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FURTHER CARDINAL ARITHMETIC

BY

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ABSTRACT

We continue the investigations in the author's book on cardinal arithmetic, assuming some knowledge of it. We deal with the cofinality of $(\mathcal{S}_{\leq\aleph_0}(\kappa), \subseteq)$ for κ real valued measurable (Section 3), densities of box products (Section 5,3), prove the equality $\mathrm{cov}(\lambda, \lambda, \theta^+, 2) = \mathrm{pp}(\lambda)$ in more cases even when $\mathrm{cf}(\lambda) = \aleph_0$ (Section 1), deal with bounds of $\mathrm{pp}(\lambda)$ for λ limit of inaccessible (Section 4) and give proofs to various claims I was sure I had already written but did not find (Section 6).

Annotated Contents

[We try to characterize when, say, λ has few countable subsets; for a given $\theta \in (\aleph_0, \lambda)$, we try to translate to expressions with pcf's the cardinal

$$\mathrm{Min}\Big\{|\mathcal{P}|\colon \mathcal{P} \subseteq \mathcal{S}_{<\mu}(\lambda) \text{ and every } a \in \mathcal{S}_{\leq\theta}(\lambda) \text{ is } \bigcup_{n<\omega} a_n, \text{ such that every } b \in \bigcup_n \mathcal{S}_{\leq\aleph_0}(a_n) \text{ is included in a member of } \mathcal{P}\Big\}.$$

This continues and improves [Sh410,§6].]

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Notation: Let $J_\lambda[\mathfrak{a}]$ be $\{\mathfrak{b} \subseteq \mathfrak{a} : \lambda \notin \text{pcf}(\mathfrak{b})\}$, equivalently $J_{<\lambda}[\mathfrak{a}] + \mathfrak{b}_\lambda[\mathfrak{a}]$. See more in [Sh513], [Sh589].

* There is a paper in preparation on independence results by Gitik and Shelah.

1. Equivalence of Two Covering Properties

1.1 CLAIM: *If* $\mathrm{pp}\,\lambda = \lambda^+$, $\lambda > \mathrm{cf}(\lambda) = \kappa > \aleph_0$ *then* $\mathrm{cov}(\lambda, \lambda, \kappa^+, 2) = \lambda^+$.

Proof: Let $\chi = \beth_3(\lambda)^+$; choose $\langle \mathfrak{B}_\zeta : \zeta < \lambda^+ \rangle$ increasing continuous, such that $\mathfrak{B}_\zeta \prec (H(\chi), \in, <^*_\chi)$, $\lambda + 1 \subseteq \mathfrak{B}_\zeta$, $\|\mathfrak{B}_\zeta\| = \lambda$ and $\langle \mathfrak{B}_\xi : \xi \leq \zeta \rangle \in \mathfrak{B}_{\zeta+1}$. Let $\mathfrak{B} =: \bigcup_{\zeta<\lambda^+} \mathfrak{B}_\zeta$ and $\mathcal{P} =: \mathcal{S}_{<\lambda}(\lambda) \cap \mathfrak{B}$. Let $a \in \mathcal{S}_{\leq\kappa}(\lambda)$; it suffices to prove $(\exists A \in \mathcal{P})[a \subseteq A]$. Let f_ξ be the $<^*_\chi$-first $f \in \prod(\mathrm{Reg} \cap \lambda)$ such that $(\forall g)[g \in \prod(\mathrm{Reg} \cap \lambda) \& g \in \mathfrak{B}_\zeta \Rightarrow g < f \bmod J^{bd}_\lambda]$, such f exists as $\prod(\mathrm{Reg} \cap \lambda)/J^{bd}_\lambda$ is λ^+-directed.

By [Sh420, 1.5, 1.2] we can find $\langle C_\alpha : \alpha < \lambda^+ \rangle$ such that: C_α is a closed subset of α, $\mathrm{otp}\, C_\alpha \leq \kappa^+$, $[\beta \in \mathrm{nacc}\, C_\alpha \Rightarrow C_\beta = C_\alpha \cap \beta]$ and $S =: \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \kappa^+$ and $\delta = \sup C_\delta\}$ is stationary.

Without loss of generality $\bar{C} \in \mathfrak{B}_0$.

Now we define for every $\alpha < \lambda^+$ elementary submodels N^0_α, N^1_α of $\mathfrak{B}$:

N^0_α is the Skolem Hull of $\{f_\zeta : \zeta \in C_\alpha\} \cup \{i : i \leq \kappa\}$ and N^1_α is the Skolem Hull of $a \cup \{f_\zeta : \zeta \in C_\alpha\} \cup \{i : i \leq \kappa\}$, both in $(H(\chi), \in, <^*_\chi)$.

Clearly:

(a) $N^0_\alpha \subseteq N^1_\alpha \subseteq \mathfrak{B}_\alpha \subseteq \mathfrak{B}$ [why? as $f_\zeta \in \mathfrak{B}_{\zeta+1}$ because $\mathfrak{B}_\zeta \in \mathfrak{B}_{\zeta+1}$],

(b) $\|N^\ell_\alpha\| \leq \kappa + \|C_\alpha\|$,

(c) $N^0_\alpha \in \mathfrak{B}_{\alpha+1}$.

[Why? As $\alpha \subseteq \mathfrak{B}_\alpha$ (you can prove it by induction on α) clearly $\alpha \in \mathfrak{B}_{\alpha+1}$, but $\bar{C} \in \mathfrak{B}_0 \subseteq \mathfrak{B}_{\alpha+1}$; hence $C_\alpha \in \mathfrak{B}_{\alpha+1}$, also $\langle \mathfrak{B}_\gamma : \gamma \leq \alpha \rangle \in \mathfrak{B}_{\alpha+1}$ hence $\langle f_\gamma : \gamma \leq \alpha \rangle \in \mathfrak{B}_{\alpha+1}$, hence $\langle f_\gamma : \gamma \in C_\alpha \rangle \in \mathfrak{B}_{\alpha+1}$. Now $N^0_\alpha \subseteq \mathfrak{B}_\alpha \in \mathfrak{B}_{\alpha+1}$ and the Skolem Hull can be computed in $\mathfrak{B}_{\alpha+1}$.]

(d) for each α with $\kappa^+ > \mathrm{otp}(C_\alpha)$, for some $\gamma_\alpha < \lambda^+$, letting $\mathfrak{a}_\alpha =: N^0_\alpha \cap \mathrm{Reg} \cap \lambda \backslash \kappa^{++}$ clearly $\mathrm{Ch}_\alpha \in \prod \mathfrak{a}_\alpha$ where $\mathrm{Ch}_\alpha(\theta) =: \sup(\theta \cap N^1_\alpha)$, and we have: $\mathrm{Ch}_\alpha < f_{\gamma_\alpha} \restriction \mathfrak{a}_\alpha \bmod J^{bd}_{\mathfrak{a}_\alpha}$.

[Why? $\mathfrak{a}_\alpha \in \mathfrak{B}_{\alpha+1}$ as $N^0_\alpha \in \mathfrak{B}_{\alpha+1}$, and $\prod \mathfrak{a}_\alpha / J^{bd}_{\mathfrak{a}_\alpha}$ is λ^+-directed (trivially) and has cofinality $\leq \max \mathrm{pcf}_{J^{bd}_{\mathfrak{a}_\alpha}}(\mathfrak{a}_\alpha) \leq \mathrm{pp}(\lambda) = \lambda^+$, so there is $\langle f^{\mathfrak{a}_\alpha}_\beta : \beta < \lambda^+ \rangle$, $<_{J^{bd}_{\mathfrak{a}_\alpha}}$-increasing cofinal sequence in $\prod \mathfrak{a}_\alpha$, so without loss of generality $\langle f^{\mathfrak{a}_\alpha}_\beta : \beta < \lambda^+ \rangle \in \mathfrak{B}_{\alpha+1}$; also by the "cofinal" above, for some $\beta \in (\alpha, \lambda^+)$, $\mathrm{Ch}_\alpha < f^{\mathfrak{a}_\alpha}_\beta \bmod J^{bd}_{\mathfrak{a}_\alpha}$. We can use the minimal β, now obviously $\beta \in \mathfrak{B}_{\beta+1}$ so $f^{\mathfrak{a}_\alpha}_\beta \in \mathfrak{B}_{\beta+1}$, hence $f^{\mathfrak{a}_\alpha}_\beta < f_{\beta+2} \bmod J^{bd}_\lambda$. Together $\gamma_\alpha =: \beta + 2$ is as required.]

(d)$^+$ for each α with $\mathrm{otp}(C_\alpha) < \kappa^+$ for some $\gamma_\alpha \in (\alpha, \lambda^+)$, for any $\mu \in \mathrm{Reg} \cap N^0_\alpha$, letting $N^{0,\mu}_\alpha =: \mathrm{Ch}_{\mathfrak{B}_\alpha}(N^0_\alpha \cup \mu)$, $\mathfrak{a}_{\alpha,\mu} = N^{0,\mu}_\alpha \cap \mathrm{Reg} \cap \lambda \setminus \mu^+$ and $\mathrm{Ch}_{\alpha,\mu} \in$

$\Pi\mathfrak{a}_{\alpha,\mu}$ be

$$\mathrm{Ch}_{\alpha,\mu}(\theta) = \begin{cases} \sup(\theta \cap N^1_\alpha) & \text{if } \theta \in N^1_\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

we have: $\mathrm{Ch}_\alpha < f_{\gamma_\alpha} \restriction \mathfrak{a}_{\alpha,\mu} \bmod J^{bd}_{\mathfrak{a}_{\alpha,\mu}}$.

[Why? Clearly $\mathrm{Ch}_{\mathfrak{B}_\alpha}(N^0_\alpha \cup \mu) \in \mathfrak{B}_{\alpha+1}$, so $\mathfrak{a}_{\alpha,\mu} \in \mathfrak{B}_{\alpha+1}$, hence there are in $\mathfrak{B}_{\alpha+1}$ elements $\langle \mathfrak{b}_\theta[\mathfrak{a}_{\alpha,\mu}]: \theta \in \mathrm{pcf}(\mathfrak{a}_{\alpha,\mu})\rangle$ and $\langle\langle f^{\mathfrak{a}_{\alpha,\mu},\theta}_\alpha: \alpha < \theta\rangle: \theta \in \mathrm{pcf}(\mathfrak{a}_{\alpha,\mu})\rangle$ as in [Sh 371, 2.6, §1]. So for some $\gamma_{\alpha,\mu} \in (\alpha, \lambda^+)$ we have $\mathrm{Ch}_\alpha \restriction \mathfrak{b}_{\lambda^+}[\mathfrak{a}_{\alpha,\mu}] < f_{\gamma_\alpha}$, so it is enough to prove $\mathfrak{a}_{\alpha,\mu} \setminus \mathfrak{b}_{\lambda^+}[\mathfrak{a}_{\alpha,\mu}]$ is bounded below μ but otherwise $\mathrm{pp}(\lambda) = \lambda^+$ will be contradicted. Let $\gamma_\alpha = \sup\{\gamma_{\alpha,\mu}: \mu \in N^0_\alpha\}$.]

(e) $E^* =: \{\delta < \lambda^+: \alpha < \delta \;\&\; |C_\alpha| \le \kappa \Rightarrow \gamma_\alpha < \delta \text{ and } \delta > \lambda\}$ is a club of λ.

Now as S is stationary, there is $\delta(*) \in S \cap E^*$. Remember $\mathrm{otp}\, C_{\delta(*)} = \kappa^+$.

Let $C_{\delta(*)} = \{\alpha_{\delta(*),\zeta}: \zeta < \kappa^+\}$ (in increasing order).

Let (for any $\zeta < \kappa^+$) M^0_ζ be the Skolem Hull of $\{f_{\alpha_{\delta(*),\xi}}: \xi < \zeta\} \cup \{i : i \le \kappa\}$, and let M^1_ζ be the Skolem Hull of $a \cup \{f_{\alpha_{\delta(*),\xi}}: \xi < \zeta\} \cup \{i : i \le \kappa\}$. Note: for $\zeta < \kappa^+$ non-limit $\{f_{\alpha_{\delta(*),\xi}}: \xi < \zeta\} = \{f_\xi: \xi \in C_{\alpha_{\delta(*),\zeta}}\}$. Clearly $\langle M^0_\zeta: \zeta < \kappa^+\rangle$, $\langle M^1_\zeta: \zeta < \kappa^+\rangle$ are increasing continuous sequences of countable elementary submodels of $\mathfrak{B}$ and $M^0_\zeta \subseteq M^1_\zeta$ and for $\zeta < \kappa^+$ a successor ordinal, $N^\ell_{\alpha_{\delta(*),\zeta}} = M^\ell_\zeta$.

Now for each successor ζ, for some $\epsilon(\zeta) \in (\zeta, \omega_1)$ we have $\gamma_{\alpha_{\delta(*),\zeta}} < \alpha_{\delta(*),\epsilon(\zeta)}$ (by the choice of $\delta(*)$) hence $f_{\gamma_{\alpha_{\delta(*),\zeta}}} < f_{\alpha_{\delta(*),\epsilon(\zeta)}} \bmod J^{bd}_\lambda$ hence $\mathrm{Ch}_{\alpha_{\delta(*),\zeta}} < f_{\alpha_{\delta(*),\epsilon(\zeta)}} \bmod J^{bd}_\lambda$.

Let $E =: \{\delta < \omega_1$: for every successor $\zeta < \delta$, $\epsilon(\zeta) < \delta\}$, clearly E is a club of κ^+. Let $\lambda = \sum_{i<\kappa} \lambda_i$, $\lambda_i < \lambda$ singular increasing continuous with i, wlog $\{\lambda_i: i < \kappa\} \subseteq \mathrm{Ch}_{\mathfrak{B}}(\{i: i \le \kappa\} \cup \{\lambda\})$. So for some $\mu_{\zeta,i} < \lambda$, we have:

$$(*) \qquad \begin{aligned} &i < \kappa, \quad \zeta = \xi + 1 < \kappa^+ \;\&\; \theta \in \mathrm{Reg} \cap \lambda \setminus \mu_{\zeta,i} \;\&\; \theta \in N^{0,\lambda_i}_{\alpha_{\delta(*),\zeta}} \cap N^1_{\alpha_{\delta(*),\zeta}} \\ &\qquad \Rightarrow \sup\left(N^1_{\alpha_{\delta(*),\zeta}} \cap \theta\right) < f_{\alpha_{\delta(*),\epsilon(\zeta)}}(\theta) \in \theta \cap N^{0,\lambda_i}_{\alpha_{\delta(*),\zeta+1}}. \end{aligned}$$

So for some limit $i(\zeta) < \kappa^+$ we have $\lambda_{i(\zeta)} = \sup\{\mu_{\zeta,j}: j < i(\zeta)\}$. Now as $\mathrm{cf}\,\lambda \le \kappa^+$ for some $i(*) < \lambda$

$$W =: \{\zeta < \kappa^+: \zeta \text{ successor ordinal and } i(\zeta) = i(*)\}$$

is unbounded in κ^+. So

$$\otimes \qquad \begin{aligned} &\text{if } \xi < \kappa^+,\ \xi \in E,\ \xi = \sup(\xi \cap W) \text{ and } \theta \in M^1_\xi \,\mathrm{Reg} \cap \lambda \cap M^{0,\lambda_{i(*)}}_\xi \setminus \lambda_{i(*)} \\ &\text{then } M^{0,\lambda_i}_\xi \cap \theta \text{ is an unbound subset of } M^1_\xi \cap \theta. \end{aligned}$$

Hence by [Sh400] 5.1A(1), remembering $M^0_{\zeta+1} = N^0_{\alpha_{\delta(*),\zeta+1}}$, we have: $M^1_\xi \subseteq$ Skolem Hull$\left[\bigcup_{\zeta<\xi} N^0_{\zeta+1} \cup \lambda_{i(*)}\right] \subseteq$ Skolem Hull $\left(N^0_{\alpha_{\delta(*),\xi+1}} \cup \lambda_{i(*)}\right)$ whenever $\xi \in E$ is an accumulation point of W. But $a \subseteq M^1_\xi$ and the right side belongs to $\mathfrak{B}$ (as we can take the Skolem Hull in $\mathfrak{B}_{\delta(*)}$). So we have finished. $\blacksquare_{1.1}$

Remark: Alternatively note: $\mathrm{cov}(\lambda,\lambda,\kappa,2) \le \mathrm{cov}(\theta,\lambda,\sigma,2)$ when $\sigma = \mathrm{cf}(\lambda) < \kappa < \lambda$, $\sigma => \aleph_0$, $\theta = \mathrm{pp}_{\Gamma(\kappa,\sigma)}(\lambda)$; remember $\mathrm{cf}(\lambda) < \kappa < \lambda$ & $\mathrm{pp}(\lambda) < \lambda^{+\kappa^+} \Rightarrow \mathrm{pp}_{<\lambda}(\lambda) = \mathrm{pp}(\lambda)$.

1.2 Claim: *For $\lambda > \mu = \mathrm{cf}(\mu) > \theta > \aleph_0$, we have $\lambda(0) \le \lambda(1) \le \lambda(2) = \lambda(3)$ and if $\mathrm{cov}(\theta,\aleph_1,\aleph_1,2) < \mu$ they are all equal, where:*

$\lambda(0) =:$ *is the minimal κ such that: if $\mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda^+ \setminus \mu$, $|\mathfrak{a}| \le \theta$ then we can find $\langle \mathfrak{a}_\ell : \ell < \omega\rangle$ such that $\mathfrak{a} = \bigcup_{\ell<\omega} \mathfrak{a}_\ell$ and* $(\forall \mathfrak{b})\left[\mathfrak{b} \in \mathcal{S}_{\le\aleph_0}(\mathfrak{a}_n) \Rightarrow \max \mathrm{pcf}(\mathfrak{b}) \le \kappa\right]$.

$\lambda(1) =: \mathrm{Min}\,\{|\mathcal{P}| : \mathcal{P} \subseteq \mathcal{S}_{<\mu}(\lambda)$, *and for every $A \subseteq \lambda$, $|A| \le \theta$ there are $A_n \subseteq A$ $(n < \omega)$, $A = \bigcup_{n<\omega} A_n$, $A_n \subseteq A_{n+1}$ such that: for $n < \omega$, every $a \in \mathcal{S}_{\le\aleph_0}(A_n)$ is a subset of some member of $\mathcal{P}$*$\}$.

$\lambda(2)$ is defined similarly to $\lambda(1)$ as:

$$\mathrm{Min}\Big\{|\mathcal{P}| : \mathcal{P} \subseteq \mathcal{S}_{<\mu}(\lambda) \text{ and for every } A \in \mathcal{S}_{\le\theta}(\lambda) \text{ for some } A_n \subseteq A(n<\omega)\ A = \bigcup_{n<\omega} A_n \text{ and for each } n<\omega \text{ for some } \mathcal{P}_n \subseteq \mathcal{P},\ |\mathcal{P}_n| < \mu,\ \sup_{B\in\mathcal{P}_n} |B| < \mu \text{ and every } a \in \mathcal{S}_{\le\aleph_0}(A_n) \text{ is a subset of some member of } \mathcal{P}_n\Big\}.$$

$\lambda(3)$ *is the minimal κ such that: if $\mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda^+ \setminus \mu$, $|\mathfrak{a}| \le \theta$, then we can find $\langle \mathfrak{a}_\ell : \ell < \omega\rangle$, $\mathfrak{a}_\ell \subseteq \mathfrak{a}_{\ell+1} \subseteq \mathfrak{a} = \bigcup_{\ell<\omega} \mathfrak{a}_\ell$ such that: there is $\{\mathfrak{b}_{\ell,i} : i < i_\ell < \mu\}$, $\mathfrak{b}_{\ell,i} \subseteq \mathfrak{a}_\ell$ such that $\max \mathrm{pcf}\, \mathfrak{b}_{\ell,i} \le \kappa$ and $(\forall \mathfrak{c})[\mathfrak{c} \subseteq \mathfrak{a}_\ell \,\&\, |\mathfrak{c}| \le \aleph_0 \Rightarrow \bigvee_i \mathfrak{c} \subseteq \mathfrak{b}_{\ell,i}]$; equivalently: $\mathcal{S}_{\le\aleph_0}(\mathfrak{a}_n)$ is included in the ideal generated by $\{\mathfrak{b}_\sigma[\mathfrak{a}_n] : \sigma \in \mathfrak{d}\}$ for some $\mathfrak{d} \subseteq \kappa^+ \cap \mathrm{pcf}\, \mathfrak{a}_n$ of cardinality $< \mu$.*

1.2A Remark: (1) We can get similar results with more parameters: replacing $\aleph_0$ and/or $\aleph_1$ by higher cardinals.

