$, $\|N_2^i\| < \mu$

(b) N_1^i is $\leq_{\mathbf{K}}$ -increasing continuous in i

(c) N_2^i is $\leq_{\mathbf{K}}$ -increasing continuous in i

(d) f_i is a $\leq_{\mathbf{K}}$ -embedding of N_1^i into M

(e) f_i is increasing in i

(f) $a_i \in N_1^{i+1}$

(g) $N_1^0 < N_1$, $N_2^0 < N_2$, $f_1 = id_{N_1}$.

For $i = 0$, (g) gives the definition. For i limit let $N_1^i = \bigcup_{j<i} N_1^j$, $N_2^i = \bigcup_{j<i} N_2^j$, $f_i = \bigcup_{j<i} f_j$ Now (a)-(f) continue to hold by continuity.

For i successor we use our assumption; [more elaborately, let $M_1^{i-1} \leq M$ be $f_{i-1}(N_1^{i-1})$ and M_2^{i-1}, g_{i-1} be such that g_{i-1} is an isomorphism from N_2^{i-1} onto M_2^{i-1} extending f_{i-1} , so $N_1^{i-1} \leq M_2^{i-1}$, now apply the assumption with M, M_1^{i-1}, M_2^{i-1}, $g_{i-1}(a_{i-1})$ here standing for M, N_1, N_2 there; so there are M_3^{i*}, M_2^{i*}, f_i^* such that:

$$M_1^{i-1} \leq M_2^{i*} \leq M_3^{i*}, \quad \|M_3^{i*}\| < \mu$$

$$M_1^{i-1} \leq M_2^{i-1} \leq M_3^{i*}, g_{i-1}(a_{a-1}) \in M_2^{i*}$$

$$f_i^* \, a \leq_{\mathbf{K}} - \text{ embedding } of \, M_2^{i,j*} \text{ into } M, f_i^* \restriction M_i^{i-1} = \text{ id }.$$

Let N_2^i, h_i be such that $N_2^{i-1} < N_2^i, h_i$ an isomorphism from N_2^i on $M_3^{i,*}$ extending g_{i-1}. Let $N_1^i = h_i^{-1}(M_2^{i,*}), f_i = f_i^* \circ (h_i \restriction N_1^i)$].

We have carry the induction. Now f_κ is a $\leq_{\mathbf{K}}$ -embedding of N_1^κ into M over N_1, but $|N_2| = \{a_i : \lambda < \kappa\} \subseteq N_1^\kappa$, so $f_\kappa \restriction N_2 : N_2 \to M$ is as required.

3.11 **Fact:** Assume $LSP(<\lambda)$. If $M \leq \mathbf{C}$ is $(\mathcal{D}_{\mathbf{K}}, \lambda^+)$ - homogeneous, $A \subseteq M, |A| \leq \lambda$, $h \in AUT(\mathbf{C}))$ *then* for some $g \in AUT(\mathbf{C}), g \restriction M \in AUT(M), g \restriction A = h \restriction A$

Proof: We can find first $N_0 \leq M$, $A \subseteq N_0$, $|||N_0||| \leq \lambda$ and then N_1 closed under h and $N_0 \leq N_1 \leq \mathbf{C}$, As M is $(\mathcal{D}_{\mathbf{K}}, \lambda^+)$ -homogeneous there is an automorphism g_0 of $\mathbf{C}, g_0 \restriction N_0 = \text{id}, g_0(N_1) < M$. Now $g_1 = g_0 \circ h \circ g_0^{-1}$ is clearly an automorphism of $g_0(N_1)$, As $g(N_1) \leq M$, $|||g(N_1)||| = |||N_1||| < \lambda$, [and M is strongly $(\mathcal{D}_{\mathbf{K}}, \lambda)$ -homogeneous) g_1 can be extended to an automorphism g_2 of M, which can be extended to an automorphism g of $\mathbf{C}$. Now g is as required.

3.12 Definition: $K_{\mu,\kappa}^{us} = \{M :$ there is a $(<\kappa)$-directed I and $(\mathcal{D}_{\mathbf{K}}, \mu)$-homogeneous models $M_t \in K$ for $t \in I$ such that $M = \bigcup_{t \in I} M_t\}$

If $\kappa = \aleph_0$, we omit it.

3.12A Remark: E.g. in 2.1 above, $K_{|T|}^{us}$ is included in the class of models of T.

III Constructions of many non isomorphic models

§0 Introduction

For a reasonable structure/non structure theory, we need ways to build many and/or complicated structures. Though they were developed mainly for proving $I(\lambda,K) = 2^{\lambda}$ (see Definition 1.2 and [Sh] Chapter VII, VIII) they may be used to build rigid or indecomposable or $L_{\infty,\lambda}$-equivalent and isomorphic, non pairwise non embeddable models (see 1.3).

We have tried several times to separate the "set theoretic "parts from the" specific algebraic construction". This was done in [Sh 136] (for [Sh A1] (see §2 here for explanation and presentation (though not complete)); in the various black boxes - see here §4,5,6 [Sh 172] [Sh 227] [Sh 229], Gobel and Shelah [GbSh 190], [GbSh 219], Eklof Mekler [EkMk D16], Grossberg and Shelah [GrSh 312] (less related, but with similar applications are the papers on "Models with second order properties", [Sh 72], [Sh 73], [Sh 82], [Sh 107], [Sh 162], [Sh 128] (construction from $\diamondsuit_{\aleph_1}$), Shelah and Stanley [Sh St 112], [Sh St 167].)

We want to explain the theory and how to apply it but our main aim in this chapter is to proved abstract non structure theorems so that in this work, when we want to prove that a class K which happens to be in the "non structure" side, have many complicated models. For this we prove some non structure theorem with various degrees of abstractness. Some are just abstract versions of theorems from [Sh, VIII] with essentially the same proof, while others give more information even for cases dealt with before, e.g.

0.1 Theorem: If $\psi \in L_{\chi^+,\omega}$, $\varphi(\bar{x},\bar{y}) \in L_{\chi^+,\omega}$, $\ell g(\bar{x}) = \ell g(\bar{y}) = \sigma$ and ψ has the $\varphi(\bar{x},\bar{y})$-order property *then* $I(\lambda,\psi) = 2^{\lambda}$ provided that e.g. $\lambda \geq \chi + \aleph_1$, $\sigma < \aleph_0$ or $\lambda \geq \chi + \aleph_1$, $\lambda^{\sigma} = \lambda$ or $\lambda > \chi + \sigma^{+}$.

Proof: When $\lambda \geq \chi + \aleph_1, \sigma < \aleph_0$, by Theorem 3.9.

Generally our construction of many models in $K_{\lambda}(= \{M \in K : \|M\| = \lambda\})$ goes as follows. We have a class K^1 of "index models" (this just indicates their role; supposedly they are well understood). By the "non structure property of K", for some formulas φ_ℓ, for every $I \in K^1_\lambda$ there is $M_I \in K$ and $\bar{a}_t \in M_I$ for $t \in I$, which satisfies (in M_I) some instances of $\pm\varphi_\ell$.

We may demand on M_I:

(0) nothing more.

(1) $\langle \bar{a}_t : t \in I \rangle$ behave like a skeleton (see 3.1(1)) or even

(2) M_I is built from I in a simple way (Δ-represented - see Definition 2.2(c)).

Now even for (0) we can have meaningful theorems (see 3.9 and 4.2).

We would like to stress that the formulas φ_ℓ need not be first order L, they just have to have the right vocabulary (but in results on "no M_i embeddable in M_j" this usually means embedding preserving $\pm\,\varphi_\ell$ (but see 2.5).

Another point is that though it would be nice to prove $I \not\cong J \Rightarrow M_I \not\cong M_J$; this does not seem realistic. What we do is to construct a family $\{I_\alpha : \alpha < 2^\lambda\} \subseteq K^1_\lambda$ such that for $\alpha \neq \beta$, I_α is not isomorphic to (or not embeddable into) I_β in a strong sense (see 2.3, 3). We are thus led to the task of constructing such I_α's, which unfortunately split to cases.

A point central to [Sh 136] but incidental here, is the construction of a model which is e.g. rigid or have few endomorphisms etc. Using the methods of §2 see [Sh 136 §3], using §4–5 (black boxes) see e.g. [Sh 220].

The methods here can be combined with [Sh 220] or [Sh 188] to get non isomorphic $L_{\infty,\lambda}$-equivalent models of cardinality λ.

In the next few paragraphs we quickly survey the results of this chapter. In this survey we omit some parameters at various defined notions. These parameters are essential for an accurate statement of the theorems. We suppress them here to emphasize what seems to be the most essential points.

In Section III.2 we discuss a method of "representability". We introduce two strongly contradictory notions, the Δ-representability of a structure M in the "polynomial algebra" of an index model (Definition 2.2) and the $\varphi(\bar{x},\bar{y})$ un-embeddability of one index model in another. Now to show a class K has many models one first shows that for some formula φ an index class K_1 has many pairwise φ- unembeddable structures, then that for each $I \in K$, there is a model M_I which is Δ-representable in the free algebra on I, and finally that if $M_I \cong M_J$ and M_J is represented in the free algebras on J then I is φ-embeddable in J.

In Section III.3 we extend and simplify the argument showing that an unstable first order theory T has 2^λ models of power λ if $\lambda \geq |T| + \aleph_1$. Rather than constructing Ehrenfeucht-Mostowski models we consider a weaker notion - that a linear order J indexes a weak (κ,φ)-skeleton like sequence in a model M. In this section K^1 is the class of linear orders. The formula $\varphi(\bar{x},\bar{y})$ need not be first order and after 3.10 may have infinitely many arguments. Most significantly we make *no* requirement on the means of definition of the class K of models (e.g. first order, $L_{\infty,\infty}$ etc.). We require only that for each linear order J there be an $M_J \in K$ and a sequence $\langle \bar{a}_s : s \in J\rangle$ which is weakly (κ,φ)-skeleton like in M_J.

If you get lost in §3, you can jump to §4.

In the rest we deal with black box, and generalizations of "an unsuperstable T has many models".

III 1.1

§1 Models from Indiscernibles

Our aim in [Sh Ch.VIII] was to prove: (in ZFC!)

1.1 Theorem. If T is a complete first order theory, unsuperstable

and $\lambda \geq |T| + \lambda_1$, then $I(\lambda,T) = 2^\lambda$ where

1.2 Definition: For a theory T

$I(\lambda,T)$ = number of models of T of power λ, up to isomorphism.

For a class K of models

$I(\lambda,K)$ = number of model in K of power λ, up to isomorphism.

$IE_\Delta(\lambda,K) = \sup\{\mu$: there are $M_i \in K_\lambda$, for $i < \mu$, such that for $i \neq j$ there is no Δ-embedding of M_i to $M_j\}$.

However, we feel (see also [Sh 31] [Sh 44], [Sh 51], [Sh 54], [Sh 136]):

1.3 Thesis

(A) The methods are enough to build many complicated, very different, models of suitable powers, for many classes, not necessarily elementary.

(B) Moreover in reasonable situations we can make them rigid, indecomposable etc., according to circumstances.

Essentially this (A) + (B) was an advice to use a device. If you need such a construction, try to imitate one of the proofs (note that the theorem was proved by partition to cases, with various proofs.) Generally the hint was not taken. As an illustration we have done various such works.

1.4(1) Examples: (A) In every $\lambda > \aleph_0$ there is a rigid dense linear order see [Ba 76,2];

(B) in every $\lambda > \aleph_0$ there is a rigid Boolean Algebra. (See [Sh 51]).

(C) In every $\lambda > \aleph_0$ there are 2^λ non-isomorphic reduced separable abelian p-groups (see [Sh 44]. §1 and p.244 $^{9-13}$).

(D) In every $\lambda > \aleph_0$ there are 2^λ u.l.f. (universal locally finite) groups up to isomorphism. (see Macintyre and Shelah [MaSh 55]).

(E) theorems on representation of rings as endomorphism rings of abelian groups (see [Sh 172], [Sh 227], Gobel and Shelah, [GbSh 224] [GbSh 219]).

(F) There are Boolean algebras rigid and complete, having few endomorphism (see various results [Sh 136], [Sh 229]).

(G) There are for most λ's, 2^λ, u.l.f. with non-inner automorphism (see Grossberg and Shelah [Gr Sh 312].)

1.4(2) Discussion: Note that M is rigid if and only if $(\forall a \neq b \in M)$ $[(M,a) \not\cong (M,b)]$. Clearly the theorems of [Sh, VIII] does not apply directly. However if we have freedom enough in constructing M, knowing constructions of many non- isomorphic model should help in constructing rigid models. Note that for general first order theory T, maybe e.g. there are definable automorphisms (or more subtle problems). See the series "Models with Second order Properties": I [Sh 72], II [Sh 73], III [Sh 82], IV [Sh 107], V [Sh 162] for different constructions. We construct there (assuming instances of GCH) models with only definable automorphism, assuming strengthening of unstability. This kind of assumption is natural, giving us enough freedom in the construction. In [Sh 136] we tried to separate the combinatorics and applications of [Sh, VIII], and advance our combinatorial knowledge. (The applications we had in mind there were to Boolean algebras).

1.5 Definition: 1) $\langle \bar{a}_t : t \in I \rangle$ is Δ-indiscernible (in M) if

(a) I is an index model (usually linear order or tree); i.e. it can be any model but its role will be as an index set.

(b) The Δ-type in M of $\bar{a}_{t_1} \wedge \cdots \wedge \bar{a}_{t_n}$ (for any $n < \omega$) depends only on the quantifier free type of $\langle t_1, \ldots, t_n \rangle$ in I.

(2) For a logic $\mathcal{L}$, " $\mathcal{L}$-indiscernible" will mean Δ-indiscernible for the set of $\mathcal{L}$-formulas in the vocabulary of M.

3) Remember that if $\bar{t} = \langle t_i : i < \alpha \rangle$ then $\bar{a}_{\bar{t}} = \bar{a}_{t_0} \wedge \bar{a}_{t_1} \wedge \cdots$

Many of the following definitions are appropriate for counting the number of models in a pseudo elementary class. Thus, we work with a pair of vocabularies, $\tau \subseteq \tau_1$. Often τ_1 will contain Skolem functions for a theory T which is $\subseteq \mathcal{L}(\tau)$.

In this section all predicates and function symbols have finite number of places, (and similarly $\varphi(\bar{x})$ means $\ell g(\bar{x}) < \omega$)

1.6 Definition: 1) $M^1 = EM^1(I, \Phi)$ if for some vocabulary $\tau = \tau^\Phi$ or $L_1 = L_1^\Phi$, and $\bar{a}_t (t \in I)$:

(i) M^1 is generated by $\{\bar{a}_t : t \in I\}$.

(ii) $\langle \bar{a}_t : t \in I \rangle$ is quantifier free indiscernible in M^1

(iii) Φ is a function, taking (for $n < \omega$) the quantifier free type of $\bar{t} = \langle t_1, \ldots, t_n \rangle$ in I to the quantifier free type of $\bar{a}_{\bar{t}} = \bar{a}_{t_1} \wedge \cdots \wedge \bar{a}_{t_n}$ in M^1.

2) A function Φ is proper for I if (iii) of 1.6(1) holds, proper for K if Φ is proper for every $I \in K$, and lastly it is proper for (K_1, K_2) if it is proper for K_1 and $EM(I, \Phi) \in K_2$

for $I \in K_1$.

3) For a logic $\mathcal{L}$, or even a set $\mathcal{L}$ of formulas in the vocabulary of M^1,Φ is almost $\mathcal{L}$-nice (for K) if:

(*) For every $I \in K$, $\langle \bar{a}_t : t \in I \rangle$ is $\mathcal{L}$-indiscernible in $EM^1(I,\Phi)$.

4) In 3), Φ is $\mathcal{L}$-nice if it is almost $\mathcal{L}$-nice and

(**) For $J \subseteq I$

$$EM^1(J,\Phi) <_{\mathcal{L}} EM^1(I,\Phi)$$

In the book [ShA1], always $L_{\omega,\omega}(\tau^{\Phi})$-nice Φ were used.

1.7 Notation: 1) $EM_{\tau}(I,\Phi) = (EM^1(I,\Phi) \restriction \tau$ (where $\tau \subseteq \tau^{\Phi}$) (we omit τ when clear from context).

2) We identify $I \subseteq {}^{\kappa \geq}\lambda$ which is closed under initial segments, with the model

$$(I, P_{\alpha}, \wedge, <_{\ell x}, \triangleleft)_{\alpha \leq \kappa}$$

where

$$P_{\alpha} = I \cap {}^{\alpha}\lambda,$$

$\rho = \eta \wedge \nu$ if $\rho = \eta \restriction \alpha$ for the maximal α such that $\eta \restriction \alpha = \nu \restriction \alpha$

$\trianglelefteq =$ being initial segment of, (including equality)

$<_{\ell x} =$ lexicographic order

3) Similarly $I \subseteq {}^{\kappa \geq}J$ for any linear order J ($\leq_{\ell x}$ is still well defined.)

4) K^{κ}_{tr} is the class of such models i.e. models isomorphic to I i.e. to $(I, P_{\alpha}, \wedge, <_{\ell x}, \triangleleft)_{\alpha \leq \kappa}$ for (tr stand for tree) some $I \subseteq {}^{\kappa \geq}J$, J a linear order.

5) K_{or} is the class of linear order.

1.7A Remark: The main case is $\kappa = \aleph_0$. We need such trees for $\kappa > \aleph_0$, e.g. if we want to build many κ-saturated models of T, $\kappa(T) > \kappa$, κ regular. If $\kappa(T) \leq \kappa$ there may be few κ-saturated models of T. In [Sh, Ch. VIII, VIIII] we have proved:

1.8 Lemma: If T is unsuperstable, *then* there are first order $\varphi_n(\bar{x},\bar{y}_n) \in L(T)$ and Φ proper for every $I \subseteq {}^{\omega\geq}\lambda$ such that:

$$\eta \in {}^{\omega}\lambda,\ \nu \in {}^{n}\lambda \Rightarrow$$

$$EM^1(I,\Phi) \models \varphi_n[\bar{a}_\eta,\bar{a}_\nu] \Leftrightarrow \eta \restriction n = \nu$$

(also $EM^1(I,\Phi) \models T$) and Φ is $\tau^{\Phi}_{\omega,\omega}$-nice, $|\tau^\Phi| = |T| + \aleph_0$ (note that for η_1,η_2 of the same length, $\eta_1 \neq \eta_2 \Rightarrow \bar{a}_{\eta_1} \neq \bar{a}_{\eta_2}$). In [ShA1,, VIII §2] we actually proved:

1.9 Theorem: 1) If $\lambda > |\tau^\Phi|$ is regular, Φ, τ^Φ,$\langle\varphi_n : n < \omega\rangle$ as in lemma 1.8 (Φ almost $L_{\omega,\omega}$-nice) then: we can find $I_\alpha \subseteq {}^{\omega\geq}\lambda$ (for $\alpha < 2^\lambda$), $|I_\alpha| = \lambda$ such that for $\alpha \neq \beta$ there is no one to one function from $EM^1(I_\alpha,\Phi)$ onto $EM^1(I_\beta,\Phi)$ preserving the $\pm\varphi_n$ for $n < \omega$.

2) The φ_n's do not need to be first order, just their vocabularies should be $\subseteq\tau^\Phi$. But instead "Φ is almost $L^{\Phi}_{\omega,\omega}(\tau)$-nice" we need "$\Phi$ is almost $\{\varphi_n(\ldots,\bar{\sigma}_\ell(\bar{x}_\ell),\ldots)_{\ell<\ell(n)} : n < \omega$, σ_ℓ terms of $\tau^\Phi\}$-nice" and we should still demand

(*) the $\bar{a}_\eta$ are finite.

3) So if as in Lemma 1.8, $\varphi_n \in \mathcal{L}(\tau)$ then $\{M_\alpha \restriction \tau : \alpha < 2^\lambda\}$ are 2^λ non- isomorphic models of T of power λ.

Proof: This is proved in [Sh] section 2 of Ch. VIII (though it is not formally claimed there is no need fo the proofs).

1.9A Remark: In [Sh] VIII §2 existence of many models in λ is proved for some

$\lambda = |\tau^\Phi|$ and there "T_1, T first order" is used.

1.10 Definition: Fix a class K (of index models) and logic $\mathcal{L}$.

1) An index model $I \in K$ is called (μ,λ)-large for $\mathcal{L}$ if:

(a) Every *qf*(in $\tau(K)$) type p which is realized in some $J \in K$ is realized in I.

(b) *for every* vocabulary τ^1 of cardinality $\leq \mu$ and τ^1-model M^1 and $\bar{a}_t \in {}^{\omega>}|M^1|$ for $t \in I$ *there is* Φ, proper for K, with $|\tau^\Phi| \leq \lambda$ such that ($\tau^1 \subseteq \tau^\Phi$ and):

(*) *for every* $\tau(K)$–*qf* type p, $I^1 \in K$ and $s_1, \ldots, s_n \in I^1$ such that $\langle s_1, \ldots, s_n \rangle$ realize p in I^1, *for some* $t_1, \ldots, t_n \in I$, $\langle t_1, \ldots, t_n \rangle$ realize p in I and

(**) for every formula $\varphi = \varphi(x_1, \ldots, x_m) \in \mathcal{L}(L^\Phi)$ and τ^Φ-terms $\sigma_\ell(\bar{y}_1, \ldots, \bar{y}_n)$ for $\ell = 1,n$;

$$M^1 \models \varphi[\sigma_1(\bar{a}_{t_1}, \ldots, \bar{a}_{t_n}), \sigma_2(\bar{a}_{t_1}, \ldots, \bar{a}_{t_n}), \ldots, \sigma_m(\bar{a}_{t_1}, \ldots, \bar{a}_{t_n})]$$

implies

$$EM^1(I^1,\Phi) \models \varphi[\sigma_1(\bar{a}_{s_1}, \ldots, \bar{a}_{s_n}), \sigma_2(\bar{a}_{s_1}, \ldots, \bar{a}_{s_n}), \ldots, \sigma_m(\bar{a}_{s_1}, \ldots, \bar{a}_{s_n})]$$

2) The class K of index models is called (μ,λ)-Ramsey for $\mathcal{L}$ if some $I \in K$ is (μ,λ)-large for $\mathcal{L}$.

3) If in 1.10(2) $\mathcal{L}$ is first order logic, we omit it.

4) For $f : Card \to Card$, K is f-Ramsey if it is $(\mu, f(\mu))$ -Ramsey for $\mathcal{L}$ for every μ. We say K if Ramsey for $\mathcal{L}$ if it is (μ,μ)-Ramsey for $\mathcal{L}$ for every μ.

5) We add to Ramsey "(almost) $\mathcal{L}$-nice" if we can get such Φ.

6) We say K is *-Ramsey if it is f-Ramsey for some $f : Card \to Card$.

1.11 Theorem: 1) For $L_{\omega,\omega}$, the class of linear orders is Ramsey.
[**Proof:** This follows from the Eherefeucht-Mostowski proof that E.M. models exist].

2) For $L_{\omega_1,\omega}$ the class of linear orders is *-Ramsey.

[**Proof:** essentially repeating the proof of Morely's omitting type theorem.]

3) For any fragment of $L_{\lambda^+,\omega}$ or $\Delta(L_{\lambda^+,\omega})$ the class of linear orders is f-Ramsey by $f(\mu) = \beth_{(2^\mu)^+}$.

[**Proof:** Like 1.1(2); see [Sh 16] and more in [GrSh 222] [GrSh 251]].

By Grossberg Shelah [GSh 238] (improving [Sh VII], where compactness of the logic $\mathcal{L}$ was used, but no large cardinals) (K_{tr}^{ω} was defined above.):

1.12 Theorem: K_{tr}^{ω} has the *-Ramsey property if e.g. there are arbitrarily large measurable cardinals.

We shall not repeat the proof.

1.13 Lemma: Suppose K_1, K_2, K_3 are classes of models, Φ is proper for (K_1, K_2), Ψ proper for (K_2, K_3) *then* for a unique Θ

a) Θ is proper for (K_1, K_3)

b) for $I \in K_1$

$$EM^1(I,\Theta) = EM^1(EM^1(I,\Phi),\Psi))$$

We write this as $\Theta = \Psi \circ \Phi$.

Proof: Straight forward.

1.14 Lemma: 1) Suppose K is a class of index models, $\tau = \tau(K)$ and

(*) there is Ψ proper for (K, K), such that for $I \in K$, $EM_\tau(I,\Psi) \in K$ and $J = EM_\tau(I,\Psi)$ is $(\aleph_0, qf)$-homogeneous over I, i.e. *if* $\bar{t} = \langle t_1, \ldots, t_n \rangle$, $\bar{s} = \langle s_1, \ldots, s_n \rangle$ realize the same qf-type in I *then* some automorphism of J take $\bar{a}_{\bar{t}}$ to $\bar{a}_{\bar{s}}$.

We conclude that:

If K is (μ,λ)-Ramsey for $\mathcal{L}$ *then* K is almost $\mathcal{L}$-nice (μ,λ)-Ramsey for $\mathcal{L}$.

2) E.g. for $\mathcal{L} \subseteq L_{\omega_1,\omega}$ we get in (1) even $\mathcal{L}$-nice.

3) The assumption (*) of (1) holds for K_{or}, K_{tr}^{ω}, K_{tr}^{κ} (as well as the others from [Sh 136].)

1.15 Conclusion: Suppose K is (μ,λ)-Ramsey for $\mathcal{L}$, T is an $\mathcal{L}$-theory (in the vocabulary $\tau(T)$), $|\tau(T)| \leq \mu$, $\varphi_\ell(\bar{R}_\ell, \bar{x},\bar{y}) \in \mathcal{L}(\tau(T) \cup \{\bar{R}_\ell\})$ (and $\bar{R}_\ell$ disjoint to $\tau(T)$ and to $\bar{R}_{3-\ell}$) and $T \cup \{\varphi_1(\bar{R}_1, \bar{x},\bar{y}), \varphi_2(\bar{R}_\ell, \bar{x},\bar{y})\}$ has no model. Suppose further that for $I \in K_{or}$ there is a model M_I of T, and $\bar{a}_t \in {}^{\omega>}M$ for $t \in I$ such that:

$$t < s \Rightarrow M \models (\exists \bar{R}_1)\varphi_1(\bar{R}_1,\bar{a}_t,\bar{a}_s)$$

$$s < t \Rightarrow M \models (\exists \bar{R}_2)\varphi_2(\bar{R}_2,\bar{a}_s,\bar{a}_t)$$

then for $\lambda \geq \mu + \aleph_1$, $I(\lambda,T) = 2^\lambda$.