(2) Of course, by assumptions as in [Sh410, §6] (e.g. $|\text{pcf}\,\mathfrak{a}| \le |\mathfrak{a}|$) we get $\lambda(0) = \lambda(3)$. This (i.e. Claim 1.2) will be continued in [Sh513].

Proof:

$\lambda(1) \le \lambda(2)$: Trivial.

$\lambda(2) \le \lambda(3)$: Let $\chi = \beth_3(\lambda(3))^+$ and for $\zeta \le \mu^+$ we choose $\mathfrak{B}_\zeta \prec (H(\chi), \in, <_\chi)$, $\{\lambda, \mu, \theta, \lambda(2), \lambda(3)\} \in \mathfrak{B}_\zeta$, $\|\mathfrak{B}_\zeta\| = \lambda(3)$ and $\lambda(3) \subseteq \mathfrak{B}_\zeta$, $\mathfrak{B}_\zeta$ ($\zeta \le \mu^+$) increasing continuous and $\langle \mathfrak{B}_\xi \colon \xi \le \zeta \rangle \in \mathfrak{B}_{\zeta+1}$ and let $\mathfrak{B} = \mathfrak{B}_{\mu^+}$. Lastly let $\mathcal{P} = \mathfrak{B} \cap \mathcal{S}_{<\mu}(\lambda)$. Clearly

$(*)_0$ a function $\mathfrak{a} \mapsto \langle \mathfrak{b}_\sigma[\mathfrak{a}] \colon \sigma \in \text{pcf}\,\mathfrak{a} \rangle$ as in [Sh371, 2.6] is definable in $(H(\chi), \in, <^*_\chi)$ hence $\mathfrak{B}$ is closed under it.

It suffices to show that $\mathcal{P}$ satisfies the requirements in the definition of $\lambda(2)$.

Let $A \subseteq \lambda$, $|A| \le \theta$. We choose by induction on $n < \omega$, N^a_n, (for $\ell < \omega$) and N^b_n, f_n such that:

(a) N^a_n, N^b_n are elementary submodels of $(H(\chi), \in, <^*_\chi)$ of cardinality θ,

(b) $f_n \in \prod \mathfrak{a}_n$ where $\mathfrak{a}_n =: N^a_n \cap \text{Reg} \cap \lambda^+ \setminus \mu$, and $f_n(\sigma) > \sup(N^b_n \cap \sigma)$ (for any $\sigma \in \mathfrak{a}_n$),

(c) $\theta + 1 \subseteq N^a_n \subseteq N^b_n \subseteq \mathfrak{B}$,

(d) N^b_n is the Skolem Hull of $\bigcup\{\text{Rang } f_\ell \colon \ell < n\} \cup A \cup (\theta + 1)$,

(e) N^a_0 is the Skolem Hull of $\theta + 1$ in $(H(\chi), \in, <^*_\chi)$,

(f) N^a_{n+1} is the Skolem Hull of $N^a_n \cup \text{Rang } f_n$,

(g) there are $\mathcal{P}_{n,\ell} \subseteq \mathcal{S}_{<\mu}(\lambda + 1)$ and $A_{n,\ell} \subseteq N^a_n$ (for $l < \omega$) such that:

  (α) $|\mathcal{P}_{n,\ell}| < \mu$ and $\mu_{n,\ell} =: \sup_{B \in \mathcal{P}_{n,\ell}} |B| < \mu$ and $\mathcal{P}_{n,\ell} \subseteq \mathcal{P}_{n,\ell+1}$,

  (β) $N^a_n = \bigcup_\ell A_{n,\ell}$, $\mathcal{P}_n = \bigcup_{\ell<\omega} \mathcal{P}_{n,\ell} \subseteq \mathfrak{B}$ and $A_{n,\ell} \subseteq A_{n,\ell+1}$,

  (γ) for every countable $a \subseteq \lambda \cap A_{n,\ell}$ there is $b \in \mathcal{P}_{n,\ell}$ satisfying $a \subseteq b$,

  (δ) $\mathcal{P}_{n,\ell} = \mathcal{S}_{\le \mu_{n,\ell}}(\lambda + 1) \cap (\text{Skolem Hull of } A_{n,\ell} \cup \mathcal{P}_{n,\ell} \cup (\theta + 1))$.

As in previous proofs, if we succeed to carry out the definition, then $\bigcup_n (N^a_n \cap \lambda) = \bigcup_n N^b_n \cap \lambda$, but the former is $\bigcup_{n,\ell} A_{n,\ell} \cap \lambda$, hence $A \subseteq \bigcup_n \bigcup_\ell A_{n,\ell}$, by (g)($\alpha$), ($\beta$) the $\mathcal{P}'_{n,\ell} = \{a \cap \lambda \colon a \in \mathcal{P}_{n,\ell}\}$ are of the right form and so by (g)(γ) we finish.

Note that without loss of generality: if $a \in \mathcal{P}_{n,\ell}$ then $a \cap \text{Reg} \cap (\lambda + 1) \setminus \mu \in \mathcal{P}_{n,\ell}$.

For $n=0$ we can define N_0^a, N_0^b, $A_{n,\ell}$ trivially. Suppose N_m^a, N_m^b, $A_{m,\ell}$, $\mathcal{P}_{m,\ell}$ are defined for $m \le n$, $\ell < \omega$ and f_m $(m<n)$ are defined. Now $\mathfrak{a}_n$ is well defined and $\subseteq \mathrm{Reg} \cap \lambda^+ \backslash \mu \subseteq \mathfrak{B}$ and $|\mathfrak{a}_n| \le \theta$. So $\mathfrak{a}_n = \bigcup_\ell \mathfrak{a}_{n,\ell}$ and $\mathfrak{a}_{n,\ell} \subseteq \mathfrak{a}_{n,\ell+1}$ where $\mathfrak{a}_{n,\ell} =: \mathfrak{a}_n \cap A_{n,\ell}$ and, of course, $\mathfrak{a}_{n,\ell} \subseteq \mathrm{Reg} \cap \lambda^+ \backslash \mu$ has cardinality $\le \theta$. Note that $\mathfrak{a}_{n,\ell}$ is not necessarily in $\mathfrak{B}$ but

$(*)_1$ every countable subset of $\mathfrak{a}_{n,\ell}$ is included in some subset of $\mathfrak{B}$ which belongs to $\mathcal{P}_{n,\ell}$ and is $\subseteq \mathrm{Reg} \cap \lambda^+ \backslash \mu$.

By the definition of $\lambda(3)$ (see "equivalently" there), for each n, ℓ we can find an increase sequence $\langle \mathfrak{a}_{n,\ell,k} \colon k < \omega \rangle$ of subsets of $\mathfrak{a}_{n,\ell}$ with union $\mathfrak{a}_{n,\ell}$ and $\mathfrak{d}_{n,\ell,k} \subseteq [\mu, \lambda(3)] \cap \mathrm{pcf}(\mathfrak{a}_{n,\ell,k})$, $|\mathfrak{d}_{n,\ell,k}| < \mu$ such that:

$(*)_2$ if $\mathfrak{b} \subseteq \mathfrak{a}_{n,\ell,k}$ is countable then $\mathfrak{b}$ is included in a finite union of some members of $\{\mathfrak{b}_\sigma[\mathfrak{a}_{n,\ell,k}] \colon \sigma \in \mathfrak{d}_{n,\ell,k}\}$ (hence $\max \mathrm{pcf}(\mathfrak{b}) \le \lambda(3)$).

By the properties of pcf:

$(*)_3$ for each $\ell, k < \omega$ and $\mathfrak{c} \subseteq \mathrm{Reg} \cap \lambda^+ \backslash \mu$ such that $\mathfrak{c} \in \mathcal{P}_{n,\ell}$ we can find $\mathfrak{e} = \mathfrak{e}_\mathfrak{c}^{\ell,k} \subseteq \lambda(3)^+ \cap \mathrm{pcf}\, \mathfrak{c}$, $|\mathfrak{e}| \le |\mathfrak{d}_{n,\ell,k}| < \mu$ such that for every $\sigma \in \mathfrak{d}_{n,\ell,k}$ we have: $\mathfrak{c} \cap \mathfrak{b}_\sigma[\mathfrak{a}_{n,\ell,k}]$ is included in a finite union of members of $\{\mathfrak{b}_\tau[\mathfrak{c}] \colon \tau \in \mathfrak{e}_\mathfrak{c}\}$.

By [Sh371, 1.4] we can find $f_n \in \prod_{\sigma \in \mathfrak{a}_n} \sigma$ such that:

$(*)_4$ (α) $\sup(N_n^b \cap \sigma) < f_n(\sigma)$;

(β) if $\mathfrak{c} \in \mathcal{P}_{n,\ell}$, $\ell, k < \omega$, $\mathfrak{c} \subseteq \mathrm{Reg} \cap \lambda^+ \backslash \mu$ and $\sigma \in \mathfrak{e}_\mathfrak{c}^{\ell,k} \subseteq \mathrm{pcf}(\mathfrak{c}) \cap [\mu, \lambda(3)]$ (where $\mathfrak{e}_\mathfrak{c}^{\ell,k}$ is from $(*)_3$) *then* for some $m < \omega$, $\sigma_p \in \sigma^+ \cap \mathrm{pcf}(\mathfrak{c})$ and $\alpha_p < \sigma_p$, (for $p \le m$) the function $f_n \restriction (\mathfrak{b}_\sigma[\mathfrak{c}])$ is included in $\mathrm{Max}_{p \le m} f_{\alpha_p}^{\mathfrak{c},\sigma_\ell} \restriction \mathfrak{b}_{\sigma_p}[\mathfrak{c}]$ (the Max taken pointwise).

Note

$(*)_5$ if $\mathfrak{b} \subseteq \mathfrak{a}_{n,\ell,k}$ is countable (where $\ell, k < \omega$) *then* there is $\mathfrak{c} \in \mathcal{P}_{n,\ell}$, $|\mathfrak{c}| < \mu$, $\mathfrak{c} \subseteq \mathrm{Reg} \cap \lambda^+ \backslash \mu$ such that $\mathfrak{b} \subseteq \mathfrak{c}$.

By $(*)_4$:

$(*)_6$ if $\ell, k < \omega$, $\mathfrak{c} \in \mathcal{P}_{n,\ell}$, $\mathfrak{c} \subseteq \mathrm{Reg} \cap \lambda^+ \backslash \mu$, and $\sigma \in \mathfrak{d}_{n,\ell,k} \cap \lambda(3)^+ \cap \mathrm{pcf}\, \mathfrak{c} \backslash \mu$ then $f_n \restriction \mathfrak{b}_\sigma[\mathfrak{c}] \in \mathfrak{B}$.

You can check that (by $(*)_2 - (*)_6$) :

$(*)_7$ if $\mathfrak{b} \subseteq \mathfrak{a}_{n,\ell,k}$ is countable *then* there is $f_\mathfrak{b}^{n,\ell,k} \in \mathfrak{B}$, $|\mathrm{Dom}\ f_\mathfrak{b}^{n,\ell,k}| < \mu$ such that $f_n \restriction \mathfrak{b} \subseteq f_\mathfrak{b}^{n,\ell,k}$.

Let $\tau_i (i < \omega)$ list the Skolem function of $(H(\chi), \in, <^*_\chi)$. Let

$$A_{n+1,\ell} = \bigcup \{ \mathrm{Rang}\,(\tau_i \restriction (A_{n,j} \cup \mathrm{Rang}\ f_n \restriction \mathfrak{a}_{n,j,k})) : i < \ell, \quad j < \ell, \quad k < \ell \},$$

$$\mathcal{P}'_{n+1,\ell} = \bigcup_{m \le \ell} \mathcal{P}_{n,m} \cup \{ f_n \restriction \mathfrak{a}' \colon \mathfrak{a}' \in \bigcup_{m \le \ell} \mathcal{P}_{n,m} \text{ and } f_n \restriction \mathfrak{a}' \in \mathfrak{B} \},$$

and $\mathcal{P}_{n+1,\ell} = \mathcal{S}_{<\mu}(\lambda+1) \cap (\text{Skolem Hull of } A_{n+1,\ell} \cup \mathcal{P}'_{n+1,\ell} \cup (\theta+1))$.

So f_n, $\mathcal{P}_{n+1,\ell}$ are as required.

Thus we have carried the induction.

$\lambda(3) \le \lambda(2)$: Let $\mathcal{P}$ exemplify the definition of $\lambda(2)$. Let $\mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda^+ \backslash \mu$, $|\mathfrak{a}| \le \theta (<\mu)$. Let $J = J_{\le \lambda(2)}[\mathfrak{a}]$, and let

$J_1 = \{\mathfrak{b}$: $\mathfrak{b} \subseteq \mathfrak{a}$ and there is $\langle \mathfrak{b}_i \colon i < i^* \rangle$, satisfying: $\mathfrak{b}_i \subseteq \mathfrak{b}$, $i^* < \mu$, $\max \mathrm{pcf}\, \mathfrak{b}_i \le \lambda(2)$ and any countable subset of $\mathfrak{b}$ is in the ideal which $\{\mathfrak{b}_i \colon i < i^*\}$ generates$\}$.

Clearly J_1 is an ideal of subsets of $\mathfrak{a}$ extending J. Let

$$J_2 = \left\{ \mathfrak{b} \colon \text{ for some } \mathfrak{b}_n \in J_1 \text{ (for } n < \omega), \ \mathfrak{b} \subseteq \bigcup_n \mathfrak{b}_n \right\}.$$

Clearly J_2 is an $\aleph_1$-complete ideal extending J_1 (and J). If $\mathfrak{a} \in J_2$ we have that $\mathfrak{a}$ satisfies the requirement thus we have finished so we can assume $\mathfrak{a} \notin J_2$. As we can force by Levy $(\lambda(2)^+, 2^{\lambda(2)})$ (alternatively, replacing $\mathfrak{a}$ by [Sh355, §1]) without loss of generality $\lambda(2)^+ = \max \mathrm{pcf}\, \mathfrak{a}$ and so $\mathrm{tcf}(\prod \mathfrak{a}/J_2) = \mathrm{tcf}(\prod \mathfrak{a}/J) = \lambda(2)^+$. Let $\bar{f} = \langle f_\alpha \colon \alpha < \lambda(2)^+ \rangle$ be $<_J$-increasing, $f_\alpha \in \prod \mathfrak{a}$, cofinal in $\prod \mathfrak{a}/J$. Let $\mathfrak{B} \prec (H(\chi), \in, <^*_\chi)$ be of cardinality $\lambda(2)$, $\lambda(2)+1 \subseteq \mathfrak{B}$, $\mathfrak{a} \in \mathfrak{B}$, $\bar{f} \in \mathfrak{B}$ and $\mathcal{P} \in \mathfrak{B}$. Let $\mathcal{P}' =: \mathfrak{B} \cap \mathcal{S}_{<\mu}(\lambda)$.

For $B \in \mathcal{P}'$ (so $|B| < \mu$) let $g_B \in \prod \mathfrak{a}$ be $g_B(\sigma) =: \sup(\sigma \cap B)$, so for some $\alpha_B < \lambda$, $g_B <_J f_{\alpha_B}$. Let $\alpha(*) = \sup\{\alpha_B \colon B \in \mathcal{P}\}$, clearly $\alpha(*) < \lambda(2)^+$. So $\bigwedge_{B \in \mathcal{P}} g_B <_J f_{\alpha(*)}$. Note: $\mathcal{P} \subseteq \mathcal{P}'$ (as $\mathcal{P} \in \mathfrak{B}$, $|\mathcal{P}| \le \lambda(2)$, $\lambda(2)+1 \subseteq \mathfrak{B}$) and for each $B \in \mathcal{P}$, $\mathfrak{c}_B =: \{\sigma \in \mathfrak{a} \colon g_B(\sigma) \ge f_{\alpha(*)}(\sigma)\}$ is in J and $J \subseteq J_1 \subseteq J_2$. Apply the choice of $\mathcal{P}$ (i.e. it exemplifies $\lambda(2)$) to $A =: \mathrm{Rang}\, f_{\alpha(*)}$, get $\langle A_n, \mathcal{P}_n \colon n < \omega \rangle$ as there. Let $\mathfrak{a}_n =: \{\sigma \in \mathfrak{a} \colon f_{\alpha(*)}(\sigma) \in A_n\}$, so $\mathfrak{a} = \bigcup_n \mathfrak{a}_n$, hence for some m, $\mathfrak{a}_m \notin J_2$ (as $\mathfrak{a} \notin J_2$, J_2 is $\aleph_1$-complete) hence $\mathfrak{a}_m \notin J_1$. As $\mathfrak{a} \in \mathfrak{B}$, $\mathcal{P} \in \mathfrak{B}$ clearly $\mathcal{P}_m \subseteq \mathfrak{B}$. So $\{\mathfrak{c}_B \colon B \in \mathcal{P}_m\}$ is a family of $< \mu$ subsets of $\mathfrak{a}$, each in J and every countable $\mathfrak{b} \subseteq \mathfrak{a}_m$ is included in at least one of them (as for some $B \in \mathcal{P}_m$, $\mathrm{Rang}(f_{\alpha(*)} \restriction \mathfrak{b}) \subseteq B$, hence $\mathfrak{b} \subseteq \mathfrak{c}_B$). Easy contradiction.

$\lambda(3) \le \lambda(0)$ IF $\mathrm{cov}(\theta, \aleph_1, \aleph_1, 2) < \mu$: Let $\mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda^+ \backslash \mu$, $|\mathfrak{a}| \le \kappa$, let $\langle \mathfrak{a}_\ell \colon \ell < \omega \rangle$ be as guaranteed by the definition of $\lambda(0)$, let $\mathcal{P}_\ell \subseteq \mathcal{S}_{<\aleph_1}(\mathfrak{a}_\ell)$ exemplify $\mathrm{cov}(\theta, \aleph_1, \aleph_1, 2) < \mu$, for each $\mathfrak{b} \in \mathcal{P}_\ell$ we can find a finite $\mathfrak{e}_\mathfrak{b} \subseteq (\mathrm{pcf}\, \mathfrak{a}_\ell) \cap \lambda^+ \backslash \mu$ such that $\mathfrak{b} \subseteq \bigcup \{\mathfrak{b}_\sigma[\mathfrak{a}_\ell] \colon \sigma \in \mathfrak{e}_\mathfrak{b}\}$ and $\{\mathfrak{b}_{\ell,i} \colon i < i^*\}$ enumerates $\{\mathfrak{e}_\mathfrak{b} \colon \mathfrak{b} \in \mathcal{P}_\ell\}$.

$\lambda(0) \le \lambda(1)$: Similar to the proof of $\lambda(3) \le \lambda(2)$. $\blacksquare_{1.2}$

1.3 CLAIM: *Assume* $\aleph_0 < \mathrm{cf}\,\lambda \le \theta < \lambda < \lambda^*$, $\mathrm{pp}(\lambda) \le \lambda^*$ *and*

$$\mathrm{cov}(\lambda^*, \lambda^+, \theta^+, 2) < \lambda^*.$$

Then $\mathrm{cov}(\lambda, \lambda, \theta^+, 2) < \lambda^*$.

Proof: Easy.

1.3A Definition: Assume $\lambda \ge \theta = \mathrm{cf}\,\theta > \kappa = \mathrm{cf}\,\kappa > \aleph_0$.

(1) $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^{\oplus}[\theta, \kappa]$ if $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^*[\theta, \kappa]$ (see [Sh420, Def 2.1(1)]), and $\delta \in S(\bar{C}) \Rightarrow \delta = \sup(\mathrm{acc}\ C_\delta)$ (note: $\mathrm{acc}\, C_\delta \subseteq C_\delta$), and we do not allow (viii)$^-$ (in [Sh420, Definition 2.1(1)]), or replace it by:

(viii)* for some list $\langle a_i : i < \theta \rangle$ of $\bigcup_{\alpha \in S(\bar{C})} \mathcal{P}_\alpha$, we have: $\delta \in S(\bar{C})$, $\alpha \in \mathrm{acc}\ C_\delta$ implies $\{a \cap \beta : a \in \mathcal{P}_\delta, \beta \in a \cap \alpha\} \subseteq \{a_i : i < \alpha\}$.