Proof: By previous theorem and 3.9.

§2 Models represented in free algebras and applications

2.1 Discussion: 1) We sometimes need τ^Φ with function symbols with infinitely many places and deal with logics $\mathcal{L}$ with formulas with infinitely many variables.

2.1A Example: We want to build complete Boolean algebras with no non-trivial 1-1 endomorphisms. How do we get completeness? We build a Boolean algebra, B_0 and take its completion. Even when B_0 satisfies the c.c.c. we need the term $\bigcup_{n<\omega} x_n$ to represent elements of the Boolean algebra from the "generators" $\{\bar{a}_t : t \in I\}$.

2) We also sometimes want to rely on a well ordered construction i.e. on the universe of $EM^1(I,\Phi)$ there is a well ordering which is involved in the definition of indiscernibility (see 2.2). This means that we have in addition an arbitrary well-order relation. E.g. we want to build many non-isomorphism $\aleph_1$-saturated' models, we have a family $\{\bar{a}_\alpha : \alpha < \lambda\}$ of sequences of length ω with $EM_{\tau(T)}(I,\Phi) \models \varphi[\bar{a}_s,\bar{a}_t] \Leftrightarrow s < t$ (< a relevant order) *but* we need to make them $\aleph_1$-saturated. Ultrapowers will probably destroy the order. The natural thing is to make M_I $\aleph_1$-primary over $EM_{\tau(T)}(I,\Phi)$. So not only are the $\bar{a}_t$ infinite, the construction involves infinitary functions but the quite arbitrary order of the constructions may play a role.

With some work we can eliminate the last for this example (using symmetry) but there is no guarantee generally and certainly it is not convenient. Moreover,

3) It is better to delete the requirement that the universe of the model is so well defined.

This motivates the following definition.

2.2 Definition: (a) $\tau(\mu,\kappa)$ is the vocabulary with function symbols

$\{F_{i,j} : i < \mu, j < \kappa\}$ where $F_{i,j}$ is a j-place function symbol and κ is always regular.

b) $\mathcal{M}_{\mu,\kappa}(I)$ is the free τ-algebra generated by I for $\tau = \tau(\mu,\kappa)$.

We use the following notation in the remainder of this definition. Let $f : M \to \mathcal{M}_{\mu,\kappa}(I)$. For $\bar{a} \in {}^\alpha M$ and for $i < \alpha$, $f(a_i) = \sigma_i(\bar{t}_i)$ with $\bar{t}_i$ is $< \kappa$ sequence from I and σ_i a term for $\tau(\mu,\kappa)$.

c) M is Δ-*represented* in $\mathcal{M}_{\mu,\kappa}(I)$ if there is a function $f : M \to \mathcal{M}_{\mu,\kappa}(I)$ such that the

Δ-type of $\bar{a} \in M(tp_\Delta(\bar{a},\emptyset,M))$ can be calculated from the sequence of terms $\langle \sigma_i : i < \alpha \rangle$ and $tp_{qf}(\langle \bar{t}_i : i < \alpha \rangle, \emptyset, I)$.

d) M is weakly Δ-represented in $\mathcal{M}_{\mu,\kappa}(I)$ if for some function $f: M \to \mathcal{M}_{\mu,\kappa}(I)$, there is a well-ordering of the image of f such that for $\bar{a} \in {}^\alpha M$ the Δ-type of $\bar{a}$ can be computed from the information described in c) and the ordering <-imposes on the subterms of the terms $\langle \sigma_i(\bar{t}_i) : i < \alpha \rangle$ in the image of f.

We introduce weak representability to deal with the dependence on the order of a construction, (cf. 2.1 (2)).

e) For $j = 1,2$ if $\bar{a}^j = \langle \sigma_i^j(\bar{t}_i^j) : i < \alpha \rangle$, $\sigma_i^1 = \sigma_i^2$ and $tp_{qf}(\langle \bar{t}_i^1 : i < \alpha \rangle, \emptyset, I) = tp_{qf}(\langle \bar{t}_i^2 : i < \alpha \rangle, \emptyset, I)$ we write $\bar{a}^1 \sim \bar{a}^2$ *mod*($\mathcal{M}_{\mu,\kappa}(I)$. For the case of weak representability we write $\bar{a}^1 \sim \bar{a}^2$ *mod*($\mathcal{M}_{\mu,\kappa}(I), <)$ if in addition the mapping $\{\langle \sigma(t_i^1), \sigma(t_i^2) \rangle : i < \alpha, \sigma$ a subterm of $\sigma_2^1 = \sigma_i^2\}$ is a <-isomorphism (and both sides are linear orders). We write $\bar{a}^1 \sim_A \bar{a}^2$ *mod* $\cdots$ if $\bar{a}^1 \wedge \bar{b} \sim \bar{a}^2 \wedge \bar{b}$ *mod* $\cdots$ when $\bar{b} \in {}^{\kappa >}A$, $A \subseteq M$. (This latter is especially important when we work over a set of parameters. We might, for instance, insist that t_i^1 and t_j^1 realize the same Dedekind cut in $I_0 \subseteq I$.)

(So M is Δ-represented in $\mathcal{M}_{\mu,\kappa}(I)$ just if $f(\bar{a}^1)$ similar to $f(\bar{a}^2)$ *mod* $\mathcal{M}_{\mu,\kappa}$ *implies* $\bar{a}^1$ and $\bar{a}^2$ realize the same Δ-type in M.).

f) We say the [weak] representation is *full* if: $c_1 \sim c_2$ *mod* $\mathcal{M}_{\kappa,\mu}(I))$ implies $[c_1 \in Rang(f) \Leftrightarrow c_2 \in Rang(f).]$

g) If $\Delta = qf$, it is omitted.

h) For $f : M \to \mathcal{M}_{\mu,\kappa}$, $\bar{a} \sim \bar{b}$ *mod*$(f, \mathcal{M}_{\mu,\kappa})$ means $f(\bar{a}) \sim f(\bar{b})$ *mod* $\mathcal{M}_{\mu,\kappa}$. Similarly, $\bar{a} \sim \bar{b}$ *mod*$(f, \mathcal{M}_{\mu,\kappa}, <)$ means $f(\bar{a}) \sim f(\bar{b})$ mod ($\mathcal{M}_{\mu,\kappa}, <)$.

Now we define a very strong negation (when φ is "right") to even weak representability.

2.3 Definition: I is $\varphi(\bar{x},\bar{y})$-unembeddable for $\tau(\mu,\kappa)$ into J if for every

$f: I \to \mathcal{M}_{\mu,\kappa}(J)$ and well ordering $<$ (of $f(I)$) there are sequences $\bar{x},\bar{y}$ of members of I, $I \models \varphi[\bar{x},\bar{y}]$ such that $\bar{x},\bar{y}$ have "similar" (2.2(c)) images in $\mathcal{M}_{\mu,\kappa}(J)$.

2.3A Remark: This definition is used in proving that the model constructed from I is not isomorphic to (or not embeddable in) the model constructed from J.

* * *

2.4 Discussion: The following example illustrates the application of this method. We first fix K^{ω}_{tr} as the class of index models and fix a formula φ_{tr} (see 2.4) such that for many pairs $I,J \in K^{\omega}_{tr}$, I is $\varphi_{tr}(\bar{x},\bar{y})$-unembeddable in J. In 2.5A we show that for each $I \in K^{\omega}_{tr}$ there is a reduced abelian p-group G_I which is representable in $\mathcal{M}_{\omega,\omega}(I)$. In 2.5B we show that [I φ_{tr}- unembeddable in J implies $G_I \equiv G_J$]; thus the number of reduced separable abelian of power λ is at least a great as the number of trees in K^{ω}_{tr} with power λ which are pairwise φ_{tr}- unembeddable. We showed in [Sh 136] that this number is 2^{λ} (for regular λ and many singulars). (but by 1.9 we get 2^{λ} pairwise non isomorphic such groups in λ, using G_I as below). We may want to strengthen "$G_I \not\equiv G_J$" to "G_I not embeddable into G_J". This depends on the exact notion of embeddability we use

2.4 Example: Class of K^{ω}_{tr}, $I \in K^{\omega}_{tr}$

$$\varphi_{tr}(x_0,x_1;y_0,y_1) \stackrel{def}{=} [x_0 = y_0] \wedge P_{\omega}(x_0) \wedge \bigvee_{n<\omega} [P_n(x_1) \wedge P_n(y_1) \wedge P_{n-1}(x_1 \wedge y_1] \wedge$$

$$\wedge [x_1 \not< x_0 \wedge y_1 \not< y_0]$$

2.4A Definition: A separable reduced abelian p-group G is a group G which satisfies (we use additive notation):

(1) G is commutative (this is "abelian")

(2) for every $x \in G$ for some n x has order p^n (i.e. $p^n x$ is the zero);

(3) G has no divisible non trivial subgroup (= reduced)

(4) every $x \in G$ belongs to some 1-generated subgroup which is a direct summand (= separable)

Any such group is a norm space:

$$|||x||| = \inf\{2^{-n} : (\exists y \in G)p^n y = x\}$$

2.5 Subexample: separable reduced abelian p-groups.
For a tree I, G_I is generated (as an abelian group) by

$$\{x_\eta : \eta \in \bigcup_{n<\omega} P^I_n\} \cup \{y^n_\eta : \eta \in P^I_\omega\}$$

freely except the relations:

$p^n x_\eta = 0$ for $\eta \in P^I_n$; and $py^{n+1}_\eta - y^n_\eta = x_{\eta \restriction n}$ and $p^n y^n_\eta = 0$ for $\eta \in P^I_\omega$, and we have essentially say $y^n_\eta = \sum\{p^{\ell-n} x_{\nu_\ell} : n \le \ell < \omega,\ \nu_\ell \in P^I_\ell$ and $\nu_\ell \lessdot \eta\}$ (infinitary sum may be well defined as G_I is a norm space).

It is easy to see (by addition relation to divisibility) that

2.5 A Fact: (*) G_I is represented into $\mathcal{M}_{\omega,\omega}(I)$.

We shall prove below:

2.5B Fact: *If I is φ_{tr}-unembeddable into J then $G_I \not\cong G_J$.*

Proof: Let $g : G_I \cong G_J \underset{h}{\to} \mathcal{M}_{\omega,\omega}(J)$ where h witnesses that G_J is representable in $\mathcal{M}_{\omega,\omega}(J)$). Let $f : I \to G_I$ be:

$$f(\eta) = \begin{cases} \sum_{1 \le \ell < \ell g(\eta)} p^{\ell-1} x_{\eta \restriction \ell} & \text{if } \eta \in \bigcup_{n<\omega} P^I_n \\ y^1_\eta & \text{if } \eta \in P^I_\omega \end{cases}$$

So $(h \circ g \circ f) : I \to \mathcal{M}_{\omega,\omega}(J)$. Now we use the fact that I is φ_{tr}-unembeddable into J. So suppose

$$I \models \varphi_{tr}[\eta_0, \nu_0, \eta_1, \nu_1] \quad \text{and} \quad h \circ g \circ f(\eta_0, \nu_0) \sim h \circ g \circ f(\eta_1, \nu_1).$$

Invoking the definition of φ_{tr}:

$\eta \stackrel{def}{=} \eta_0 = \eta_1 \in P^I_\omega$ and for some n $\nu_1 \lessdot \eta_1$, $\nu_1 \in P^I_n$, $\nu_0 \in P^I_n$ and $\nu_0 \neq \nu_1$. For $i = 0,1$ let $z_{\nu_i} = \Sigma\{p^{\ell-1} x_\nu : \nu \lessdot \nu_i, \nu \in P^I_\ell, 1 \le \ell \le n\}$.

Now $G_I \models$ "p^n divides $(y^1_\eta - z_{\nu_1})$".

Hence as g is an isomorphism,

$G_J \models$ "p^n divides $(g(y^1_\eta) - g(z_{\nu_0}))$"

i.e.

$G_J \models$ "p^n divides $(g \circ f(\eta) - g \circ f(\nu_0))$".

Similarly $G_J \models$ "p^n does not divide $(g \circ f(\eta) - p^{n-1} g \circ f(\nu_1)$" but $h \circ g \circ f(\langle \eta, \nu_0 \rangle) \sim h \circ g \circ f(\langle \eta, \nu_1 \rangle)$ *mod* $\mathcal{M}_{\omega,\omega}(J)$ the contradiction, proving 2.5B.

2.6 Discussion continued: But really G_J is $\mathcal{L}$- represented in $\mathcal{M}_{\omega,\omega}(J)$ if for $\mathcal{L}$ we take the set of formulas $\{$"p^{n+1} divides $y - \sum_{\ell=1}^{n} p^\ell y_\ell$" $: n < \omega\}$ (Of course, we do not use the full power of $\mathcal{L}$-representation, only some specific instances). So the above proves that G_I is not $\mathcal{L}$- embeddable into G_J.

More precisely still, we have shown above that there is no pure embedding (pure $= \Sigma^0_1$) of G_I into G_J. We can improve this to show there is no embedding in the algebraic sense. (see [Sh 136 pg 106_{10} - 107^7] and below). Unfortunately for the coherence of the theory the proof does not imply this directly. Rather we need (for $\mu = \aleph_0$)

2.6A Definition: 1) $Pr_\mu(I, J)$ means: (letting χ be large enough) for every $x \in H(\chi)$ there is M, $x \in M$ such that:

$$M \prec (H(\chi), \in),\ \mu + 1 \subseteq M \ \text{and}\ I, J \in M,$$

and for every $\eta \in P^J_\omega$, $[\{\eta \restriction n : n < \omega\} \subseteq M \Rightarrow \eta \in I]$.

but for some $\eta \in P^I_\omega$, $\{\eta \restriction n : n < \omega\} \subseteq M$ but $\eta \notin M$

2) $Pr_{\mu,\kappa}(I,J)$ is defined similarly, replacing K^ω_{tr}, with K^κ_{tr}, κ.

3) $Pr^+_{\mu,\kappa}(I,J)$ is defined similarly adding $M = \bigcup_{i<\delta} M_i$ where δ is a limit ordinal, $M_i \prec M$, $\langle M_i : i < \delta\rangle$ is increasing continuous and $\langle M_j : j \leq i\rangle \in M_{i+1}$, and for some $\eta \in P^I_\omega$ $\{\eta \restriction i : i < \kappa\}$ included in M but in no M_i, $i < \delta$.

4) $Pr^*_{\mu,\kappa}(I,J)$ is defined like (2) with $\delta = \kappa$.

2.6B Theorem: Suppose $\lambda > \mu$, and

(i) λ is regular, or

(ii) $\lambda = \lambda^{\aleph_0}$ strong limit or

(iii) $(\exists\chi)[\mu \leq \chi \wedge (\chi^{\aleph_0})^+ < \lambda \leq 2^\chi]$ *or*

(iv) $\lambda = \sum_{i<cf\,\lambda} \lambda_i$, $cf\,\lambda < \lambda$, each λ_i a regular cardinal and for each $i < cf(\lambda)$ there is $S_i \subseteq \{\delta < \lambda : cf\,\delta = \aleph_0\}$ such that $(\forall\delta < \lambda)[\bigvee_{j<i} cf(\delta) = \lambda_j \rightarrow S_i \cap \delta$ not stationary].

Then (A) there are $I_\alpha \in K^\omega_{tr}$, $|I_\alpha| = \lambda$ for $\alpha < 2^\lambda$, such that $Pr_\mu(I_\alpha, I_\beta)$ for $\alpha \neq \beta < 2^\lambda$

(B) there are for $\alpha < \lambda$, $I_\alpha \in K^\omega_{tr}$, $|I_\alpha| = \lambda$, such that

$$Pr_\mu(I_\alpha, \sum_{\substack{\beta<\lambda \\ \beta\neq\alpha}} I_\beta)$$

2.6C Fact: If $\lambda > \mu$ and the conclusion (A) of Theorem 2.6B holds *then* there are 2^λ separable reduced abelian p-groups of cardinality λ no one embedded into another.

2.6D Discussion: We still can get considerable amounts of information by the general theory. When we want many models of K (no one embeddable into the others) we need

(*) there are 2^λ index models I of power λ each $\varphi_K(\bar{x},\bar{y})$-unembeddable into any other.

* * *

But when you want rigid, indecomposable, etc. you need

(*) there are $\{I_\alpha \in K : \alpha < \lambda\}$, I_α, φ_K-unembeddable into

$\sum\limits_{\beta \neq \alpha} I_\beta$ (and I_α has cardinality λ).

Why?

2.7 Example: Constructing Rigid Boolean Algebras. For $I \in K$ let $BA(I)$ is the Boolean Algebra freely generated by $\{a_\eta : \eta \in I\}$ except the relations $a_\eta \leq a_\nu$ when $\nu \in P^I_\omega$, $n < \omega, \eta = \nu \restriction n$. Start with $B_0 = \{0,1\}$, successively for some $a_i \in B_i$, $0 < a_i < 1$, take

$$B_{i+1} = (B_i \restriction (1-a_i)) + ((B_i \restriction a_i) * BA(I_\alpha))$$

$$B_\lambda = \bigcup_{i<\lambda} B_i = \{a_i : i < \lambda\}, \quad |I_\alpha| = \lambda$$

Of course we chose $\{I_\alpha : \alpha < \lambda\}$ such that I_α is φ_{tr}-unembeddable into $\sum\limits_{\beta \neq \alpha} I_\beta$. The point is that each $a \in B_\lambda - \{0,1\}$ was "marked" by some I_α, (the α such that $a_\alpha = a$). Now $BA(I_\alpha)$ is embeddable into $B_\lambda \restriction a_\alpha$; but $B_\lambda \restriction (1-a_\alpha)$ is weakly $L_{\omega,\omega}$- represented in $\mathcal{M}_{\omega,\omega}(\sum\limits_{\beta<\alpha} I_\beta)$. So for no automorphism f of B_λ, $f(a_\alpha) \leq 1-a_\alpha$ which suffice to get "B_λ is rigid"; in fact it has no one to one endomorphism. If we want stronger rigidity and/or $B_\lambda \models$ c.c.c, and/or B_λ is complete we may have to change K^ω_{tr} and/or φ_{tr}. See [Sh 136] (e.g. 0.2, 0.3).

This illustrates some of the complications in definition 2.1. E.g. the weak representation and the uncountable κ (for complete BA.)

§3. Order implies many non-isomorphic models

In this section we prove that not only any unsable T has in any $\lambda \geq |T| + \aleph_1$, the maximal number (2^λ) of pairwise non-isomorphic models, but that for any Φ proper for linear orders, if the formula $\varphi(\bar{x},\bar{y})$ with vocabulary τ order $\{\bar{a}_s : s \in I\}$ in $EM_\tau(I,\Phi)$ (Ehrenfeucht-Mostowski model) for any I, *then* the number of non isomorphic models $EM_\tau(I,\Phi)$ of power λ up to isomorphism is 2^λ when $\lambda \geq |\tau^\Phi| + \aleph_1$. In previously dealing with this problem, the author in the first attempt [Sh 12] excludes some cardinal λ when $\lambda = |\tau^\Phi| + \aleph_1$ and in the second [ShA1, VIII §3] replaces the $EM_\tau(I,\Phi)$ with some kind of restricted ultrapowers. As subsequently ([Sh 100]) we prove that $PC(T_1,T) = \{M \restriction \tau(T) : M \models T_1\}$ (T an unsuperstable theory, T_1 first order $|T_1| = \aleph_1$, $|T| = \aleph_0$) may be categorical in $\aleph_1$ and for $T =$ the theory of dense linear order, may have a universal model in $\aleph_1$ even though CH fail, we thought that the use of ultrapower was necessary.

Now we can get the theorem also for the number of models of $\psi \in L_{\lambda^+,\omega}$ in λ ($> \aleph_0$) when ψ is unstable. Incidentally the proof is considerably easier.

Note that we do not need to demand $\varphi(\bar{x},\bar{y})$ to be first-order; a formula in any logic is O.K.; it is enough to demand $\varphi(\bar{x},\bar{y})$ to have a suitable vocabulary. This is because an isomorphism from N onto M preserve satisfaction of such φ and its negation. However the length of $\bar{x}$ (and $\bar{y}$) is crucial. Naturally we concentrate on the finite (in 3.1-3.11). But when we are not assuming this, we can, "almost always" save the result. In first reading, it may be advisable to concentrate on the case "λ is regular".

3.1 Definition: Let M be a model I an index model for $s \in I$, $\bar{a}_s$ is a sequence from M, the length of $\bar{a}_s$ depend on $tp_{qf}(s, \emptyset, I)$ only ; Ψ is a set of formulas of the form $\varphi(\bar{x},\bar{a})$, $\bar{a}$ from M, φ has a vocabulary contained in $\tau(M)$.

1) We say $\langle \bar{a}_s : s \in I\rangle$ is weakly κ-skeleton-like for Ψ when: for every $\varphi(\bar{x},\bar{a}) \in \Psi$, there is $J \subseteq I$, $|J| < \kappa$ such that:

(*) if $s,t \in I$, $tp_{qf}(t, J, I) = tp_{qf}(s, J, I)$ *then* $M \models \varphi[\bar{a}_s,\bar{a}] \equiv \varphi[\bar{a}_t,\bar{a}]$

2) If $\Psi = \{\varphi(\bar{x},\bar{a}):\varphi(\bar{x},\bar{y}_\varphi) \in \Delta , \bar{a} \in \mathbf{J}\}$ we write $(\Delta,\mathbf{J})$ instead Ψ; if $\Delta = \{\varphi(\bar{x},\bar{y})\}$ we write $\varphi(\bar{x};\bar{y})$ instead Δ, if $\mathbf{J} = \{\bar{a}:\bar{a}$ from A, and for some $\varphi(\bar{x},\bar{y}) \in \Delta$, $\ell g(\bar{a}) = \ell g(\bar{y})\}$ we write A instead of $\mathbf{J}$.

3) Supposing $\psi(\bar{x},\bar{y}) \stackrel{def}{=} \varphi(\bar{y}, \bar{x})$, I a linear order we say $\langle \bar{a}_s:s \in I\rangle$ is weakly $(\kappa,\varphi(\bar{x},\bar{y}))$-skeleton like for $\mathbf{J}$ if : $\varphi(\bar{x},\bar{y})$ is asymmetric with vocabulary contained in $\tau(M)$, $\ell g(\bar{a}_s) = \ell g(\bar{x}) = lg(\bar{y})$, $\langle \bar{a}_s:s \in I\rangle$ is weakly κ-skeleton like for $(\{\varphi(\bar{x},\bar{y}),\psi(\bar{x},\bar{y})\},\mathbf{J})$ and for $s,t \in M$, $M \models \varphi[\bar{a}_s,\bar{a}_t]$ iff $I \models s < t$.

4) In part (3) if $\mathbf{J} = {}^{\alpha}|M|$, $\alpha = \ell g(\bar{x}) = \ell g(\bar{y})$ we write "inside M" or, "for M" instead "for $\mathbf{J}$".

Note that Definition 3.1 requires considerably more than "the $\bar{a}_s$ are ordered by φ" and even than "the $\bar{a}_s$ are order indiscernibles ordered by φ."

We now want to assign invariants to linear orders. We quote proofs from the Appendix to [Sh] where different terminology was employed. Speaking very roughly, we there discussed only INV^{α}_{κ} where $\kappa = \aleph_0$. The assertion in the appendix that two linear orders are contradictory corresponds to the assertion here that the invariants are defined and different.

In the following, for any regular cardinal $\mu > \aleph_0$, D_μ denotes the filter on μ generated by the closed unbounded sets. If E is a filter on μ and $F \subseteq \mu$ intersects each member of E, then $E + F$ denotes the filter generated by $E \cup \{F\}$.

For a linear order I and a cardinal κ, let $D = D(\kappa, I)$ be $D_{cf(I)} + \{\delta < cf(I) : \kappa \leq cf(\delta)\}$. Two functions f and g from $cf(I)$ to some set X, are equivalent *mod* D if $\{\delta : f(\delta) = g(\delta)\} \in D$. We write f/D for the equivalence class of f for this equivalence relations.

3.2 Definition: For κ a regular cardinal, α an ordinal, we define $INV^{\alpha}_{\kappa}(I)$ for linear orders I, by induction on α:

$\alpha = 0$: $INV^{\alpha}_{\kappa}(I)$ is the cofinality of I if $cf(I)$ is $\geq\kappa$, and is undefined otherwise.

$\alpha = \beta + 1$: Let $I = \bigcup_{i<cf\, I} I_i$, with I_i increasing and continuous in i and I_i a proper initial segment

of I. For $\delta < cf(I)$ let $J_\delta = (I - I_\delta)^*$ (where X^* denotes the inverse order of X).

If $cf(I) > \kappa$ and for some C club of $cf(I)$

(*) if $\delta \in C$ have cofinality at least κ, *then* $INV^0_\kappa(J_\delta)$ *is defined*

then we let

$$INV^\alpha_\kappa(I) = \langle INV^\beta_\kappa(J_\delta) : cf(\delta) \geq \kappa,\ \ \delta \leq cf(I) \rangle / D(\kappa, I)$$

Otherwise $INV^\alpha_\kappa(I)$ is not defined.

α-limit: $INV^\alpha_\kappa(I) = \langle INV^\beta_\kappa(I) : \beta < \alpha)$

Remark: Really just $\alpha = 0,1,2$ are used. For regular $\lambda, \alpha = 1$ suffice but for singulår λ, $\alpha = 2$ is used (see 3.4).

3.3 Lemma: Suppose κ is regular and I, J are linear orders, $\bar{a}_s (s \in I)$, $\bar{b}_t (t \in J)$ are from M, $\varphi(\bar{x}, \bar{y})$ an $\tau(M)$- formula ($\kappa > \ell g(\bar{x}) = \ell g(\bar{y}) = \ell g(\bar{a}_s) = \ell g(\bar{b}_t)$) $\psi(\bar{x}, \bar{y}) \stackrel{def}{=} \varphi(\bar{y}, \bar{x})$. Assume:

(a) (α) for every $s \in I$ for every large enough $t \in J$ $M \models \varphi[\bar{a}_s, \bar{b}_t]$.

(β) for every $t \in J$ for every large enough $s \in I$ $M \models \varphi[\bar{b}_t, \bar{a}_s]$.

(b) (α) $\langle \bar{a}_s : s \in I \rangle$ is weakly $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like in M.

(β) $\langle \bar{b}_t : t \in J \rangle$ is weakly $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like in M.

(c) $INV^\alpha_\kappa(I)$, $INV^\alpha_\kappa(J)$ are defined.

Then $INV^\alpha_\kappa(I) = INV^\alpha_\kappa(J)$.

Proof: Just like [Sh,AP 3.3].

3.4 Lemma: 1) If λ, κ are regular, $\lambda > \kappa$, *then* there are 2^λ linear orders $I_\alpha (\alpha < 2^\lambda)$, each of power λ, with pairwise distinct $INV^1_\kappa(I_\alpha)(\alpha < 2^\lambda)$, each well defined.

2) If $\lambda > \kappa$, κ a regular *then* there are linear orders $I_\alpha(\alpha < 2^\lambda)$, each of power λ with pairwise distinct $INV^2_\kappa(I_\alpha)(\alpha < 2^\lambda)$, each well defined.

Proof: By [Sh,AP 3.3].

* * *

Now we want to attach the invariants of a linear order I to a model M which has a skeleton-like sequence indexed by I. In α) (in Definition 3.5 below) we define what it means for a sequence I to (κ,θ)-represent the $\{\varphi,\psi\}$-type of c over A. (In the simplest case I has cofinality θ from below and the same cofinality as I^* from below with respect to a weakly $(\kappa,\varphi(\bar{x},\bar{y}))$-skeleton like sequence its index set in M.) In β) we say that the type of c over A has a (κ,θ,α) invariant if

(1) all sequences with defined invariants agree.

(2) some representing sequence, its index set (which is a clear order) has a defined INV^α_κ.

More fully and formally:

3.5 Definition: Let $A \subseteq M$, $\bar{c} \in M$ and $\varphi(\bar{x},\bar{y})$ an asymmetric formula with vocabulary contained in $\tau(M)$ and $\psi(\bar{x},\bar{y})=\varphi(\bar{y},\bar{x})$.

(α) We say $\langle \bar{a}_s : s \in I\rangle$ (κ, θ)-represent $(\bar{c}, A, M, \varphi(\bar{x},\bar{y}))$ *if:*

I is a linear order, and for some linear order J of cofinality θ, $J \cap I = \phi$, and $\bar{a}_t \in {}^{\ell g(\bar{x})}A$ for $t \in J$, such that for every large enough $s \in I$, $\bar{a}_s$ realizes $tp_{\{\varphi(\bar{x},\bar{y}),\psi(\bar{x},\bar{y})\}}(\bar{c}, A, M)$ and $\langle \bar{a}_s : s \in J+(I)^*\rangle$ is weakly $(\kappa,\varphi(\bar{x},\bar{y}))$-skeleton like for M (I^*-the inverse of I). [if $\theta \neq \kappa$, less suffice].

(β) We say $(\bar{c}, A, M, \varphi(\bar{x},\bar{y}))$ has a (κ,θ,α)-invariant when:

(i) if for $\ell = 1,2$, $\langle \bar{a}^\ell_s : s \in I^*_\ell\rangle$ (κ,θ)-represent $(\bar{c}, A, M, \varphi(\bar{x},\bar{y}))$ and $INV^\alpha_\kappa(I_\ell)$ are defined *then*

$$INV^{\alpha}_{\kappa}(I_1) = INV^{\alpha}_{\kappa}(I_2).$$

(ii) some $\langle \bar{a}_s : s \in I \rangle$ (κ,θ)-represent $(\bar{c}, A, M, \varphi(\bar{x},\bar{y}))$, with $INV^{\alpha}_{\kappa}(I)$ well defined.

$(\beta)'$ Let "(κ,α)-invariant "means" (κ,θ,α)-invariant for some regular $\theta \geq \kappa$. Similarly for "κ-represent".

(γ) Let $INV^{\alpha}_{\kappa}(\bar{c},A,M,\varphi(\bar{x},\bar{y}))$ be $INV^{\alpha}_{\kappa}(I)$ when $(\bar{c},A,M,\varphi(\bar{x},\bar{y}))$ has (κ,θ,α)-invariant and $\langle \bar{a}_s : s \in I \rangle$ (κ,θ)-represent it.

3.6⁻ Discussion: Each of Definition 3.6, Lemmas 3.7 and 3.8, and the proof of Theorem 3.9 have 3 cases. In the easiest case $\lambda = \|M\|$ is regular. When λ is singular the computation of $INV^{\alpha}_{\kappa}(\kappa\ ,\ \varphi(\bar{x},\bar{y}))$ is easier when $cf(\lambda) > \kappa$ (case 2). The third case arises when $\lambda > \kappa > cf(\lambda)$.

The easiness of the regular case is caused by the fact that any two continuous increasing representations of a model with power λ must "agree" on a club. In the second case we are able to restrict the first argument to a cofinal sequence of M. For the third case we must construct a "dual argument", noticing that much of a long sequence must concentrate on one member of the representation.

3.6 Definition: Let $\varphi(\bar{x},\bar{y})$ be a formula with vocabulary $\subseteq \tau(M)$ $(\ell g(\bar{x}) = \ell g(\bar{y}))$, M a model of power λ, $\lambda > \kappa$, κ regular, α an ordinal.

(0) If M is a model of power λ, $\overline{M}$ is a representation of M if:

$\overline{M} = \langle M_i : i < cf\,\lambda \rangle$, it is increasing continuous $\|M_i\| < \lambda$, $M = \bigcup_{i<\lambda} M_i$ (and $M_i \subseteq M$).

Similarly for sets.

1) For λ regular:

$INV^{\alpha}_{\kappa}(M, \varphi(\bar{x},\bar{y})) = \{e$: for *every* representation $\langle A_i : i < \lambda \rangle$ of

$|M|$, *there are* $\delta < \lambda$ and $\bar{c} \in M$, such that $cf\,\delta \geq \kappa$ and $e = INV^{\alpha}_{\kappa}(\bar{c}, A_{\delta}, ,\varphi(\bar{x},\bar{y}))$ (so the latter is well defined)$\}$.

2) For regular cardinals $\theta > \kappa$, $\lambda > cf\,\lambda = \theta$, a model M of cardinal λ and an asymmetric formula $\varphi(\bar{x},\bar{y})$ (in $\tau(M)$) let

$$D_{\theta,\kappa} = D_{\theta} + \{\delta < \theta : cf\,\delta \geq \kappa\}$$

$INV^{\alpha}_{\kappa,\theta}(M) = \{\langle e_i : i < \theta\rangle / D_{\theta,\kappa}$: for every representation $\langle A_i : i < \theta\rangle$ of $|M|$, *there are* $S \in D_{\theta,\kappa}$ *and for* every $\delta \in S$ there is $\bar{c}_{\delta} \in M$ such that $e_{\delta} = INV^{\alpha}_{\kappa}(\bar{c}_{\delta}, A_{\delta}, M, \varphi(\bar{x},\bar{y}))\}$.

3) For a regular cardinal θ, $\lambda > \theta > \kappa + cf\,\lambda$ and function h with domain a stationary subset of $\{\delta < \theta : cf\,\delta \geq \kappa\}$ and range a set of regular cardinals $< \lambda$, let $D_{h,\lambda} = D_{\theta} + \{\{\delta : h(\delta) \geq \mu\} : \mu < \lambda\}$, and assuming $D_{h,\lambda}$ is a proper filter let :

$INV^{\alpha,h}_{\kappa,\theta}(M, \varphi(\bar{x},\bar{y})) = \{\langle e_i : i < \theta\rangle / D_{h,\lambda}$: for *every* representation

$\langle A_i : i < cf\,\lambda\rangle$, of $|M|$ *there are* $\gamma < cf\,\lambda$ and $S \in D_{h,\lambda}$, $S \subseteq Dom\ h$, and for each $\delta \in S$, for some $\bar{c} \in M$, $e_i = INV^{\alpha}_{\kappa}(\bar{c}, A, M, \varphi(\bar{x},\bar{y}))\}$.

3.6A Remark: Of course, also in 3.6(1) we could have used $\langle e_i : i < \lambda\rangle / D_{\lambda}$ as invariants.

3.7 Lemma: Suppose $\varphi(\bar{x},\bar{y})$ a formula in the vocabulary of M, $\ell g(\bar{x}) = \ell g(\bar{y}) < \omega$.

1) If $\lambda > \aleph_0$ is regular, M a model of cardinal λ, κ regular $<\lambda$, *then* $INV^{\alpha}_{\kappa}(M, \varphi(\bar{x},\bar{y}))$ *has* power $\leq\lambda$.

2) If λ is singular, $\theta = cf\,\lambda > \kappa$, *then* $INV^{\alpha}_{\kappa,\theta}(M, \varphi(\bar{x},\bar{y}))$ almost has power $\leq\lambda$, which means: there are no e^{ζ}_i ($i < \theta, \zeta < \lambda^+$) such that

(i) for $\zeta < \lambda^+$, $\langle e^{\zeta}_i : i < \theta\rangle / D_{\theta,\kappa} \in INV^{\alpha}_{\kappa,\theta}(M, \varphi(\bar{x},\bar{y}))$

(ii) for $i < \theta$, $\zeta < \xi < \lambda^+$, $e^{\zeta}_i \neq e^{\xi}_i$

3) If λ is singular, θ regular, $\kappa + cf\,\lambda < \theta < \lambda$, h a function from some stationary subset

of

$\{i < \theta : cf\, i \geq \kappa\}$, into $\{\mu : \mu < \lambda$ is a regular cardinal$\}$, $D_{\theta,h}$ a proper filter, *then* $INV^{\alpha,h}_{\kappa,\theta}(M, \varphi(\bar{x},\bar{y}))$ almost has power $\leq\lambda$, which means: there are no e^{ζ}_i $(i < \theta,\ \zeta < \lambda^+)$ such that

(i) for $\zeta < \lambda^+$, $\langle e^{\zeta}_i : i < \theta\rangle / D_{\theta,h} \in INV^{\alpha,h}_{\kappa,\theta}(M, \varphi(\bar{x},\bar{y}))$

(ii) for $i < \theta, \zeta < \xi < \lambda^+$, $e^{\zeta}_i \neq e^{\xi}_i$.

Proof: Straightforward.

We now show that if $|I| \leq \lambda$ and $INV^{\alpha}_{\kappa}(I)$ is defined then there is a linear order J such that if a model M has a weakly (κ,φ)-skeleton like sequence inside M of order-type J then $INV^{\alpha}_{\kappa}(I) \in INV^{\alpha}_{\kappa}(M, \varphi)$. Again the proof splits into three cases depending on the cofinality of λ. The following result provides a detail needed for the proof.

3.7A Claim: Suppose $\langle \bar{a}_t : t \in J\rangle$ is a weakly (κ, φ)-skeleton like inside M and $I \subseteq J$. *If* for each $s \in J$ either $\{t \in I : t < s\}$ or the inverse order on $\{t \in I : t > s\}$ has cofinality less than κ *then* $\langle \bar{a}_t : t \in I\rangle$ is weakly (κ, φ)- skeleton like for M.

Proof: We must show that for every $\varphi(x,\bar{a})$ there is an $I_{\bar{a}} \subseteq I$ with $|I_{\bar{a}}| < \kappa$ such that if $s,t \in I$ and $tp_{qf}(s, I_{\bar{a}}, I) = tp_{qf}(t, I_{\bar{a}}, I)$ then $\theta(\bar{a}_s,\bar{a}) \equiv \theta(\bar{a}_t,\bar{a})$ for $\theta = \varphi,\psi$. We know there is such a set $J_{\bar{a}}$ for J and $\varphi(\bar{x},\bar{a})$. For each $s \in J_{\bar{a}}$ choose a set X_s of $<\kappa$ elements of I such that X_s tends to s, i.e. to the cut that s induce in I (either from above or below). (so if $s \in I$, $X_s = \{s\}$, otherwise use the assumption). Let $I_{\bar{a}} = \bigcup_{s \in J_{\bar{a}}} X_s$. Now it is easy to see that if t_1 and $t_2 \in I$ have the same qf-type over $I_{\bar{a}}$ they have the same qf type over $J_{\bar{a}}$ and the claim follows.

3.8 Lemma: Assume $\ell g(\bar{x}) = \ell g(\bar{y}) < \aleph_0$, $\varphi = \varphi(\bar{x},\bar{y})$.

1) Let $\lambda > \aleph_0$ be regular. If I is a linear order of power $\leq\lambda$, and $INV^{\alpha}_{\kappa}(I)$ is well defined, then for some linear order J of power λ the following hold:

(*) if M is a model of power λ, $\bar{a}_s \in M$, $\langle \bar{a}_s : s \in J\rangle$ is weakly $(\kappa,\varphi(\bar{x},\bar{y}))$-skeleton like inside M ($\varphi(\bar{x},\bar{y})$ asymmetric), *then* $INV^{\alpha}_{\kappa}(I) \in INV^{\alpha}_{\kappa}(M, \varphi(\bar{x},\bar{y}))$.

2) Let λ be singular, $\theta = cf\,\lambda > \kappa$, $\lambda = \sum_{i<\theta} \lambda$, λ_i increasing continuous for $i < \theta$, I_i is a linear order of cofinality $> \lambda_i$ and cardinality $\leq \lambda$, $INV_\kappa^\alpha(I_i)$ well defined, then for some linear order J of power λ the following holds

(**) if M is a model of power λ, $\overline{a}_s \in M$, $\langle a_s{:}s \in J\rangle$ is weakly $(\kappa,\varphi(\overline{x},\overline{y}))$-skeleton like for M, $(\varphi(\overline{x},\overline{y})$ asymmetric) *then* $\langle INV_\kappa^\alpha(I_i){:}i < \theta\rangle/D_{\theta,\kappa}$ belongs to $INV_\kappa^\alpha(M,\varphi(\overline{x},\overline{y}))$.

3) Let λ be singular, θ,κ regular, $\lambda > \theta > (cf(\lambda) + \kappa)$, $\lambda = \sum_{i<cf\,\lambda} \lambda_i$, λ_i increasing continuous, and for $i < \theta$, I_i a linear order, $INV_\kappa^\alpha(I_i)$ is well defined. Then for some linear order J of power λ the following holds:

(***) *if* M is a model of power λ, $\overline{a}_s \in M$, $\langle \overline{a}_s{:}s \in J\rangle$ is weakly $(\kappa,\varphi(\overline{x},\overline{y}))$-skeleton like for M, $(\varphi(\overline{x},\overline{y})$ asymmetric), h a function from a stationary subset of $\{\delta < \theta{:}cf\,\delta \geq \kappa\}$ and range a set of regular cardinals $\langle\lambda$ but $>\theta$, $D_{\theta,h}$ *then* $\langle INV_{\kappa,\theta}^{\alpha,h}(I_i){:}i < \theta\rangle/D_{\theta,h}$ belongs to $INV_{\kappa,\theta}^{\alpha,h}(M,\varphi(\overline{x},\overline{y}))$.

Proof: 1) We must choose a linear order J of power λ such that: if J indexes a weakly $(\kappa,\varphi(\overline{x},\overline{y}))$-skeleton like sequence inside M then $INV_\kappa^\alpha(I) \in INV_\kappa^\alpha(M,\varphi(\overline{x},\overline{y}))$. For this we must find for any continuous increasing decomposition a $\overline{c} \in M$ and a δ with $INV_\kappa^\alpha(\overline{c}, A_\delta, M, \varphi(\overline{x},\overline{y})) = INV_\kappa^\alpha(I)$. To obtain $\overline{c}$, we use a function $s{:}\lambda \to J$. Let for $\alpha < \lambda$, I_α be pairwise disjoint linear orders isomorphic to I^*.

Let $J = \sum_{\alpha<\lambda} I_\alpha^*$ (I^* means we use the inverse of as an ordered set). Suppose $\langle \overline{a}_s{:}s \in J\rangle$ is weakly $(\kappa,\varphi(\overline{x},\overline{y}))$-skeleton like inside M, $(\varphi(\overline{x},\overline{y}))$ asymmetric), M has cardinality λ. Let $s(\alpha) \in I_\alpha$ and $M = \bigcup_{\alpha<\lambda} A_\alpha$, $|A_\alpha| < \lambda$, $\langle A_\alpha{:}\alpha < \lambda\rangle$ increasing continuous. By the definition of weak $(\kappa,\varphi(\overline{x},\overline{y}))$ skeleton (3.1(1)), for every (finite) $\overline{a} \in M$, there is a subset $J_{\overline{a}}$ of J of power $< \kappa$ such that: *if* $s,t \in J - J_{\overline{a}}$ induces the same Dedekind cut on $J_{\overline{a}}$, *then* $M \models \varphi[\overline{a}_s,\overline{a}] \equiv \varphi[\overline{a}_t,\overline{a}]$ and $M \models \varphi[\overline{a},\overline{a}_s] \equiv \varphi[\overline{a},\overline{a}_t]$. Since λ is regular for some closed unbounded subset C of λ, for $\delta \in C$:

(*) (i) $\overline{a}_{s(\alpha)} \in A_\delta$ for $\alpha < \delta$

(ii) $J_{\overline{a}} \subseteq \sum_{\beta<\delta^*} I_\beta^*$ for $\overline{a} \in A_\delta$

So it is enough to prove that for $\delta \in C$ of cofinality $\geq \kappa$, $INV^{\alpha}_{\kappa}(I) = INV^{\alpha}_{\kappa}(\bar{a}_{s(\delta)}, A_{\delta}, M, \varphi(\bar{x},\bar{y}))$. It is easy to see that $\langle \bar{a}_s : s \in I_{\delta} \rangle$ κ-represent $(\bar{a}_{s(\delta)}, A_{\delta}, M, \varphi(\bar{x},\bar{y}))$. The required θ and J in Definition 3.5(α) are $cf(\delta)$ and $\langle \bar{a}_{s(\beta)} : \beta < \delta \rangle$. Now use claim 3.7A. So (see Definition 3.5(γ)) it is enough to show that $(\bar{a}_{s(\delta)}, A_{\delta}, M, \varphi(\bar{x},\bar{y}))$ has a (κ,α)-invariant. Now in Definition 3.5(β), part (ii) is obvious by the above; so it remains to prove (i).

Let $\theta \overset{def}{=} cf\, \delta$.

So assume that for $\ell = 1,2$, $\langle \bar{a}^{\ell}_s : s \in I^{\ell} \rangle$ (κ,θ)-represent $(\bar{a}_{s(\delta)}, A_{\delta}, M, \varphi(\bar{x},\bar{y}))$, let J^{ℓ}, $\langle a^{\ell}_t : t \in J^{\ell} \rangle$ exemplify this and let $J^*_{\ell} = J^{\ell} + (I^{\ell})^*$ and assume $INV^{\alpha}_{\kappa}(I^{\ell})$ are well defined. We have to prove that $INV^{\alpha}_{\kappa}(I^1) = INV^{\alpha}_{\kappa}(I^2)$. We shall use Lemma 3.3 (with I^1, I^2 here standing for I, J there).

Remark: The following observation underlies the next step in the proof. It follows easily from Definition 3.1 (1).

3.8A Fact: Suppose $\langle \bar{a}_s : s \in J + I^* \rangle$ is weakly (κ,φ)-skeleton like inside M and both J and I have cofinality $\geq \kappa$. Then for every $\bar{b} \in M$ there exist $s_0 \in J$ and $s_1 \in I^*$ such that *if* $s_0 < s,t < s_1$ (in $J + I^*$) then

$$M \models \psi(\bar{a}_s,\bar{b}) \equiv \psi(\bar{a}_t,\bar{b}).$$

$$M \models \psi(\bar{a}_s,\bar{b}) \equiv \psi(\bar{a}_t,\bar{b}).$$

Now we return to the proof of Theorem 3.8.

Let us prove (a)(α) from 3.3. So suppose it fail, i.e. $s \in I^1$ but for arbitrarily large $t \in (I^2)^*$, $M \models \neg\varphi[\bar{a}^1_s, \bar{a}^2_t]$.

Since $\langle \bar{a}^2_t : t \in J^2 + I^2 \rangle$ is weakly (κ,φ)-skeleton like inside M the preceding Fact 3.8A yields that for arbitrarily large $t \in J^2$, $M \models \neg\varphi[\bar{a}^1_s, \bar{a}^2_t]$. Since $\bar{a}^1_s$ and $\bar{a}_{s(\delta)}$ realize the same $\{\varphi,\psi\}$-type over A_{δ}, (Definition 3.5 (α) this implies $M \models \neg\varphi[\bar{a}_{s(\delta)}, \bar{a}^2_t]$ for arbitrarily large $t \in J^2$. Choose such $t_0 \in J_i$. This quickly contradicts the choice of J^2 and I^2. For, it implies that for every $t \in I^2$, we have $M \models \neg\varphi(\bar{a}^2_t, \bar{a}^2_{t_0})$ which is impossible if $J^2 + I^2$ is weakly (κ,φ)-

skeleton like (Definition 3.1(3)).

2),3) Left to the reader (or see the proof of case (d) in Theorem 3.11.)

3.9 Theorem: Suppose $\lambda > \kappa$, K_λ a family of τ-models, each of power λ, $\varphi(\bar{x},\bar{y})$ an asymmetric formula with vocabulary $\subseteq\tau$ and $\ell g(\bar{x})=\ell g(\bar{y}) < \aleph_0$. Suppose further that for every linear order J there is $M \in K_\lambda$, and $\bar{a}_s \in M$ for $s \in J$ such that $\langle \bar{a}_s : s \in J\rangle$ is weakly $(\kappa,\varphi(\bar{x},\bar{y}))$-skeleton like in M.

Then, in K_λ, there are 2^λ pairwise non-isomorphic models.

Proof: Let first $\lambda > \aleph_0$ be regular.

By 3.4 (1) there are linear order I_ζ ($\zeta < 2^\lambda$) each of power λ, such that $INV^1_\kappa(I_\zeta)$ are well defined and distinct. Let J_ζ relate to I_ζ as guarantee by 3.8(1). Let $M_\zeta \in K_\lambda$ be such that there are $\bar{a}^\zeta_s \in M_\zeta$ for $s \in J_\zeta$ such that $\langle \bar{a}_s : s \in J_\zeta\rangle$ is weakly $(\kappa,\varphi(\bar{x},\bar{y}))$-skeleton like inside M_ζ (exists by assumption). By 3.8(1) $INV^\alpha_\kappa(I_\zeta) \in INV^\alpha_\kappa(M_\zeta,\varphi(\bar{x},\bar{y}))$.

Clearly $M_\zeta \cong M_\xi \Rightarrow INV^\alpha_\kappa(M_\zeta,\varphi(\bar{x},\bar{y})) = INV^\alpha_\kappa(M_\xi,\varphi(\bar{x},\bar{y}))$, hence $M_\zeta \cong M_\xi \Rightarrow INV^\alpha_\kappa(I_\zeta) \in INV^\alpha_\kappa(M_\xi, \varphi(\bar{x},\bar{y}))$. So if for some $\xi < 2^\lambda$, the number of $\zeta < 2^\lambda$ for which $M_\zeta \cong M_\xi$ is $> \lambda$, *then* $INV^1_\kappa(M_\xi,\varphi(\bar{x},\bar{y}))$ has power $> \lambda$ (remember $INV^1_\kappa(I_\zeta)$, $\zeta < 2^\lambda$, were distinct). But this contradicts 3.7(1). So $\{(\zeta,\xi):\zeta,\xi < 2^\lambda, M_\zeta \cong M_\xi\}$, which is an equivalence relation, satisfies: each equivalence class has power $\leq\lambda$; hence there are 2^λ equivalence classes and we finish.

For λ singular the proof is similar. If $cf\,\lambda > \kappa$, we can choose $\theta = (cf\,\lambda)$ and use $INV^2_{\kappa,\theta}$, 3.4(1), (3.8(2), 3.7(2) instead of INV^1_κ, 3.4(1), 3.8(1), 3.7(1) respectively.

If $cf\,\lambda \leq \kappa$, let $\theta = \kappa^+$, so $\lambda > \theta > \kappa + cf\,\lambda$; hence we can find $h : \{\delta < \theta : cf\,\delta \geq \kappa\} \to \{\mu : \mu + cf\,\mu < \lambda\}$ such that for each $\mu = cf\,\mu < \lambda$, $\{\delta < \theta : cf\,\delta \geq \kappa$ and $h(\delta) = \mu\}$ is stationary. Now we can use $INV^{2,h}_{\kappa,\theta}$, 3.4(2), 3.8(3), 3.7(3) instead INV^1_κ, 3.4(1), 3.8(1), 3.7(1) respectively.

Alternatively for λ singular see proof of 3.16 and 3.11.

3.10 Conclusion: 1) If T_1 is the first order $T \subseteq T_1$, T is unstable, and complete, $\lambda \geq |T_1| + \aleph_1$ *then* there are 2^λ pairwise non-isomorphic models of T of power λ which are reducts of models of T_1.

2) If $T \subseteq T_1$ are as above, $\lambda \geq |T_1| + \kappa^+$, $\lambda = \lambda^{<\kappa}$, κ regular, *then* there are 2^λ pairwise non-isomorphic models of T of power λ which are reducts of models M_i^1 of T_1 such that M_i, M_i^1 are κ-compact and κ-homogeneous. [really we can get strongly homogeneous]

3) If $\psi \in L_{\lambda^+,\omega}(\tau_1)$, $\tau \subseteq \tau^1$, ψ has the order property for $L_{\lambda^+,\omega}$ [i.e. for some formula $\varphi(\bar{x},\bar{y}) \in L_{\lambda^+,\omega}$ for arbitrarily large μ there is a model M of ψ and $\bar{a}_i \in M$ for $i < \mu$ such that $M \models \varphi[\bar{a}_i,\bar{a}_j]$ iff $i < j$ and $\ell g(\bar{x}) = \ell g(\bar{y}) < \aleph_0$].

Then for $\mu \geq \lambda + \aleph_1$, ψ has 2^λ models of power μ, with pairwise non-isomorphic τ-reducts.

Proof: 1) By [Sh] VIII 2.4 (and see assumption V just before it, p. $394^{11,14}$) we have the assumption of 3.9.

2) By [Sh] VII 3.1 or case II of the proof of Theorem 3.2 (there) we have the assumption of 3.9.

3) See e.g. Grossberg and Shelah [GrSh 222] why the assumption of 3.9 holds.

* * *

Now we turn out attention to the case the sequences are infinitary (see more in the latter version):

3.11 Theorem: Suppose $\sigma < \kappa < \lambda$ are cardinals, κ regular, and in 3.9's hypothesis we have $\ell g(\bar{a}_s) = \sigma \geq \aleph_0$ *then* 3.9's conclusion statement holds, if at least one of the following holds:

(a) $\lambda = \lambda^\sigma$

(b) $\lambda^\kappa < 2^\lambda$

(c) Replace 3.9's assumption by:

$(*)_1$ $\lambda^\sigma < 2^\lambda$, $cf\,\lambda > \sigma$.