(2) For $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^{\oplus}[\theta, \kappa]$ we define a filter $\mathcal{D}^{\oplus}_{(\bar{C},\bar{\mathcal{P}})}(\lambda)$ on $[\mathcal{S}_{<\kappa}(\lambda)]^{<\kappa}$ (rather than on $\mathcal{S}_{<\kappa}(\lambda)$ as in [Sh420, 2.4]) (let $\chi = \beth_{\omega+1}(\lambda)$) :

$Y \in \mathcal{D}^{\oplus}_{(\bar{C},\bar{\mathcal{P}})}(\lambda)$ iff $Y \subseteq (\mathcal{S}_{<\kappa}(\lambda))^{<\kappa}$ and for some $x \in H(\chi)$ for every $\langle N_\alpha, N^*_a : \alpha < \theta,\ a \in \bigcup_{\delta \in S} \mathcal{P}_\delta \rangle$ satisfying condition $\otimes$ from [Sh420, 2.4], and also $[a \in \mathcal{P}_\delta\ \&\ \delta \in S\ \&\ \alpha < \theta \Rightarrow x \in N^*_a\ \&\ x \in N_\alpha]$ there is $A \in \mathrm{id}^a(\bar{C})$ such that $\delta \in S(\bar{C}) \backslash A \Rightarrow \langle \bigcup_{a \in \mathcal{P}_\delta} N^*_a \cap \lambda \cap N_\alpha : \alpha \in \mathrm{acc}\ C_\delta \rangle \in Y$.

Remark: For 1.3B below, see Definition of $\mathcal{T}^\ell(\theta, \kappa)$ and compare with [Sh420, Definition 2.1(2), (3)].

1.3B CLAIM:

(1) *If* $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^{\oplus}[\theta, \kappa]$ *(so* $\lambda > \kappa$ *are regular uncountable) then* $D^{\oplus}_{(\bar{C},\bar{\mathcal{P}})}(\lambda)$ *is a non-trivial ideal on* $[\mathcal{S}_{<\kappa}(\lambda)]^{<\kappa}$.

(2) *If* $\bar{C} \in \mathcal{T}^0[\theta, \kappa]$, $[\delta \in S(\bar{C}) \Rightarrow \delta = \sup \mathrm{acc}\ C_\delta]$, $\mathcal{P}_\delta = \{C_\delta \cap \alpha : \alpha \in C_\delta\}$ *then* $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^{\oplus}[\theta, \kappa]$. *If* $\bar{C} \in \mathcal{T}^1[\theta, \kappa]$, $[\delta \in S(\bar{C}) \Rightarrow \delta = \sup \mathrm{acc}\ C_\delta]$ *and* $\mathcal{P}_\delta = \mathcal{S}_{<\aleph_0}(C_\delta)$ *then* $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^{\oplus}[\theta, \kappa]$.

(3) *If* θ *is successor of regular,* $\sigma = \mathrm{cf}\,\sigma < \kappa$, *there is* $\bar{C} \in \mathcal{T}^0[\theta, \kappa] \cap \mathcal{T}^1[\theta, \kappa]$ *with: for* $\delta \in S(\bar{C})$, C_δ *is closed,* $\mathrm{cf}\,\delta = \sigma$ *and* $\mathrm{otp}\, C_\delta$ *divisible by* ω^2 *(hence* $\delta = \sup \mathrm{acc}\ C_\delta$*).*

(4) *Instead of "θ successor of regular", it suffices to demand*

(∗) $\theta > \kappa$ *regular uncountable, and* $\bigwedge_{\alpha < \theta} \bigvee_{\kappa_1 \in [\kappa, \theta)} \mathrm{cov}(\alpha, \kappa_1, \kappa, 2) < \hat{\vartheta}$.

Replacing 2 by σ, "C_δ closed" is weakened to "$\{\mathrm{otp}(\alpha \cap C_\delta) : \alpha \in C_\delta\}$ is stationary".

Proof: Check.

1.3C CLAIM: *Let $\lambda > \kappa = \mathrm{cf}\,\kappa > \aleph_0$, $\theta = \kappa^+$, $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^{\oplus}[\theta, \kappa]$ then the following cardinals are equal:*

$\mu(0) = \mathrm{cf}\,(\mathcal{S}_{<\kappa}(\lambda), \subseteq)$,

$\mu(4) = \mathrm{Min}\left\{|Y|: Y \in \mathcal{D}^{\oplus}_{(\bar{C},\bar{\mathcal{P}})}(\lambda)\right\}$.

Proof: Included in the proof of [Sh420, 2.6].

1.3D CLAIM: *Let $\lambda_1 \geq \lambda_0 > \kappa = \mathrm{cf}\,\kappa > \aleph_0$, $\theta = \kappa^+$ and $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^{\oplus}[\theta, \kappa]$. Let $\mathfrak{B}_{\lambda_1}$ be a rich enough model with universe λ_1 and countable vocabulary which is rich enough (e.g. all functions (from λ_1 to λ_1) definable in $(H(\beth_\omega(\lambda_1)^+), \in, <^*)$ with any finite number of places). Then the following cardinals are equal:*

$\mu^*(0) = \mathrm{cov}(\lambda_1, \lambda_0^+, \kappa, 2)$,

$\mu^+(4) = \mathrm{Min}\left\{|Y/\approx^{\lambda_0}_{\mathfrak{B}_{\lambda_1}}|: Y \in \mathcal{D}^{\oplus}_{(\bar{C},\bar{\mathcal{P}})}(\lambda_1)\right\}$ *where* $\langle a'_i\colon i \in \mathrm{acc}\ C_\delta\rangle \approx^{\lambda_0}_{\mathfrak{B}} \langle a''_i\colon i \in \mathrm{acc}\,C_\delta\rangle$ *iff* $\bigwedge_{i\in \mathrm{acc} C_\delta}$ *Skolem Hull* ${}_{\mathfrak{B}_{\lambda_1}}(a'_i \cup \lambda_0) =$ *Skolem Hull* ${}_{\mathfrak{B}_{\lambda_1}}(a'_i \cup \lambda_0)$.

Proof: Like the proof of [Sh420], 2.6, but using [Sh400, 3.3A].

2. Equality Relevant to Weak Diamond

It is well known that:

$$\kappa = \mathrm{cf}\,\kappa \,\&\, \theta > 2^{<\kappa} \Rightarrow \mathrm{cov}(\theta, \kappa, \kappa, 2) = \theta^{<\kappa} = \mathrm{cov}(\theta, \kappa, \kappa, 2)^{<\kappa}.$$

Now we have

2.1 CLAIM:

(1) *If $\mu > \lambda \geq \kappa$, $\theta = \mathrm{cov}(\mu, \lambda^+, \lambda^+, \kappa)$, $\mathrm{cov}(\lambda, \kappa, \kappa, 2) \leq \mu$ (or $\leq \theta$) then*

$$\mathrm{cov}(\mu, \lambda^+, \lambda^+, 2) = \mathrm{cov}(\theta, \kappa, \kappa, 2).$$

(2) *If in addition $\lambda \geq 2^{<\kappa}$ (or just $\theta \geq 2^{<\kappa}$) then*

$$\mathrm{cov}(\mu, \lambda^+, \lambda^+, 2)^{<\kappa} = \mathrm{cov}(\mu, \lambda^+, \lambda^+, 2).$$

2.1A Remark:

(1) A most interesting case is $\kappa = \aleph_1$.

(2) This clarifies things in [Sh-f,AP1.17].

Proof: (1) Note that $\theta \geq \mu$ (because $\mu > \lambda \geq \kappa$). First we prove " $\leq$ ". Let $\mathcal{P}_0$ be a family of θ subsets of μ each of cardinality $\leq \lambda$, such that every subset of μ of cardinality $\leq \lambda$ is included in the union of $< \kappa$ of them (exists by the definition of $\theta = \text{cov}(\mu, \lambda^+, \lambda^+, \kappa)$). Let $\mathcal{P}_0 = \{A_i : i < \theta\}$. Let $\mathcal{P}_1$ be a family of $\text{cov}(\theta, \kappa, \kappa, 2)$ subsets of θ, each of cardinality $< \kappa$ such that any subset of θ of cardinality $< \kappa$ is included in one of them.

Let $\mathcal{P} =: \{\bigcup_{i \in a} A_i : a \in \mathcal{P}_1\}$; clearly $\mathcal{P}$ is a family of subsets of μ each of cardinality $\leq \lambda$, $|\mathcal{P}| \leq |\mathcal{P}_1| = \text{cov}(\theta, \kappa, \kappa, 2)$, and every $A \subseteq \mu$, $|A| \leq \lambda$ is included in some union of $< \kappa$ members of $\mathcal{P}_0$ (by the choice of $\mathcal{P}_0$), say $\bigcup_{i \in b} A_i$, $b \subseteq \theta$, $|b| < \kappa$; by the choice of $\mathcal{P}_1$, for some $a \in \mathcal{P}_1$ we have $b \subseteq a$, hence $A \subseteq \bigcup_{i \in b} A_i \subseteq \bigcup_{i \in a} A_i \in \mathcal{P}$. So $\mathcal{P}$ exemplify $\text{cov}(\mu, \lambda^+, \lambda^+, 2) \leq \text{cov}(\theta, \kappa, \kappa, 2)$.

Second we prove the inequality $\geq$. If $\kappa \leq \aleph_0$ then $\text{cov}(\mu, \lambda^+, \lambda^+, 2) = \theta$ and $\text{cov}(\theta, \kappa, \kappa, 2) = \theta$ so $\geq$ trivially holds; so assume $\kappa > \aleph_0$. Obviously $\text{cov}(\mu, \lambda^+, \lambda^+, 2) \geq \theta$. Note, if κ is singular then, as $\text{cf}\,\lambda^+ > \lambda \geq \kappa$ for some $\kappa_1 < \kappa$, we have $\theta = \text{cov}(\mu, \lambda^+, \lambda^+, \kappa) = \text{cov}(\mu, \lambda^+, \lambda^+, \kappa')$ whenever $\kappa' \in [\kappa_1, \kappa]$ is a successor (by [Sh355, 5.2(8)]); also $\text{cov}(\theta, \kappa, \kappa, 2) \leq \sup\{\text{cov}(\theta, \kappa, \kappa', 2) : \kappa' \in [\kappa_1, \kappa]$ is a successor cardinal$\}$ and $\text{cov}(\theta, \kappa, \kappa', 2) \leq \text{cov}(\theta, \kappa', \kappa', 2)$ when $\kappa' < \kappa$, so without loss of generality κ is regular uncountable. Hence for any $\theta_1 < \theta$ we have

$(*)_{\theta_1}$ we can find a family $\mathcal{P} = \{A_i : i < \theta_1\}$, $A_i \subseteq \mu$, $|A_i| \leq \lambda$, such that any subfamily of cardinality $\leq \lambda^+$ has a transversal. [Why? By [Sh355, 5.4], $(=^+)$ and [Sh355,1.5A] even for $\leq \mu$.]

Hence if $\theta_1 \leq \theta$, $\text{cf}\,\theta_1 < \lambda^+$ (or even $\text{cf}\,\theta_1 \leq \mu$) then $(*)_{\theta_1}$. Now we shall prove below

$$(\otimes_1) \qquad (*)_{\theta_1} \Rightarrow \text{cov}(\theta_1, \kappa, \kappa, 2) \leq \text{cov}(\mu, \lambda^+, \lambda^+, 2)$$

and obviously

$$(\otimes_2) \qquad \text{if } \text{cf}\,\theta \geq \kappa \text{ then } \text{cov}(\theta, \kappa, \kappa, 2) = \sum_{\alpha < \theta} \text{cov}(\alpha, \kappa, \kappa, 2)$$

together; (as $\theta \leq \text{cov}(\theta, \lambda^+, \lambda^+, 2)$ which holds as $\lambda < \mu \leq \theta$) we are done.

Proof of $\otimes_1$: Let $\{A_i : i < \theta_1\}$ exemplify $(*)_{\theta_1}$ and $\mathcal{P}_2$ exemplify the value of $\text{cov}(\mu, \lambda^+, \lambda^+, 2)$. Now for every $a \subseteq \theta_1$, $|a| < \kappa$, let $B_a =: \bigcup_{i \in a} A_i$; so $B_a \subseteq \mu$, $|B_a| \leq \lambda$ hence there is $A_a \in \mathcal{P}_2$ such that: $B_a \subseteq A_a$. Now for $A \in \mathcal{P}_2$ define $b[A] =: \{i < \theta_1 : A_i \subseteq A\}$; it has cardinality $\leq \lambda$ (as any subfamily of $\{A_i : A_i \subseteq A\}$ of cardinality $\leq \lambda^+$ has a transversal). Note $a \subseteq b[A_a]$ (just read the definitions of $b[A]$ and A_a; note $a \in \mathcal{S}_{<\kappa}(\theta_1)$). For $A \in \mathcal{P}_2$ let $\mathcal{P}_A$ be a family of $\leq \mathrm{cov}(\lambda, \kappa, \kappa, 2)$ subsets of $b[A]$ each of cardinality $< \kappa$ such that any such set is included in one of them (exists as $|b[A]| \leq \lambda$ by the definition of $\mathrm{cov}(\lambda, \kappa, \kappa, 2)$). So for any $a \in \mathcal{S}_{<\kappa}(\theta_1)$ for some $c \in \mathcal{P}_{A_a}$, $a \subseteq c$. We can conclude that $\bigcup\{\mathcal{P}_A : A \in \mathcal{P}_2\}$ is a family exemplifying $\mathrm{cov}(\theta_1, \kappa, \kappa, 2) \leq \mathrm{cov}(\mu, \lambda^+, \lambda^+, 2) + \mathrm{cov}(\lambda, \kappa, \kappa, 2)$ but the last term is $\leq \mu$ (by an assumption) whereas the first is $\geq \mu$ (as $\mu > \lambda$) hence the second term is redundant.

(2) By the first part it is enough to prove $\mathrm{cov}(\theta, \kappa, \kappa, 2)^{<\kappa} = \mathrm{cov}(\theta, \kappa, \kappa, 2)$, which is easy and well known (as $\theta \geq \mu > \lambda \geq 2^{<\kappa}$). $\blacksquare_{2.1}$

2.1B Remark: So actually if $\mu > \lambda \geq \kappa$, $\theta = \mathrm{cov}(\mu, \lambda^+, \lambda^+, \kappa)$ then ($\theta \geq \mu > \lambda \geq \kappa$ and)

$$\begin{aligned}\mathrm{cov}(\mu, \lambda^+, \lambda^+, 2) &\leq \mathrm{cov}(\mu, \lambda^+, \lambda^+, \kappa) + \mathrm{cov}(\theta, \kappa, \kappa, 2)\\ &= \theta + \mathrm{cov}(\theta, \kappa, \kappa, 2) = \mathrm{cov}(\theta, \kappa, \kappa, 2)\end{aligned}$$

and

$$\mathrm{cov}(\theta, \kappa, \kappa, 2) \leq \mathrm{cov}(\mu, \lambda^+, \lambda^+, 2) + \mathrm{cov}(\lambda, \kappa, \kappa, 2),$$

hence, $\mathrm{cov}(\theta, \kappa, \kappa, 2) = \mathrm{cov}(\mu, \lambda^+, \lambda^+, 2) + \mathrm{cov}(\lambda, \kappa, \kappa, 2)$.

3. Cofinality of $\mathcal{S}_{\leq\aleph_0}(\kappa)$ for κ Real Valued Measurable and Trees

In Rubin–Shelah [RuSh117] two covering properties were discussed concerning partition theorems on trees, the stronger one was sufficient, the weaker one necessary so it was asked whether they are equivalent. [Sh371, 6.1, 6.2] gave a partial positive answer (for λ successor of regular, but then it gives a stronger theorem); here we prove the equivalence.

In Gitik–Shelah [GiSh412] cardinal arithmetic, e.g. near a real valued measurable cardinal κ, was investigated, e.g. $\{2^\sigma : \sigma < \kappa\}$ is finite (and more); this section continues it. In particular we answer a problem of Fremlin: for κ real valued measurable, do we have $\mathrm{cf}(\mathcal{S}_{<\aleph_1}(\kappa), \subseteq) = \kappa$? Then we deal with trees with many branches; on earlier theorems see [Sh355, §0], and later [Sh410, 4.3].

3.1 THEOREM: *Assume λ, θ, κ are regular cardinals and $\lambda > \theta = \kappa > \aleph_0$. Then the following conditions are equivalent:*

(A) *for every $\mu < \lambda$ we have $\mathrm{cov}(\mu, \theta, \kappa, 2) < \lambda$,*

(B) *if $\mu < \lambda$ and $a_\alpha \in \mathcal{S}_{<\kappa}(\mu)$ for $\alpha < \lambda$ then for some $W \subseteq \lambda$ of cardinality λ we have $|\bigcup_{\alpha \in W} a_\alpha| < \theta$.*

3.1A Remark: (1) Note that (B) is equivalent to: if $a_\alpha \in \mathcal{S}_{<\kappa}(\lambda)$ for $\alpha < \lambda$, then for some unbounded $S \subseteq \{\delta < \lambda\colon \mathrm{cf}(\delta) \geq \kappa\}$ and $b \in \mathcal{S}_{<\theta}(\lambda)$, for $\alpha \neq \beta$ in S, $a_\alpha \cap a_\beta \subseteq b$ (we can start with any stationary $S_0 \subseteq \{\delta < \lambda\colon \mathrm{cf}\,\delta \geq \kappa\}$, and use Fodour Lemma).

(2) We can replace everywhere θ by κ, but want to prepare for a possible generalization. By the proof we can strengthen "$W \subseteq \lambda$ of cardinality λ" to "$W \subseteq \lambda$ is stationary" (for $\neg$(A) $\rightarrow$ $\neg$(B) this is trivial, for (A) $\rightarrow$ (B) real), so these two versions of (B) are equivalent.

Proof:

(A)$\Rightarrow$(B):

Trivial [for $\mu < \lambda$ let $\mathcal{P}_\mu \subseteq \mathcal{S}_{<\theta}(\mu)$ exemplify $\mathrm{cov}(\mu, \theta, \kappa, 2) < \lambda$; suppose $\mu < \lambda$ and $a_\alpha \in \mathcal{S}_{<\kappa}(\mu)$ for $\alpha < \lambda$ are given, for each α for some $A_\alpha \in \mathcal{P}_\mu$ we have $a_\alpha \subseteq A_\alpha$; as $|\mathcal{P}_\mu| < \lambda = \mathrm{cf}\,\lambda$ for some A^* we have $W =: \{\alpha < \lambda\colon A_\alpha = A^*\}$ has cardinality λ, so S is as required in (B)].

$\neg$(A) $\Rightarrow$ $\neg$(B):

First Case: *For some $\mu \in [\theta, \lambda)$, $\mathrm{cf}\,\mu < \kappa < \mu$ and $pp^+_{<\kappa}(\mu) > \lambda$.* Then we can find $\mathfrak{a} \subseteq \mathrm{Reg} \cap \mu \backslash \theta$, $|\mathfrak{a}| < \kappa$, $\sup \mathfrak{a} = \mu$ and $\max \mathrm{pcf}_{J^{bd}_{\mathfrak{a}}} \mathfrak{a} \geq \lambda$. So by [Sh355, 2.3] without loss of generality $\lambda = \max \mathrm{pcf}\,\mathfrak{a}$; let $\langle f_\alpha\colon \alpha < \lambda\rangle$ be $<_{J_{<\lambda}[\mathfrak{a}]}$-increasing cofinal in $\prod \mathfrak{a}$.

Let $a_\alpha = \mathrm{Rang}(f_\alpha)$, so for $\alpha < \lambda$, a_α is a subset of $\mu < \lambda$ of cardinality $< \kappa$. Suppose $W \subseteq \lambda$ has cardinality λ, hence is unbounded, and we shall show that $\mu = |\bigcup_{\alpha \in W} a_\alpha|$; as $\mu \geq \theta$ this is enough. Clearly $a_\alpha = \mathrm{Rang}\ f_\alpha \subseteq \sup \mathfrak{a} = \mu$, hence $\bigcup_{\alpha \in W} a_\alpha \subseteq \mu$. If $|\bigcup_{\alpha \in W} a_\alpha| < \mu$ define $g \in \prod \mathfrak{a}$ by: $g(\sigma)$ is $\sup \left(\sigma \cap \bigcup_{\alpha \in W} a_\alpha\right)$ if $\sigma > |\bigcup_{\alpha \in W} a_\alpha|$ and 0 otherwise. So $g \in \prod \mathfrak{a}$ hence for some $\beta < \lambda$ $g < f_\beta$ mod $J_{<\lambda}[\mathfrak{a}]$. As the f_β's are $<_{J_{<\lambda}[\mathfrak{a}]}$-increasing and $W \subseteq \lambda$ unbounded, without loss of generality $\beta \in W$, hence by g's choice $\left[\sigma \in \mathfrak{a} \backslash |\bigcup_{\alpha \in W} a_\beta|^+ \Rightarrow f_\beta(\sigma) \leq g(\sigma)\right]$ but $\left\{\sigma\colon \sigma \in \mathfrak{a}, \sigma > |\bigcup_{\theta \in W} a_\alpha|^+\right\} \notin J_{<\lambda}[\mathfrak{a}]$ (as μ is a limit cardinal and $\max \mathrm{pcf}_{J^{bd}_{\mathfrak{a}}}(\mathfrak{a}) \geq \lambda$), contradiction.