$(*)_2$ for every linear order J of cardinality λ there is $M_J \in K_\lambda$ and $\langle a_s{:}s \in J\rangle$ ($\bar{a}_s \in {}^\sigma|M|$) which is weakly ($\kappa$, $<\lambda$, $\varphi(\bar{x},\bar{y})$)-skeleton like inside M (see definition 3.12 below).

(d) Replace 3.9's assumption by: for some regular $\mu(0) \leq 2^\lambda$:

$(*)_1$ $cf\ \lambda > \sigma$.

$(*)_2$ as in (c).

$(*)_3^{\mu(0)}$ for $J \in K_\lambda^{or}$, ($= (K_{or})_\lambda$) and a set $A \subseteq M_J$ (from $(*)_2$) if $|A| < \lambda$ then:
(i) $\mu(0) > |S^\sigma_{\{\varphi,\psi\}}(A, M)|$ or at least
(ii) $\mu(0) > |\{Av_{\{\varphi,\psi\}}(\langle \bar{b}_i{:}i < \kappa\rangle, A, M) : \bar{b}_i \in A$ for $i < \kappa$, the average is well defined and is realized in $M\}$ and if $cf\,\lambda < \lambda$, $|A| \leq (cf\,\lambda) + \kappa$ is enough.

(e) Replace 3.9's assumption by:

for some regular $\mu(0) \leq 2^\lambda$

$(*)_4^{\mu(0)}$ for every $J \in K_\lambda^{or}$ there is $M_J \in K_\lambda$ with $\langle \bar{a}_s{:}s \in I\rangle$ weakly ($\kappa,\varphi(\bar{x},\bar{y})$)-skeleton like inside M (so $\bar{a}_s \in {}^\sigma|M|$), such that:

(i) $\mu(0) > |\{Av_{\{\varphi,\psi\}}(\langle \bar{b}_i : i < \kappa\rangle, M, M)$: for $i < \kappa$, $\bar{b}_i \in {}^\sigma|M|$, and $\langle \bar{b}_i : i < \kappa\rangle$ is weakly (κ, $\varphi(\bar{x},\bar{y})$)-skeleton like inside $M\}|$ (on Av see Ch I, §2, we can even restrict further the set of $\langle \bar{b}_i : i < \kappa\rangle$ which we consider).

(f) for some $\mu < \lambda$, there is a linear order of power μ with $\geq\lambda$ Dedekind cuts with upper and lower cofinality $\geq \kappa$ and $2^{\mu+\sigma} < 2^\lambda$.

3.12 Definition: We say $\langle \bar{a}_s : s \in I \rangle$ is weakly $(\kappa, <\lambda, \varphi(\bar{x},\bar{y}))$- skeleton like in M *if:* Definition 3.1(3) holds, and for each $A \subseteq M$, $|A| < \mu$, there is $J \subseteq I$, $|J| < \lambda$ such that for every $\bar{c} \in {}^{\sigma}A$, 3.1(1)(*) holds.

If $\mu = \lambda$ we omit λ.

Proof of 3.11.

Case (a): We can in Definition 3.5 replace A by **J**, a set of sequences of length σ from M. Thus in Definition 3.6, replace $\langle A_i : i < \lambda \rangle$ by $\langle \mathbf{J}_i : i < cf(\lambda) \rangle$, ${}^{\sigma}|M| = \bigcup_i \mathbf{J}_i$, $|\mathbf{J}_i| < \lambda$, $\mathbf{J}_i$ increasing continuous. No further change in 3.1- 3.9 is needed.

Alternatively, we can define $N = F_\sigma(M)$ as the model with universe $|M| \cup {}^{\sigma}|M|$, $\tau(N) = \tau(M) \cup \{F_i : i < \sigma\}$, $R^N = R^N$ for $R \in \tau(M)$,

$$G^N(x_1, \ldots, x_n) = \begin{cases} G^M(x_1, \ldots, x_n) & \text{if } x_1, \ldots, x_n \in |M| \\ x_1 & \textit{otherwise} \end{cases}$$

for function symbol $G \in \tau(M)$ which has n-places and

$$F_i^N(x) = \begin{cases} x(i) & \text{if } x \in {}^{\sigma}|M| \\ x & \text{if } x \in M \end{cases}$$

for $i < \sigma$.

Note that $M_1 \cong M_2$ if and only if $F_\sigma(M_1) \cong F_\sigma(M_2)$, $|||F_\sigma(M)||| = |||M|||^\sigma$, etc. So we can apply 3.9 to the class $\{F_\sigma(M) : M \in K_\lambda\}$ and get the desired conclusion.

Case (b): Left to the reader [use weakly $(\kappa, \varphi(\bar{x},\bar{y}))$-skeleton like sequences $\langle \bar{a}_s : s \in \kappa + (I_\zeta)^* \rangle$ in $M_\zeta \in K_\lambda$ for $\zeta < 2^\lambda$, with $\langle INV_\kappa^2(I_\zeta) : \zeta < 2^\lambda \rangle$ pairwise distinct, and count the number of models $(M_\zeta, \langle \bar{a}_s : s \in \kappa \rangle)$ up to isomorphism, then "forget the $\bar{a}_s$, $s \in \kappa$", i.e. use 3.13 below (= [Sh, VIII 1.3]))].

Case (c): Repeat the proof of 3.9 (the only difference is that the cardinality of the invariant of M_I is $\leq \lambda^\sigma$ rather than $\leq \lambda$).

Case (d): If λ is regular use case (c). So let us assume $cf\,\lambda < \lambda$, and let $\theta \stackrel{def}{=} \kappa + cf\,\lambda$

(which is regular and $<\lambda$. Now for μ regular $>\kappa$, $\mu\le\lambda$, let $\{I^{\mu}_{\alpha}:\alpha<2^{\lambda}\}$ be such that :

(i) $INV^2_{\kappa}(I^{\mu}_{\alpha})\ne INV^2_{\kappa}(I^{\mu}_{\beta})$ for $\alpha\ne\beta$

(ii) the order I^{μ}_{α} has cofinality μ and cardinality λ.

Let $I^{\mu}_{\alpha,\zeta}\cong I^{\mu}_{\alpha}$ for $\zeta<\kappa^+$ and w.l.o.g. the members of $\{I^{\mu}_{\alpha,\zeta}:\mu\le\lambda$ is regular, $\zeta<\kappa^+$, $\alpha<2^{\lambda}\}$ are pairwise disjoint. Now let $h:\theta^+\to\{\mu:\mu\le\lambda,\mu$ regular$\}$ be such that for every regular $\chi<\theta$, $\{\delta<\kappa^+: cf\,\delta=\kappa,\ h(\delta)\ge\chi\}$ is stationary. We define for $\alpha<2^{\lambda}$, $\zeta<\kappa^+$ the linear order $J_{\alpha,\zeta}$ as $(I^{h(\zeta)}_{\alpha,\zeta})^*$, $J_{\alpha}=\sum_{\zeta<\kappa^+}J_{\alpha,\zeta}$, and set $s(\alpha,\zeta)\in J_{\alpha,\zeta}$.

So by $(*)_2$ there is, for $\alpha<2^{\lambda}$, a model $M_{\alpha}\in K_{\lambda}$, and $\langle\bar a^{\alpha}_s:s\in J_{\alpha}\rangle$, $\bar a^{\alpha}_s\in{}^{\sigma}|M_{\alpha}|$, $\langle\bar a^{\alpha}_s:s\in J_{\alpha}\rangle$ is weakly $(\{\kappa,<\lambda,\varphi(\bar x,\bar y))$-skeleton like inside M.

Let for $M\in K_{\lambda}$, $G(M)$ be the set of $\langle e_i:i<\kappa^+\rangle$, e_i is $INV^2_{\kappa}(J)$ for some J of cofinality $h(i)$ and cardinality λ, such that:

$(*)$ *for every* $\langle A^0_i:i<\kappa^+\rangle$, $|A^0_i|\le\theta$, A^0_i, increasing continuous in i *there is* $\langle A^1_i:i<\theta^+\rangle$, A^1_i increasing continuous in i, $|A^1_i|\le\theta$, $A^0_i\subseteq A^1_i$ such that (if (i) of $(*)^{\mu(0)}_3$ of Case (d)) :

$\{i$: for some $\bar c\in{}^{\sigma}|M|$, $e_i=INV^2_{\kappa}(\bar c,A^1_i,M,\ \varphi(\bar x,\bar y))\}\in D_{h,\lambda}$ (see 3.6(2).

(we leave (ii) of case (d) to the reader.

Now if $M=M_{\alpha}$ let $A^1_i=A^0_i\cup\bigcup_{\alpha<i}\bar a_{s(\alpha)}$; now we know that for A^1_i there is $J^a_i\subseteq J_{\alpha}$, $|J^a_i|<\lambda$, as in Definition 3.11A. So $\{i<\theta^+: cf\,i=\theta,\ \bigcup_{j<i}J^a_i\cap I^{h(\zeta)}_{\alpha,\zeta}$ is bounded in $I^{h(\zeta)}_{\alpha,\zeta}\}\in D_{h,\lambda}$ (why?: as $\kappa^+<\lambda$ apply 3.12 to $\bigcup_j A^1_j$). So easily $\langle INV^2_{\kappa}(I^{h(\zeta)}_{\alpha,\zeta}):\zeta<\theta^+\rangle\in G(M_{\alpha})$. Easily by $(*)^{\mu(0)}_3$ for every M_{α}

$$|\{\beta:\langle e^{\beta}_{\zeta}:\zeta<\theta^+\rangle\in G(M_{\alpha})\}|<\mu(0)$$

and $M_{\alpha}\cong M_{\beta}\Rightarrow G(M_{\alpha})=G(M_{\beta})$. As $\mu(0)\le 2^{\lambda}$ is regular, we can finish easily.

Case e: Like case (b).

Case (f): By the following variant of [ShA1, VII 1.3].

3.13 Fact: If $\tau_2 = \tau_1 \cup \{c_i : i \in I\}$, c_i-individual constants, K a class of τ_ℓ-models (for $\ell = 1,2$) $M \in K_2 \Rightarrow M \restriction \tau_1 \in K_1$ and $\mu = I(\lambda, K_2) > \lambda^{|I|}$ then $I(\lambda, K_1) \geq \mu$ (so if $\mu = 2^{\lambda + |\tau_1|}$, equality holds.)

3.14 Conclusion: 1) Suppose $\psi \in L_{\chi^+,\omega}(\tau_1)$, $\tau \subseteq \tau_1$, $\varphi(\bar{x},\bar{y}) \in L_{\chi^+,\omega}(\tau)$, $\ell g(\bar{x}) = \ell g(\bar{y}) = \sigma \leq \chi$, and for every μ for some model M of ψ there are $\bar{a}_i \in {}^{\sigma}M (i<\mu)$ such that $M \models \varphi[\bar{a}_i, \bar{a}_j]$ iff $i < j$. *Then* for every $\lambda > \chi + \sigma^+$, ψ has 2^λ models of power λ with pair-wise non-isomorphic τ-reducts.

2) Suppose $\psi \in L_{\chi^+,\omega}(\tau_0)$, $\varphi_\ell(\bar{x},\bar{y}) \in L_{\chi^+,\omega}(\tau_\ell)$ for $\ell = 1,2$, $\ell g(\bar{x}) = \ell(\bar{y}) = \sigma$, $\tau_0 = \tau_1 \cap \tau_2$, $\{\psi, \varphi_1(\bar{x},\bar{y}), \varphi_2(\bar{x},\bar{y})\}$ has no model and

(*) for every α there is a τ_0-model M and $\bar{a}_\beta \in {}^{\sigma}|M|$ for $\beta < \alpha$, such that: if $\beta < \gamma < \alpha$ then

(i) for some expansion M' of M, $M' \models \varphi_1[\bar{a}_\beta, \bar{a}_\gamma]$,

(ii) for some expansion M' of M, $M' \models \varphi_2[\bar{a}_\gamma, \bar{a}_\beta]$.

Then for $\lambda > \chi + \sigma^+$, $I(\lambda, \psi) = 2^\lambda$ (i.e. there are 2^λ non isomorphic τ_0-models of ψ of cardinality λ).

Proof: 1) follows from (2).

2) We know that for some Φ proper for K_{or}, for every $I \in K_{or}$, $EM^1(I,\Phi)$ is a model of ψ and for $s,t \in I$, if $I \models s < t$ *then*

$$EM^1(I,\Phi) \models \varphi_1[\bar{a}_s, \bar{a}_t] \qquad EM^1(I,\Phi) \models \neg \varphi_2[\bar{a}_s, \bar{a}_t] .$$

(see [Sh16, Th. 2.5], [Gr Sh 222]).

So we can use 3.16 below cases (C), (D) (E) (as $\theta = \aleph_0$)

* * *

We may want in e.g. 3.10 to get not just non isomorphic models, but non isomorphic because some nice invariant is different.

3.15 Definition: (1) Let μ be a regular uncountable cardinal, h a function from some stationary $S \subseteq \mu$ to a set of regular cardinal $\leq \lambda$, M a τ-model, $\varphi(\bar{x},\bar{y})$ a formula in the vocabulary τ, $\ell g(\bar{x}) = \ell g(\bar{y}) = \sigma$. Now M obeys (h, φ) *if* the following holds:

(*) there is a function H from $(\mathbf{S}_{<\mu}(M))^{<\mu}$ to $\mathbf{S}_{<\mu}(M)$ such that:

if $\langle A_i : i < \mu \rangle$ is an increasing continuous sequence of subsets of M, $|A_i| < \lambda$, $H(\langle A_i : i \leq j \rangle) \subseteq A_{j+1}$ *then* for some club $C \subseteq \mu$, for $\delta \in C \cap S$ the following holds:

$\oplus$ if for $i < cf(\delta)$, $\bar{a}_i \subseteq A_{\alpha_i}$ for some $\alpha_i < \delta$, $\langle \bar{a}_i : i < cf\,\delta \rangle$ is weakly $(\kappa, \varphi(\bar{x},\bar{y}))$-skeleton like inside M, for each $\alpha < \delta$ $\langle tp_{\{\varphi,\psi\}}(\bar{a}_i, A_\alpha) : i < cf(\delta) \rangle$ is eventually constant and p is a subset of $p^* \stackrel{def}{=} \{\theta(\bar{x},\bar{c}) : \bar{c} \subseteq M$, and for every $i < cf(\delta)$ large enough $\models \theta[\bar{a}_i, \bar{c}]$ and $\theta(\bar{x}, \bar{y})$ is $\in \{\varphi(\bar{x}, \bar{y}), \neg\varphi(\bar{x}, \bar{y}), \varphi(\bar{y}, \bar{x}), \neg\varphi(\bar{y},\bar{x})\}\}$ of power $< h(\delta)$ and $p^* \restriction A_\delta$ is realized in M *then* p is realized in M.

2) In (1), we say that M obeys $(h, \varphi(\bar{x},\bar{y}))$ exactly, if in (*), for $\delta \in C \cap S$, $\oplus$ fail for $h(\delta)^+$ (i.e. for some $\bar{a}_i$, p a is there, $|p| = h(\delta)$, p is not realized in M.

3.16 Theorem: Assume $\lambda > \sigma$, $\varphi(\bar{x},\bar{y})$ an asymmetric $\tau(K)$-formula, $\sigma = \ell g(\bar{x}) = \ell g(\bar{y})$. Suppose that for $I \in K_\chi^{tr}$ there is a τ-model $M_I \in K_\lambda$, weakly full $\varphi(\bar{x},\bar{y})$-represented in $\mathcal{M}_{\chi,\theta}(I)$ where $\lambda > \chi + \sigma^+ + \theta$ and for $s \in I$, $\bar{a}_s = \langle F_i(s) : i < \sigma \rangle \in {}^\sigma|M_I|$: $M_I \models \varphi[\bar{a}_s, \bar{a}_t]$ iff $s < t$ (for $s,t \in I$).

Then $I(\lambda, K) = 2^\lambda$ in the cases listed below and in some we get reasonable invariants.

Proof: Note that, letting $\kappa \stackrel{def}{=} \sigma^+ + \theta$:

(*) in M_I, $\langle \bar{a}_s : s \in I \rangle$ is weakly $(\kappa, <\mu, \varphi(\bar{x},\bar{y}))$-skeleton like in M_I, whenever $\mu \geq \kappa$, $\mu > \chi, \mu$ is regular.

Case A: $\lambda^\sigma = \lambda$

As $\kappa \stackrel{def}{=} \sigma^+ + \theta < \lambda$ we can apply 3.11 case (a), so we can assume $\lambda = \lambda < \lambda^\sigma$, from now on.

Case B: $\lambda^\sigma < 2^\lambda$, $\kappa < \lambda$ $cf\,\lambda > \sigma$. By 3.11(c).

Case C: λ is regular $(\forall \mu < \lambda)[\mu^{<\theta} < \lambda]$, $\lambda \geq \kappa^{++}$, $\chi < \lambda$. Let $S_0 = \{\delta < \lambda : cf\,\delta \geq \kappa\}$. For a function $h : S_0 \to \{\mu : \mu$ a regular cardinal, $\kappa \leq \mu < \lambda\}$ let I_h be the linear order, with set of elements $\{(\alpha,\beta) : \alpha < \lambda + \kappa, \beta < h(\alpha)$ if $\alpha \in S_0$ and $\beta < \kappa$ otherwise$\}$. Order is: $(\alpha_1,\beta_1) \leq (\alpha_2,\beta_2)$ *if and only if* $\alpha_1 < \alpha_2$ or $\alpha_1 = \alpha_2$, $\beta_1 \geq \beta_2$. Now

(a) M_{I_h} obeys $(h, (\varphi(\bar{x},\bar{y}))$ exactly (see Definition 3.16).

This clearly suffices and is easy.

Case D: Like case C but $\lambda = \kappa^+$, like case C but $h : S_0 \to \{\kappa^+, \kappa\}$. Using 3.17 below, we let $J = J^{[\lambda]} \times J^{[\kappa]} \in K^{or}_\chi$. Let $J = \bigcup_{\zeta<\lambda} J^0_\zeta$, J^0_ζ increasing continuous, $|J^0_\zeta| < \lambda$, $cf\,J = cf(J)^* = \lambda$, and for $\zeta < \lambda$: (W.l.o.g. by 3.17(2))

(*) if $s \in J - J^0_\zeta$ *then*

$cf(J \restriction \{t \in J : (\forall v \in J^0_\zeta) v < s \equiv v < t\}) = \lambda$ and $cf(J \restriction \{t \in J : t \in J^0_\zeta, t < s\}) \geq \kappa$

or $cf[J \restriction \{t \in J : (\forall v \in J_\zeta) v < s \equiv v < t)]^* = \lambda$

and $cf[J \restriction \{t \in J : t \in J^0_\zeta, s < t]^* \geq \kappa$

or $(\forall t \in J^0_\zeta)[t < s]$ and then $cf\,\zeta \geq \kappa \Rightarrow cf\,J^0_\zeta \geq \kappa$

or $(\forall t \in J^{0}_{\xi})[s < t]$ and then $cf\,\zeta \geq \kappa \Rightarrow cf[J^{0}_{\xi}]^{*} \geq \kappa$

Let $J^{\zeta}_{a}(\zeta < \lambda)$ be pairwise disjoint, each isomorphic to J^{*}. Let $J^{\zeta}_{b}(\zeta < \lambda)$ be pairwise disjoint, each isomorphic to $(J \times \kappa)^{*}$. Now for $h{:}S_0 \rightarrow \{\kappa^{+},\kappa\}$, $(h(\alpha+1) = \kappa^{+})$ let

$$I_h = (\sum_{\zeta<\lambda} J^{\zeta}_{h} + J^{[\kappa]}) \text{ where } J^{\zeta}_{h} = \begin{cases} J^{\zeta}_{b} & \text{if } h(\zeta) = \kappa^{+} \\ J^{\zeta}_{a} & \textit{otherwise} \end{cases}$$

Case E: $0 < \alpha < \mu_1 < \lambda$, $\mu_i (i < \alpha)$ strictly increasing, each μ_i regular $\mu_{i+1} > \mu_i^{+++}$, $\mu_i > \chi + \sigma^{+} + \theta$, $(\forall \mu < \mu_i)\mu^{<\theta} < \mu_i$, $\prod_i 2^{\mu_i} = 2^{\lambda}$ (without the last assumption we just get a smaller number of models)

We just sum things, where for each i we imitate case (C).

Let $J^{i} = J^{[\mu_i^{++}]}$ for $i < \lambda$ be from Fact 3.17, and for each i define $J_h \in K^{or}_{\mu_i^{+3}}$ for $h{:}\{\delta < \mu_i^{++}{:}cf\,\delta = \mu_i^{+}\} \rightarrow \{\mu_i^{+},\mu_i^{++}\}$ as $\sum_{\zeta<(\mu_i^{+3}+\kappa)} J^{i}_{\zeta}$, where: $\mu_i^{+3} + \kappa$ is ordinal addition, the J^{i}_{ζ} are pairwise disjoint, J^{i}_{ζ} is isomorphic to J^{i} except when $h(\zeta)$ is well defined an equal to μ_i^{+}, then J^{i}_{ζ} is isomorphic to $J^{i}_{\zeta} \times (\mu_i^{+})$. Lastly for every $\bar{h} \in \prod_i\{h{:}\text{Dom } h = \{\delta < \mu_i^{++} : cf\,\delta = \mu_i^{+}\}$, h as above$\}$, $I_{\bar{h}} \overset{def}{=} \sum_i J_{h_i}$.

More details in second version.

The fact we need is

3.17 Fact: For each regualr λ we can define a linear order $J^{[\lambda]}$ such that

(1) $J^{[\lambda]}$ is a dense linear order of cardinality λ.

(2) There is an algebra $N^{[\lambda]}$ with universe $J^{[\lambda]}$ and $\aleph_0$ finitary functions such that:

(*) if $I \subseteq J^{[\lambda]}$ is a subalgebra, $t \in J^{[\lambda]} - I$ then

$$cf[I \upharpoonright \{s \in I : s < t\}] = \lambda \quad or$$

$$cf[(I \upharpoonright \{s \in I : t < s\}]^*] = \lambda$$

(**) if $I \subseteq J^{[\lambda]}$ is a subalgebra then $I \subseteq_{dc} J^{[\lambda]}$, where:

Let $I \subseteq_{dc} J$ means I is a submodel of J as a linear order, and for $t \in J-I$, there is a maximal s, $s < t \wedge s \in I$ *or* there is a minimal s, $t < s \wedge s \in I$, or $(\forall s \in J)[s < t]$ or $(\forall s \in J)[t < s]$.

3) for each $t \in J^{[\lambda]}$

$$cf[J^{[\lambda]} \upharpoonright \{s \in J^{[\lambda]} : s < t\}] = \lambda \quad or$$

$$cf[(J^{[\lambda]} \upharpoonright \{s \in J^{[\lambda]} : t < s\})^*] = \lambda$$

4) if (J^1, J^2) is a Dedekind cut of $J^{[\lambda]}$ then $(cf(J^1),\ cf((J^2)^*))$ is one of

$$\{(\mu,\lambda) : \mu \leq \lambda\} \cup \{(\lambda,\mu) : \mu \leq \lambda\} \cup \{(\aleph_0, \aleph_0)\}$$

5)) if $\alpha < \lambda^+$, then $J^{[\lambda]} \times (\alpha+1)$ and $J^{[\lambda]}((\alpha+1)^*)$ are isomorphic to $J^{[\lambda]}$.

6) If $\lambda > \kappa$, every submodel of $J^{[\lambda]}$ of cardinality $\leq \kappa$ can be embedded into $J^{[\kappa]}$ (we use it just for a fixed pair $\lambda = \kappa^+$).

Proof: See [Sh. 220] (appendix) which relays a work of Galvin and Laver cited there.

§4 The easy black box and an easy application

4.1 Discussion:

The non structure theorem we have discused so far rests usually on some freedom on finite sequences and on a kind of order. When our freedom is related to infinite ones, and to trees, our work is sometimes harder. In particular, we have to consider, for $(\lambda \geq \chi, \chi$ regular):

(*) We have for $S \subseteq {}^{\chi \geq}\lambda$ a model M_S, $\bar{a}_\eta(\eta \in {}^{\chi >}\lambda, \ell g(\bar{a}_\eta) = \ell g(\bar{x}_{\ell g(\eta)})$ such that for $\eta \in {}^{\chi}\lambda : M_S \models \varphi(\cdots \bar{a}_{\eta \restriction \alpha})_{\alpha < \chi}$ if and only if $\eta \in S$

(and M_S is quite "simply defined" from S). Of course, if we do not ask more from M_S, we can get nowhere: we certainly restrict its power and usually it is φ-representable in a variant $\mathcal{M}_{\mu,\kappa}(S)$ (for suitable μ, κ). Certainly for T unsuperstable we have such a formula φ

$$\varphi = (\exists \bar{x}) \bigwedge_n \varphi_n(\bar{x}, \bar{a}_{\eta \restriction n})$$

Here we do not try to get the best results, just exemplify some (i.e. we do not present the results when $\lambda = \lambda^\chi$ is replaced by $\lambda = \lambda^{<\chi}$) By the proof of [Sh, VIII 2.5] (see later a complete proof).

4.2 Theorem: Suppose $\lambda = \lambda^\chi$ and (*) of 4.1 holds for φ and $|||M_S||| = \lambda$ for

$${}^{\chi >}\lambda \subseteq S \subseteq {}^{\chi \leq}\lambda$$

and $\ell g(\bar{a}_\eta) \leq \chi$, or just $\lambda^{\ell g(\bar{a}_\eta)} = \lambda$: *then* (using ${}^{\chi \geq}\lambda \subseteq S \subseteq {}^{\chi \geq}\lambda$).

1) there is no model M of power λ into which every M_S can be $(\pm\varphi)$-embedded (i.e. by a function preserving φ and $\neg\varphi$).

2) For any $M_i (i < \lambda)$, $|||M_i||| = \lambda$, for some S, $({}^{\chi >}\lambda \subseteq S \subseteq {}^{\chi \geq}\lambda)$, M_S cannot be $\pm\varphi$-embedded into any M_i.

4.3 Example: Look at Boolean Algebras.

$$\varphi(\cdots, a_n, \cdots) \stackrel{def}{=} (\bigcup_n a_n) \neq 1 \equiv \text{there is no } x \neq 0, \chi \cap a_n = 0 \text{ for each } n$$

Let for ${}^{\omega >}\lambda \subseteq S \subseteq {}^{\omega \geq}\lambda$, M_S be the Boolean Algebra generated freely by χ_η $(\eta \in S)$ except the relations: for $\eta \in S$, if $n < \ell g(\eta) = \omega$ then $x_\eta \cap x_{\eta \restriction n} = 0$. So $|||M_S||| = |S| \in [\lambda, \lambda^{\aleph_0}]$, in M_S for $\eta \in {}^{\omega}\lambda$, $M_S \models (\bigvee_n x_{\eta \restriction n}) = 1$ *if and only if* $\eta \notin S$ (work a little in Boolean Algebra). So

4.4 Conclusion: If $\lambda = \lambda^{\aleph_0}$, there is no Boolean Algebra B of power λ universal under σ-embeddings. (See [Sh 2.2, VII Ex. 2.2]).

For another application on locally finite groups-usual embeddings, see Grossberg and Shelah - [GrSh 174]. A related work is Dugas, Fay and Shelah [DFSh 262].

Proof of the Theorem 4.2: It is enough to prove (2), w.l.o.g. $|M_i|$ are pairwise disjoint subsets of λ. Now

4.5 Fact: Assume $\lambda = \lambda^\chi$. Let $\bar{a}_\eta (\eta \in {}^{\chi>}\lambda)$ be given, each of length $\leq \chi$.

There are functions $f_\eta (\eta \in {}^\chi\lambda)$ such that

(i) Dom $f_\eta = \bigcup_{\alpha<\chi} \bar{a}_{\eta \restriction \alpha}$

(ii) *Rang* $f_\eta \subseteq \lambda$

(iii) if $f: \bigcup_{\eta \in {}^{\chi>}\lambda} \bar{a}_\eta \to \lambda$, *then* for some $\eta \in {}^\chi\lambda$, $f_\eta \subseteq f$.

Remark: We prove this in 1969/70 (for lower bounds on $I(\lambda,T,T)$, T unsuperstable, but it was superseded, eventually the method was used in one of the cases in [Sh VIII §2]: for strong limit singular [Sh VIII 2.6]. It was developed in [Sh 172] [Sh 227] for constructing abelian groups with prescribed endomorphism groups and further see Eklof and Mekler [EkMk], this version was developed for a proof of the existence of abelian (torsion free $\aleph_1$-free) group G with $G^{***} = G^* \oplus A$ $(G^* \stackrel{def}{=} Hom(G,Z)$ in a work by Mekler and Shelah.

Proof of Fact 4.5: Let $\{\langle \bar{b}^i_\alpha : \alpha < \gamma \rangle : i < \lambda\}$ list all sequences of the form $\langle \bar{b}_\alpha : \alpha < \gamma \rangle$ such that $\gamma < \chi$, $\bar{b}_\alpha \subseteq \lambda$, $\ell g(\bar{b}_\alpha) \leq \chi$.

For $\eta \in {}^\chi\lambda$, f_η is the function (with domain $\bigcup \bar{a}_{\eta \restriction \alpha}$) such that:

$$f_\eta(\bar{a}_{\eta \restriction \alpha}) \equiv \bar{b}^{\eta(\alpha)}_\alpha \text{ if it is defined and } f_\eta(\bar{a}_{\eta \restriction \alpha}) = \langle 0 : i < \ell g(\bar{a}_\eta) \rangle \text{ otherwise.}$$

So $\langle f_\eta : \eta \in {}^\chi\lambda \rangle$ is well defined. Properties (i),(ii) are straightforward, so let us prove (iii). Let $f: \bigcup_{\eta \in {}^{\chi>}\lambda} \bar{a}_\eta \to \lambda$. We define $\eta_\alpha = \langle \beta_i : i < \alpha \rangle$ by induction on α. $\alpha = 0$ or α limit - no problem.

$\alpha+1$: be β_α be minimal such that $\bar{b}^{\beta_\alpha}_\alpha = f(\bar{a}_{\eta_\alpha})$.

So $\eta \stackrel{def}{=} \langle \beta_i : i < \chi \rangle$ is as required.

Remark: We can present it as a game. (See in the book [Sh, VIII 2.5]).

Continuation of the proof of the Theorem 4.2:

Now define

$S = ({}^{\chi >}\lambda) \cup \{\eta \in {}^{\chi}\lambda$: for some $i < \lambda$, $Rang(f_\eta) \subseteq |M_i|$ and $M_i \models \neg\varphi(\ldots, f_\eta(\bar{a}_{\eta \restriction \alpha}), \ldots)\}$.

Look at M_S. Clearly

(a) no $\pm\varphi$-embedding of M into M_i extends $f_\eta, \eta \in {}^{\chi}\lambda$.

For if $f : M \to M_i$ is a $(\pm\varphi)$-embedding we have by Fact 4.5 that for some $\eta \in {}^{\chi}\lambda$, $f \restriction \bigcup_{\alpha < \chi} \bar{a}_{\eta \restriction \alpha} = f_\eta$.

§5 An application of a better black box, enough usually for $I(\lambda, \mathbf{K}) = 2^\lambda$ for most λ for a complicated $\mathbf{K}$

5.0 More Discussion

Next we consider:

Assume λ is regular, $(\forall \mu < \lambda)\mu^{<\chi} < \lambda$. Let $T_\alpha \subseteq \{\delta < \lambda : cf\, \delta = \chi\}$ be pairwise disjoint stationary sets. For $A \subseteq \lambda$

$$T_A = \bigcup_{i \in A} T_i.$$

We want to define S_A

$${}^{\chi >}\lambda \subseteq S_A \subseteq {}^{\chi \geq}\lambda$$

such that

$$A \not\subseteq B \to M_{S_A} \not\equiv M_{S_B}$$

Of course we have to strengthen the restrictions on M_S. For $\eta \in S_A \cap {}^{\chi}\lambda$, if η is increasing converging to some $\delta \in T_A$, denote this δ by $\delta(\eta)$.

The decision whether $\eta \in S_A$ will be done by induction on $\delta(\eta)$. Arriving to η, we are assuming we know quite a lot on

$$f \restriction \bigcup_{\alpha<\chi} \bar{a}_{\eta\restriction\alpha}$$

which we are trying to kill, in particular that (if $M_S = \bigcup_{i<\lambda} M_S^i$, $\| M_S^i \| < \lambda$, M_S^i increasing-continuous in i and we can assume $\delta(\eta) \notin T_B$ because we can use a club of $\delta(\eta)$'s.).

5.0A Notation: 1) Let, for an ordinal α and a regular $\theta \geq \aleph_0$, $H_{<\theta}(\alpha)$ be the smallest set Y such that

(i) $i \in Y$ for $i < \alpha$

(ii) $x \in Y$ for $x \subseteq Y$ of cardinality $< \theta$

2) We can agree that $\mathcal{M}_{\lambda,\theta}(\alpha)$ is interpretable. in $(H_{<\theta}(\alpha),\in)$ and in particular its universe is a definable subset of $H_{<\theta}(\alpha)$, and also R is where:

$$R = \{(\sigma^*,\langle t_i : i < \gamma_x\rangle,x) : x \in \mathcal{M}_{\mu,<\theta}({}^{\theta>}\lambda)$$

$$\leq \mu \leq \alpha, x = \sigma^*(\langle t_i^* : i < \gamma_x\rangle)\}.$$

etc.

The main theorem of the section is:

5.1 Theorem: $IE_{\pm\varphi}(\lambda,K) = 2^\lambda$ provided that:

(a) $\lambda = \lambda^\chi$:

(b) $\varphi = \varphi(\cdots \bar{x}_\alpha)_{\alpha<\chi})$ with vocabulary τ_K.

(c) for every $S, {}^{\chi>}\lambda \subseteq S \subseteq {}^{\chi\geq}\lambda$, there is a model $M_S \in K_\lambda$, and $\bar{a}_\eta \in (M_S)$ for $\eta \in {}^{\chi>}\lambda$ $\ell g(\bar{x}_\eta) = \ell g(\bar{x}_{y(\eta)})$ such that

(α) for $\eta \in {}^{\chi}\lambda$: $M_S \models \varphi(\cdots a_{\eta \restriction \alpha} \cdots)$ if and only if $\eta \in S$.

(β) there are $f = f_S : M_S \to \mathcal{M}_{\mu,\kappa}(S)$ where $\mu \leq \lambda, \kappa = \chi^+$ such that

(*) if $\bar{b}_\alpha \in M_S$, $\ell g(\bar{b}_\alpha) = \ell g(\bar{a}_\eta)$ for $\eta \in {}^{\alpha}\kappa$, $f(\bar{b}_\alpha) = \bar{\sigma}_\alpha(\bar{t}_\alpha)$ then the truth value of $M_S \models \varphi[\cdots \bar{b}_\alpha \cdots]_{\alpha<\chi}$ can be computed from $\langle \bar{\sigma}_\alpha : \alpha < \chi \rangle$, the q.f. type of $\langle \bar{t}_\alpha : \alpha < \chi \rangle$ in S and the truth values or $(\exists \nu \in {}^{\chi}\lambda)[\bigwedge_{i<\chi} \nu \restriction \alpha_i = \bar{t}_{\beta_i}(\gamma_i) \restriction \varepsilon_i]$ for $\alpha_i, \beta_i, \gamma_i, \varepsilon_i < \chi$ (i.e. in a way not depending on S). [we can weaken this]

5.2 Fact: Suppose

(*) $\lambda = \lambda^{2^\chi}$, $cf\,\lambda > \chi$

Then there are $\{(M^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}$ such that

(i) *for every* model M with universe $H_{<\chi^+}(\lambda)$, $|\tau(M)| \leq \chi$ for some α, $M^\alpha \prec M$.

(ii) $\eta^\alpha \in {}^{\chi}\lambda$, $(\forall i < \chi)[\eta^\alpha \restriction i \in M^\alpha]$, $\eta^\alpha \notin M^\alpha$ and $\alpha \neq \beta \Rightarrow \eta^\alpha \neq \eta^\beta$.

(iii) for every $\beta < \alpha$, $\{\eta^\alpha \restriction i : i < \chi\} \not\subseteq M^\beta$

(iv) for $\beta < \alpha$ if $\{\eta^\beta \restriction i : i < \chi\} \subseteq M^\alpha$ then $|M^\beta| \subseteq |M^\alpha|$.

(v) $\|M^\alpha\| = \chi$, $|M^\alpha| \subseteq H_{<\chi^+}(\delta(\eta^\alpha))$

Proof of 5.2: See 6.x.

5.3 Proof of 5.1 from the conclusion of 5.2:

W.l.o.g. $|M_S| = \lambda$ in 5.1.

We shall define for every $A \subseteq \lambda$ a set $S[A]$, ${}^{\chi>}\lambda \subseteq S[A] \leq {}^{\chi\geq}\lambda$.

Let $T_\alpha = \{\eta \in {}^{\chi}\lambda : \{\eta \restriction i : i < \chi\} \subseteq M^\alpha\}$. We shall define by induction on α, for every A, $S[A] \cap T_\alpha$ so that on the one hand those restriction are compatible, (so that we can define $S[A]$ in the end, for each $A \subseteq \lambda$) and on the other hand they guarantee the non $\pm\varphi$-embeddability

For each α:

Case I: *if* there are distinct subsets A_1, A_2 of λ, and ${}^{\chi>}\lambda \subseteq S_1, S_2 \subseteq {}^{\chi\geq}\lambda$ and a $\pm\varphi$-embedding f of M_{S_1} into M_{S_2} and

$$M^\alpha \prec (H_{<\chi^+}(\lambda \in R), A_1, A_2, S_1, S_2, M_{S_1}, M_{S_2}, f_{S_1}, f_{S_2}, f)$$

where $R = \{(\sigma, \sigma_x, x), (1+i, t_i^x, x) : \ x$ has the form $\sigma_x(\langle t_i^x : i < \sigma_x\rangle)\}$ (we choose for each x a unique such term σ and $S_2 \cap T_\alpha \subseteq S_2 \cap (\bigcup_{\beta<\alpha} T_\beta)$ and S_2 satisfies the restriction imposed for each $\beta < \alpha$, and computing according to (*) of 5.1 the truth value $\mathbf{t}^\alpha$ of $M_{S_2} \models \varphi[\cdots, f(\bar{a}_{\eta^\alpha \restriction i}), \cdots)]_{i<\chi}$, *then* we restrict:

(i) if $B \subseteq \lambda$, $B \cap |M^\alpha| = A_2 \cap M^\alpha$ then

$$S[B] \cap (T^\alpha - \bigcup_{\beta<\alpha} T^\beta) = \emptyset$$

(ii) if $B \subseteq \lambda$, $B \cap |M^\alpha| = A_1 \cap |M^\alpha|$ and $\mathbf{t}^\alpha$ is truth then

$$S[B] \cap (T^\alpha - \bigcup_{\beta<\alpha} T^\beta) = \emptyset$$

(iii) if $B \subseteq \lambda$, $B \cap |M^\alpha| = A_1 \cap |M^\alpha|$ and $\mathbf{t}^\alpha$ is false then

$$S[B] \cap (T^\alpha - \bigcup_{\beta<\alpha} T^\beta) = \{\eta^\alpha\}$$

Case II: not I.

No restriction is imposed.

The point is

5.3A Fact: The choice of A_1, A_2, S_1, S_2, f is immaterial (any two candidates lead to the same decision).

5.3B Fact: $M_{S[A]}$ $(A \subseteq \lambda)$ are pairwise non isomorphic, moreover for $A \neq B \subseteq \lambda$ there is no $(\pm\varphi)$-embedding of $M_{S[A]}$ into $M_{S[B]}$.

* * *

Still the assumption of 5.2 is too strong. However a statement weaker than the conclusion of 5.2 holds under weaker cardinality restrictions and the proof 5.3 of 5.1 above works using it, thus we finish the proof of 5.1.

5.4 Fact: Suppose $\lambda = \lambda^\chi$

Then there are $\{(M^\alpha, A_1^\alpha, A_2^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}$ such that:

(*) (i) *for every* model M with universe $H_{<\chi^+}(\lambda)$, $|\tau(M)| \leq \chi$ (arity of relations and functions finite) and sets $A_1 \neq A_2 \subseteq \lambda$ *for some* $\alpha < \alpha(*)$, $(M^\alpha, A_1^\alpha, A_2^\alpha) \prec (M, A_1, A_2)$

(ii) $\eta^\alpha \in {}^\chi\lambda$, $\{\eta^\alpha \restriction i : i < \chi\} \subseteq |M^\alpha|$, $\eta^\alpha \notin M^\alpha$, and $\alpha \neq \beta \Rightarrow \eta^\alpha \neq \eta^\beta$.

(iii) for every $\beta < \alpha$, if $\{\eta^\alpha \restriction i : i < \chi\} \subseteq M^\beta$ *then* $\alpha < \beta + 2^\chi$, $|M^\alpha| \subseteq |M^\beta|$, and $A_1^\alpha \cap |M^\alpha| \neq A_2^\beta \cap |M^\alpha|$.

(iv) for every $\beta < \alpha$ if $\{\eta^\beta \restriction i : i < \chi\} \subseteq M^\alpha$ *then* $|M^\beta| \subseteq |M^\alpha|$

Proof: See 6.x.

Hint:: for λ regular.

Let $\langle S_\zeta : \zeta < \lambda\rangle$ be pairwise disjoint stationary subsets of $\{\delta < \lambda : cf(\delta) = \chi\}$. We define for each $\zeta < \lambda$, $\{(M^\alpha, A^\alpha, B^\alpha, \eta^\alpha)$, $\alpha \in \{(M^\zeta_1, A^\zeta_\alpha, B^\zeta_\alpha, \eta^\zeta_\alpha) : \alpha < \alpha_\zeta\}$ such that from (*) of 5.4, (i) holds when $\zeta \in A_1 - A_2$, as well as (ii), (iii), and $\sup(M^\zeta_\alpha \cap \lambda) < \lambda$. See 6.x.

Then we combine those sets (no serious problems).

Section 6 will appear in the second version.

Chapter IV: K is not smooth or not χ- based

We deal in this chapter with two dividing lines: smoothness and being χ-based both absent in the first order case (but the second is somewhat parallel to stability).

We do some positive theory without them, just enough to show that their negation has strong nonstructure consequences. Once they are out of the way, much of the theory for stable theories can be redone.

Recall that we work in (AxFr_1) (in particular limits exists but smoothness may fail: $\langle M_i : i \le \delta \rangle$ is $\le_{\mathbf{K}}$-increasing, but $\bigcup_{i<\delta} M_i \not\le_{\mathbf{K}} M_\delta$.)

More on smoothness see Chapter VIII §x. In later versions we will remove the regularity assumption from the non structure theorems and restrict Ax(A4) to smooth chains.

§1 Non Smoothness implies Non Structure

1.0 Context: AxFr_1.

Our main aim in this section is told by its title. Remember *that* **K** *is smooth if:* $\bigcup_{i<\delta} M_i \le M$ *when* $\langle M_i : i < \delta \rangle$ *is* $\le$*-increasing, and for every* $i < \delta$, $M_i \le M$. *The main theorem is 1.11: if* λ *is* regular and K-inaccessible, and there is a counterexample to smoothness by $\langle M_i : i < \delta \rangle$, M, with $|\delta| + \sum_{i<\delta} \|M_i\| < \lambda$ then $I(\lambda, K) = 2^\lambda$ (usually there are 2^λ models no one $\le_{\mathbf{K}}$-embeddable into another.)

Note that we may tend to accept smoothness "without saying", as it is trivial for first order theories, hence should be careful with claims being proved without it. However, the phenomenon occurs also for first order T, if we look at *{M : M* a $|T|^+$-saturated model of *T}* under a suitable order $<^*$ (as in e.g. [Sh 48]) and then such a property was called didip (dimensional discontinuity property, see [Sh 132], [ShA1 Ch X]). But there we always have sequences of length $<\kappa_r(T)$.

Our main theorem 1.11 has some defect: first the requirement that λ is regular and **K**-inaccessible. By our "adopted rules of the game" this is not serious. More troublesome is that we have no theorem showing that if κ-smoothness fails then $(<\kappa_m(\mathbf{K}))$-smoothness fail for some reasonably small $\kappa_m(\mathbf{K})$. The remedy we have is to use V1.1; by it $(\leq\chi)$-smoothness + NF is χ-based implies smoothness.

So "if **K** is not $(\leq LS(\mathbf{K}))$-smooth or NF is not $LS(\mathbf{K})$-based *then* $I(\lambda,K) = 2^\lambda$ for every regular $\lambda = \lambda^{LS(\mathbf{K})}$ etc". See end of the section.

Context: Axiomatic Framework 1. of II §1.

The next several results are Lemmas for the proof of Theorem 1.11. Specifically Claim 1.7 carries out a major step in the construction; Claims 1.1 and 1.6 are used to prove Claim 1.7 .

One of the basic tools of first order stability theory is the " transitivity of non forking": let $A \subseteq B \subseteq C$, if $tp(a,C)$ does not fork over B and $tp(a,B)$ does not fork over A then $tp(a,C)$ does not fork over A. Claim 1.1 is a slightly disguised version of this principle in framework AxFr_1. (Let M_1 play the role of a and M_0, M_2, M_2 play the role of A, B, C resp; the second hypothesis of Claim 1.1 is then apparently stronger than a direct translation. However replacing M_3 by the model generated by M_1 and M_2 yields the original situation).

1.1 Claim: If $NF(M_0,M_1,M_2,M_3)$ and $NF(M_2,M_3,M_4,M_5)$ then $NF(M_0,M_1,M_4,M_5)$.

1.1A Definition: We call this claim transitivity of NF. Ax (E4).

Proof: Let $M'_3 = \langle M_1 \cup M_2 \rangle^{gn}_{M_3}$, so by Axiom (C4), (and Ax (C1)) $M'_3 \leq M_3$, so by Ax (C3) (c) (a monotonicity) $NF(M_0,M_1,M_2,M'_3)$. So by Ax (C1), $M_2 \leq M'_3 \leq M_3$, and by Axiom (C3)(a) + (C6) (symmetry), [alternatively, by (C3) (a)^d] we get $NF(M_2,M'_3,M_4,M_5)$.

Similarly, letting $M'_5 = \langle M'_3 \cup M_4 \rangle^{gn}_{M_5}$ we get $M'_5 \le M_5, NF(M_2, M'_3, M_4, M'_5)$.

By Axiom (C2) (existence), there are M''_4, M''_5 and an isomorphism g from M_4 onto M''_4 over M_0, such that $NF(M_0, M_1, M''_4, M''_5)$, and w.l.o.g. (by Ax (C3)(c), and Ax (C4)) $M''_5 = \langle M_1 \cup M''_4 \rangle^{gn}_{M''_5}$. Let $M''_2 = g(M_2)$, so $M_0 \le M''_2 \le M''_4$.

Let $M''_3 = \langle M_1 \cup M''_2 \rangle^{gn}_{M''_5}$. By the base enlargment axiom (C4) (and (C1)) $M''_3 \le M''_5$ so by Ax (C3), (first (a), then (c)) $NF(M_0, M_1, M''_2, M''_3)$. By Ax (C4) $NF(M''_2, M''_3, M''_4, M''_5)$, and clearly $M''_5 = \langle M''_3 \cup M''_4 \rangle^{gn}_{M''_5}$, $M''_3 = \langle M_1 \cup M''_2 \rangle^{gn}_{M''_3}$. Applying twice the uniqueness (Axiom (C5)) we can extend g to an isomorphism g'' from M'_5 onto M''_5, $g''(M'_3) = M''_3$, g the identity over M_1. As everything is preserved by isomorphism, clearly $NF(M_0, M_1, M_4, M'_5)$. By Ax (C3) (b) $NF(M_0, M_1, M_4, M_5)$.

1.2 Fact: Suppose that for $\ell = 0,1$, $\langle M_{\ell,i} : i \le \delta \rangle$ is increasing continuous and for each $i \le \delta$, $NF(M_{0,i}, M_{1,i}, M_2, M)$ (δ of course, is a limit ordinal). Then $\langle M_{1,\delta} \cup M_2 \rangle^{gn}_M = \bigcup_{i<\delta} \langle M_{1,i} \cup M_2 \rangle^{gn}_M$.

Proof: We prove this by induction on the ordinal δ. Let for $i \le \delta$, $N_i = \langle M_{1,i} \cup M_{0,\delta} \rangle^{gn}_M$, by Axiom (C4) $NF(M_{0,\delta}, N_i, M_2, M)$, and clearly $[i < j \Rightarrow N_i \subseteq N_j]$ (by Ax (B2)). We prove by induction on $\alpha \le \delta$ that $\langle N_i : i \le \alpha \rangle$ is increasing and continuous. If α is not limit, this is trivial; if α is limit $<\delta$ use the induction hypothesis. Lastly if $\alpha = \delta$ by Ax (B2) $[i < \delta \Rightarrow N_i \subseteq N_\delta)$ hence $\bigcup_{i<\delta} N_i \subseteq N_\delta$; on the other hand $M_{0,\delta} \subseteq M_{1,\delta}$ hence $N_\delta = \langle M_{1,\delta} \cup M_{0,\delta} \rangle^{gn}_M = \langle M_{1,\delta} \rangle^{gn}_M = M_{1,\delta} = \bigcup_{i<\delta} M_{1,i} \subseteq \bigcup_{i,\delta} N_i$. Together $N_\delta = \bigcup_{i,\delta} N_i$ as required, so $\langle N_i : i < \delta \rangle$ is really $\le$-icreasing continuous. Now apply Ax C7 with $M_{0,\delta}$ as M_0, N_i as $M_{1,i}, M_2$ as M_2 and M as M to conclude $\bigcup_{i<\delta} \langle N_i \cup M_2 \rangle^{gn}_M = \langle N_\delta \cup M_2 \rangle^{gn}_M$. Untangling our notation note that $M_{1,\delta} = N_\delta$ and $\langle N_i \cup M_2 \rangle^{gn}_M = \langle M_{1,i} \cup M_2 \rangle^{gn}_M$ (since $M_{0,\delta} \subseteq M_2$, by Ax (B0), (B1), (B2)). Substituting we conclude $\bigcup_{i<\delta} \langle M_{1,i} \cup M_2 \rangle^{gn}_M = \langle N_\delta \cup M_2 \rangle^{gn}_M$ as required.

Remark: Fact 1.2 is a natural strengthening of axiom (C7). Instead of fixing an M_0 such that NF (M_0, $M_{1,i}$, M_2, M) we have allowed the base $M_{0,i}$ to vary with i.

* * *

The next two lemmas are easier to understand as part of the proof of Lemma 1.1 of Chapter V. Specifically Lemma 1.4 is the core of the proof of the μ-based implies μ'-based (for $\mu' > \mu$ when **K** is $(\leq\mu,\leq\mu)$-smooth). Lemma 1.3 is used to prove Lemma 1.4 (and the proof of 1.4 is used in the proof of 1.6).

Lemma 1.3 asserts that if $\langle M_i : i \leq \delta \rangle$ is an $\leq$-increasing continuous sequence, $N_i = \langle M_i \cup N_0 \rangle_{N_i}^{gn}$ is also $\leq$-increasing continuous and for $i < j$, $NF(M_i, N_i, M_j, N_j\rangle$ then $M_\delta \leq N_\delta$ and some further corollaries. If, in the nonforking condition, we could replace M_i by M_0, M_j by M_δ, and N_j by N_δ we would be in the situation of axiom (C7). The proof proceeds by showing that we achieve this happy situation by replacing M_δ, N_δ by isomorphic copies which are independent from N_0 over M_0. After applying axiom (C7) we return to the original models by the invariance of nonforking under isomorphism.

1.3 Claim: Suppose $\langle M_i : i \leq \delta \rangle$, $\langle N_i : i \leq \delta \rangle$ are $\leq$-increasing continuous and for $i < j < \delta$, $NF(M_i, N_i, M_j, N_j)$ and $N_i = \langle M_i \cup N_0 \rangle^{gn}_{N_i}$. *Then* $M_\delta \leq N_\delta$ and for $i < \delta$, $NF(M_i, N_i, M_\delta, N_\delta)$, $N_\delta = \langle M_\delta \cup N_0 \rangle^{gn}_{N_\delta}$.

Proof: There are M'_δ, N'_δ and g such that $NF(M_0, N_0, M'_\delta, N'_\delta)$, and g is an isomorphism from M_δ onto M'_δ over M_0, $N'_\delta = \langle M'_\delta \cup N_0 \rangle^{gn}_{N'_\delta}$. Let $N'_i = \langle M'_i \cup N_0 \rangle^{gn}_{N'_\delta}$ where $M'_i = g(M_i)$. By Axiom (C3), (C4) for $i < j < \delta$, $NF(M'_i, N'_i, M'_j, N'_j)$, $\langle N'_i : i \leq \delta \rangle$ is increasing and by Fact 1.2 also continuous. So by Axiom (C5) we can define by induction on $i \leq \delta$, g_i, an isomorphism from N_i onto N'_i extending $(g \restriction M'_i) \cup id_{N_0}$ and every $g_j (j < i)$. Now g_δ shows that $NF(M_i, N_i, M_\delta, N)$ (as $NF(M'_i, N'_i, M'_\delta, N'_\delta)$) and $N = \langle M'_\delta \cup N'_0 \rangle^{gn}_{N_\delta}$ (as $N'_\delta = \langle M_\delta \cup N_0 \rangle^{gn}_{N'_\delta}$).