The main case is:

Second Case: *For no $\mu \in [\theta, \lambda)$ is $\mathrm{cf}\,\mu < \kappa < \mu$, $pp^+_{<\kappa}(\mu) > \lambda$.* Let $\chi =: \beth_2(\lambda)^+$, $\mathfrak{B}$ be the model with universe λ and the relations and functions definable in $(H(\chi), \in, <^*_\chi)$ possibly with the parameters κ, θ, λ. We know that $\lambda > \theta^+$ (otherwise $\lambda = \theta^+$ and (A) holds). Let $S \subseteq \{\delta < \lambda\colon \mathrm{cf}\,\delta = \theta\}$ be stationary and in $I[\lambda]$ (see [Sh420, 1.5]) and let $S \subseteq S^+$, $\bar{C} = \langle C_\alpha : \alpha \in S^+ \rangle$ be such that: C_α closed, $\mathrm{otp}\, C_\alpha \leq \theta$, $[\beta \in \mathrm{nacc}\ C_\alpha \Rightarrow C_\beta = C_\alpha \cap \beta]$, $[\mathrm{otp}\ C_\alpha = \kappa \Leftrightarrow \alpha \in S]$ and for $\alpha \in S^+$ limit, C_α is unbounded in α (see [Sh420, 1.2]).

Without loss of generality $\bar{C}$ is definable in $(\mathfrak{B}, \kappa, \theta, \lambda)$. Let $\mu_0 \in [\theta, \lambda)$ be minimal such that $\mathrm{cov}(\mu_0, \theta, \kappa, 2) \geq \lambda$, so $\mu_0 > \theta$, $\kappa > \mathrm{cf}\, \mu_0$. We choose by induction on $\alpha < \lambda$, $\mathfrak{A}_\alpha$, a_α such that:

(α) $\mathfrak{A}_\alpha \prec (H(\chi), \in, <^*_\chi)$, $\|\mathfrak{A}_\alpha\| < \lambda$ and $\mathfrak{A}_\alpha \cap \lambda$ is an ordinal and $\{\lambda, \mu_0, \theta, \kappa, \mathfrak{B}, \bar{C}\} \in \mathfrak{A}_\alpha$.

(β) $\mathfrak{A}_\alpha (\alpha < \lambda)$ is increasing continuous and $\langle \mathfrak{A}_\beta : \beta \leq \alpha \rangle \in \mathfrak{A}_{\alpha+1}$.

(γ) $a_\alpha \in \mathcal{S}_{<\kappa}(\mu_0)$ is such that for no $A \in \mathcal{S}_{<\theta}(\mu_0) \cap \mathfrak{A}_\alpha$ is $a_\alpha \subseteq A$.

(δ) $\langle a_\beta : \beta \leq \alpha \rangle \in \mathfrak{A}_{\alpha+1}$.

There is no problem to carry the definition and let $\mathfrak{A} = \bigcup_{\alpha<\lambda} \mathfrak{A}_\alpha$. Clearly it is enough to show that $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$ contradict (B). Clearly $\mu_0 \in (\theta, \lambda)$ and $a_\alpha \in \mathcal{S}_{<\kappa}(\mu_0)$. So let $W \subseteq \lambda$, $|W| = \lambda$ and we shall prove that $|\bigcup_{\alpha \in W} a_\alpha| \geq \theta$. Note:

($*$) if $\mathfrak{a} \subseteq [\theta, \lambda)$, $|\mathfrak{a}| < \kappa$, $\mathfrak{a} \in \mathfrak{A}_\alpha$ (and $\mathfrak{a} \subseteq \mathrm{Reg}$, of course) then $(\prod \mathfrak{a}) \cap \mathfrak{A}_\alpha$ is cofinal in $\prod \mathfrak{a}$ (as $\max \mathrm{pcf}\, \mathfrak{a} < \lambda$).

Let $R = \{(\alpha, \beta) : \beta \in a_\alpha, \alpha < \lambda\}$ and

$$E =: \{\delta < \lambda : (\mathfrak{A}_\delta, R \restriction \delta, W \cap \delta, \mu_0) \prec (\mathfrak{A}, R, W, \mu_0) \quad \text{and} \quad \mathfrak{A}_\delta \cap \lambda = \delta\}.$$

Clearly E is a club of λ, hence we can find $\delta(*) \in S \cap \mathrm{acc}(E)$. Let $C_{\delta(*)} = \{\gamma_i : i < \theta\}$ (in increasing order). We now define by induction on $n < \omega$, M_n, $\langle N^n_\zeta : \zeta < \theta \rangle$, f_n such that:

(a) M_n is an elementary submodel of $(\mathfrak{A}, R, W)$, $\|M_n\| = \theta$,

(b) $\langle N^n_\zeta : \zeta < \theta \rangle$ is an increasing continuous sequence of elementary submodels of $\mathfrak{B}$,

(c) $\|N^n_\zeta\| < \theta$,

(d) $N^n_\zeta \in \mathfrak{A}_{\delta(*)}$,

(e) $\bigcup_{\zeta<\kappa} |N^n_\zeta| \subseteq |M_n|$,

(f) $f_n \in \prod(\mathrm{Reg} \cap M_n)$,

(g) $f_n(\sigma) > \sup(M_n \cap \sigma)$ for $\sigma \in \mathrm{Dom}(f_n) \backslash \theta^+$,

(h) for every $\zeta < \theta$, $f_n \restriction (\mathrm{Reg} \cap N^n_\zeta \backslash \theta^+) \in \mathfrak{A}_{\delta(*)}$,

(i) N^0_ζ is the Skolem Hull in $\mathfrak{B}$ of $\{\gamma_i, i : i < \zeta\}$,

(j) N^{n+1}_ζ is the Skolem Hull in $\mathfrak{B}$ of $N^n_\zeta \cup \{f_n(\sigma) : \sigma \in \mathrm{Reg} \cap N^n_\zeta \backslash \theta^+\}$,

(k) M_n is the Skolem Hull in $(\mathfrak{A}, R, W)$ of $\bigcup_{\ell<n} M_\ell \cup \bigcup_{\zeta<\theta} N^n_\zeta$.

There is no problem to carry the definition: for $n = 0$ define N^0_ζ by (i) [trivially (b) holds and also (c), as for (d), note that $\bar{C} \in \mathfrak{A}_0 \prec \mathfrak{A}_{\delta(*)}$ and $\{\gamma_i: i < \zeta\} \in \mathfrak{A}_{\delta(*)}$ as $\bar{C}$ is definable in $\mathfrak{B}$ hence $\{\langle \alpha, \gamma, \zeta\rangle: \alpha \in S^+, \zeta < \theta$, and γ is the ζ-th member of $C_\alpha\}$ is a relation of $\mathfrak{B}$ hence each $C_{\gamma_{\zeta+1}}(\zeta < \theta)$ is in $\mathfrak{A}_{\delta(*)}$ hence each $\{\gamma_i: i < \zeta\}$ is and we can compute the Skolem Hull in $\mathfrak{A}_{\gamma_j}$ for $j < \theta$ large enough].

Next, choose M_n by (k), it satisfies (e) + (a). If $\langle N^n_\zeta: \zeta < \theta\rangle$, M_n are defined, we can find f_n satisfying (f) + (g) + (h) by [Sh371,1.4] (remember (*)). For $n + 1$ define N^n_ζ by (j) and then M_{n+1} by (k).

Next by [Sh400, 3.3A or 5.1A(1)] we have

$$(*) \quad \bigcup_{n<\omega} M_n \cap \delta(*) = \bigcup_{\substack{n<\omega \\ \zeta<\theta}} N^n_\zeta \cap \delta(*) \quad \text{hence} \quad \bigcup_{\substack{n<\omega \\ \zeta<\theta}} N^n_\zeta \cap W \ \text{ is unbounded in } \delta(*),$$

hence for some n

$$(*)_n \qquad \bigcup_{\zeta<\theta} N^n_\zeta \cap W \ \text{ is unbounded in } \delta(*).$$

Remember $N^n_\zeta \in \mathfrak{A}_{\delta(*)} = \bigcup_{\alpha<\delta(*)} \mathfrak{A}_\alpha = \bigcup_{i<\theta} \mathfrak{A}_{\gamma_i}$. So for some club e of θ we have:

$$(\otimes) \qquad \text{if } \zeta \in e, \quad \xi < \zeta \ \text{ then: } N^n_\xi \in \mathfrak{A}_{\gamma_\zeta}, \text{ and } \gamma_\zeta \in E \cap C_{\delta(*)}$$

(remember $\delta(*) \in \operatorname{acc}(E)$).

Hence, for $\zeta \in e$, we have: $\mathfrak{A}_{\gamma_\zeta} \cap \lambda = \gamma_\zeta$, and $W \cap N^n_\zeta \backslash \sup N^n_\xi \neq \emptyset$ for every $\xi < \zeta$. Let $e = \{\zeta(\epsilon): \epsilon < \theta\}$, $\zeta(\epsilon)$ strictly increasing continuous in ϵ. Now for every $\epsilon < \theta$, $N^n_{\zeta(\epsilon)} \in \mathfrak{A}_{\gamma_{\zeta(\epsilon+1)}}$ (and $\langle a_\beta: \beta \leq \sup(\lambda \cap N^n_{\zeta(\epsilon)}\rangle \in \mathfrak{A}_{\gamma_{\zeta(\epsilon+1)}}$) hence $A_1 =: \bigcup\{a_\beta: \beta \in W \cap N^n_{\zeta(\epsilon)}\} \subseteq A_2 =: \bigcup\{a_\beta: \beta \in N^n_{\zeta(\epsilon+1)}\} \cap \mu_0 \in \mathfrak{A}_{\gamma_{\zeta(\epsilon+1)}}$ and A_2 is a subset of μ_0 of cardinality $< \theta$ hence (by the choice of the a_γ's above) $a_{\gamma_{\zeta(\epsilon+1)}} \not\subseteq A_2$ hence $a_{\gamma_{\zeta(\epsilon+1)}} \not\subseteq \bigcup\{a_\beta: \beta \in W \cap N^n_{\zeta(\epsilon)}\}$; moreover, similarly $\gamma_{\zeta(\epsilon+1)} \leq \gamma < \lambda \Rightarrow a_\gamma \not\subseteq \bigcup\{a_\beta: \beta \in W \cap N^n_{\zeta(\epsilon)}\}$.

But $W \cap N^n_{\zeta(\epsilon+2)} \backslash \gamma_{\zeta(\epsilon+1)} \neq \emptyset$, hence $\langle \bigcup\{a_\beta: \beta \in W \cap N^n_{\zeta(\epsilon)}\}: \epsilon < \theta\rangle$ is not eventually constant, hence

$$\bigcup\left\{a_\beta: \beta \in W \cap \bigcup_{\epsilon<\theta} N^n_{\zeta(\epsilon)}\right\} = \bigcup\left\{a_\beta: \beta \in W \cap \bigcup_{\zeta<\theta} N^n_\zeta\right\}$$

has cardinality θ. Hence $\bigcup_{\beta\in W} a_\beta$ has cardinality $\geq \theta$, as required. $\blacksquare_{3.1}$

3.2 Conclusion: (1) If λ is real valued measurable then $\kappa = \mathrm{cf}\,[\mathcal{S}_{<\aleph_1}(\lambda), \subseteq]$ (equivalently, $\mathrm{cov}(\lambda, \aleph_1, \aleph_1, 2) = \lambda$).
(2) Suppose λ is regular $> \kappa = \mathrm{cf}\,\kappa > \aleph_0$, I is a λ-complete ideal on λ extending J^{bd}_λ and is κ-saturated (i.e. we cannot partition λ to κ sets not in I). Then for $\alpha < \lambda$, $\mathrm{cf}(\mathcal{S}_{<\kappa}(\alpha), \subseteq) < \lambda$, equivalently $\mathrm{cov}(\alpha, \kappa, \kappa, 2) < \lambda$.

3.2A Remark: (1) So for regular $\theta \in (\kappa, \lambda)$ (in the above situation) we have $\bigwedge_{\alpha<\lambda} \mathrm{cov}(\alpha, \theta, \theta, 2) < \lambda$; actually $\kappa \leq \mathrm{cf}\,\theta \leq \theta < \lambda$ suffices by the proof.

Proof: (1) Follows by (2).
(2) The conclusion is (A) of Theorem 3.1, hence it suffices to prove (B). Let $\mu < \lambda$ and $a_\alpha \in \mathcal{S}_{<\kappa}(\mu)$ for $\alpha < \lambda$ be given. As $\kappa < \lambda = \mathrm{cf}\,\lambda$ without loss of generality for some $\sigma < \kappa$, $\bigwedge_{\alpha<\lambda} |a_\alpha| = \sigma$. Let f_α be a function from σ onto a_α, so $\mathrm{Rang}\, f_\alpha \subseteq \mu$. Now for each $i < \sigma$, $\langle \{\alpha < \lambda\colon f_\alpha(i) = \gamma\}\colon \gamma < \mu\rangle$ is a partition of λ to μ sets; as I is κ-saturated, $b_i =: \{\gamma < \mu\colon \{\alpha < \lambda\colon f_\alpha(i) = \gamma\} \notin I\}$ has cardinality $< \kappa$, hence $b =: \bigcup_{i<\sigma} b_i$ has cardinality $< \kappa + \sigma^+ \leq \kappa$ (remember $\sigma < \kappa = \mathrm{cf}\,\kappa$). For each $i < \sigma$, $\gamma \in \mu \backslash b_i$ the set $\{\alpha < \lambda\colon f_\alpha(i) = \gamma\}$ is in I; so as I is λ-complete, $\lambda > \mu$ we have: $\{\alpha < \lambda\colon f_\alpha(i) \notin b_i\}$ is in I. Now let

$$W =: \{\alpha < \lambda\colon \text{ for some } i < \sigma, f_\alpha(i) \notin b_i\} \subseteq \bigcup_{i<\sigma} \{\alpha < \lambda\colon f_\alpha(i) \notin b_i\}.$$

This is the union of $\leq \sigma < \lambda$ sets each in I, hence is in I, so $|\lambda \backslash W| = \lambda$, and clearly

$$\bigcup_{\alpha\in\lambda\backslash W} a_\alpha = \{f_\alpha(i)\colon \alpha \in \lambda\backslash W, i < \sigma\} \subseteq \{f_\alpha(i)\colon \alpha < \lambda, \neg f_\alpha(i) \notin b_i, i < \sigma\} \subseteq b,$$

and $|b| < \kappa$ so $\lambda \backslash W$ is as required in (B) of Theorem 3.1. $\blacksquare_{3.2}$

3.3 Lemma: *For every λ there is μ, $\lambda \leq \mu < 2^\lambda$ such that* (A) *or* (B) *or* (C) *below holds (letting $\kappa = \mathrm{Min}\{\theta\colon 2^\theta = 2^\lambda\}$)* :

- (A) *$\mu = \lambda$ and for every regular $\chi \leq 2^\lambda$ there is a tree T of cardinality $\leq \lambda$ with $\geq \chi$ $\mathrm{cf}(\kappa)$-branches (hence there is a linear order of cardinality $\geq \chi$ and density $\leq \lambda$).*
- (B) *$\mu > \lambda$ is singular, and:*
  - (α) $\mathrm{pp}(\mu) = 2^\lambda$ *(even $\lambda = \kappa \Rightarrow \mathrm{pp}^+(\mu) = (2^\lambda)^+$)*, $\mathrm{cf}\,\mu \leq \lambda$, $(\forall\theta)[\mathrm{cf}\,\theta \leq \lambda < \theta < \mu \Rightarrow \mathrm{pp}_\lambda\,\theta < \mu]$ *(and $\mu \leq 2^{<\kappa}$)*

hence

(α)′ *for every successor** $\chi \le 2^\lambda$ *there is a tree from [Sh355, 3.5]:* $\mathrm{cf}\,\mu$ *levels, every level of cardinality* $< \mu$ *and* χ $(\mathrm{cf}\,\mu)$*-branches,*

(β) *for every* $\chi \in (\lambda, \mu)$*, there is a tree* T *of cardinality* λ *with* $\ge \chi$ *branches of the same height,*

(γ) $\mathrm{cf}\,\mu \ge \mathrm{cf}\,\kappa$ *and even* $\mathrm{cf}\,\kappa > \aleph_0 \Rightarrow pp_{\Gamma(\mathrm{cf}\,\mu)}(\mu) =^+ 2^\lambda$.

(C) *Like (B) but we omit* (α) *and retain* (α)′.

Proof:

FIRST CASE: $\kappa = \aleph_0$. Trivially (A) holds.

SECOND CASE: *κ is regular uncountable.* So $\kappa \le \lambda$ and $2^\kappa = 2^\lambda$ and $[\theta < \kappa \Rightarrow 2^\theta < 2^\kappa]$ hence $2^{<\kappa} < 2^\kappa$ (remember $\mathrm{cf}(2^\kappa) > \kappa$). Try to apply [Sh410, 4.3], its assumptions (i) + (ii) hold (with κ here standing for λ there) and if possibility (A) here fails then the assumption (iii) there holds, too; so there is μ as there; so (α), (γ) of (B) of 3.3 holds** and let us prove (β), so assume $\chi \in (\lambda, \mu)$, without loss of generality, is regular, and we shall prove the statement in (β) of 3.3(B). Without loss of generality χ is regular and $\mu' \in (\lambda, \chi) \,\&\, \mathrm{cf}\,\mu' \le \lambda \Rightarrow \mathrm{pp}_\lambda(\mu') < \chi$; i.e. χ is $(\lambda, \lambda^+, 2)$-inaccessible. [Why? If χ is not as required, we shall show how to replace χ by an appropriate regular $\chi' \in [\chi, \mu)$.]

Let $\mu' \in (\lambda, \chi)$ be minimal such that $\mathrm{pp}_\lambda(\mu') \ge \chi$, (so $\mathrm{cf}\,\mu' \le \lambda$) now $\mathrm{pp}(\mu') < \mu$ (by the choice of μ) and $\chi' =: \mathrm{pp}(\mu')^+$, by [Sh355, 2.3] is as required].

Let θ be minimal such that $2^\theta \ge \chi$. So trivially $\theta \le \kappa \le \lambda < \chi$ and $(2^{<\kappa})^\kappa = 2^\kappa$ hence $\mu \le 2^{<\kappa}$ hence $\chi < 2^{<\kappa}$; as χ is regular $< 2^{<\kappa}$ but $> \lambda \ge \kappa$, clearly $\theta < \kappa \le \lambda$; also trivially $2^{<\theta} \le \chi \le 2^\theta$ but χ is regular $> \lambda \ge \kappa > \theta$ and $[\sigma < \theta \Rightarrow 2^\sigma < \chi]$, so $2^{<\theta} < \chi \le 2^\theta$. Try to apply [Sh410, 4.3] with θ here standing for λ there; assumptions (i), (ii) there hold, and if assumption (iii) fails we get a tree with $\le \theta$ nodes *and* $\ge \chi$ θ-branches as required. So assume (iii) holds and we get there μ'; if $\mu' \le \lambda$ we have a tree as required; if

* If $\lambda = \kappa$, just regular, and we can change λ for this.

** Alternatively to quoting [Sh410, 4.3], we can get this directly, if $\mathrm{cov}(2^{<\kappa}, \lambda^+, (\mathrm{cf}\,\kappa)^+, \mathrm{cf}\,\kappa) < 2^\lambda$ we can get (A); otherwise by [Sh355, 5.4] for some $\mu_0 \in (\lambda, 2^{<\kappa}]$, $\mathrm{cf}(\mu_0) = \mathrm{cf}\,\kappa$ and $\mathrm{pp}(\mu_0) = (2^\lambda)$. Let $\mu \in (\lambda, 2^{<\kappa}]$ be minimal such that $\mathrm{cf}\,\mu \le \lambda \,\&\, \mathrm{pp}_\lambda(\mu) > 2^{<\kappa}$. Necessarily ([Sh355, 2.3] and [Sh371, 1.6(2), (3), (5)]) $\mathrm{pp}_\lambda(\mu) = \mathrm{pp}\,\mu = \mathrm{pp}(\mu_0) = (2^\lambda)$ and (again using [Sh355, 2.3]) we have $(\forall\theta)[\mathrm{cf}\,\theta \le \lambda < \theta < \mu \Rightarrow \mathrm{pp}_\lambda(\theta) < \mu]$; together ($\alpha$) of (B) holds. Also $\mu \le 2^{\le\kappa}$, hence $\mathrm{cf}(\mu) < \kappa \Rightarrow \mathrm{pp}\,\mu \le \mu^{<\kappa} \le 2^{<\kappa}$, contradiction, so ($\gamma$) of (B) follows from ($\alpha$). Note that if we replace λ by κ (changing the conclusion a little; or $\lambda = \kappa$) then by [Sh355, 5.4(2)] if 2^λ is regular the conclusion holds for $\chi = 2^\lambda$ too.