1.4 Claim: Suppose $\langle N_i : i \leq \delta \rangle$, $\langle M_i : i \leq \delta \rangle$ are increasing continuous, and for $i < j < \delta$, $NF(M_i, N_i, M_j, N_j)$. Then $M_\delta \leq N_\delta$ and for $i < \delta$, $NF(M_i, N_i, M_\delta, N_\delta)$.

Proof: The proof will proceed by applying the following subclaim first to the given $\langle M_i : i < \delta \rangle$, $\langle N_i : i < \delta \rangle$ and then to a second set. We use the following notation.

Let for $i \le j < \delta$, $N_{i,j} = \langle M_j \cup N_i \rangle^{gn}_{N_j}$.

Let $N_{i,\delta} = \bigcup_{i \le j < \delta} N_{i,j}$.

1.4A Subclaim: Let $\langle M_i : i < \delta \rangle$, $\langle N_i : i < \delta \rangle$ satisfy the hypothesis of Claim 1.9. Then (for $i < \delta$):

a) $M_\delta \le N_{i,\delta}$.

b) $NF(M_j, N_{i,j}, M_\delta, N_{i,\delta})$ (when $i \le j < \delta$)

c) $N_{i,\delta} = \langle M_\delta \cup N_i \rangle^{gn}_{N_{i,\delta}}$

d) For $(i <) j_1 < j_2 < \delta$, $NF(N_{i,j_1}, N_{j_1}, N_{i,j_2}, N_{j_2})$.

e) for each $i < \delta$, $\langle N_{i,j} : i \le j < \delta \rangle$ is increasing continuous.

Proof of 1.4A: By Axiom (C4), $N_{i,j} \le N_j$, and (together with Axiom (C3)) for $i \le j_1 < j_2 < \delta$, $NF(M_{j_1}, N_{i,j_1}, M_{j_2}, N_{i,j_2})$, and clearly $N_{i,j_2} = \langle M_{j_2} \cup N_{i,j_1} \rangle^{gn}_{N_{i,j_2}}$. By Fact 1.2 for each i $\langle N_{i,j} : i \le j < \delta \rangle$ is not only $\le$-increasing but also continuous [i.e. (e) holds]. Remember $N_{i,\delta} = \bigcup_{i \le j < \delta} N_{i,j}$. So by 1.8 $M_j \le N_{i,\delta}$ [so (a) holds] and $NF(M_j, N_{i,j}, M_\delta, N_{i,\delta})$ [so (b) holds] and $N_{i,\delta} = \langle M_\delta \cup N_i \rangle^{gn}_{N_{i,\delta}}$ [so (c) holds]. By Axiom (C4) if $i \le j_1 < j_2 < \delta$, then $NF(N_{i,j_1}, N_{j_1}, N_{i,j_2}, N_{j_2})$ [so (d) holds].

Proof: We return to the proof of 1.4. Applying the subclaim to the original sequences $\langle M_i : i < \delta \rangle$ and $\langle N_i : i < \delta \rangle$ we see by e) and d) that for each i the sequences $\langle N_{i,j} : i \le j < \delta \rangle$ as $\langle M_j : 0 \le j < \delta \rangle$ and $\langle N_j : i \le j \le \delta \rangle$ as $\langle N_j : 0 \le j < \delta \rangle$ satisfy the hypothesis of 1.4 and thus 1.4A (now indexed by j). Applying the subclaim to these sequences we conclude by (a) that for $i_1 < i_2 < \delta$, $N_{i_1,\delta} \le N_{i_2,\delta}$. Applying 1.3 with M_i as M_i and $N_{0,i}$ as N_i we conclude $M_\delta \le N_{0,\delta}$.

Now note that $\bigcup_{i<\delta} N_{i,\delta}$ includes each N_i $(i < \delta)$ hence includes $\bigcup_{i<\delta} N_i$, but this is N_δ (as $\langle N_i : i \le \delta \rangle$ is increasing continuous so $N_\delta = \bigcup_{i<\delta} N_{i,\delta}$. As we have noted above that $\langle N_{i,\delta} : i < \delta \rangle$ is $\le$-increasing by Ax (A4) we know that for $i < \delta$, $N_{i,\delta} \le \bigcup_{\zeta<\delta} N_{\zeta,\delta}$. [by 1.4B below we can apply Ax (A4) for smooth chains only]. By the last sentence, this says $N_{i,\delta} \subseteq N_\delta$. As we have noted above that $M_\delta \le N_{0,\delta}$, we get $M_\delta \le N_\delta$, one of our desired conclusions. Note that also for limit $\alpha < \delta$, $N_{\alpha,\delta} = \bigcup_{i<\alpha} N_{i,\alpha}$. We prove this by induction on δ. For one inclusion, for $i < \alpha$, $N_i \subseteq N_\delta$ hence $\alpha < j < \delta$: $N_{i,j} = \langle M_j \cup N_j \rangle^{gn}_{N_j} \subseteq \langle M_j, N_\alpha \rangle^{gn}_{N_\alpha} = N_{\alpha,j}$ so taking unions $N_{i,\delta} \subseteq N_{\alpha,\delta}$. For the other inclusion clearly when $i, \alpha \le j < \delta$, $N_{i,j} = \langle M_j, N_{i,\alpha} \rangle$ Now, the first application of the subfact yielded $NF(M_j, N_{i,j}, M_\delta, N_{i,\delta})$, by definition $N_{i,i} = N_i$ and by the second application of the subfact $N_{i,\delta} \le N_\delta$ (and $M_\delta \le N_{i,\delta}$, $N_{i,i} \le N_{i,\delta}$). Substituting and applying the monotonicity axiom we have $NF(M_i, N_i, M_\delta, N_\delta)$ (the second conclusion of 1.4).

1.4B Subfact: $\langle N_{i,\delta} : i < \delta \rangle$ is ($\le$-increasing and) continuous.

Proof: We prove this by induciton on δ. Let $\alpha < \delta$ be a limit ordinal and we should show that $N_{\alpha,\delta} = \bigcup_{i<\alpha} N_{i,\delta}$. For one inclusion, for $i < \alpha$, $N_i \le N_\delta$ hence when $i, \alpha \le j < \delta$, $N_{i,j} = \langle M_j \cup N_i \rangle^{gn}_{N_j} \subseteq \langle M_j \cup N_\alpha \rangle^{gn}_{N_\alpha} = N_{\alpha,j}$, so taking unions $N_{i,\delta} \subseteq N_{\alpha,\delta}$. Hence; $\bigcup_{i<\alpha} N_{i,\delta} \subseteq N_{\alpha,\delta}$; for the other inclusion clearly when $i, \alpha \le j < \delta$, $N_{i,\delta} \subseteq N_{\alpha,\delta}$ for the $\bigcup_{i<\alpha}$ $N_{i,j} = \langle M_j \cup N_{i,\alpha} \rangle^{gn}_{N_\delta}$ and $\langle N_{i,\alpha} : i < \alpha \rangle$ is increasing continuous (by the induction hypothesis on δ). Easily $\bigcup_{i<\alpha} N_{i,\alpha} = N_\alpha$. Also we know for $i < \alpha$, $NF(M_\alpha, N_{i,\alpha}, M_\delta, N_{i,\delta})$, hence $NF(M_\alpha, N_{i,\alpha} M_\delta, N_{\alpha,\delta})$. By Ax (C7) we conclude that $\langle N_\alpha, M_\delta \rangle^{gn}_{N_{\alpha,\delta}} = \bigcup_{i<\alpha} \langle N_{i,\alpha}, M_\delta \rangle^{gn}_{N_{\alpha,\delta}} = \bigcup_{i<\alpha} N_{i,\delta}$ but $N_{\alpha,\delta} = \bigcup_{\alpha \le j < \delta} N_{\alpha,j}$, and clearly $N_{\alpha,j} \subseteq \langle N_\alpha, M_\delta \rangle^{gn}_{N_{\alpha,\delta}}$ so $N_{\alpha,j} \subseteq \bigcup_{i<\alpha} N_{i,\delta}$, hence $N_{\alpha,\delta} = \bigcup_{\alpha \le j < \delta} N_{\alpha,j} \subseteq \bigcup_{i<\alpha} N_{i,\delta}$ the other inclusion having been proved we finish.

From Claim 1.3 we can derive the "local character of dependence". Specifically

Lemma 1.5: Axiom (C8)_ holds if smoothness holds (and more). That is, assume ($cf\,\delta$)-smoothness; *if* $\langle M_{1,i} : i \le \delta \rangle$ is $\le$-increasing continuous and for each $i < \delta$, $NF(M_0, M_{1,i}, M_2, M)$ *then* $NF(M_0, M_{1,\delta}, M_2, M)$.

Proof: By the choice of the way Claim 1.3 was written we must first apply symmetry to rewrite the hypothesis as $NF(M_0, M_2, M_{1,i}, M)$. Now for each $i < \delta$, let N_i denote $\langle M_{1,i} \cup M_2 \rangle^{gn}_M$ and let $N_\delta = \bigcup_{i<\delta} N_i$. By Ax(C4) (and monotonicity) we have $NF(M_{1,i}, N_i, M_{1,j}, N_j)$ if $i < j < \delta$. Now Claim 1.3 yields $NF(M_{1,i}, N_i, M_{1,\delta}, N_\delta)$. By monotonicity the original hypothesis gives $NF(M_0, M_2, M_{1,i}, N_i)$. Now Claim 1.1 yields $NF(M_0, M_2, M_{1,\delta}, N_\delta)$; $(cf\,\delta)$-smoothness gives $N_\delta \leq M$, so by monotonicity this implies $NF(M_0, M_2, M_{1,\delta}, M)$ as required.

1.6 Claim: 1) Suppose $\langle M_i : i \leq \delta + 1\rangle$, $\langle N_i^a : i \leq \delta\rangle$, $\langle N_i^b : i \leq \delta\rangle$ are $\leq$-increasing continuous sequences and $NF(M_i, N_i^a, M_{\delta+1}, N_i^b)$, $N_i^b = \langle M_{\delta+1} \cup N_i^a\rangle^{gn}_{N_i^b}$ for $i < \delta$. Then $NF(M_\delta, N_\delta^a, M_{\delta+1}, N_\delta^b)$.

2) If **K** satisfies $(cf\,\delta)$-smoothness, we can omit the assumption "$N_i^b = \langle M_{\delta+1} \cup N_i^a\rangle^{gn}_{N_i^b}$".

Proof: We use the proof of 1.4 with M_i $(i \leq \delta)$, N_i^a $(i \leq \delta)$, N_δ^b here corresponding to M_i $(i \leq \delta)$ $N_i (i \leq \delta)$, M there. Using its notation $\langle N_{i,\delta} : i < \delta\rangle$ is $\leq$-increasing continuous, (see 1.4B) $N_\delta = \bigcup_{i<\delta} N_{i,\delta}$. By Ax (C4) for $i < \delta$, $NF(M_\delta, N_{i,\delta}, M_{\delta+1}, M)$. Let for $i < \delta$, $N'_i = \langle M_2 \cup N_{i,\delta}\rangle^{gn}_M$ and $N'_\delta = \bigcup_{i<\delta} N_i$; so clearly (Ax(C4)) for $i < j < \delta$, $NF(N_{i,\delta}, N'_i, N_{j,\delta}, N'_j)$ and $NF(M_\delta, M_{\delta+1}, N_0, N'_0)$. By 1.4 $NF(N_{0,\delta}, N'_0, N_\delta, \bigcup_{j<\delta} N'_j)$, and as $NF(M_\delta, M_{\delta+1}, N_0, N'_0)$ we get (by 1.1) $NF(M_\delta, M_{\delta+1}, N_\delta, \bigcup_{j<\delta} N'_j)$, i.e. $NF(M_\delta, M_{\delta+1}, N_\delta, N'_\delta)$.

So it is enough to prove that $N'_\delta \leq M$. If **K** is $(cf\,\delta)$-smooth this is obvious (as $N'_i \leq M$ for $i < \delta$ by (Ax(C4)). In the other case

$$M = N_\delta^b = \bigcup_{i<\delta} N_i^b = \bigcup_{i<\delta} \langle M_{\delta+1} \cup N_i^a\rangle^{gn}_{N_i^b} = \bigcup_{i<\delta} \langle M_{\delta+1} \cup N_i\rangle^{gn}_M,$$

$$\bigcup_{i<\delta} N'_i = N'_\delta.$$

1.7 Claim: Suppose $\langle M_i : i < \delta\rangle$, $\langle N_i : i < \delta\rangle$ are $\leq$-increasing continuous, and for $i < j < \delta$, $NF(M_i, N_i, M_j, N_j)$. If $M_i \leq M$ and $i < \delta$, *then* we can find N, $N_i \leq N$ for $i < \delta$ and M can be embedded into N over $\bigcup_{i<\delta} M_i$.

1.7A Remark: 1) This is a strengthened version of the existence of an amalgamation.

2) Note that for a successor ordinal instead of a limit δ, the proof is trivial - use Axiom (C2).

Proof: We define by induction on $i \leq \delta$ models N_i^a, N_i^b and functions f_i such that:

(a) f_i is an isomorphism from N_i onto N_i^a over M_i;

(b) $\langle N_i^a : i \leq \delta \rangle$ is increasing continuous;

(c) $\langle N_i^b : i \leq \delta \rangle$ is increasing continuous;

(d) f_i is increasing continuous in i;

(e) $NF(M_i, N_i^a, M, N_i^b)$;

(f) $N_i^b = \langle M \cup N_i^a \rangle^{gn}_{N_i^b}$.

For $i = 0$ let $N_0^a = N_0$, $f_0 = id_{N_0}$, and so we just have to define N_0^b such that (a),(e) and (f) holds. This is possible by Axiom (C2) (which follows from Axiom (C4)).

For $i = j+1$: let $N_i^y = \langle M_{j+1}, N_j^a \rangle^{gn}_{N_j^b}$. As $NF(M_j, N_j^a, M, N_j^b)$, by Axiom (C4), $N_i^y \leq N_j^b$ and as $NF(M_i, N_i^y, M, N_{j+1})$; by Axiom (C3) $NF(M_j, N_j^a, M_{j+1}, N_i^y)$. Let $N_i^x = \langle M_{j+1}, N_j \rangle^{gn}_{N_{j+1}}$, so by Axiom (C4), $N_i^x \leq N_{j+1}$ and $NF(M_j, N_j, M_i, N_i^x)$ and by Axiom (C3), $NF(M_j, N_j, M_{j+1}, N_i^x)$.

By Axiom (C5) (uniqueness) there is an isomorphism g_i from N_i^x onto N_i^y, extending $f_j \cup id_{M_{j+1}}$. By Axiom (C2) (existence) there are N_i^a, N_i^b, f_i such that f_i is an isomorphism from N_i onto N_i^a extending g_i and $NF(N_i^y, N_i^a, N_j^b, N_i^b)$, and (by Axiom (C3), (C4) w.l.o.g.) $N_i^b = \langle N_j^b \cup N_i^a \rangle^{gn}_{N_i^b}$. By 1.1 $NF(M_i, N_i^a, M, N_i^b)$.

For i limit $< \delta$: let $N_i = \bigcup_{j<i} N_j^b$, $f_i = \bigcup_{j<i} f_j$, $N_i^a = \bigcup_{j<i} N_j^a$. As $\langle N_j : j \leq i \rangle$, $\langle M_j : j \leq i \rangle$ are increasing continuous, clearly (a)-(d) holds. As for (f), for each $j < i$,

$$N_j^b = \langle M \cup N_j^a \rangle^{gn}_{N_j^b} = \langle M \cup N_j^a \rangle^{gn}_{N_i^b} \subseteq \langle M \cup N_i^a \rangle^{gn}_{N_i^b},$$

hence $N_i^b = \bigcup_{j<i} N_j^b \subseteq \langle M \cup N_i^a \rangle^{gn}_{N_i^b} \subseteq N_i^b$ so $N_i^b = \langle M \cup N_i^a \rangle^{gn}_{N_i^b}$ as required. As for (e) use 1.2(1).

So we can carry the definition. In the end using $f_\delta = \bigcup_{i<\delta} f_i$, $N_\delta^b = \bigcup_{i<\delta} N_i^b$, $N_\delta^a = \bigcup_{i<\delta} N_i^a$ and chasing arrows, we finish.

* * *

Here is a rough prescription for deducing the existence of many models of power λ from the failure of smoothness at some $\kappa < \lambda$ for models of cardinality $< \lambda$ (i.e. the existence of a sequence $\langle M_i : i \le \kappa \rangle$ with $\bigcup M_i \not\le M_\kappa$). For each $\eta \in 2^\lambda$ build a sequence of models $\langle M_{\eta \restriction \alpha} : \alpha < \lambda \rangle$ such that $M_\eta = \bigcup \{ M_{\eta \restriction \alpha} : \alpha < \lambda \}$ has power λ and $smth(M_\eta) = \{\delta : M_{\eta \restriction \delta} \subseteq M_\eta\}/D_\lambda$ is a subset of $\eta^{-1}(1)$. (Cf. Definition 1.12). 2^λ of the M_η will be nonisomorphic since if $M_\eta \approx M_{\eta'}$, then $smth(M_\eta) = smth(M_{\eta'})$. The failure of smoothness should allow us to decide for δ of cofinality κ whether $\bigcup_{\beta<\delta} M_{\eta \restriction \delta} \le M_\delta$ depending on the value of $\eta(\delta)$.

But there is a fly in the ointment. If $T \subseteq {}^{\kappa>}\lambda$, $|T| = \lambda$, $\langle T_i : i < \lambda \rangle$ a representation of T (i.e. $T = \bigcup_{i<\delta} T_i$, T_i increasing continuous, $|T_i| < \lambda$), we do not know whether for "many" $\delta < \lambda$, $cf\,\delta = \kappa$ and there is $\eta_\delta \in {}^\kappa\lambda$ such that $\{\eta_\delta \restriction \zeta < \kappa\} \subseteq T_\delta$, but $(\forall \alpha < \delta)[\{\eta_\delta \restriction \zeta : \zeta < \kappa\} \not\subseteq T_\delta]$. Under mild cardinality restrictions we can circumvent this difficulty by working on a "good" stationary subset of λ. The required definition and background facts are laid out in 1.8 and 1.10 .

1.8 Definition: For a regular $\lambda > \aleph_0$, $S \subseteq \lambda$ is called *good* if we can find $\langle C_i : i < \lambda \rangle$ where C_i is a subset of i and for some a closed unbounded $C \subseteq \lambda$ for every limit $\delta \in C \cap S$, for some closed unbounded $C^*_\delta \subseteq \delta$ of order type $< \delta$, $(\forall \alpha < \delta)[C^*_\delta \cap \alpha \in \{C_i : i < \delta\}]$.

1.8A Remark: 1) We can weaken the definition by replacing C_i by $< \lambda$ candidates, and modulo a club we get an equivalent definition. More exactly, let $S \subseteq \lambda$ be called *-good if there are $\langle \langle C_{i,\xi} : \xi < \xi(i) \rangle : i < \lambda \rangle$, $C_{i,\xi} \subseteq \lambda$, $\xi(i) < \lambda$ and for every limit $\delta \in S$, for some closed unbounded $C^*_\delta \subseteq \delta$ of order type $< \delta$ $(\forall \alpha < \delta)[C^*_\delta \cap \alpha \in \{C_{i,\xi} : i < \alpha,\ \xi < \xi(i)\}]$.

Easily (for $S \subseteq \lambda$, λ regular), S is good if and only if S is *-good.

By [Sh 108], (or see [Sh 88, Appendix]):

1.10 Lemma: Let $\lambda > \kappa$ be regular, $S = \{\delta < \lambda : cf\ \delta = \kappa\}$.

1) S is good if $(\forall \mu < \lambda)\mu^{<\kappa} < \lambda$;

2) some stationary $S' \subseteq S$ is good *if:* $\lambda = \lambda^{<\kappa}$ *or* $\lambda = \mu^+$, $(\forall \chi < \mu)\chi^\kappa < \mu$,

3) If there is a good stationary $S \subseteq \{\delta < \lambda : cf\ \delta = \kappa\}$ and $\mu < \kappa$ is regular *then* there is a good stationary $S \subseteq \{\delta < \lambda : cf\ \delta = \mu\}$;

4) In Definition 1.4 , without loss of generality, we can demand that for limit $\delta \in S$, $C^*_\delta = C_\delta$ has order type $cf\ \delta$, $(\forall \gamma \in C_\alpha)$ [γ limit $\Leftrightarrow otp(\gamma \cap C_\alpha)$ is limit], $i \neq j \Rightarrow C_i \neq C_j$ and let $C_i \lessdot C_j$ mean C_i is an initial segment of C_j, w.l.o.g. it implies $i < j$ and $otp\ C_\alpha$ is limit if and only if α is limit. We may demand: $C_i \lessdot C_j \Rightarrow C_i = C_j \cap i]$ and $[otp\ C_i < \sup\{cf\ (b) : \delta \in S\}$ but shall not use them.

1.11 Theorem: 1) Assume λ is regular and **K**-inaccessible and there is a good stationary $S \subseteq \{i < \lambda : cf\ i = \kappa\}$. Suppose $\lambda > \kappa$, $M_i (i \le \kappa)$ are models from K of cardinality $< \lambda$, $\langle M_i : i \le \kappa \rangle$ is $\le$-increasing, but $\bigcup_{i<\kappa} M_i \not\le M_\kappa$. Then $I(\lambda, K) = 2^\lambda$,

2) Moreover, if $\lambda^{<\chi} + 2^{\chi(\mathbf{K}) + |\tau(\mathbf{K})|} = \lambda$, then K has 2^λ, $(\mathcal{D}_{\mathbf{K}}, \chi)$-homogeneous pairwise non-isomorphic models of power λ.

1.11A Remark: 1) Not only do we get 2^λ [$(\mathcal{D}_{\mathbf{K}}(,\chi)$-homogeneous] models in K_λ, which are pairwise non isomorphic but the construction yields usually that one has a $\le_{\mathbf{K}}$-embedding into any other. (See Fact 1.13).

2) In the proof below, we can retain the same κ, if we assume that for some stationary $S \subseteq \{i < \lambda : cf\ i = \kappa\}$ we have square (i.e. there is S', $S \subseteq S' \subseteq \{i : cf\ i \le \kappa\}$ and C_δ a club of δ

of order type $\leq \kappa$ for $\delta \in S'$ such that $[\delta_1 \in C_{\delta_2} \Rightarrow C_{\delta_1} = \delta_1 \cap C_{\delta_2}]$); see III 6.3.

Proof of 1.11 : 1) Without loss of generality, for our λ, and under the assumptions on $\langle M_i : i \leq \kappa \rangle$, κ is minimal (see 1.10(3)).

So without loss of generality, $\langle M_i : i < \kappa \rangle$ is $\leq$-increasing continuous.

Let $\langle C_i : i < \lambda \rangle$ exemplify that $S \subseteq \lambda$ is good (see Definition 1.4), and (by 1.5(4)) without loss of generality $[i \in \lambda - S \Rightarrow |C_i| < \kappa]$. Let $C'_\delta = \{\alpha \in C_\delta : \alpha = \sup(\alpha \cap C_\delta)\}$.

Now we define by induction on $\alpha < \lambda$, for every

$\eta \in T_\alpha \stackrel{def}{=} \{h \ : h$ a function from $\alpha + 1$ to $\{0,1\}$, and $[i \notin S \Rightarrow h(i) = 0]\}$

a model M_η and also a function f_η such that:

(a) $M_\eta \in K$ has as universe some ordinal $\alpha_\eta < \lambda$;

(b) for $\beta < \alpha$, $M_{\eta \restriction \beta} \leq M_\eta$;

(c) if α is a limit ordinal, $\alpha \notin S$ *then* $M_\eta = \bigcup_{\beta < \alpha} M_{\eta \restriction \beta}$;

(d) if $\alpha \in \lambda - S$ *then* f_η is a $\leq$-embedding of $M_{otp(C_{\lg(\eta)})}$ into M_η;

(e) if $\alpha \in \lambda - S$, $C_\beta \lessdot C_\alpha$ *then* $f_{\eta \restriction \beta} \subseteq f_\eta$;

(f) if $\alpha \in S$, $\eta(\alpha) = 0$ *then* $M_\eta = \bigcup_{\beta < \alpha} M_{\eta \restriction \beta}$;

(g) if $\alpha \in S$, $\eta(\alpha) = 1$ then $\bigcup_{\beta < \alpha} M_{\eta \restriction \beta} \not\leq M_\eta$;

(h) if $\alpha \notin S$, $\beta < \alpha$, $\eta \in T_\alpha$, $C_\beta \lessdot C_\alpha$, *then*

$$NF(f_{\eta \restriction \beta}(M_{otp(C_\beta)}), M_{\eta \restriction \beta}, f_\eta(M_{otp(C_\alpha)}), M_\eta)$$

The definition is by cases:

Case 1: α is a limit ordinal, and if $\alpha \in S$ then $\eta(\alpha) = 0$.

We let $M_\eta = \bigcup_{\beta<\alpha} M_{\eta\restriction\beta}$, and when $\alpha \notin S$, $f_\eta = \bigcup\{f_{\eta\restriction\beta} : \beta < \alpha, C_\beta \lessdot C_\alpha$.

Note that (h) holds by 1.4 (using monotonicity).

Case 2: $\alpha = \beta + 1$.

So C_α has a last element, say $\gamma_\alpha = \gamma(\alpha) < \alpha$, so $C_\alpha - \{\gamma_\alpha\} = C_\zeta$, $\zeta < \alpha$ (see Lemma 1.5(4)). By Axiom (C2) there is an extension f_η of $f_{\eta\restriction\zeta}$, and models N_η, M_η such that f_η is an isomorphism from $M_{otp(C_\alpha)}$ onto N_η satisfying
$NF(f_{\eta\restriction\zeta}(M_{otp(C_\zeta)}), M_{\eta\restriction\beta}, N_\eta, M_\eta)$. W.l.o.g. the universe of M_η is an ordinal $< \lambda$ (we use "λ is **K**-inaccessible").