$\mu' \in (\lambda, 2^{<\theta}] \subseteq (\lambda, \chi)$ we get contradiction to "χ is $(\lambda, \lambda^+, 2)$-inaccessible" which, without loss of generality, we have assumed above.

THIRD CASE: *κ is singular (hence $2^{<\kappa}$ is singular,* $\mathrm{cf}(2^{<\kappa}) = \mathrm{cf}\,\kappa$). Let $\mu =: 2^{<\kappa}$ and we shall prove (C); easily (B)(γ) holds. Now ${}^{\kappa>}2$ is a tree with $2^{<\kappa} = \mu$ nodes and $2^\kappa = 2^\lambda$ κ-branches, so $(\alpha)'$ of (C) holds. As for (β) of (B), if κ is strong limit checking the conclusion is immediate, otherwise it follows from 3.4 part (3) below.

Clearly if $\mathrm{cf}\,\kappa > \aleph_0$, also (B) holds. $\blacksquare_{3.3}$

3.4 CLAIM:

(1) *Assume $\theta_{n+1} = \mathrm{Min}\{\theta\colon 2^\theta > 2^{\theta_n}\}$ for $n < \omega$ and $\sum_{n<\omega} \theta_n < 2^{\theta_0}$ (so θ_{n+1} is regular, $\theta_{n+1} > \theta_n$). Then: for infinitely many $n < \omega$, for some $\mu_n \in [\theta_n, \theta_{n+1})$ (so $2^{\mu_n} = 2^{\theta_n}$) we have:*

$(*)_{\mu_n,\theta_n}$ *for every regular $\chi \le 2^{\theta_n}$ there is a tree of cardinality μ_n with $\ge \chi$ θ_n-branches; if $\mu_n > \theta_n$ then $\mathrm{cf}(\mu_n) = \theta_n$, μ_n is $(\theta_n, \theta_n^+, 2)$-inaccessible.*

(2) *Moreover*

(α) *for every $n < \omega$ large enough for some μ_n :*

$$\theta_n \le \mu_n < \sum_{m<\omega} \theta_m \quad \text{and } (*)_{\mu_n,\theta_n} \quad \text{and } \mathrm{cf}(\mu_n) = \theta_n,$$

$$[\mu_n > \theta_n \Rightarrow \mu_n \ \text{ is } [(\theta_n, \theta_n^+, 2)\text{-inaccessible}, \mathrm{pp}(\mu_n) = 2^{\theta_n}].$$

(β) *Moreover, for infinitely many m we can demand: for every $n < m$, $\chi = \mathrm{cf}\,\chi \le 2^{\theta_n}$ the tree T^n_χ (witnessing $(*)_{\mu_n,\theta_n}$ for χ) has cardinality $< \theta_{m+1}$ (i.e. $\mu_m < \theta_{m+1}$).*

(3) *If κ is singular, $\kappa < 2^{<\kappa} < 2^\kappa$ then for every regular $\chi \in (\kappa, 2^{<\kappa})$, there is a tree with $< \kappa$ nodes and $\ge \chi$ branches (of same height). Also for some $\theta^* \in (\kappa, \mathrm{pp}^+(\kappa)) \cap \mathrm{Reg}$, for every regular $\chi \le 2^\kappa$ there is a tree T, $|T| \le \kappa^{\mathrm{cf}\,\kappa}$, with $\ge \chi$ θ^*-branches.*

Proof: Clearly (2) implies (1) and (3) (for (3) second sentence use ultraproduct). Let $\theta =: \sum_{n<\omega} \theta_n$. Let $S_0 =: \{n < \omega\colon (*)_{\theta_n,\theta_n} \text{ fails}\}$. Let for $n \in \omega \backslash S_0$, $\mu_n = \theta_n$ and note that (α) of 3.4(2) holds and if S_0 is co-infinite, also (β) of 3.4(2) holds. We can assume that S_0 is infinite (otherwise the conclusion of 3.4(2) holds). By [Sh355, 5.11], fully [Sh410, 4.3] for $n \in S_0$ there is μ_n such that:

$(\alpha)_n$ $\theta_n = \mathrm{cf}\,\mu_n < \mu_n \le 2^{<\theta_n}$,

$(\beta)_n$ $\mathrm{pp}_{\Gamma(\theta_n)}(\mu_n) \ge 2^{\theta_n}$ (hence equality holds and really $\mathrm{pp}^+_{\Gamma(\theta_n)}(\mu_n) = (2^{\theta_n})^+$) and

$(\gamma)_n$ $\theta_n < \mu' < \mu_n \& \mathrm{cf}\,\mu' \leq \theta_n \Rightarrow \mathrm{pp}_{\leq\theta_n}(\mu') < \mu_n$ hence $\mathrm{pp}^+_{\theta_n}(\mu_n) = \mathrm{pp}^+_{\Gamma(\theta_n)}(\mu_n)$ $= (2^{\theta_n})$.

Note that $2^{<\theta_n} = 2^{\theta_{n-1}}$ so $\mu_n \leq 2^{\theta_{n-1}}$. By [Sh355, 5.11] for $n \in S_0$, part (α) (of 3.4(2)) holds except possibly $\mu_n < \theta$.

Remember $\mathrm{cf}(\mu_n) = \theta_n$.

Let $n < m$ be in S_0 and $\mu_n > \theta_m$, so $\mathrm{Max}\{\mathrm{cf}\,\mu_n, \mathrm{cf}\,\mu_m\} = \mathrm{Max}\{\theta_n, \theta_m\} < \mathrm{Min}\{\mu_n, \mu_m\}$ so by $(\gamma)_n$ (and [Sh355, 2.3(2)]) we have $\mu_n \geq \mu_m$. Note $\mathrm{cf}\,\mu_n = \theta_n$, $\mathrm{cf}\,\mu_m = \theta_m$ (which holds by $(\alpha)_n$, $(\alpha)_m$) hence $\mu_n > \mu_m$. As the class of cardinals is well ordered we get $S_1 =: \{n < \omega: n \in S_0, \mu_n \geq \theta_{n+1}\}$ is co-infinite and $S =: \{n: \mu_n \geq \theta\}$ is finite (so (α) of 3.4(2)(b) holds).

So for some $n(*) < \omega$, $S \subseteq n(*)$ hence for every $n \in [n(*), \omega)$ for some $m \in (n, \omega)$, $\mu_n < \theta_m$. Note: $n \neq m \Rightarrow \mu_n \neq \mu_m$ (as their cofinalities are distinct) and $[n \notin S_0 \Rightarrow \mu_n \notin \{\theta_m: m < \omega\}]$. Assume $n \geq n(*)$, if $\mu_n > \theta_{n+1}$, let $m = m_n = \mathrm{Min}\{m: \mu_{m+1} > \mu_n$ and $m \geq n\}$ (it is well defined as $\bigvee_k \mu_n < \theta_k$ and $\theta_k < \mu_k < \theta = \bigcup_{\ell<\omega} \theta_\ell$) and we shall show $\mu_m < \theta_{m+1}$; assume not, hence $m \in S_0$; so $\mu_{m+1} \leq 2^{\theta_m} = \mathrm{pp}_{\Gamma(\theta_m)}(\mu_m) \leq \mathrm{pp}_{\theta_{m+1}}(\mu_m)$ but $\mu_m \leq \mu_n$ (by the choice of m) so as $\mathrm{cf}(\mu_m) = \theta_m \neq \theta_{m+1}$, necessarily $\mu_m > \theta_{m+1}$ and if $m+1 \notin S_0$ trivially and if $m+1 \in S_0$ by one of the demands on μ_{m+1} (in its choice) and [Sh355, 2.3] we have $\mu_{m+1} \leq \mu_m$; but $\mu_m < \mu_n$, so $\mu_{m+1} < \mu_n$ contradicting the choice of m. So by the last sentence, $n \geq n(*) \Rightarrow \mu_{m_n} < \theta_{m_n+1}$. By [Sh355, 5.11] we get the desired conclusion (i.e. also part (β) of 3.4(2)). $\blacksquare_{3.4}$

Remark: It seemed that we cannot get more as we can get an appropriate product of a forcing notion as in Gitik and Shelah [GiSh344].

4. Bounds for $\mathrm{pp}_{\Gamma(\aleph_1)}$ for Limits of Inaccessibles*

4.1 Convention: For any cardinal μ, $\mu > \mathrm{cf}\,\mu = \aleph_1$ we let $\mathcal{Y}_\mu$, Eq_μ be as in [Sh420, 3.1], $\bar{\mu}$ is a strictly increasing continuous sequence of singular cardinals of cofinality $\aleph_0$ of length ω_1, $\mu = \sum_{i<\aleph_1} \mu_i$.

So μ stands here for μ^* in [Sh420, §3, §4, §5]. (Of course, $\aleph_1$ can be replaced by "regular uncountable".)

* In previous versions these sections have been in [Sh410], [Sh420] hence we use $\mathcal{Y}$, etc. (and not the context of [Sh386]); see 4.2B below.

4.2 THEOREM (Hypothesis [Sh420, 6.1C]*):

(1) *Assume*

(a) $\mu > \mathrm{cf}\,\mu = \aleph_1$, $\mathcal{Y} = \mathcal{Y}_\mu$, $Eq'_\mu \subseteq Eq_\mu$,

(b) *every* $D \in \mathrm{FIL}(\mathcal{Y})$ *is nice (see [Sh420, 3.5]),* $E = \mathrm{FIL}(\mathcal{Y})$ *(or at least there is a nice* $\mathcal{E}$ *(see [Sh420, 5.2–5],* $E = \bigcup \mathcal{E} = \mathrm{Min}\ \mathcal{E}$, $\mathcal{E}$ *is* μ*-divisible having weak* μ*-sums, but we concentrate on the first case),*

(c) $\mu < \lambda < \mathrm{pp}^+_E(\mu)$, λ *inaccessible.*

Then there are $e \in Eq_\mu$ *and* $\langle \lambda_x : x \in \mathcal{Y}/e \rangle$, *a sequence of inaccessibles* $< \mu$ *and a* $D \in \mathrm{FIL}(e, \mathcal{Y}) \cap E$ *nice to* μ, $D \in \mathrm{FIL}(e, \mathcal{Y}_\mu)$ *such that:*

(α) $\prod_{x \in \mathcal{Y}_\mu / e} \lambda_x / D$ *has true cofinality* λ,

(β) $\mu = \mathrm{tlim}_D \langle \lambda_x : x \in \mathcal{Y}_\mu \rangle$.

(2) *We can weaken "(b)" to "*$E \subseteq \mathrm{FIL}(Eq, \mathcal{Y})$ *and for* $D \in E$, *in the game* $wG(\mu, D, e, \mathcal{Y})$ *the second player wins choosing filters only from* E.

(3) *Moreover, for given* e_0, D_0, $\langle \lambda^0_x : x \in \mathcal{Y}/e_0 \rangle$, *if* $\prod_{x \in \mathcal{Y}/e_0} \lambda^0_x / D^e_0$ *is* λ*-directed, then without loss of generality* $e_0 \leq e$, $D_0 \leq D$ *and* $\lambda_x \leq \lambda_{x[e_0]}$.

4.2A Remark: (1) We could have separated the two roles of μ (in the definition of $\mathcal{Y}$, etc. and in $\lambda \in (\mu, \mathrm{pp}^+_E(\mu))$) but the result is less useful; except for the unique possible cardinal appearing later.

(2) Compare with a conclusion of [Sh386] (see in particular 5.8 there):

THEOREM: *Suppose* $\lambda > 2^{\aleph_1}$, λ *(weakly) inaccessible.*

(1) *If* $\aleph_1 < \lambda_i = \mathrm{cf}\,\lambda_i < \lambda$ *for* $i < \omega_1$, D *is a normal filter on* ω_1, $\prod_{i<\omega_1} \lambda_i / D$ *is* λ*-directed, then for some* λ'_i, $\aleph_1 < \lambda'_i = \mathrm{cf}\,\lambda'_i \leq \lambda_i$ *and normal filter* D' *extending* D, $\lambda = \mathrm{tcf}\left(\prod_{i<\omega_1} \lambda'_i / D'\right)$ *and* $\{i : \lambda_i$ *inaccessible*$\} \in D'$.

(2) *If* $\aleph_1 = \mathrm{cf}\,\mu < \mu < \lambda$, $\mathrm{pp}_{\Gamma(\aleph_1)}(\mu) \geq \lambda$ *then for some* $\langle \lambda_i : i < \omega_1 \rangle$, $\aleph_1 < \lambda_i = \mathrm{cf}\,\lambda_i < \mu$, *each* λ_i *inaccessible and* $\lambda \in \mathrm{pcf}_{\Gamma(\aleph_1)}\{\lambda_i : i < \omega_1\}$.

Proof of 4.2: (1) By the definition of $\mathrm{pp}^+_E(\mu)$ (and assumption (c), and [Sh355, 2.3 (1) + (3)]) there are $D \in E$ and $f \in {}^{\mathcal{Y}_\mu/e}\mu$ such that:

$(\mathrm{A})_f$ $\mu > f(x) = \mathrm{cf}[f(x)] > \mu_{\iota(x)}$,

$(\mathrm{B})_{f,D}$ $\lambda = \mathrm{tcf}\left[\prod_{x \in \mathcal{Y}/e} f(x)/D\right]$.

Let $K_0 =: \{(f, D) : D \in E, f \in {}^{\mathcal{Y}_\mu/e}\mu$ and conditions $(\mathrm{A})_f$ and $(\mathrm{B})_{f,D}$ hold$\}$, so $K_0 \neq \emptyset$. Now if $(f, D) \in K_0$, for some γ

$(\mathrm{C})_{f,D,\gamma}$ in $G^\gamma(D, f, e, \mathcal{Y})$ the second player wins (see [Sh420, 3.4(2)])

---

* I.e.: if $\mathfrak{a} \subset \mathrm{Reg}$, $|\mathfrak{a}| < \min(\mathfrak{a})$, λ inaccessible then $\lambda > \sup(\lambda \cap \mathrm{pcf}\,\mathfrak{a})$.

hence $K_1 \neq \emptyset$ where $K_1 =: \{(f, D, \gamma) \in K_0$ condition $(C)_{f,D,\gamma}$ holds$\}$. Choose $(f^1, D_1, \gamma_{\langle\rangle}) \in K_1$ with $\gamma_{\langle\rangle}$ minimal. By the definition of the game

$(*)$ for every $A \neq \emptyset \bmod D_1$ we have $(f^1, D_1 + A, \gamma_{\langle\rangle}) \in K_1$.

Let $e_1 = e(D_1)$.

Case A: $\{x\colon f^1(x)$ *inaccessible*$\} \neq \emptyset \bmod D_1$. We can get the desired conclusion (by increasing D_1).

Case B: $\{x\colon f^1(x)$ *successor cardinal*$\} \neq \emptyset \bmod D_1$. By $(*)$, without loss of generality $f^1(x) = g(x)^+$, $g(x)$ a cardinal (so $\geq \mu_{\iota(x)}$) for every $x \in \mathcal{Y}_\mu/e$. By [Sh355, 1.3] for every regular $\kappa \in (\mu, \lambda)$ there is $f_\kappa \in {}^{(\mathcal{Y}/e)}\mathrm{Ord}$ satisfying:

(a) $f_\kappa < f^1$, each $f_\kappa(x)$ regular,

(b) $\mathrm{tlim}_{D_1} f_\kappa = \mu$,

(c) $\prod_x f_\kappa(x)/D_1$ has true cofinality κ.

By (a) we get

(d) $f_\kappa \leq g$.

By (b) we get, by the normality of D_1, that for the D_1-majority of $x \in \mathcal{Y}/e$, $f_\kappa(x) \geq \mu_{\iota(x)}$; as $f_\kappa(x)$ is regular (by (a)) and $\mu_{\iota(x)}$ singular (see 4.1) we get

(e) for the D_1-majority of $x \in \mathcal{Y}/e$, we have $f_\kappa(x) > \mu_{\iota(x)}$.

Let χ be large enough, let N be an elementary submodel of $(H(\chi), \in, <^*_\chi)$, $\lambda \in N$, $D_1 \in N$, $N \cap \lambda$ is the ordinal $\|N\|$ (singular for simplicity) and $\{\mu, \langle f^1, g, f_\kappa\colon \kappa \in \mathrm{Reg} \cap (\mu, \lambda)\rangle\}$ belongs to N. Choose $\kappa \in \mathrm{Reg} \cap \lambda \setminus (\sup \lambda \cap N)$, now in $\prod_{x \in \mathcal{Y}/e_1} f_\kappa(x)/D_1$, there is a cofinal sequence $\langle f_{\kappa,\zeta}\colon \zeta < \kappa\rangle$; as $\kappa > \sup(\lambda \cap N)$, so for some $\zeta(*) < \kappa$:

$$\otimes \quad h \in N \cap {}^{\mathcal{Y}/e_1}\mathrm{Ord} \Rightarrow \big\{x \in \mathcal{Y}/e_1\colon f_{\kappa,\zeta(*)}(x) \leq h(x) < f_\kappa(x)\big\} = \emptyset \bmod D_1.$$

[Why? For any such h define $h' \in {}^{\mathcal{Y}/e_1}\mathrm{Ord}$ by: $h'(x)$ is $h(x)$ if $h(x) < f_\kappa(x)$ and zero otherwise, so for some $\zeta_h < \kappa$, $h' < f_{\kappa,\zeta_h} \bmod D_1$. Let $\zeta(*) = \sup\big\{\zeta_h\colon h \in N \cap^{\mathcal{Y}/e_1} N\big\}$; it is $< \kappa$ as $\|N\| < \kappa$, and it is as required.]

Let $f_* = f_{\kappa,\zeta(*)}$. The continuation imitates [Sh371, §4], [Sh410, §5].

Let

$$\begin{aligned} K_2 = \Big\{(D, \bar{B}, \langle j_x\colon x \in \mathcal{Y}/e_1\rangle)\colon & D_1 \subseteq D \in E, \quad \text{player II wins } G_E^{\gamma_{\langle\rangle}}(f^1, D), \\ & e_1 = e(D), \bar{B} = \langle < B_{x,j}\colon j < j_x^0 \leq \mu_{\iota(x)} > : x \in \mathcal{Y}/e_1\rangle \in N, \\ & |B_{x,j_x}| \lneq g(x) \text{ and } j_x < j_x^0 \leq \mu_{\iota(x)}, \\ & \{x \in \mathcal{Y}/e_1\colon f_*(x) \text{ is in } B_{x,j_x}\} \in D\Big\}. \end{aligned}$$

Clearly $K_2 \neq \emptyset$. For each $(D, \bar{B}, \langle j_x: x \in \mathcal{Y}/e_1\rangle) \in K_2$:

$(*)_1$ letting $h \in {}^{\mathcal{Y}/e_1}\mathrm{Ord}$, $h(x) = |B_{x,j_x}|$, for some $\bar{h} = \langle\langle\langle\rangle, f^1\rangle, \langle\langle 0\rangle, h\rangle\rangle$, for some $\gamma_{<0>} < \gamma_{<>}$ and D player II wins in $G_E^{\langle\gamma_{<>},\gamma_{<0>}\rangle}(D, \bar{h}, e_1, \mathcal{Y}_\mu)$.

So choose $(D, \bar{B}, \langle j_x: x \in \mathcal{Y}/e_1\rangle, \gamma_{\langle 0\rangle})$ such that:

$(*)_2$ $(D, \bar{B}, \langle j_x: x \in \mathcal{Y}/e_1\rangle) \in K_2$, $(*)_1$ for $\gamma_{\langle 0\rangle}$ holds and (under those restrictions) $\gamma_{\langle 0\rangle}$ is minimal.

So (as player I can "move twice"), for every $A \in D^+$, if we replace D by $D + A$, then $(*)_2$ still holds.

So without loss of generality (for the first and third members use normality):

$(*)_3$ one of the following sets belongs to D:

$$A_{0,\zeta} = \{x \in \mathcal{Y}/e_1\colon \mathrm{cf}\,|B_{x,j_x}| > \mu_{\iota(x)} \text{ and } j_x^0 < \mu_\zeta\}$$
$$\text{(for some } \zeta < \omega_1 \text{ such that } |\mathcal{Y}/e_1| < \mu_\zeta),$$
$$A_1 = \{x \in \mathcal{Y}/e_1\colon \mathrm{cf}\,|B_{x,j_x}| < \mu_{\iota(x)} \leq |B_{x,j_x}|\},$$
$$A_{2,\zeta} = \{x \in \mathcal{Y}/e_1\colon |B_{x,j_x}| \leq \mu_\zeta \text{ and } j_x < \mu_\zeta\} \quad \text{(for some } \zeta < \omega_1).$$

If $A_{2,\zeta} \in D$ then (for $x \in \mathcal{Y}/e_1$)

$$B_x^* =: \bigcup\{B_{x,j}\colon x \in \mathcal{Y}/e_1, j < j_x^0 \text{ and } |B_{x,j_x}| < \mu_\zeta \text{ and } j < \mu_\zeta\}$$

is a set of $\leq \mu_\zeta$ ordinals and

$$\{x \in \mathcal{Y}/e_1\colon f_*(x) \in B_x^*\} \in D$$

and $\langle B_x^*\colon x \in \mathcal{Y}/e_1\rangle$ belongs to N (as $(D, \bar{B}, \langle j_x\colon x \in \mathcal{Y}/e_1\rangle) \in K_2$ and the definition of K_2), contradiction to the choice of f_* (see $\otimes$, remember $D_1 \subseteq D$ by the definition of K_2).

If $A_1 \in D$, we can find $\bar{B}^1 \in N$, $\bar{B}^1 = \langle\langle B_{x,j}^1\colon j < j_x^1 \leq \mu_{\iota(x)}\rangle\colon x \in \mathcal{Y}/e_1\rangle$, $|B_{x,j}^1| \leq g(x)$ and $\bigwedge_{j<j_x^1}\left[\mathrm{cf}\,|B_{x,j}^1| \geq \mu_{\iota(x)} \vee |B_{x,j}^1| = 1\right]$ and each $B_{x,j}$ satisfying $\mathrm{cf}\,|B_{x,j}| < \mu_{i(x)}$ is a union of $\mathrm{cf}\,|B_{x,j}|$ sets of the form B_{x,j^1}^1 of smaller cardinality and so for some $j_x^2 < j_x^1$, $f_*(x) \in B_{x,j_x} \Rightarrow f_*(x) \in B_{x,j_x^2}$ & $|B_{x,j_x^2}| < |B_{x,j_x}|$. Now playing one move in $G_E^{\langle\gamma_{<>},\gamma_{<0>}\rangle}(D, \bar{h}, e, \mathcal{Y})$ we get contradiction to choice of $\gamma_{\langle 0\rangle}$.

We are left with the case $A_{0,\zeta} \in D$, so without loss of generality $\bigwedge_{x,j} \mathrm{cf}\,|B_{x,j}| > \mu_{\iota(x)}$. Let

$$\mathfrak{a} = \{\mathrm{cf}\,|B_{x,j}|\colon \mathrm{cf}\,|B_{x,j}| > \mu_{\iota(x)}, x \in \mathcal{Y}/e_1, j < j_x^0, j < \mu_\zeta \text{ and } \iota(x) > \zeta\},$$

so $\mathfrak{a}$ is a set of regular cardinals, and (remember $|\mathcal{Y}/e_1| < \mu_\zeta$) we have $|\mathfrak{a}| < \operatorname{Min}\mathfrak{a}$, so let $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\theta[\mathfrak{a}] \colon \theta \in \operatorname{pcf}\mathfrak{a}\rangle$ be as in [Sh371, 2.6]. So as (by the Definition of K_2), $\langle\langle B_{x,j} \colon j < j^0_x\rangle \colon x \in \mathcal{Y}/e_1\rangle \in N$, clearly $\mathfrak{a} \in N$ hence without loss of generality $\bar{\mathfrak{b}} \in N$. Let $\lambda^* = \sup[\lambda \cap \operatorname{pcf}\mathfrak{a}]$, so by Hypothesis [420, 6.1(C)], $\lambda^* < \lambda$, but $\lambda^* \in N$, so $\lambda^* + 1 \subseteq N$.

By the minimality of the rank we have for every $\theta \in \lambda^* \cap \operatorname{pcf}\mathfrak{a}$, $\{x \in y/e_1 \colon \operatorname{cf}|B_{x,j_x}| \in \mathfrak{b}_\theta\} = \emptyset \bmod D$ hence $\prod_x \operatorname{cf}|B_{x,j_x}|/D$ is λ-directed, hence we get contradiction to the minimality of the rank of f_1.

(2), (3) Proof left to the reader. $\blacksquare_{4.2}$

4.2B Remark:

1. The proof of 4.3 below shows that in [Sh386] the assumption of the existence of nice filters is very weak, removing it will cost a little for at most one place.
2. We could have used the framework of [Sh386] but not for 4.3 (or use forcing).

4.3 CLAIM (Hypothesis 6.1(C) of [Sh420] even in any $K[A]$): *Assume $\mu > \operatorname{cf}\mu = \aleph_1$, $\mu > \theta > \aleph_1$, $\mathrm{pp}_{\Gamma(\theta,\aleph_1)}(\mu) \geq \lambda > \mu$, λ inaccessible. Then for some $e \in Eq_\mu$, $D \in \mathrm{FIL}(e, \mathcal{Y}_\mu)$ and sequence of inaccessibles $\langle \lambda_x \colon x \in \mathcal{Y}_\mu/e\rangle$, we have $\operatorname{tlim}_D \lambda_x = \mu$ and $\lambda = \operatorname{tcf}(\prod \lambda_x/D)$ except perhaps for a unique λ in V (not depending on μ) and then $\mathrm{pp}^+_{\Gamma(\theta,\aleph_1)}(\mu) \leq \lambda^+$.*

Proof: By the Hyp. (see [Sh513, 6.12]) for some $\mathfrak{a} \subsetneq \mathrm{Reg} \cap \mu$, $|\mathfrak{a}| < \operatorname{Min}(\mathfrak{a})$, $\lambda = \max\operatorname{pcf}(\mathfrak{a})$, and

$$(\forall \lambda' < \lambda)(\exists \mathfrak{b})[\mathfrak{b} \subseteq \mathfrak{a} \;\&\; |\mathfrak{b}| < \theta >]\&\lambda > \sup \operatorname{pcf}_{\aleph_1-\text{complete}}(\mathfrak{b}) > \lambda'],$$

$J = J_{<\lambda}[\mathfrak{a}]$. First assume "in $K[A]$ there is a Ramsey cardinal $> \lambda^\theta$ when $A \subseteq \lambda^\theta$". Choose $A \subseteq \lambda^\theta$ such that ${}^\theta\lambda \subseteq L[A]$ and for every $\alpha < \lambda^\theta$, there is a one to one function f_α from $|\alpha|$ (i.e. $|\alpha|^V$) onto α, $f_\alpha \in L[A]$, so $\mathrm{Card}^{L[A]} \cap (\lambda^\theta + 1) = \mathrm{Card}^V$, and apply 4.2 to the universe $K[A]$ (its assumption holds by [Sh420, 5.6]).

Second assume $(*)_\lambda$ "in $K[A]$ there is a Ramsey cardinal $> \lambda$ when $A \subseteq \lambda^+$" and assume our desired conclusion fails. Let $S \subseteq \lambda$ be stationary $[\delta \in S \Rightarrow \operatorname{cf}\delta = \theta^+]$, $\langle a_\alpha \colon \alpha < \lambda\rangle$, exemplify $S \in I[\lambda]$ (exist by [Sh420, §1]). We can find $\mathfrak{a}$, J as described above. Let $\langle f_\alpha \colon \alpha < \lambda\rangle$ exemplify $\lambda = \operatorname{tcf}(\prod \mathfrak{a}/J)$, now by [Sh355, 1.3] without loss of generality $\lambda = \max\operatorname{pcf}\mathfrak{a}$. Let $A_0 \subseteq \lambda$ be such that $\mathfrak{a}$, $\langle f_\alpha \colon \alpha < \lambda\rangle$, $\langle \mathfrak{b}_\sigma[\mathfrak{a}] \colon \sigma \in \operatorname{pcf}\mathfrak{a}\rangle$ are in $L[A_0]$. Hence in $L[A_0]$ for suitable J, $\langle f_\alpha/J \colon \alpha < \lambda\rangle$ is increasing, and without loss of generality for some $\langle\langle \mathfrak{c}^\delta_\alpha \colon \alpha \in a_\delta\rangle \colon \delta \in S\rangle \in L[A_0]$, we have: for $\delta \in S$, $\mathrm{cf}\,\delta = |\mathfrak{a}|^+$, a_δ a club of δ and $\langle f_\alpha \restriction (\mathfrak{a}\backslash \mathfrak{c}^\delta_\alpha)\colon \alpha \in a_\delta\rangle$ is $<$-increasing (see [Sh345b, 2.5] ("good point")) and $\mathfrak{c}^\delta_\alpha \in J$ and S is stationary in V, so the assumption of 4.3 holds in V^1 whenever $L[A_0] \subseteq V^1 \subseteq V$; hence for $A \subseteq \lambda^+$, in $K[A_0, A]$ the conclusion of 4.2 holds as we are assuming $(*)_\lambda$.

Note: if $A \subseteq \lambda$, in $K[A]$, $\lambda^{<\lambda} = \lambda$ hence if $\alpha < \lambda^+$, $A \subseteq \alpha$ then $K[A] \models$ "$\lambda^{<\lambda} < (\lambda^+)^V$".

Choose by induction on $\alpha < \lambda^+$ a set $A_\alpha \subseteq [\lambda\alpha, \lambda(\alpha+1))$ such that: A_0 is as above and for $\alpha > 0$: if $\langle \lambda_x\colon x \in \mathcal{Y}/e\rangle$, J exemplify the conclusion of 4.2 in $K\left[\bigcup_{\beta<\alpha} A_\beta\right]$, and $\langle f_i\colon i < \lambda\rangle$ exemplify the $\lambda = \mathrm{tcf}\left(\prod_{x\in\mathcal{Y}/e} \lambda_x/J\right)$, without loss of generality J canonical (all in $K\left[\bigcup_{\beta<\alpha} A_\beta\right]$, canonical means: the normal ideal generated by $\{x\colon \lambda_x \in \mathfrak{b}_{<\lambda}[\{\lambda_y\colon y \in \mathcal{Y}/e\}]\}$), then in $K\left[\bigcup_{\beta\le\alpha} A_\beta\right]$ we can find f, $\bigwedge_{\alpha<\lambda} f <_J \langle \lambda_x\colon x \in \mathcal{Y}/e\rangle$, $\bigwedge_\alpha f \not<_J f_\alpha$ (as they cannot exemplify the conclusion of 4.5 in V — otherwise we have finished).

Let $A = \bigcup_{\alpha<\lambda^+} A_\alpha$.

Now in $K[A]$ there are e, $\langle \lambda_x\colon \lambda \in \mathcal{Y}/e\rangle$, $\langle f_i\colon i < \lambda\rangle$ (and J) exemplifying the conclusion of 4.2 (by $(*)$ and [Sh513, 6.12(3)]). By 4.5 below, for some $\delta < \lambda^+$, e, $\langle \lambda_x\colon x \in \mathcal{Y}/e\rangle$, $\langle \mathfrak{b}_\sigma[\{\lambda_x\colon x \in \mathcal{Y}/e\}]\colon \sigma \in \mathrm{pcf}\{\lambda_x\colon x \in \mathcal{Y}/e\}\rangle$, $f_\alpha(\alpha < \lambda)$ all belongs to $K\left[\bigcup_{\gamma<\delta} A_\gamma\right]$, and in $K\left[\bigcup_{\gamma\le\delta} A_\gamma\right]$ we get a contradiction.

If $(*)_\lambda$ holds for every λ we are done. If not, let λ_0 be minimal such that $(*)_{\lambda_0}$ fails; so if $\lambda < \lambda_0$ the conclusion holds, and if $\lambda > \lambda_0$ then let $A \subseteq \lambda_0^+$ be such that in $K[A]$ there is no Ramsey, hence ([DoJ]) for $\mu \ge \lambda_0^+$ in V, $\mathrm{cov}(\mu, \theta, \theta, 2) \le \mu$, so the assumptions of 4.3 fail. Similarly $\mu > \theta$, $\mathrm{cf}(\mu) = \aleph_1$, $\mathrm{pp}_{\Gamma(\theta,\aleph_1)}(\mu) > \lambda_0^+$ bring a contradiction. $\blacksquare_{4.3}$

4.4 Conclusion: Hypothesis [Sh420, 6.1(C)] in any $K[A]$. (1) Assume $\mu > \mathrm{cf}\,\mu = \aleph_1$, $\mu_0 < \mu$, $\sigma \ge |\{\lambda\colon \mu_0 < \lambda < \mu,\ \lambda \text{ inaccessible}\}| < \mu$. Then

$$\sigma^{+4} > \left|\left\{\lambda\colon \mu < \lambda < \mathop{\mathrm{pp}}_{\Gamma(\sigma,\aleph_1)}(\mu) \text{ and } \lambda \text{ is inaccessible}\right\}\right|.$$

(2) The parallel of [Sh400, 4.3].

Proof: See [Sh410, 3.5] and use 4.2(3). $\blacksquare$

By [DoJe]

4.5 THEOREM: *If λ is regular ($> \aleph_1$) $A \subseteq \lambda$, $Z \in K[A]$ a bounded subset of λ then for some $\alpha < \lambda$, $Z \in \bigcup_{\alpha<\lambda} K[A \cap \alpha]$.*

We shall return to this elsewhere.

5. Densities of Box Products

5.1 Definition: $d_{<\kappa}(\lambda, \theta)$ is the density of the topological space ${}^{\lambda}\theta$ where the topology is generated by the following family of clopen sets:

$$\{[f]: f \in {}^{a}\theta \text{ for some } a \subseteq \lambda, |a| < \kappa\}$$

where

$$[f] = \{g \in {}^{\lambda}\theta: g \subseteq f\}.$$

So

$$d_{<\kappa}(\lambda,\theta) = \\ \text{Min}\left\{|F|: F \subseteq {}^{\lambda}\theta \text{ and if } a \in \mathcal{S}_{<\kappa}(\lambda) \text{ and } g \in {}^{a}\theta \text{ then } (\exists f \in F) g \subseteq f\right\}.$$

If $\theta = 2$ we may omit it, if $\kappa = \aleph_0$ we may omit it (i.e. $d(\lambda, \theta) = d_{<\aleph_0}(\lambda, \theta)$). Always we assume $\lambda \geq \aleph_0$, $\kappa \geq \aleph_0, \theta > 1$ and $\lambda^+ \geq \kappa$. We write $d_\kappa(\lambda, \theta)$ for $d_{<\kappa^+}(\lambda, \theta)$.

5.1A Discussion: Note: for $\kappa = \aleph_0$ this is the Tichonov product, for higher κ those are called box products and d has obvious monotonicity properties.

$d\left(2^{\aleph_0}\right) = \aleph_0$ by the classical Hewitt–Marczewski–Pondiczery theorem [H], [Ma], [P]. This has been generalized by Engelking–Karlowicz [EK] and by Comfort–Negrepontis [CN1], [CN2] to show, for example, that $d_{<\kappa}(2^\alpha, \alpha) = \alpha$ if and only if $\alpha = \alpha^{<\kappa}$ ([CN1] (Theorem 3.1)). Cater–Erdős–Galvin [CEG] show that every non-degenerate space X satisfies $\text{cf}(d_{<\kappa}(\lambda, X)) \geq \text{cf}(\kappa)$ when $\kappa \leq \lambda^+$, and they note (in our notation) that "$d_{<\kappa}(\lambda)$ is usually (if not always) equal to the well-known upper bound $(\log \lambda)^{<\kappa}$". It is known (cf. [CEG], [CR]) that SCH $\Rightarrow d_{<\aleph_1}(\lambda) = (\log \lambda)^{\aleph_0}$, but it is not known whether $d_{<\aleph_1}(\lambda) = (\log \lambda)^{\aleph_0}$ is a theorem of ZFC.

The point in those theorems is the upper bound, as, of course, $d_{<\kappa}(\mu, \theta) > \chi$ if $\mu > 2^\chi$ & $\theta > 2$ [why? because if $F = \{f_i: i < \chi\}$ exemplify $d_{<\kappa}(\mu, \theta) \leq \chi$, the number of possible sequences $\langle \text{Min}\{1, f_i(\zeta)\}: i < \chi \rangle$ (where $\zeta < \mu$) is $\leq 2^\chi$, so for some $\zeta \neq \xi$ they are equal and we get contradiction by g, $g(\zeta) = 0$, $g(\xi) = 1$, $\mathrm{Dom}\, g = \{\zeta, \xi\}$].

Also trivial is: for κ limit, $d_{<\kappa}(\lambda, \theta) = \kappa + \sup_{\sigma<\kappa} d_{<\sigma}(\lambda, \theta)$, so we only use κ regular; $d_{<\kappa}(\lambda, \theta) \geq \sigma^\theta$ for $\sigma < \kappa$.

Also if $\mathrm{cf}(\lambda) < \kappa$, λ strong limit then $d_{<\kappa}(\lambda) > \lambda$. The general case (say $2^{<\mu} < \lambda < 2^\mu$, $\mathrm{cf}\, \mu \leq \theta$) is similar; we ignore it in order to make the discussion simpler.

So the main problem is:

5.2 Problem: *Assume λ is strong limit singular, $\lambda > \kappa > \mathrm{cf}(\lambda)$, what is $d_{<\kappa}(\lambda)$? Is it always 2^λ? Is it always $> \lambda^+$ when $2^\lambda > \lambda^+$?*

In [Sh93] this question was raised (later and independently) for model theoretic reasons. I thank Comfort for asking me about it in the Fall of '90.

5.3 Lemma: *Suppose λ is singular strong limit, $\mathrm{cf}(\lambda) = \mathrm{cf}(\delta^*) \leq \delta^* < \mathrm{cf}(\kappa) \leq \kappa < \lambda$, $2 \leq \theta < \lambda$, $\lambda \leq \chi < 2^\lambda$ and $\langle \lambda_\alpha, \mu_\alpha, \chi_\alpha, \chi^*_\alpha : \alpha < \delta^* \rangle$ is such that:*

$\chi_\alpha = \theta^{\mu_\alpha}$, $\chi^*_\alpha = \mathrm{cov}(\chi_\alpha, \lambda_\alpha, \lambda_\alpha, 2)$,

$\alpha < \beta \Rightarrow \mu_\alpha < \mu_\beta$,

$\lambda = \bigcup_{\alpha<\delta^*} \mu_\alpha = \mathrm{tlim}_{\alpha<\delta} \lambda_\alpha$, $\theta < \mu_\alpha$,

$d_{<\kappa}(\mu_\alpha, \theta) \geq \lambda_\alpha$ *(this holds e.g. if* $(\forall \lambda' < \lambda_\alpha)[2^{\lambda'} < \mu_\alpha]$*)*,

$A_\alpha = [\mu_\alpha, \mu_\alpha + \mu_\alpha]$,

$G_\alpha = \{g$: g *a partial function from some* $a \in \mathcal{S}_{<\kappa}(A_\alpha)$ *to* $\theta\}$,

for $g \in G_\alpha$,

$[g] = \{f \in X_\alpha \colon g \subseteq f\}$ *where* $X_\alpha =: {}^{(A_\alpha)}\theta$, *so* $|X_\alpha| = \chi_\alpha$,

h_α *is a function from* $\mathcal{S}_{<\lambda_\alpha}\left({}^{(A_\alpha)}\theta\right)$ *to* G_α *such that* $h_\alpha(a)$ *"exemplifies" that* a *is not dense in* ${}^{(A_\alpha)}\theta$, *i.e.* $[f \in a \,\&\, g = h_\alpha(a) \Rightarrow g \not\subseteq f]$.