Case 3: $\alpha \in S$, $\eta(\alpha) = 1$.

We apply Claim 1.7 twice. In each case the $\langle M_i : i < \kappa\rangle$ from Claim 1.7 is $\langle M_{\eta\restriction\beta} : C_\beta \lessdot C_\alpha, \beta < \alpha\rangle$ and the $\langle N_i : i < \kappa\rangle$ is $\langle f_{\eta\restriction\beta}(M_i) : C_\beta \lessdot C_\alpha, \beta < \alpha\rangle$. In the first case M is $\bigcup_{i<\kappa} M_i$ and in the second case M is M_κ. We find models N^1, N^2 in K such that:

(i) $M_{\eta\restriction\beta} \leq N^\ell$ for $\beta < \alpha$, $\ell = 1,2$.

(ii) $\bigcup\{f_{\eta\restriction\beta} : \beta < \alpha, C_\beta \lessdot C_\alpha\}$ is a $\leq$-embedding of $\bigcup_{i<\kappa} M_i$ into N^1; we call this embedding by g^1.

(iii) there is an embedding g^2 of M_κ into N^2 which extends

$$\bigcup\{f_{\eta\restriction\beta} : \beta < \alpha, C_\beta \lessdot C_\alpha\}.$$

Condition i) is satisfied because $\{M_{\eta\restriction\beta} : \beta < \alpha, C_\beta \lessdot C_\alpha\}$ is cofinal in $\{M_{\eta\restriction\beta} : \beta < \alpha\}$ as $\alpha \in S$. Now we will show $\bigcup_{\beta<\alpha} M_{\eta\restriction\beta}$ is not $\leq$ one of N^1 and N^2.

If for $\ell = 1,2$, $\bigcup_{\beta<\alpha} M_{\eta \restriction \beta} \leq N^\ell$, then we can find $N \in K$ and $\leq$- embeddings f^ℓ of N^ℓ into N over $\bigcup_{\beta<\alpha} M_{\eta \restriction \beta}$. So $(f^1 \circ g^1)$ is a $\leq$-embedding of $\bigcup_{\beta<\kappa} M_\beta$ into N so $(f^1 \circ g^1)(\bigcup_{\beta<\kappa} M_\beta) \leq N$.

Also $f^2 \circ g^2$ is a $\leq$-embedding of M_κ into N so $(f^2 \circ g^2)(M_\kappa) \leq N$.

But $(f^1 \circ g^1)(\bigcup_{\beta<\kappa} M_\beta) \subseteq (f^2 \circ g^2)(M_\kappa)$ hence (by Axiom (A3)) $(f^1 \circ g^1)(\bigcup_{\beta<\alpha} M_\beta) \leq (f^2 \circ g^2)(M_\kappa)$, hence (by invariance) $\bigcup_{\beta<\kappa} M_\beta \leq M_\kappa$, contradicting the that $\langle M_i : i \leq \kappa \rangle$ is a counterexample to smoothness.

So for some ℓ, $\bigcup_{\beta<\alpha} M_{\eta \restriction \beta} \not\leq N^\ell$, and (as λ is **K**-inaccessible) without loss of generality $\| N^\ell \| < \lambda$, so without loss of generality N^ℓ has universe an ordinal $< \lambda$, and let $M_\eta = N^\ell$.

We finish by:

1.11B Fact: If $\eta \in T_\lambda (= \{h : \lambda \to \{0,1\},\ [i \in \lambda - S \Rightarrow h(i) = 0]\})$ $M_\eta = \bigcup_{i<\lambda} M^i$, $\langle |M^i| : i < \lambda \rangle$ increasing continuous, $\| M^i \| < \lambda$, then $Smth(M_\eta) = \eta^{-1}(\{1\}) \bmod D_\lambda$ where

1.12 Definition: For $M \in K_\lambda$, λ regular, $|M| = \bigcup_{i<\lambda} A_i$, A_i increasing continuous, $|A_i| < \lambda$, $M_i \overset{def}{=} M \restriction A_i$, then $Smth(M) = \{i : M_i \leq_{\mathbf{K}} M\} / D_\lambda$ (D_λ-the club filter).

End of the Proof of 1.11 : 2) Now Theorem 1.6(2) is an easy variant: for α successor ordinal, by any reasonable bookkeeping, take care to make all the $M_\eta (\eta \in T_\lambda) - (\mathcal{D}_{\mathbf{K}}, \chi)$-homogeneous.

1.13 Fact: 1) We can conclude in 1.11 that in K_λ there are 2^λ models, no one $\leq_{\mathbf{K}}$-embeddable into another (and when $\lambda = \lambda^{<\chi} + 2^{\chi(\mathbf{K}) + (\tau(\mathbf{K})}$, each $(\mathcal{D}_{\mathbf{K}}, \chi)$-homogeneous) provided that

(*) if M, $N \in K_\lambda$ and M is $\leq_{\mathbf{K}}$-embeddable into N then $Smth(N) \subseteq Smth(M)$.

2) The statement (*) above holds if $\leq_{\mathbf{K}}$ (i.e. $\{(M, N) : N \leq_{\mathbf{K}} N\}$ is a $PC_{\mu,\omega}$ class, where

$\mu < \lambda$ or is a $PC_{\mu,\theta}$-class where $\mu < \lambda$, $(\forall \sigma < \lambda)\ \sigma^{<\theta} < \lambda$ or is as $PC_{\mu,\kappa}$-class where $\mu < \kappa$

3) Assume (in 1.11) that S, as a subset of λ, is not small (see [DvSh 65] or see [Sh A2, Ch XIV]). Let $\mu(\lambda)$ be as in [Sh 87] (so it is "usually" 2^λ). We can find $M_i \in K_\lambda$ for $i < \mu(\lambda)$ such that

(a) for $i \neq j$, M_i cannot be $\leq_{\mathbf{K}}$-embedded into M_j

(b) if $\lambda = \lambda^{<\chi} + 2^{\chi(\mathbf{K})+|\tau(\mathbf{K})|}$ then each M_i is $(\mathcal{D}_{\mathbf{K}}, \chi)$- homogeneous.

Proof: 1) Trivial.

2) So suppose w.l.o.g. $M \leq N$. Let $\langle M_i : i < \lambda \rangle$, $\langle N_i : i < \lambda \rangle$ be representations of M, N respectively. As $M \leq N$ by the assumption $C = \{\delta < \lambda : N_\delta \cap M = M_\delta$ and $M_\delta \leq N_\delta\}$ contains a closed unbounded subset of λ.

3) See [Sh 87].

1.14 Remark: See a work, in preparation, by Baldwin and Shelah for attempts to weaken the framework from $AxFr_1$ to $AxFr_3$. That is , dealing with "prime models" rather then "generated substructures."

§2 Non χ- base

2.1 Hypothesis: $AxFr_1$ (of course) and χ is such that K has the χ-LSP.

Under a smoothness hypothesis we will show this implies K has the λ-LSP for all larger λ.

Remark: We can through §2–4 replace χ^+ by a regular uncountable cardinal.

2.2 Convention: $\mathbf{C}$ is a large homogeneous universal model.

* * *

We did not assuem an axiom bounding the cardinality of $\langle A\rangle_M^{gn}$ in terms of $|A|$. Thus even if K has Lowenheim Skolem property down to κ ($LSP(\kappa)$) it may not have it down to $\lambda > \kappa$. This problem disappears in the presence of smoothness.

2.3 Claim: 1) For $\lambda \geq \chi$, $LSP(\lambda)$ holds if $(\leq\lambda, <\infty)$-smoothness holds (see Definition II. 1.12(3),(4)).

2) If **K** is $(<\mu, <\mu)$-smooth and has $LSP(\leq\lambda,\chi)$ *then* for every λ, $\chi \leq \lambda < \mu$, **K** has $LSP(\mu,\lambda)$. (see Definition II 1.11.(4)).

See proof below, as we need the following observation.

2.4 Claim: 1) Suppose $\langle M_t : t \in I\rangle$, is given where I is a directed partial order,

(a) if $[I \models t < s \Rightarrow M_t \leq M_s]$ *then* for $s \in I$, $M_s \leq_K M \overset{def}{=} \bigcup_{t\in I} M_t = \langle \bigcup_{t\in I} M_t\rangle_M^{gn}$,

(b) if $[t \in I \Rightarrow M_t \leq M]$ and $[I \models t < s \Rightarrow M_t \subseteq M_s]$ *then* for $s \in I$, $M_s \leq \bigcup_{t\in I} M_t = \langle \bigcup_{t\in I} M_t\rangle_M^{gn} \leq M$ provided that $(\leq \sup_t \|| M_t \||, \leq |I|)^+$-smoothness holds (or $(\||M\||, \leq |I|)$-smoothenss holds).

2) If $A \subseteq M \in$ **K**, LSP (or jsut $LSP(\||M\||, |A|)$, *then* we can find a directed I and $M_t \leq M$, $\|| M_t \|| = |A|$ for $t \in I$, $A \subseteq M_t \subseteq M_s$ for $t \leq s$ from I and $M = \bigcup_{t\in I} M_t$.

3) In (1) if $NF(M^a, M_t, N^a, M^*)$ whenever $t \in I$ (so $M^a \leq M_t$ for every t) *then* $NF(M^a, M, N^a, M^*)$.

Proof of Claim 2.4: 1) By induction on $|I|$.

(i) If $|I|$ is finite the result is trivial, use maximal member.

(ii) $|I| \geq \aleph_0$. Let $I = \bigcup_{\alpha<|I|} I_\alpha$, I_α increasing, $|I_\alpha| < |I|$, and each I_α directed.

Let $M_\alpha = \bigcup_{t \in I_\alpha} M_t$. For (a) by smoothness $M_\alpha \le \bigcup_\alpha M_\alpha = \bigcup_t M_t$ so by transitivity of $\le_{\mathbf{K}}$ we finish. For (b), by the induction hypothesis $M_\alpha \le M$ for each α and clearly for $\beta < \alpha$ $M_\beta \subseteq M_\alpha$ hence $\beta < \alpha \Rightarrow M_\beta \le M_\alpha$. So by smoothness $\bigcup_{\alpha<\delta} M_\alpha \le M$. Easily it is equal to $\bigcup_{t \in I} M_t$.

2) See proof of 2.3.

3) Like the proof of (1), using Claim 1.5 in the induction step.

2.4A Remark: In some circumstances, e.g. Banach models or $|T|^+$-saturated models of T, where smoothness fails, if still we have a prime model on (or closure of) the union of increasing chains, we can "save" $(\forall \mu \ge \chi)\ LSP(\mu)$ by replacing the cardinality of a model M by e.g. the density character i.e. the minimal cardinality μ, such for some $A \subseteq M$ $|A| = \mu$, M the closure of A (for Banach models) or is $|T|^+$-primary over M (for $|T|^+$-saturated models) or by $pscard(M)$ as in II 1.17.

Proof of Claim 2.3: 1) Let $A \subseteq M$, $|A| \le \lambda$. Define by induction on $n < \omega$ for every finite $u \subseteq A$ of power n, a model N_u such that: $N_u \le M$, $\|| N_u \|| \le \chi$ and $w \subseteq u \Rightarrow N_w \subseteq N_u$. There is no problem to do it, $A \subseteq \bigcup_u N_u \subseteq M$, $\|| \bigcup_u N_u \|| \le \lambda$ and $\bigcup_u N_u \le M$ by Claim 2.4.

2) Let $A \subseteq M$ with $|A| = \lambda$. For each finite sequence $\bar a \in {}^{\omega>}|M|$ choose $N_{\bar a} \le M$ with $\||N_{\bar a}\|| \le \chi$ such that $[\bar a \subseteq N_{\bar a},\ \bar b \subseteq \bar a$ implies $N_{\bar b} \subseteq N_{\bar a}]$ (so they form a directed indexed set of models). Since as $\mathbf{K}$ is $(<\mu,<\mu)$-smooth, for each $B \subseteq M$ of cardinality $<\mu$. $N_B \overset{def}{=} \bigcup \{N_{\bar a} : \bar a \in {}^{\omega>} B\}$ is in K and $[\bar a \in {}^{\omega>} A \Rightarrow N_{\bar a} \le_{\mathbf{K}} N_A]$ and $\||N_B\|| \le |B| + \chi$ (all by 2.4(1) (a)). It remains to show $N_A \le M$.

Note again by $(<\mu,<\mu)$-smoothness (*) $[C \subseteq B \subseteq M \wedge |B| < \lambda \Rightarrow N_C \le N_B]$ (use 2.14(1)(b)). Write M as $\bigcup_{i<\mu} A_i$ with $A = A_0$, the A_i increasing continuous and $|A_i| < \mu$. Then $M = \bigcup_{i<\mu} N_{A_i}$, and by (*) $\langle N_{A_i} : i < \mu \rangle$ is $\le$-increasing continuous. So for $j < \mu$, $N_{A_j} \le \bigcup_{i<\lambda} N_{A_i}$ i.e. $N_{A_j} \le M$; taking $j = 0$, we finish.

2.5 Definition: 1) NF is *κ-based when: if* $M \leq M^*$ and $A \subseteq M^*$ where $|A| \leq \kappa$ *then* for some $N_0, N_1, \| N_1 \| \leq \kappa$, $N_0 \subseteq M \cap N_1, A \subseteq N$, and N_0, M, N_1 are in stable amalgamation (inside M^* of course). We define "(<κ)-based" similarly. We may say **K** is κ-based.

2) NF is (λ, κ)-based if (1) holds when $\| M \| = \lambda$ (similarly we define "NF is $(\leq\lambda, \kappa)$-based", etc).

The following lemma will lead via Section 3 to the conclusion in Theorem 4.1, that if **K** is not χ-based then κ has 2^λ non-isomorphic homogeneous models in many powers λ.

2.6 Lemma: Assume **K** is $(\leq\lambda, \leq\chi)$-smooth, **K** has χ-LSP, *NF* is not $(\leq\lambda, \chi)$-based; as exemplified by M, A, M^* where $\| M \| \leq \lambda$, $|A| \leq \chi$; *then there are* $M_i, N_i (i < \chi^+)$ *such that:*

(a) $\| M_i \|, \| N_i \| \leq \chi$;

(b) $A \subseteq N_0$;

(c) $M_i = M \cap N_i$;

(d) $M_i \leq N_i \leq M^* : M_i \leq M \leq M^*$;

(e) M_i, M_{i+1}, N_i not in stable amalgamation (inside M^*);

(f) $\langle M_i : i < \chi^+ \rangle$ is continuous, increasing;

(g) $\langle N_i : i < \chi^+ \rangle$ is continuous, increasing.

Proof: We define by induction on *i*.

Case 1: $i = 0$: We choose by induction on $\zeta < \chi$, A_ζ, B_ζ such that $|A_\zeta| + |B_\zeta| \leq \chi$, $A_\zeta < M^*$, $B_\zeta < M$, $B_\zeta \supseteq \bigcup_{\xi<\zeta} B_\xi \cup (A_\zeta \cap M)$, $A_\zeta \supseteq A \bigcup_{\xi<\zeta} A_\xi \bigcup_{\xi<\zeta} B_\xi$. Now $N_0 \stackrel{def}{=} \bigcup_{\xi<\chi} A_\zeta$ is as required: $\bigcup_{\zeta<\chi} A_\zeta < M^*$, (by smoothness) and $(\bigcup_{\zeta<\chi} A_\zeta) \cap M = \bigcup_{\zeta<\chi} (A_\zeta \cap M) = \bigcup_{\delta<\chi} B_\zeta \leq M$. (by definition). Let $M_0 = M \cap N_0 = \bigcup_{\zeta<\chi} B_\zeta$.

Case 2 : i limit: Take unions.

Case 3: $i = j + 1$: We can represent M as a direct limit of $\leq$-submodels including M_j of power $\leq \chi$, $M = \bigcup_{t \in I} M_t$ (use $LSP(\lambda,\chi)$ and 2.4(2)). Necessarily for some t, M_j, M_t, N_j are not in stable amalgamation. [Why? by 2.4(3)] Now define M_i, N_i as in the case $i = 0$ such that $M_t \subseteq M_i$, $N_j \subseteq N_i$ and (a),(c),(d) holds. Now and by monotonicity of NF (e) holds.

2.6A Remark: 1) In case 1 we can choose A_ζ, B_ζ only for $\zeta < \theta$ where θ is a regular cardinal $\leq \chi$. Then we shall use $(\leq\chi,\theta)$-smoothness only (and if we restrict ourselves to the case $|||N||| \geq \chi$ we can use (χ,θ)-smoothness only.

2) Let $\theta = cf\,\theta \leq \chi$, and assume only $(\leq\chi,\theta)$-smoothness. Then as explained above we can still prove the lemma, but in (f) and (g) we know that we get continuity only for $\delta < \chi^+$ of cofinality θ. This complicates the combinatorics in section 4.

2.7 Claim: 1) Suppose **K** is (χ,θ)-smooth and (λ,χ)-based, $\theta \leq \chi$. If $M \leq M^*$, $|||M||| = \lambda$, $A \subseteq M$, $|A| \leq \chi$ *then* there is $N \subseteq M^*$, such that $A \subseteq N$, $|||N||| \leq \chi$ and $NF(N \cap M, M, N, M^*)$.

2) Suppose $\theta \leq \chi$, $LSP(\leq\chi^+, \chi)$ and **K** is $(\leq\chi, \leq\theta)$-smooth. Then the existence of M_i, N_i $(i < \chi^+)$ as in 2.6 is equivalent to "**K** is not (χ^+,χ)-based".

Proof: 1) This is proved in case 1 of the proof of 2.6.

2) Easy to (use 1.6).

2.8 Remark: In Definition 2.5 we may ask that N_0,N_1 exist not as submodels of M^* but of some M^{**}, where $M^* \leq M^{**}$. This is apparently weaker definition . However assuming e.g. $(\leq\chi,\theta)^+$-smoothness for some $\theta \leq \chi$ is enough to get back the old definition.

§3 Stable Constructions

The following definition generalizes the notion of a construction from Chapter IV of [Sh]. More precisely, since we are demanding independence, an F^f_λ-construction.

3.1 Definition: We define by simultaneous induction on α.

1) $\mathbf{A} = \langle A, B_i, w_i : i < \alpha \rangle$ is a stable construction inside N if (letting for $u \subseteq \alpha$, $A_u = \langle A \cup \bigcup_{j \in u} B_j \rangle_N^{gn}$):

(i) $A,\ B_i \leq N$ and $A_j \leq N$ (note $A_j = A_{\{\gamma:\gamma<j\}}$) for $i < \alpha$, $j \leq \alpha$.

(ii) a) $w_i \subseteq i$

b) w_i is closed for $\mathbf{A} \restriction i$ [defined below in 3.1(2)).]

(iii) $B_i \cap A_i \leq \langle A \cup \bigcup_{j \in w_i} B_j \rangle_N^{gn} = A_{w_i}$.

(iv) $NF(B_i \cap A_i, B_i, A_i, N)$

(v) $B_i \cap A \leq A$

(vi) For each i one of the following occurs:

Case (a): $i = 0$.

Case (b): For some $\gamma_i < i$, $w_i = w_{\gamma_i} \cup \{\gamma_i\}$, $B_i \cap A_i = B_{\gamma_i}$.

Case (c): $B_i = \langle \bigcup_{j \in w_i} B_j \rangle_N^{gn}$.

2) For such $\mathbf{A}$, u is called *closed* if:

a) $u \subseteq \alpha$

b) $i \in u \Rightarrow w_i \subseteq u$.

3) **A** is a $(<\mu)$-stable construction *if* **A** is a stable construction and $|B_i| < \lambda$ for $i < \ell g(\mathbf{A})$. In this case we say $A_{\ell g(\mathbf{A})}$ is $(<\mu)$-stably constructible over A.

3.2 Notation: If $\mathbf{A} = \langle A,B_i,w_i : i < \alpha\rangle$, then $\mathbf{A} \restriction \beta \overset{def}{=} \langle A,B_i,w_i{:}i \leq \alpha \cap \beta\rangle$ and $\alpha \overset{def}{=} \ell g(\mathbf{A})$. For $w \subseteq \alpha$, $A_w = \langle A \cup \bigcup_{i \in w} B_i\rangle_N^{gn}$ (or $A_w^{\mathbf{A}}$).

3.3 Claim: 1) If **A** is a stable construction inside N then $\mathbf{A} \restriction \beta$ is a stable construction inside N.

2) If **A** is a stable construction inside N, $\alpha \leq \ell g(\mathbf{A})$ *then* α *is closed for* **A**.

3) The intersection of any family of sets each closed for **A** is closed for **A**.

4) The union of any family of subsets of $\ell g(\mathbf{A})$ closed for **A** is closed for **A**.

5) If $u \subseteq \ell g(\mathbf{a})$ is closed for **A** where **A** is a stable construction inside N *then* $A_u \leq N$.

Proof: Easy, but (5) is proved in 3.4.

3.4 Claim: If **A** is a stable construction inside N, for $\ell = 0,1,2$, $u_\ell \subseteq \alpha = \ell g(\mathbf{a})$ is closed, and $u_0 = u_1 \cap u_2$ then A_{u_0}, A_{u_1}, A_{u_2} is in stable amalgamation inside N.

Proof: Straightforward, by induction on $\ell g(\mathbf{A})$ (for successor remember 1.1, for limit use 1.9).

3.5 Claim: If $\mathbf{A} = \langle A_i,B_i,w_i : i < \alpha\rangle$ is a stable construction inside N, h a one-to-one function from α onto β, $[j \in w_i \Rightarrow h(j) < i]$ and let $w^*_{h(i)} = \{h(j) : j \in w_i\}$, $B^*_{h(i)} = B_i$ *then* $\mathbf{A}' = \langle A,\, B^*_i, w^*_i : i < \beta\rangle$ is a stable construction inside N.

Proof: Easy.

3.6 Claim: 1) If $\lambda^{<\chi} + 2^{|\tau(K)|} = \lambda$, $\chi \geq LSP(\mathbf{K})$, $M \in K$ and $\|M\| \leq \lambda$ *then* there is a stable construction $\mathbf{A} = \langle A,B_i,w_i : i < \delta\rangle$ inside some $N \in K$ such that $A = |M|$, $A_\delta = |N|$, $\|N\| \leq \lambda$ and N is $(\mathcal{D}_{\mathbf{K}},\chi)$-homogeneous.

Proof: Straightforward.

Remark: On uniqueness see §5.

§4 NonStructure from non "NF is not χ- based"

We are trying to get nonstructure from non "NF χ-based" for suitable regular χ. Remember the definition of "λ is **K**-inaccessible" (II 1.16).

4.1 Theorem: Assume $\chi^+ \geq \mu > LS(\mathbf{K}) + |\tau(\mathbf{K})|$ and $(\leq \chi^+, \leq \chi^+)$-smoothness holds but NF is not χ-based with counterexample as in 2.6, *then:* for every $\lambda = \lambda^{<\mu} + 2^\chi$ which is regular, and **K**-inaccessible such that some $S^* \subseteq \{\delta < \lambda$: cf $\delta = \chi^+\}$ is good and stationary *there are* 2^λ non-isomorphic $(\mathcal{D}_{\mathbf{K}}, \mu)$-homogeneous models.

We give, in essence, three proofs of (variants of) Theorem 4.1. Items 4.4 through 4.6 reduce the proof of the general case (arbitrary λ) to results in Chapter III. Items 4.7 through 4.9 (using the construction of 4.4) prove Theorem 4.1 as stated except for the requirement that the models be $(\mathcal{D}_{\mathbf{K}}, \mu)$-homogeneous. Item 4.10 explains how to modify this proof to demand the models to be $(\mathcal{D}_{\mathbf{K}}, \mu)$-homogeneous.

4.2 Idea of proof:

Picture:

M_i $\quad$ N_i

χ^+ $\qquad$ $\neg NF(M_i, N_i, M_{i+1}, N_{i+1})$

M_{i+1} $\quad$ N_{i+1}

M $\quad$ N

In 2.6 from a counterexample we get a canonical counterexample (as in the picture). We copy $\langle M_i : i < \chi^+ \rangle$ along the tree ${}^{\chi^+ >}\lambda$: i.e., define M_η $(\eta \in {}^{\chi^+ >}\lambda)$ and $f_\eta : M_{\ell g(\eta)} \underset{isomorphism}{\overset{onto}{\rightarrow}} M_\eta$, f_η increasing, amalgamating them freely. For $\eta \in {}^{(\chi^+)}\lambda$ we can have $N_\eta, g_\eta \supseteq \bigcup_{\alpha < \chi^+} f_{\eta \restriction \alpha}$ such that $g_\eta : N \rightarrow N_\eta$, (isomorphism onto). For $S \subseteq {}^{\chi^+}\lambda$ let $N_S = \langle N_\eta,\ M_\nu : \eta \in S,\ \nu \in {}^{(\chi^+)>}\lambda \rangle^{gn}_{N_S}$. Now in N_S over M_η there is a copy of N_η if and only if $\eta \in S$ (i.e. we shall prove this)

So we have coded S, see III 5.1 for why this implies non-structure. We shall give the proof of 4.1 after some further discussion.

NF is not χ-based generalizes (roughly) the first order notion "$|||\chi \geq |T|$, T unstable" Since in the first order case $\kappa(T) \leq |T|^+$; the case however does not appear for first order when $\chi < \kappa(T)$, as $|acl(\emptyset)| = |T|$ by the definition of $\mathbf{C}^{eq}$. But it would appear if we varied the first order notions slightly (perhaps to deal more precisely with algebra), and instead of using the cardinality of a set A in the definitions used the cardinality of a minimal set of generators for A. The following example explores this possibility.

4.3A Example: $T = T^{eq}$ is (first order complete) stable, not superstable. Now (i) if $A, B \subseteq \mathbf{C}$ are algebraically closed, $B = acl(\bar{b})$, $|\bar{b}| < \kappa$, κ regular *then* we can find $\bar{a} \in A$, $|\bar{a}| < \kappa$ such that $acl(\bar{a})$, $acl(\bar{a} \cup \bar{b})$, A are in stable amalgamation *if and only if* $\kappa > \kappa(T)$. There are two reasonable ways to define $|||A|||_{gen}$:

$|||A|||_{gen} = Min\{|B| : B \subseteq A \subseteq acl(B)\}$.

$|||A|||'_{gen} = Min\{|B| : A \subseteq acl(B)\}$.

The second is less natural but $A_1 \subseteq A_2 \Rightarrow |||A_1|||'_{gen} \leq |||A_2|||'_{gen}$ (i.e. monotonicity holds)

So "NF κ-based" is a generalization of $\kappa \geq \kappa_r(T)$.