Then (F)⇒(E)⇒(D)⇔(C)⇒(B)⇔(A); and $(E)^\sigma$ decrease with σ and $(E)^\sigma$ ⇒(G) when $\chi^*_\alpha = \chi_\alpha$; and if every λ_α is regular (G)⇒(F) and if in addition $\bigwedge_{\alpha<\delta^*} \chi^*_\alpha = \chi_\alpha$ then (G)⇔(F)⇔(E), and if $\{\alpha < \delta^* \colon \sigma \leq \lambda_\alpha\} \neq \emptyset \bmod J$ and $\sigma < \lambda$ then (E)⇔$(E)^\sigma$ (fixing J), where

(A) $d_{<\kappa}(\lambda, \theta) > \chi$;

(B) *if $x_\zeta \in \prod_{\alpha<\delta^*} X_\alpha$ for $\zeta < \chi$ then there is $\bar{g} \in \prod_{\alpha<\delta^*} G_\alpha$ such that: for every $\zeta < \chi$, $\{\alpha < \delta^* \colon x_\zeta(\alpha) \notin [g_\zeta]\} \neq \emptyset$;*

(C) *if $x_\zeta \in \prod_{\alpha<\delta^*} X_\alpha$ for $\zeta < \chi$ then for some $w_\alpha \in \mathcal{S}_{<\lambda_\alpha}(X_\alpha)$ $(\alpha < \delta^*)$ for every $\zeta < \chi$, $\{\alpha < \delta^* \colon x_\zeta(\alpha) \in w_\alpha\} \neq \emptyset$;*

(D) *for every* $x_\zeta \in \prod_{\alpha<\delta^*} \chi_\alpha$ *for* $\zeta < \chi$ *there is* $\bar{w} \in \prod_{\alpha<\delta^*} \mathcal{S}_{<\lambda_\alpha}(\chi_\alpha)$ *such that: for each* $\zeta < \chi$, $\bigvee_{\alpha<\delta^*} x_\zeta(\alpha) \in w_\alpha$;

(E)$^\sigma$ *for some ideal* J *on* δ^* *extending* $J^{bd}_{\delta^*}$ *for every* $x_\zeta \in \prod_{\alpha<\delta^*} \chi_\alpha$ *(for* $\zeta < \chi$*) there are* $\epsilon(*) < \sigma$ *and* $\bar{w}^\epsilon \in \prod_{\alpha<\delta^*} \mathcal{S}_{<\lambda_\alpha}(\chi_\alpha)$ *for* $\epsilon < \epsilon(*)$ *such that for each* ζ *we have* $\bigvee_\epsilon \{\alpha < \delta^*: x_\zeta(\alpha) \notin w^\epsilon_\alpha\} = \emptyset \bmod J$.
If $\sigma = 2$ *we may omit it;*

(F) *for some non-trivial ideal* J *on* δ^* *extending* $J^{bd}_{\delta^*}$ *we have*

$$\prod_{\alpha<\delta^*} (\mathcal{S}_{<\lambda_\alpha}(\chi_\alpha), \subseteq) / J \text{ is } \chi^+\text{-directed;}$$

(G) *for some non-trivial ideal* J *on* δ^* *extending* $J^{bd}_{\delta^*}$, *for any* $\langle \mathcal{P}_\alpha: \alpha < \delta^* \rangle$, $\mathcal{P}_\alpha$ *a* λ_α*-directed partial order of cardinality* $\leq \chi^*_\alpha$, *we have:* $\prod_{\alpha<\delta^*} \mathcal{P}_\alpha / J$ *is* χ^+*-directed.*

5.3A *Remark:*

(1) Note that the desired conclusion is 5.2(A).
(2) The interesting case of 5.3 is when $\{\mu_\alpha: \alpha < \delta^*\}$ does not contain a club of λ.
(3) Note that with notational changes we can arrange "λ is the disjoint union of $A_\alpha (\alpha < \delta^*)$, hence $\lambda_\theta = \prod_{\alpha<\delta^*} X_\alpha$".

Proof: Check. Clearly $(E)^\sigma$ decreases with σ, i.e. if $\sigma_1 < \sigma_2$ then $(E)^{\sigma_1} \Rightarrow (E)^{\sigma_2}$.

(E)⇒(D): Just for J varying on non-trivial ideals, we have monotonicity in J; and for $J = \{\emptyset\}$ we get (D).

(D)⇔(C): (C) is a translation of (D).

(C)⇒(B): If $x_\zeta \in \prod_{\alpha<\delta^*} X_\alpha$ for $\zeta < \chi$, let $\langle w_\alpha: \alpha < \delta^* \rangle$ be as in (C); for each α we know that w_α is not a dense subset of X_α (as $d_{<\kappa}(\mu_\alpha, \theta) \geq \lambda_\alpha > |w_\alpha|$) so there is $g_\alpha \in G_\alpha$ for which $[g_\alpha] \cap w_\alpha = \emptyset$, so $\bar{g} =: \langle g_\alpha: \alpha < \delta^* \rangle$ is as required in (B).

(B)⇔(A): They say the same (see 5.3A(3)).

(F)⇒(E): Note that (E) just says that in $\prod_{\alpha<\delta^*} (\mathcal{S}_{<\lambda_\alpha}(\chi_\alpha), \subseteq)$, any subset of $\{f: f \in \prod_{\alpha<\delta^*} \mathcal{S}_{<\lambda_\alpha}(\chi_\alpha)$, such that each $f(\alpha)$ is a singleton$\}$ has a $\leq_J$-upper bounded. In this form it is clearly a specific case of (F).

(E)$^\sigma$ $\Rightarrow$(G) WHEN $\chi_\alpha = \chi^*_\alpha$: where $\{\alpha < \delta^*: \sigma \le \lambda_\alpha\} \ne \emptyset \bmod J$: Easy too.

Next assume every λ_α is regular, J an ideal on δ^*.

(G)$\Rightarrow$(F): (F) is a particular case of (G), because $(\mathcal{S}_{<\lambda_\alpha}(\chi_\alpha) \subseteq)$ is λ_α-directed as λ_α is regular and $\mathcal{S}_{<\lambda_\alpha}(\chi_\alpha)$ can be replaced by any cofinal subset and there is one of cardinality χ^*_α by its definition.

The rest should be clear. $\blacksquare_{5.3}$

5.4 CLAIM: *Assume λ is strong limit, $\theta < \lambda_0$, $\langle \lambda_\alpha: \alpha < \delta^* \rangle$, $\langle \chi^*_\alpha: \alpha < \delta^* \rangle$ are (strictly) increasing with limit λ, $\delta^* < \kappa \le \mathrm{cf}(\lambda) < \lambda$, $\lambda < \chi < 2^\lambda$ and $\lambda_\alpha \le \chi^*_\alpha$, λ_α regular for each $\alpha < \delta^*$. Then (G) of 5.3 holds (hence $d_{<\kappa}(\lambda, \theta) > \chi$) in any of the following cases:*

(a) *for some μ_α strong limit, $\mathrm{cf}(\mu_\alpha) < \kappa$, $2^{\mu_\alpha} = \mu^+_\alpha$, $\lambda_\alpha = \mu^+_\alpha$, $\chi^*_\alpha = \mu^+_\alpha$ and $\prod_{\alpha<\delta^*} \mu^+_\alpha / J$ is χ^+-directed,*

(b) *$k < \omega$ and for every α, $\chi^*_\alpha \le \lambda^{+k}_\alpha$ and for some ideal J on δ^*, for $\ell \le k$, $\prod \lambda^{+\ell}_\alpha / J$ is χ^+-directed, and $d_{<\kappa}(\chi^*_\alpha, \theta) \ge \lambda_\alpha$,*

(c) *for some $\gamma < \mathrm{cf}(\lambda)$ for every $\alpha < \delta^*$, $\chi^*_\alpha \le \lambda^{+\gamma}_\alpha$ and for some ideal J on δ^* for every $\zeta < \gamma$, $\prod_{\alpha<\delta^*}, \lambda^{+(\zeta+1)}_\alpha / J$ is χ^+-directed, and $d_{<\kappa}(\chi^*_\alpha, \theta) \ge \lambda_\alpha$,*

(d) *for some ideal J on δ^* extending $J^{bd}_{\delta^*}$ for every regular $\lambda'_\alpha \in [\lambda_\alpha, \chi^*_\alpha]$ satisfying $\mathrm{tlim}_J(\mathrm{cf}\, \lambda'_\alpha) = \lambda$, we have $\prod_{\alpha<\delta^*} \lambda'_\alpha / J$ is χ^+-directed and $d_{<\kappa}(\chi^*_\alpha, \theta) \ge \lambda_\alpha$.*

Proof: Clearly (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d).

Now the statements follow from the following observations 5.4A–5.7.

5.4A *Observation:* Assume that for $\alpha < \delta$, $\mathcal{P}_\alpha$ is a (non-empty) λ_α-directed partial order of cardinality χ_α, $|\delta|^+ < \lambda_\alpha = \mathrm{cf}(\lambda_\alpha) \le \chi_\alpha$, J an ideal on δ, $\theta^* = \mathrm{Min}\{\theta$: for some A and $\bar f$: $\bar f = \langle f_i: i < \theta \rangle$, $f_i \in \prod_{\alpha<\delta} \mathcal{P}_\alpha$ is $<_{J+A}$-increasing, $A \subseteq \delta$, $\delta \backslash A \notin J$ but for no $g \in \prod_{\alpha<\delta} \mathcal{P}_\alpha$, $\bigwedge_{i<\theta} \{\alpha: \mathcal{P}_\alpha \models f_i(\alpha) \le g(\alpha)\} \ne \emptyset \bmod (J + A)\}$. *Then* $\prod_{\alpha<\delta} \mathcal{P}_\alpha / J$ is θ^*-directed.

Proof: Without loss of generality no $\mathcal{P}_\alpha$ has a maximal element. If the conclusion of 5.4A fails, let F be a subset of $\prod_{\alpha<\delta} \mathcal{P}_\alpha$ with no $<_J$-upper bound, of minimal cardinality. Let $\theta = |F|$, so let $F = \{f_i: i < \theta\}$; by the choice of F without loss of generality $\alpha < \beta \Rightarrow f_\alpha <_J f_\beta$ hence θ is necessarily regular. If $\{\alpha < \delta: \lambda_\alpha \le \theta\} \in J$ we can find an upper bound: $g(\alpha)$ is a $\mathcal{P}_\alpha$-upper bound of $\{f_i(\alpha): i < \theta\}$ when $\lambda_\alpha > \theta$, and arbitrarily otherwise. So without loss of generality $\bigwedge_\alpha \lambda_\alpha \le \theta$. Now, remember $|\delta|^+ < \lambda_\alpha$, and so $|\delta|^+ < \theta$. By [Sh420, §1] we can find $\bar C = \langle C_i: i < \theta \rangle$,

$C_i \subset i$, $j \in C_i \Rightarrow C_j = j \cap C_i$, $\mathrm{otp}(C_i) \le |\delta|^+$ and $S =: \{i < \lambda\colon \mathrm{cf}(i) = |\delta|^+, \delta = \sup(C_i)\}$ stationary: so wlog $j \in C_i \Rightarrow \bigwedge_{\alpha<\delta} \mathcal{P}_\alpha \models f_j(\alpha) < f_i(\alpha)$. Now we repeat the proof from [Sh282, 14]; better see [Sh345a, 2.6] or here 6.1.* $\blacksquare_{5.4A}$

5.5 Observation: In 5.4A, if A, $\bar f$ exemplify $\theta^* = \theta$ then

$$\theta^* \ge \min\{\operatorname*{pre}_{J+A}(\bar\chi, \bar\lambda)\colon A \subseteq \delta \text{ and } \delta \setminus A \notin J\}$$

where

5.6 Definition: For ideal I on δ and $\bar\chi = \langle \chi_\alpha\colon \alpha < \delta\rangle$, $\bar\lambda = \langle \lambda_\alpha\colon \alpha < \delta\rangle$, $\lambda_\alpha = \mathrm{cf}(\lambda_\alpha) \le \chi_\alpha$ we let $\mathrm{pre}_I(\bar\chi, \bar\lambda) =: \mathrm{Min}\{|\mathcal{P}|\colon \mathcal{P}$ is a family of sequences of the form $\langle B_\alpha\colon \alpha < \delta\rangle$, $B_\alpha \subseteq \chi_\alpha$, $|B_\alpha| < \lambda_\alpha$ such that for every $g \in \prod_{\alpha<\delta} \chi_\alpha$ for some $\bar B \in \mathcal{P}$, $\{\alpha < \delta\colon g(\alpha) \in B_\alpha\} \ne \emptyset \bmod I\}$.

Proof: Check.

5.6A Remark: We use other parts of 5.3.

5.7 Observation: Let I be an ideal on δ^*, $\chi_\alpha \ge \lambda_\alpha > \delta^*$.

(1) Define $\mathcal{J}[I] = \{I + A\colon A \subseteq \delta, \delta\backslash A \notin I\}$.

(2) If $I_1 \subseteq I_2$, $\lambda^1_\alpha \ge \lambda^2_\alpha$, $\chi^1_\alpha \le \chi^2_\alpha$ for $\alpha < \delta$ then $\mathrm{pre}_{I_1}(\bar\chi^1, \bar\lambda^1) \le \mathrm{pre}_{I_2}(\bar\chi^2, \bar\lambda^2)$.

(3) If δ^* is the disjoint union of A_1, A_2, $A_\ell \notin I$ and $I_\ell =: I + A_\ell$ then $\mathrm{pre}_I(\bar\chi, \bar\lambda) = \mathrm{Min}\left\{\mathrm{pre}_{I_1}(\bar\chi, \bar\lambda), \mathrm{pre}_{I_2}(\bar\chi, \bar\lambda)\right\}$.

(4) $\mathrm{pre}_I(\bar\chi^+, \bar\lambda) \le \mathrm{pre}_I(\bar\chi, \bar\lambda) + \sup\{\mathrm{tcf}(\prod \chi^+_\alpha / I + A)\colon A \subseteq \delta, \delta\backslash A \notin I\}$.** Moreover $\mathrm{pre}_I(\bar\chi^+, \bar\lambda) \le \mathrm{Min}\{\mathrm{pre}_{I+A}(\bar\chi, \bar\lambda) + \mathrm{tcf}(\prod_{\alpha<\delta} \chi^+_\alpha/(I + A))\colon A \subseteq \delta, \delta \setminus A \notin I$ (and the tcf is well defined)$\}$.

(5) If each χ_α is a limit cardinal, $\mathrm{cf}\,\chi_\alpha > \delta^*$, then $\sup_{J\in\mathcal{J}[I]} \mathrm{pre}_J(\bar\chi, \bar\lambda) = \sup_{\bar\chi' < \bar\chi} \sup_{J\in\mathcal{J}[I]} \mathrm{pre}_J(\bar\chi', \bar\lambda) + \sup_{J\in\mathcal{J}[I]} \mathrm{tcf}(\Pi\chi_\alpha/I)$.

(6) $2^{|\delta^*|} + \sup_{J\in\mathcal{J}[I]} \sup\{\mathrm{tcf}(\Pi_{\alpha<\delta}\chi'_\alpha/J)\colon \lambda_\alpha \le {\chi'}_\alpha = \mathrm{cf}(\chi'_\alpha) \le \chi_\alpha$ and the true cofinality is well defined$\} \le 2^{|\delta^*|} + \sup_{J\in\mathcal{J}[I]} \mathrm{pre}_J(\bar\chi, \bar\lambda) \le 2^{|\delta^*|} + \sup_{J\in\mathcal{J}[I]} \sup\{\mathrm{tcf}(\Pi_{\alpha<\delta}\chi'_\alpha/J)\colon |\delta^*| < \mathrm{cf}(\chi'_\alpha)$ and $\lambda_\alpha \le \chi'_\alpha \le \chi_\alpha\}$.

(7) In part (6), if I is a precipitous ideal then the first inequality is equality.

Proof: Straightforward.

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* In the main case here, $\bigwedge_\alpha 2^{|\delta^*|} < \lambda_\alpha$ and then trying all the possible A's, using their g's, the proof is very simple.

** Of course, $\bar\chi^+ = \langle \chi^+_\alpha\colon \alpha < \delta\rangle$.

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5.9 Observation: In several of the models of set theory in which we know "λ strong, singular, limit, $2^\lambda > \lambda^{+}$" our sufficient conditions for $d_{\mathrm{cf}\,\lambda}(\lambda, 2) = 2^\lambda$ usually hold by the sufficient condition 5.4(a) (simplest: if GCH holds below λ, $\mathrm{cf}\,\lambda = \aleph_0$).

Remark: We could prove this consistency by looking more at the consistency proofs, adding many Cohen subsets to λ in preliminary forcing; but the present way looks more informative.**

6. Odds and Ends

6.1 LEMMA: *Suppose* $\mathrm{cf}(\delta) > \kappa^+$, *$I$ an ideal on κ,* $f_\alpha \in {}^\kappa\mathrm{Ord}$ *for* $\alpha < \delta$ *is* $\le_I$*-increasing. Then there are* J_α, $\bar{s}$, $f'_\alpha(\alpha < \delta)$ *such that:*

(A) $\bar{s} = \langle s_i : i < \kappa\rangle$, *each s_i a set of* $\le \kappa$ *ordinals,*

(B) $\bigwedge_{i<\kappa} \bigwedge_{\alpha<\delta} \bigvee_{\beta\in s_i} f_\alpha(i) \le \beta$,

(C) $f'_\alpha \in \prod_{i<\kappa} s_i$ *is defined by* $f'_\alpha(i) = \mathrm{Min}[s_i \backslash f_\alpha(i)]$,

(D) $\mathrm{cf}[f'_\alpha(i)] \le \kappa$ *(e.g. $f'_\alpha(i)$ is a successor ordinal) implies* $f'_\alpha(i) = f_\alpha(i)$,

such that:

(E) *J_α is an ideal on κ extending I (for $\alpha < \lambda$), decreasing with α (in fact for some* $a_{\alpha,\beta} \subseteq \kappa$ *(for* $\alpha < \beta < \kappa$), $a_{\alpha,\beta}/I$ *decreases with β, increases with α and J_α is the ideal generated by* $I \cup \{a_{\alpha,\beta}: \alpha < \beta < \lambda\}$) *so possibly* $J_\alpha = \mathcal{P}(\kappa)$ *and possibly* $J_\alpha = I$,

(F) *if D is an ultrafilter on κ disjoint to J_α then f'_α/D is a* $<_D$*-l.u.b of* $\langle f_\beta/D: \beta < \delta\rangle$ *and* $\{i < \kappa: \mathrm{cf}[f'_\alpha(i))] > \kappa\} \in D$,

(G) *if D is an ultrafilter on κ disjoint to I but for every α not disjoint to J_α then $\bar{s}$ exemplifies* $\langle f_\alpha: \alpha < \delta\rangle$ *is chaotic for D, i.e. for some club E of δ,* $\beta < \gamma \in E \Rightarrow f_\beta \le_D f'_\beta <_D f_\gamma$,

(H) *if* $\mathrm{cf}(\delta) > 2^\kappa$ *then* $\langle f_\alpha: \alpha < \delta\rangle$ *has a* $\le_I$*-l.u.b. and even* $\le_I$*-e.u.b,*

(I) *if* $b_\alpha =: \{i: f'_\alpha(i)$ *has cofinality* $\le \kappa$ *(e.g. is a successor)*$\} \notin J_\alpha$ *then: for every* $\beta \in (\alpha, \delta)$ *we have* $f'_\alpha \restriction b_\alpha = f_\beta \restriction b_\alpha \bmod J_\alpha$.

Moreover

(F)$^+$ *if* $\kappa \notin J_\alpha$ *then f'_α is an* $<_{J_\alpha}$*-e.u.b (= exact upper bound) of* $\langle f_\beta: \beta < \delta\rangle$.

Proof: Let $S = \{j: j \le \sup \bigcup_{\alpha<\delta} \mathrm{Rang}(f_\alpha)$ has cofinality $\le \kappa\}$, $\bar{e} = \langle e_j: j \in S\rangle$ be such that $[j = i+1 \Rightarrow e_j = \{i\}]$, $[j$ limit $\& j' \in S \cap e_j \Rightarrow e_{j'} \subseteq e_j]$, $e_j \subseteq j$ $[j$ limit $\Rightarrow j = \sup e_j]$ and $|e_j| \le \kappa$.