Discussion continued: Later, in Chapter V, we shall have another notion, capturing the parallel of $\kappa(T)$ and so in particular "superstability". But remember that "stable" was captured in Chapter I and axiomatized in Chapter II. Looking carefully at universal classes (see II 2.2) we see that for this case (i.e. $\leq_{\mathbf{K}}$ is $\leq^{\aleph}_{qf,\mu^+,\chi^+}$-see II 2.2D, K a universal class without the (χ^+,qf)-order property, $\mu = 2^{2^\chi}$) "**K** is $\chi_{\mathbf{K}}$-based" follows. However this is seemingly not true for the general **K** we are dealing with. Also note that if e.g. K is the class of submodels of models of T, T first order, stable not superstable with elimination of quantifiers, so K is a universal class, *then* in II 2.2 we get $(\mathbf{K},\leq_{\mathbf{K}}, NF,\langle\ \rangle^{gn})$ satisfying Ax Fr_1 but this **K** is not $\aleph_0$-smooth (nor κ-smooth for $\kappa < \kappa_r(T)$)

After the following theorem and assumption we shall be able to generate some facts on stable theories to our context, e.g.,

$|T|^+$-primary model, parallelism. In other words, only assuming smoothness and **K** is χ-based we can generalize stability theory.

4.4 Proof of Theorem 4.1: By lemma 2.6, we got from our assumption, the sequence $\langle M_i : i \leq \chi^+\rangle, \langle N_i : i \leq \chi^+\rangle$ such that:

(i) both $\leq$-increasing continuous

(ii) $i < \chi^+ \Rightarrow |||M_i||| + |||N_i||| \leq \chi$.

(iii) $\neg NF(M_i, N_i, M_{i+1}, N_{i+1})$ for $i < \chi^+$

(iv) $M_i \leq N_i \leq N_{\chi^+}$ for $i \leq \chi^+$ (for $i = \chi^+$ use (χ^+,χ^+)-smoothness)

Let $N \overset{def}{=} N_{\chi^+}$, $M \overset{def}{=} M_{\chi^+}$.

Let $\{\eta_i : i < i^*(0)\}$ be a list of ${}^{(\chi^+)>}\lambda$ such that $[i \leq j \Rightarrow \ell g(\eta_i) \leq \ell g(\eta_j)]$.

We define by induction on $i < i^*(0)$, f_{η_i}, M_{η_i}, L_i such that:

(a) f_{η_i} is an isomorphism from $M_{\ell g(\eta_i)}$ onto M_{η_i}.

(b) $\eta_j = \eta_i \restriction \alpha \Rightarrow f_{\eta_j} \subseteq f_{\eta_i}$ (hence $M_{\eta_j} \leq M_{\eta_i}$).

(c) $M_{\eta_j} \leq L_i$ for $j < i$.

(d) L_i is increasing, continuous

(e) if $\ell g(\eta_i) = \gamma + 1$, let $\eta_j = \eta_i \restriction \gamma$ and $NF(M_{\eta_j}, L_i, M_{\eta_i}, L_{i+1})$, $L_{i+1} = \langle M_{\eta_i} \cup L_i \rangle^{gn}_{L_{i+1}}$.

(f) $M_{<>} = M_{\eta_0} = L_0$.

There is no problem.

Now for $T \subseteq {}^{(\chi^+)>}\lambda$, let $L^T = \langle \bigcup_{\eta \in T} M_\eta \rangle^{gn}_{L_{i^*(0)}}$.

4.5 Claim: 1) $\langle \emptyset, M_{\eta_i}, \{j : (\exists \alpha < \ell g(\eta_i))\ (\eta_i \restriction \alpha = \eta_j)\} : i < i^*(0) \rangle$ is a stable construction inside $L_{i^*(0)}$.

2) If $T_0, T_1, T_2 \leq {}^{(\chi^+)>}\lambda$ are closed under initial segments, $T_0 = T_1 \cap T_2$ *then* $NF(L^{T_0}, L^{T_1}, L^{T_2}, L_{i^*(0)})$.

Proof of 4.5: 1) Should be clear by comparing the construction with Definition 3.1.
2) It is immediate by 3.4.

Remark: That is, is does not matter in which order we carry out the definition.

Continuation of the proof of 4.1: We have built a tree of the $\{M_\eta : \eta \in {}^{\chi^+>}\lambda\}$. Since the original sequence $\langle M_i : i \leq \chi^+ \rangle$ was continuous any model containing this tree will contain all the $M_\eta \overset{def}{=} \bigcup_{i<\chi^+} M_{\eta \restriction i}$ for η such that $\ell g(\eta) = \chi^+$. Now we past independent copies of $N = N_{\chi^+}$ on the top of the tree. We will see that we can realize or omit a particular N_η (with $\eta \in {}^{\chi^+}\lambda$) at will.

Let $\{\nu_\alpha : \alpha < \lambda^{\chi^+}\}$ list ${}^{(\chi^+)}\lambda$, and we can easily define g_α, N_{ν_α}, L^* such that:

$$g_{\nu_\alpha} : N_{\chi^+} \to N_{\nu_\alpha}$$

is an isomorphism onto extending

$$f_{\nu_\alpha} \stackrel{def}{=} \bigcup_{\xi<\chi^+} f_{\nu_\alpha \restriction \xi}$$

such that

(α) $\langle \bigcup_{\beta<\alpha} N_{\nu_\beta} \cup L_{i^*(0)} \rangle^{gn}_{L^*} \leq L^*$

(β) $NF(\bigcup_{\xi<\chi^+} M_{\nu_\alpha \restriction \xi}\, \langle \bigcup_{\beta<\alpha} N_{\nu_\beta} \cup L_{i^*(0)} \rangle^{gn}_{L^*},\ N_{\nu_\alpha}, L^*)$

(γ) $L^* = \langle \bigcup \{ N_{\nu_\beta} : \beta < {}^{(\chi^+)>}\lambda \} \cup L_{i^*(0)} \rangle^{gn}_{L^*}$

To achieve the third condition, choose a first approximation N'_{ν_α} so that

$$NF(\bigcup_{\xi<\chi^+} M_{\nu_\alpha \restriction \xi}, L_{i^*(0)},\ N'_{\nu_\alpha}, L^*_\alpha)$$

and then when defining the N_{ν_α} by induction on α choose N_{ν_α} isomorphic to N'_{ν_α} over $L_{i^*(0)}$ so that

$$NF(L_{i^*(0)}, \langle \bigcup_{\beta<\alpha} N_{\nu_\beta} \cup L_{i^*(0)} \rangle^{gn}_{L^*}, L^*_\alpha, L^*).$$

Now, transitivity of independence gives the required result.

Let for $T \subseteq {}^{\chi^+ \geq}\lambda$

$$L^T = \langle \bigcup \{ M_\eta : \eta \in T \cap {}^{\chi^+>}\lambda \} \bigcup \{ N_\nu : \nu \in T \cap {}^{\chi^+}\lambda \} \rangle^{gn}_{L^*}.$$

The first definition of L^T did not involve the N_η and the second ones does; however tt is easy to see that the two definitions of L^T are compatible. You can use 3.3, 3.4, 3.5. Using 3.6 let, if $\lambda = \lambda^{<\mu} + 2^\chi$, L^T_* be ($\mathcal{D}_{\mathbf{K}}$,μ)-homogeneous and (<μ)-stably constructible over L^T and let $\langle L^T, B^T_i, w^t_i : i < i^T \rangle$ be such a construction. For other λ (or when proving the version without "($\mathcal{D}_{\mathbf{K}}$, μ)-homogeneous) let $L^T_* \stackrel{def}{=} L^T$.

Clearly $|||L^T_*||| = \lambda$ when $|T| \leq \lambda$.

Recall $N = N_{\chi^+}$ (beginning of proof).

4.6 Claim: If $T \subseteq {}^{\chi^+ \geq}\lambda$, $\nu \in {}^{\chi^+}\lambda, \{\nu \restriction \alpha : \alpha < \chi^+\} \subseteq T$ but $\nu \notin T$, *then:*

1) $f_\nu (= \bigcup_{\xi < \chi^+} t_{\nu \restriction \xi})$ cannot be extended to a $\leq$-embedding of N into L^T.

2) Similarly for L^T_*.

Proof: 1) Let $g : N \to L^T$ be an $\leq$-embedding, extending f_ν. W.l.o.g. T is closed under initial segments. For $\xi < \chi^+$, let

$$T_\xi = \{\rho \in T : \nu \restriction \xi \not\triangleleft \rho \quad or \quad \nu \restriction \xi = \rho\}.$$

Clearly (see 3.3, 3.4, 3.5)

(i) $L^T = \bigcup_{\xi < \chi^+} L^{T_\xi}$.

(ii) L^{T_ξ} is increasing continuous in ξ (if ξ is a limit ordinal - $M_{\nu \restriction \xi} = \bigcup_{\zeta < \xi} M_{\nu \restriction \zeta}$).

(iii) $NF(M_{\nu \restriction \zeta}, L^{T_\zeta}, M_\nu, L^T)$ remembering $M_\nu = \bigcup_{\xi < \chi^+} M_{\nu \restriction \xi}$. For every $\zeta < \chi^+$ $g''(N_\zeta)$ is $\subseteq \bigcup_{\zeta < \chi^+} L^{T_\zeta}, L^{T_\zeta}$ increasing, $|g''(N_\zeta)| \leq \chi$; *hence* for some $\xi(\zeta) < \chi^+$ $g''(N_\zeta) \subseteq L^{T_{\xi(\zeta)}}$ \$*HenceC* $\stackrel{def}{=} \{\alpha < \chi^+ : (\forall \zeta < \alpha) \xi(\zeta) < \alpha,$ *and* α *is a limit ordinal\}* is a closed unbounded subset of χ^+. Fix ζ in C. Then $g''(N_\zeta) \subset L^{T_\zeta}$, note that

$$g''(N_\zeta) \leq L^T, \ L^{T_\zeta} \leq L^T$$

Remember $NF(M_{\nu \restriction \zeta}, L^{T_\zeta}, M_\nu, L^*)$ hence by monotonicity

$$NF(M_{\nu \restriction \zeta}, g''(N_\zeta), M_\nu, L^*)$$

Again monotonicity

$$NF(M_{\nu \restriction \zeta}, g''(N_\zeta), M_{\nu \restriction (\zeta+1)}, L^*)$$

$$g''(N_\zeta) \cup M_{\nu \restriction (\zeta+1)} \subseteq g''(N_{\zeta+1}) \leq L^*$$

but

$$NF(M_{\nu\restriction\zeta}, g''(N_\zeta), M_{\nu\restriction(\zeta+1)}, g''(N_{\zeta+1}))$$

which contradicts the hypothesis on $\langle M_i, N_i : i \le \chi^+\rangle$ (and g being an $\le_{\mathbf{K}}$-embedding).

2) Similar proof.

4.7 Continuation of the proof of 4.1:

Proof without the homogeneity condition

The assumptions on λ imply that there is $T \subseteq {}^{\chi^+ >}\lambda$, $T = \bigcup_{\alpha<\lambda} T_\alpha$, T_α increasing continuous, T_α is closed under initial segments , $|T_\alpha| < \lambda$, and for $\delta \in S^*$, $\eta_\delta \in {}^{(\chi^+)}\lambda$, $\{\eta_\delta \restriction \zeta : \zeta < \chi^+\} \subseteq T_\delta$ and for no $\alpha < \delta$, $\{\eta_\delta \restriction \zeta : \zeta < \chi^+\} \subseteq T_\alpha$ (i.e. as S^* is good cf. statement of Theorem). Let for $S \subseteq S^*$, $L_{[S]} = L_*^{T \cup \{\eta_\delta : \delta \in S\}}$. Clearly $L_{[S]}$ is a model of cardinality λ which is $(\mathcal{D}_{\mathbf{K}},\mu)$-homogeneous when demanded. Decompose $L_{[S]}$ as $\bigcup_{\alpha<\lambda} L_{[S],\alpha}$, $\langle L_{[S],\alpha} : \alpha < \lambda\rangle$ is increasing continuous, $\|L_{[S],\alpha}\| < \lambda$).

4.8 Definition: For any $M \in K_\lambda$, λ regular $>LSP(\mathbf{K})$ and representation $\langle M_i : i < \lambda\rangle$ of M (i.e. it increasing continuous, $M = \bigcup_{i<\lambda} M_i$ and $\| M_i \| < \lambda\}$, we let:

$$Bs_\chi(\langle M_i : i < \lambda\rangle) \stackrel{def}{=} \{\delta < \lambda : cf(\delta) = \chi^+ \text{ and: } \textit{for every } A \subseteq M,\ |A| \le \chi \textit{ there are}$$

$$N_0 \le M_\delta,\ N_1 \le M,\ \| N_1 \| \le \chi, \text{ such that } NF(N_0, N_1, M_\delta, M)\}$$

It is a D_λ-invariant, so we can let

$$Bs_\chi(M) = Bs_\chi(\langle M_i : i < \lambda\rangle)/D_\lambda$$

Now we finish by (using our proving without the homogeneity condition)

Fact 4.9: There is a club C such that for any stationary $S \subseteq S^*$, $C \cap S^* \cap (\lambda - Bs_\chi(L_{[S]})) = C \cap S$.

Proof: We show that the required C is

$$C = \{\alpha : L_{[S],\alpha} = \langle \{ M_\eta : \eta \in T_\alpha\} \cup \{N_{\nu_\delta} : \delta < \alpha,\rangle_{L}^{gn}\}$$

It is easy to see C is a club since $L_{[S]}$ is generated by $\{ M_\eta : \eta \in T\} \cup \{N_{\nu_\delta} : \delta \in S\}$.

Case i: Consider $\delta \in (C \cap S^*) - S$. To see the left hand side of the equation is contained in the right we must show $\delta \in Bs_\chi(L_{[S]})$. Since $\delta \notin S$, the construction did not put N_{ν_δ} into $L_{[S]}$.

Let $A \subseteq L_{[S]}$ with $|A| \leq \chi$. Then there is a $t_A \subseteq T \cup \{\eta_\delta : \delta \in S\}$ with $|t_A| \leq \chi$ and

$$A \subseteq \langle \{ M_\eta : \eta \in t_A \cap {}^{\chi^+>}\lambda\} \cup \{N_{\nu_\alpha} : \nu_\alpha \in t_A\}\rangle_{L}^{gn}.$$

Let t_A^+ be the closure of t_A under the taking of initial segments. We want to find N_0 and N_1 which witness that $\delta \in BS_\chi(L_{[S]})$.

We need two auxilliary sets

$$t_0 \overset{def}{=} \{\eta \in T_\delta : (\exists \rho \in (T \cap t_A) - T_\delta)[\eta \lessdot \rho] \ \ or \ \ (\exists \rho \in \{\nu_\alpha : \alpha > \delta\} \cap t_A)\ [\eta \lessdot \rho]\}.$$

$$t_1 \overset{def}{=} \{\eta \in T : (\exists \rho \in (T \cap t_A) - T_\delta)[\eta \lessdot \rho] \ \ or \ \ (\exists \rho \in \{\nu_\alpha : \alpha > \delta\} \cap t_A)\ [\eta \lessdot \rho]\}.$$

Now let $N^0 = \langle \{ M_\eta : \eta \in t_0\rangle_{L}^{gn}$, $N^1 = \langle \{ M_\eta : \eta \in t_A \cap T\} \cup \{N_{\nu_\alpha} : \nu_\alpha \in t_A\}\rangle_{L}^{gn}$. Then $NF(N^0, N^1, L_{[S],\delta}, L^*)$. (Remember, $L_{[S],\delta} = \langle \{ M_\eta : \eta \in T_\delta\} \cup \{N_{\nu_\alpha} : \alpha < \delta\rangle_{L}^{gn}$ since $\delta \in C$). Note that $|\{M_\eta : \eta \in t_0\rangle_{L}^{gn}| \leq \chi$. For each $\eta \in {}^{\chi^+>}\lambda$ $|M_\eta| \leq \chi$, so we only have to see $|t_0| \leq \chi$. Clearly $|t_A| \leq \chi$, so $|\{\eta \in T_\delta: ((\exists \rho \in (T \cap t_A) - T_\delta)[\eta \lessdot p]\}| \leq \chi$. But also $|\{\eta \in T_\delta : (\exists \rho \in \{\nu_\alpha : \alpha > \delta\}) \cap t_A\}| \leq \chi$ since for any ν_α with $\alpha > \delta$, $|\{\eta_\alpha \restriction \xi : \xi < \chi^+\} \cap T_\delta| \leq \chi$ (See paragraph before Definition 4.8.)

Now $A \subseteq \langle N^1 \cup L_{[S],\delta}\rangle_{L}^{gn}$. So we can choose N^3 and N^4 with $N^0 \leq N^3 \leq N^1$, $N^0 \leq N^4 \leq L_{[S],\delta}$ and $|N^3|$, $|N^4| \leq \chi$ while $A \subseteq \langle N^3 \cup N^4\rangle_{L}^{gn}$ and (by monotonicity for NF) $NF(N^0, N^3, L_{[S],\delta}, L^*)$. Now applying axiom (C4) we see $NF(N^4, \langle N^3 \cup N^4\rangle_{L}^{gn}, L_{[S],\delta}, L^*\rangle$. So the required N_0, N_1 are N^4 and $\langle N^3 \cup N^4\rangle_{L}^{gn}$.

Case ii: Suppose $\delta \in C \cap S^* \cap S$. To show the right hand side of the equation is contained in the left we must show $\delta \notin Bs_\chi(L_{[S]})$. Suppose, for contradiction that $\delta \in Bs_\chi(L_{[S]})$. We will find an $i < \chi^+$ such that $NF(M_i, M_{i+1}, N_i, N_{i+1})$ contrary to our original choice of the sequences $\langle M_i : i < \chi^+\rangle$, $\langle N_i : i < \chi^+\rangle$. Finding this i will require an auxilliary construction.

We have $N_{\nu_\delta} \le L_{[S]}$. Remember g_{ν_δ} maps $N_{\chi^+} = N$ isomorphically on N_{ν_δ}. Now we define by induction on $i < \chi^+$: A_i, P_i^1, P_i^2 and t_i to satisfy the following conditions.

First, to see where things live, each of A_i, P_i^1, P_i^2 are $\le_{\mathbf{K}}$-submodels of $L_{[S]}$ of cardinality $\le \chi$. Each $t_i \subseteq {}^{\chi^+ >}\lambda \cup \{\nu_\alpha : \alpha \in S\}$ and $|t_i| \le \chi$. The sequences $\langle A_i : i < \chi^+\rangle$, $\langle P_i^1 : i < \chi^+\rangle$, $\langle P_i^2 : i < \chi^+\rangle$ are increasing and continuous.

We list the remaining properties while indicating the construction. At a successor stage we will use t_i to define A_i; then find P_i^1 and t_{i+1}.

$$t_0 = \{\nu_\delta\}$$

$$A_i = \langle \{ M_\eta : \eta \in t_i \cap T\} \cup \{f_{\nu_\alpha}(N_j) : \nu_\alpha \in t_i, j < i\}\rangle_{L^*}^{gn}$$

Now choose P_i^2, P_i^1, with $|P_i^2|, |P_i^1| \le \chi$ such that $NF(P_i^1, P_i^2, L_{[S],\delta}, L^*)$ and $A_i \subseteq P_i^2$ (using the assumption $\delta \in Bs(\langle L_{[S],\alpha} : \alpha < \lambda\rangle)$ Then choose t_{i+1} so that $P_i^2 \subseteq \langle \{ M_\eta : \eta \in t_{i+1} \cap T\rangle \cup \{N_{\nu_\alpha} : \alpha < \delta, \nu_\alpha \in t_{i+1}\}\rangle_{L^*}^{gn}$ and also

$$P_i^1 \subseteq \langle \{ M_\eta : \eta \in t_{i+1} \cap T_\delta\} \cup \{f_{\nu_\alpha}(N_j) : \nu_\alpha \in t_{t+1} \cap \{\nu_i : i < \delta\}\}, j < i\rangle_{L^*}^{gn}.$$

There is a club $C^* \subseteq \chi^+$ on which $P_\xi^2 \cap N_{\nu_\delta} = g_{\nu_\delta}(N_\xi)$ and all other requirements we shall use below (by the usual methods of constructing clubs). Fix ξ belonging to this club. By Claim 1.6, $NF(P_\xi^1, P_\xi^2, L_{[S],\delta}, L^*)$. We want to shrink $L_{[S],\delta}$ and P_ξ^1 to obtain $NF(M_{\nu_\delta \restriction \xi}, g_{\nu_\delta}(N_\xi), M_{\nu_\delta \restriction (\xi+1)}, L^*)$ which contradicts the original choice of the M_i and N_i. For this we need further definitions

$$T_\xi^* = \{\eta \in T : \nu_\delta \restriction \xi \not< \eta\} \cup \{\nu_\alpha : \alpha \in S, \nu_\alpha \restriction \xi \ne \nu_\delta \restriction \xi\}.$$

$$T_\xi^+ = \{\eta \in T : \eta < \nu_\delta \restriction \xi \text{ or } \nu_\delta \restriction \xi < \eta\} \cup \{\nu_\alpha : \nu_\delta \restriction \xi < \nu_\alpha \text{ and } \alpha \in S\}$$

Thus $T^{\ddagger}_{\xi} \cup T^{\ddagger}_{\xi} = T \cup \{\nu_\alpha : \alpha \in S\}$ and $T^{\ddagger}_{\xi} \cap T^{*}_{\xi} = \{\nu_\delta \restriction \alpha : \alpha < \xi\}$.

Now we show $t_\xi \cap T^+ = \{\nu_\delta\} \cup \{\nu_\delta \restriction \alpha : \alpha < \xi\}$. This is straightforward from the fact that the $\langle t_\alpha : \alpha < \chi^+\rangle$ are a continuous increasing chain of sets of cardinality $\leq \chi$. In more detail, let $t^* = \bigcup_{i<\chi^+} t_i$. For each $\rho \in t^*$ let $C_\rho = \{\xi : \rho \in T^*_\xi\} \cup \{\nu_\delta\}$. Then C_ρ is a club on χ^+. Since $|t_i| \leq \chi$ for each i, $\bigcap \{C_\rho : \rho \in t_\xi\}$ is also a club. Taking the diagonal intersection, $\{\xi : \forall \zeta < \xi \; \forall \rho \in t_\zeta (\xi \in C_\rho)\}$ is a club. Now if we require (w.l.o.g.) our ξ to come from this club we have $t_\xi \cap T^{\ddagger}_{\xi} = \{\nu_\delta\} \cup \{\nu_\delta \restriction \alpha : \alpha < \xi\}$. So for our limit ξ,

$$P^1_\xi \subseteq \langle M_\eta : \eta \in t_\xi \cap T_\delta\} \cup \{f_{\nu_\alpha}(N_\xi) : \alpha < \delta,\ \nu_\alpha \in t_\xi\}\rangle^{gn}_{L^*} \subseteq L^{T^*}.$$

Clearly $NF(M_{\nu_\delta \restriction \xi}, L^{T^*}, L^{T^+}, L^*)$ and $M_{\nu_\delta \restriction (\xi+1)} \leq L^{T^+}$ so by monotonicity we have $NF(M_{\nu_\delta \restriction \xi}, P^1_\xi, M_{\nu_\delta \restriction (\xi+1)}, L_{[S],\delta})$. But we also have $NF(P^1_\xi, P^2_\xi, L_{[S],\delta}, L^*)$. By transitivity (Lemma 1.1) we conclude $NF(M_{\nu_\delta \restriction \xi}, P^2_\xi, L_{[S],\delta}\, L^*)$. As $P^2_\xi \supseteq g_{\nu_\delta}(N_\xi)$ this contradicts the choice of M_i and N_i. Thus we have established Fact 4.9.

End of the proof of 4.1 without homogeneity: Theorem 4.1 easily follows from 4.9. For, if $L_{[S]} \cong L_{[S']}$ (with $S, S' \subseteq S^*$), Fact 4.9 implies that S and S' agree on a club. But there are 2^λ stationary subsets of S^* which are pairwise not equal *mod* D_λ.

4.10 Proof of 4.1 with the homogeneity condition

Suppose g is a $\leq_{\mathbf{K}}$-embedding of N into L^T_* extending f_ν. Let $\langle T^*_\zeta : \zeta < \chi^+\rangle$ be defined by $T^*_\zeta = \{\eta \in T : \eta \restriction \zeta \neq \nu_\delta \restriction \zeta\} \cup \{\nu_\alpha : \alpha \in S,\ \nu_\alpha \restriction \zeta \neq \nu_\delta \neq \zeta\}$; and let $\langle L^+, B^+_j, w^+_j : j < j(T)\rangle$ be a stable construction of L^T_* over L_T. By 3.5 w.l.o.g. there are $\langle j_\zeta(T) : \zeta < \chi^+\rangle$ increasing continuous, $\bigcup_{\zeta<\chi^+} j_\zeta(T) = j(T)$, and $B^T_j \cap L^T \subseteq L^{T^*_\zeta}$ for $j < j_\zeta(T)$, $\zeta < \chi^+$. Let $L^{T_\zeta}_* = \langle L^{T^*_\zeta} \cup \{B^T_j : j < j_\zeta(T)\}\rangle^{gn}_{L^T_*}$, so $\langle L^{T^*_\zeta}_* : \zeta < \chi^+\rangle$ is increasing continuous with union L^T_*. As in the proof of Fact 4.9 $C = \{\zeta < \chi^+ : g$ maps N_ξ into $L^{T^*_\zeta}_*$ for $\xi < \zeta$, ζ limit$\}$ is a club of χ^+. Also the rest of the proof is similar.

4.11 Remark: 1) So it was enough for 4.5 (and really 4.1) that

$$\{i < \chi^+ : \neg NF(M_i, N_i, M_{i+1}, N_{i+1})\}$$

is stationary.

2) By III 5.1, 5.1 we can get other variants of 4.1 as we have the right representation.

4.12 Fact: We can use the proof of 4.1 to get 2^λ models in $\lambda_1 \geq \lambda$. Using models which have a stable construction $\langle L^T, B^T_\alpha, w^T_\alpha : \alpha < \alpha(T)\rangle$, $|||B^T_\alpha||| \leq \chi$ (so we get something for singular λ_1).

3) We can in 4.1 omit the "$(\mathcal{D}_{\mathbf{K}},\mu)$-homogeneous" demand gaining the the omition of "$\lambda = \lambda^{<\mu}$". If we demand only $\lambda \geq 2^\mu$ we have the models in $\mathbf{K}^{us}_{\mu:\chi^+}$ (see Definition II 3. 12).

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