** See much more on independence in a paper of Gitik and Shelah.

For a set $a \subseteq \sup \bigcup_{\alpha<\delta} \mathrm{Rang}\,(f_\alpha)$ let $\bar{e}[a] = a \cup \bigcup_{j \in a \cap S} e_j$ hence $\bar{e}[\bar{e}[a]] = \bar{e}[a]$ and $[a \subseteq b \Rightarrow \bar{e}[a] \subseteq \bar{e}[b]]$ and $|\bar{e}[a]| \le |a| + \kappa$. We try to choose by induction on $\zeta < \kappa^+$, the following: α_ζ, D_ζ, g_ζ, $\bar{s}_\zeta = \langle s_{\zeta,i}: i < \kappa\rangle$, $\langle f_{\zeta,\alpha}: \alpha < \delta\rangle$ such that:

(a) $g_\zeta \in {}^\kappa\mathrm{Ord}$,

(b) $s_{\zeta,i} = \bar{e}\,[\{g_\epsilon(i): \epsilon < \zeta\} \cup \{\sup_{\alpha<\delta} f_\alpha(i) + 1\}]$ so it is a set *of* $\le \kappa$ ordinals, increasing with ζ, $\sup_{\alpha<\delta} f_\alpha(i) + 1 \in s_{\zeta,i}$,

(c) $f_{\zeta,\alpha} \in {}^\kappa\mathrm{Ord}$, $f_{\zeta,\alpha}(i) = \mathrm{Min}[s_{\zeta,i} \backslash f_\alpha(i)]$,

(d) D_ζ is an ultrafilter on κ disjoint to I,

(e) for $\alpha < \delta$, $f_\alpha \le_{D_\zeta} g_\zeta$,

(f) α_ζ is an ordinal $< \delta$,

(g) $\alpha_\zeta \le \alpha < \lambda \Rightarrow g_\zeta <_{D_\zeta} f_{\zeta,\alpha}$.

If we succeed, let $\alpha(*) = \sup_{\zeta<\kappa^+} \alpha_\zeta$, so as $\mathrm{cf}(\delta) > \kappa^+$ clearly $\alpha(*) < \delta$. Now let $i < \kappa$ and look at $\langle f_{\zeta,\alpha(*)}(i): \zeta < \kappa^+\rangle$; by its definition (see (c)), $f_{\zeta,\alpha(*)}(i)$ is the minimal member of the set $s_{\zeta,i} \backslash f_{\alpha(*)}(i)$. This set increases with ζ, so $f_{\zeta,\alpha(*)}(i)$ decreases with ζ (though not necessarily strictly), hence is eventually constant; so for some $\zeta_i < \kappa^+$ we have $\zeta \in [\zeta_i, \kappa^+) \Rightarrow f_{\zeta,\alpha(*)}(i) = f_{\zeta_i,\alpha(*)}(i)$. Let $\zeta(*) = \sup_{i<\kappa} \zeta_i$, so $\zeta(*) < \kappa^+$, hence

$$(*) \qquad \zeta \in [\zeta(*), \kappa^+) \Rightarrow \bigwedge_i f_{\zeta,\alpha(*)}(i) = f_{\zeta(*),\alpha(*)}(i) \Rightarrow f_{\zeta,\alpha(*)} = f_{\zeta(*),\alpha(*)}.$$

We know that $f_{\alpha(*)} \le_{D_{\zeta(*)}} g_{\zeta(*)} <_{D_{\zeta(*)}} f_{\zeta(*),\alpha(*)}$ hence for some i, $f_{\alpha(*)}(i) \le g_{\zeta(*)}(i) < f_{\zeta(*),\alpha(*)}(i)$, but $g_{\zeta(*)}(i) \in s_{\zeta(*)+1,i}$ hence $f_{\zeta(*)+1,\alpha(*)}(i) \le g_{\zeta(*)}(i) < f_{\zeta(*),\alpha(*)}(i)$, contradicting the choice of $\zeta(*)$.

So necessarily for some $\zeta < \kappa^+$ we are stuck, and clearly $s_{\zeta,i} (i < \kappa)$, $f_{\zeta,\alpha} (\alpha < \lambda)$ are well defined.

Let $s_i =: s_{\zeta,i}$ (for $i < \kappa$) and $f'_\alpha = f_{\zeta,\alpha}$ (for $\alpha < \lambda$). Clearly s_i is a set of $\le \kappa$ ordinals; now clearly:

$(*)_1$ $f_\alpha \le f'_\alpha$

$(*)_2$ $\alpha < \beta \Rightarrow f'_\alpha \le_I f'_\beta$,

$(*)_3$ if $b = \{i: f'_\alpha(\alpha) < f'_\beta(i)\} \notin I$, $\alpha < \beta < \delta$ then $f'_\alpha \restriction b <_I f_\beta \restriction b$.

We let for $\alpha < \delta$

$$J_\alpha = \Big\{ b \subseteq \kappa: b \in I \text{ or } b \notin I \quad \text{and for some } \beta \text{ we have: } \alpha < \beta < \delta \text{ and } f'_\alpha \restriction (\kappa \setminus b) =_I f'_\beta \restriction (\kappa \setminus b) \Big\}.$$

We let for $\alpha < \beta < \delta$, $a_{\alpha,\beta} =: \{i < \kappa: f'_\alpha(i) < f'_\beta(i)\}$. Then

$(*)_4$ J_α is an ideal on κ extending I, in fact is the ideal generated by $I \cup \{a_{\alpha,\beta}: \beta \in (\alpha, \delta)\}$.

As $\langle f'_\alpha: \alpha < \delta \rangle$ is $\leq_I$-increasing (i.e. $(*)_1$):

$(*)_5$ J_α decreases with α, in fact $a_{\alpha,\beta}/I$ increases with β, decreases with α,

$(*)_6$ if D is an ultrafilter on κ disjoint to J_α, then f'_α/D is a $<_D$-lub of $\{f_\beta/D: \beta < \delta\}$.

[Why? We know that $\beta \in (\alpha, \delta) \Rightarrow a_{\alpha,\beta} = \emptyset \bmod D$, so $f_\beta \leq f'_\beta =_D f'_\alpha$ for $\beta \in (\alpha, \delta)$, so f'_α/D is an $\leq_D$-upper bound. If it is not a least upper bound then for some $g \in {}^\kappa\mathrm{Ord}$, $\bigwedge_\beta f_\beta \leq_D g <_D f'_\alpha$ and we can get a contradiction to the choice of ζ, $\bar{s}$, f'_β as: (D, g) could serve as D_ζ, g_ζ.]

$(*)_7$ If D is an ultrafilter on κ disjoint to I but not to J_α (for every $\alpha < \lambda$) then $\bar{s}$ exemplifies $\langle f_\alpha: \alpha < \delta \rangle$ is chaotic for D.

[Why? For every $\alpha < \delta$ for some $\beta \in (\alpha, \delta)$ we have $a_{\alpha,\beta} \in D$, i.e. $\{i < \kappa: f'_\alpha(i) < f'_\beta(i)\} \in D$, so $\langle f'_\alpha/D: \alpha < \delta \rangle$ is not eventually constant, so if $\alpha < \beta$, $f'_\alpha <_D f'_\beta$ then $f'_\alpha <_D f_\beta$ (by $(*)_3$) and $f_\beta \leq_D f'_\beta$ (by (c)) as required.]

$(*)_8$ if $\kappa \notin J_\alpha$ then f'_α is an $\leq_{J_\alpha}$-e.u.b. of $\langle f_\beta: \beta < \delta \rangle$.

[Why? By $(*)_6$, f'_α is a $\leq_{J_\alpha}$-upper bound of $\langle f_\beta: \beta < \delta \rangle$; so assume that it is not a $\leq_{J_\alpha}$-e.u.b. of $\langle f_\beta: \beta < \delta \rangle$, hence there is a function g with domain κ, such that $g(i) < \mathrm{Max}\{1, f'_\alpha(i)\}$, but for no $\beta < \delta$ do we have

$$C_\beta =: \{i < \kappa: g(i) < \mathrm{Max}\{1, f_\beta(i)\} = \kappa \bmod J_\alpha.$$

Clearly $\langle C_\beta: \beta < \delta \rangle$ is increasing modulo J_α so there is an ultrafilter D on κ disjoint to $J_\alpha \cup \{C_\beta: \beta < \delta\}$. So $f_\beta \leq_D g \leq_D f'_\alpha$, so we get a contradiction to $(*)_6$ except when $g =_D f'_\alpha$ and then $f'_\alpha =_D O_\kappa$ (as $g(i) < 1 \vee g(i) < f'_\alpha(i)$). If we can demand $b^* = \{i: f'_\alpha(i) = 0\} \notin D$ we are done, but easily $b^* \setminus C_\beta \in J_\alpha$ so we finish.]

$(*)_9$ If $\mathrm{cf}[f'_\alpha(i)] \leq \kappa$ then $f'_\alpha(i) = f_\alpha(i)$.

[Why? By the definition of $s_\zeta = \bar{e}[\ldots]$ and the choice of $\bar{e}$, and $f'_\alpha(i)$.]

$(*)_{10}$ Clause (I) of the conclusion holds.

[Why? As $f_\alpha \leq_{J_\alpha} f_\beta \leq_{J_\alpha} f'_\alpha$ and $f_\alpha \restriction b =_{J_\alpha} f'_\alpha \restriction b$ by $(*)_9$.]

The reader can check the rest. $\blacksquare_{6.1}$

6.1A Example: We show that l.u.b and e.u.b are not the same. Let I be an ideal on κ, $\kappa^+ < \lambda = \mathrm{cf}(\lambda)$, $\bar{a} = \langle a_\alpha: \alpha < \lambda \rangle$ be a sequence of subsets of κ, (strictly) increasing modulo I, $\kappa \backslash a_\alpha \notin I$ but there is no $b \in \mathcal{P}(\kappa) \backslash I$ such that

$\bigwedge_\alpha b \cap a_\alpha \in I$. [Does this occur? E.g. for $I = \mathcal{S}_{<\aleph_0}(\omega)$, the existence of such $\bar{a}$ is known to be consistent; e.g. MA $\&\kappa = \aleph_0$ & $\lambda = 2^{\aleph_0}$. Moreover, for any κ and $\kappa^+ < \lambda = \mathrm{cf}\,\lambda \leq 2^\kappa$ we can find $a_\alpha \subseteq \kappa$ for $\alpha < \lambda$ such that, e.g., any Boolean combination of the a_α's has cardinality κ (less needed). Let I_0 be the ideal on κ generated by $\mathcal{S}_{<\kappa}(\kappa) \cup \{a_\alpha \backslash a_\beta \colon \alpha < \beta < \lambda\}$, and let I be maximal in $\{J \colon J$ an ideal on κ, $I_0 \subseteq J$ and $[\alpha < \beta < \lambda \Rightarrow a_\beta \backslash a_\alpha \notin J]\}$. So if G.C.H. fails, we have examples.] For $\alpha < \lambda$, we let $f_\alpha \colon \kappa \to \mathrm{Ord}$ be:

$$f_\alpha(i) = \begin{cases} \alpha & \text{if } \alpha \in \kappa \setminus a_i, \\ \lambda + \alpha & \text{if } \alpha \in a_i. \end{cases}$$

Now the constant function $f \in {}^\kappa\mathrm{Ord}$, $f(i) = \lambda + \lambda$ is a l.u.b of $\langle f_\alpha \colon \alpha < \lambda \rangle$ but not an e.u.b. (both $\bmod J$) (not e.u.b. is exemplified by $g \in {}^\kappa\mathrm{Ord}$ which is constantly λ).

6.2 Claim: *Suppose $\mu > \kappa = \mathrm{cf}\,\mu$, $\mu = \mathrm{tlim}_J\,\lambda_i$, $\delta < \mu$, $\lambda_i = \mathrm{cf}(\lambda_i) > \delta$ for $i < \delta$, J a σ-complete ideal on δ and $\lambda = \mathrm{tcf}\left(\prod_{i<\delta} \lambda_i / J\right)$, and $\langle f_\alpha \colon \alpha < \lambda \rangle$ exemplifies this.*

Then we have

- (*) *if $\langle u_\beta \colon \beta < \lambda \rangle$ is a sequence of pairwise disjoint non-empty subsets of λ, each of cardinality $\leq \sigma$ (not $< \sigma$!) and $\alpha^* < \mu$, then we can find $B \subseteq \lambda$ such that:*
  - (a) $\mathrm{otp}(B) = \alpha^*$,
  - (b) *if $\beta \in B$, $\gamma \in B$ and $\beta < \gamma$ then $\sup u_\beta < \min u_\gamma$,*
  - (c) *we can find $s_\zeta \in J$ for $\zeta \in \bigcup_{i \in B} u_i$ such that: if $\zeta \in \bigcup_{\beta \in B} u_\beta$, $\xi \in \bigcup_{\beta \in B} u_\beta$, $\zeta < \xi$ and $i \in \delta \backslash s_\zeta \backslash s_\xi$, then $f_\zeta(i) < f_\xi(i)$.*

Proof: For each regular $\theta, \theta^+ < \mu$, there is a stationary $S_\theta \subseteq \{\delta < \lambda \colon \mathrm{cf}(\delta) = \theta < \delta\}$ which is in $I[\lambda]$ (see [Sh420, 1.5]) which is equivalent (see [Sh420, 1.2(1)]) to:

- (*) there is $\bar{C}^\theta = \langle C^\theta_\alpha \colon i < \lambda \rangle$,
  - (α) C^θ_α a subset of α, with no accumulation points (in C^θ_α),
  - (β) $[\alpha \in \mathrm{nacc}(C^\theta_\beta) \Rightarrow C^\theta_\alpha = C^\theta_\beta \cap \alpha]$,
  - (γ) for some club E^0_θ of λ,

$$[\delta \in S_\theta \cap E^0_\theta \Rightarrow \mathrm{cf}(\delta) = \theta < \delta \;\&\; \delta = \sup C^\theta_\delta \;\&\; \mathrm{otp}(C^\theta_\delta) = \theta].$$

Without loss of generality $S_\theta \subseteq E^0_\theta$, and $\bigwedge_{\alpha<\delta} \mathrm{otp}(C^\theta_\delta) \leq \theta$. By [Sh365, 2.3, Def. 1.3] for some club E_θ of λ, $\langle g\ell(C^\theta_\alpha, E_\theta) \colon \alpha \in S_\theta \rangle$ guess clubs (i.e. for every club $E \subseteq E_\theta$ of λ, for stationarily many $\zeta \in S_\theta$, $\mathrm{g}\ell(C^\theta_\zeta, E_\theta) \subseteq E$) (remember $\mathrm{g}\ell(C^\theta_\delta, E_\theta) = \{\sup(\gamma \cap E_\theta): \gamma \in C^\theta_\delta; \gamma > \mathrm{Min}(E_\theta)\}$). Let $C^{\theta,*}_\alpha = \{\gamma \in C^\theta_\alpha: \gamma = \mathrm{Min}(C^\theta_\alpha \setminus \sup(\gamma \cap E_\theta))\}$, they have all the properties of the C^θ_α's and guess clubs in a weak sense: for every club E of λ for some $\alpha \in S_\theta \cap E$, if $\gamma_1 < \gamma_2$ are successive members of E then $|(\gamma_1, \gamma_2] \cap C^{\theta,*}_\alpha| \leq 1$; moreover, the function $\gamma \mapsto \sup(E \cap \gamma)$ is one to one on $C^{\theta,*}_\zeta$.

Now we define by induction on $\zeta < \lambda$, an ordinal α_ζ and functions $g^\zeta_\theta \in \prod_{i<\delta} \lambda_i$ (for each $\theta \in \Theta =: \{\theta: \theta < \mu, \theta$ regular uncountable$\}$).

For given ζ, let $\alpha_\zeta < \lambda$ be minimal such that:

$$\xi < \zeta \Rightarrow \alpha_\xi < \alpha_\zeta,$$
$$\xi < \zeta \,\&\, \theta \in \Theta \Rightarrow g^\zeta_\theta < f_{\alpha_\zeta} \bmod J.$$

Now α_ζ exists as $\langle f_\alpha: \alpha < \lambda\rangle$ is $<_J$-increasing cofinal in $\prod_{i<\lambda_i} /J$. Now for each $\theta \in \Theta$ we define g^ζ_θ as follows:

for $i < \delta^*$, $g^\zeta_\theta(i)$ is $\sup\left[\{g^\xi_\theta(i) + 1: \xi \in C^\theta_\zeta\} \cup \{f_{\alpha_\zeta}(i) + 1\}\right]$ if this number is $< \lambda_i$, and $f_{\alpha_\zeta}(i)$ otherwise.

Having made the definition we prove the assertion. We are given $\langle u_\beta: \beta < \lambda\rangle$, a sequence of pairwise disjoint non-empty subsets of λ, each of cardinality $< \sigma$ and $\alpha^* < \mu$. We should find B as promised; let $\theta =: (|\alpha^*| + |\delta|)^+$ so $\theta < \mu$ is regular $> |\delta|$. Let $E = \{\delta \in E_\theta$: for every ζ: $[\zeta < \delta \Leftrightarrow \sup u_\zeta < \delta \Leftrightarrow u_\zeta \subseteq \delta \Leftrightarrow \alpha_\zeta < \delta]\}$. Choose $\alpha \in S_\theta \cap \mathrm{acc}(E)$ such that $\mathrm{g}\ell(C^\theta_\zeta, E_\theta) \subseteq E$; hence letting $C^{\theta,*}_\alpha = \{\gamma_i: i < \theta\}$ (increasing) we know $\bigwedge_i (\gamma_i, \gamma_{i+1}) \cap E \neq \emptyset$. Now $B = \{\gamma_{5i+3}: i < \alpha^*\}$ are as required. For $\alpha \in \bigcup_{\zeta<\alpha^*} u_{5\zeta+3}$ let $s_\alpha = s^o_\alpha \cup s^1_\alpha$. For $\alpha \in u_{5\zeta+3}$, $\zeta < \alpha^*$, let $s^o_\alpha = \{i < \delta: g^{5\zeta+1}_\theta(i) < f_\alpha(i) < g^{5\zeta+4}(i)\}$, for each $\zeta < \alpha^*$; let $\langle \alpha_\epsilon: \epsilon < |u_{5\zeta+3}|\rangle$ enumerate $u_{5\zeta+3}$ and

$$s^1_{\alpha_\epsilon} = \{i: \text{ for every } \xi < \epsilon, f_{\alpha_\xi}(i) < f_{\alpha_\epsilon}(i) \Leftrightarrow \alpha_\xi < \alpha_\epsilon \Leftrightarrow f_{\alpha_\xi}(i) \leq f_{\alpha_\epsilon}(i)\}. \qquad \blacksquare_{6.2}$$

6.2A Remark: In 6.2: (1) We can avoid guessing clubs.

(2) Assume $\sigma < \theta_1 < \theta_2 < \mu$ are regular and there is $S \subseteq \{\delta < \lambda: \mathrm{cf}(\delta) = \theta_1\}$ from $I[\lambda]$ such that for every $\zeta < \lambda$ (or at least a club) of cofinality θ_2, $S \cap \zeta$ is stationary and $\langle f_\alpha: \alpha < \lambda\rangle$ obey suitable $\bar{C}^\theta$ (see [Sh345a, §2]). Then for some $A \subseteq \lambda$ unbounded, for every $\langle u_\beta: \beta < \theta_2\rangle$ sequence of pairwise disjoint non-empty subsets of A, each of cardinality $< \sigma$ with $[\min u_\beta, \sup u_\beta]$ pairwise disjoint we have: for every $B_0 \subseteq A$ of order type θ_2, for some $B \subseteq B_0$, $|B| = \theta_1$, (c) of (*) of 6.2 holds.

(3) In $(*)$ of 6.2, "$\alpha^* < \mu$" can be replaced by "$\alpha^* < \mu^+$" (prove by induction on α^*).

6.3 OBSERVATION: *Assume $\lambda < \lambda^{<\lambda}$, $\mu = \mathrm{Min}\{\mu: 2^\mu > \lambda\}$. Then there are δ, χ and $\mathcal{T}$, satisfying the condition $(*)$ below for $\chi = 2^\mu$ or at least arbitrarily large regular $\chi \leq 2^\mu$.*

$(*)$ *$\mathcal{T}$ a tree with δ levels, (where $\delta \leq \mu$) with a set X of $\geq \chi$ δ-branches, and for $\alpha < \delta$, $\bigcup_{\beta<\alpha} |\mathcal{T}_\beta| < \lambda$.*

Proof of Observation: So let $\chi \leq 2^\mu$ be regular, $\chi > \lambda$.

CASE 1: $\bigwedge_{\alpha<\mu} 2^{|\alpha|} < \lambda$. Then $\mathcal{T} = {}^{\mu>}2$, $\mathcal{T}_\alpha = {}^{\alpha}2$ are O.K. (the set of branches ${}^{\mu}2$ has cardinality 2^μ).

CASE 2: *Not Case 1.* So for some $\theta < \mu$, $2^\theta \geq \lambda$, but by the choice of μ, $2^\theta \leq \lambda$, so $2^\theta = \lambda$, $\theta < \mu$ and so $\theta \leq \alpha < \mu \Rightarrow 2^{|\alpha|} = 2^\theta$. Note $|{}^{\mu>}2| = \lambda$ as $\mu \leq \lambda$.

SUBCASE 2A: $\mathrm{cf}(\lambda) \neq \mathrm{cf}(\mu)$. Let ${}^{\mu>}2 = \bigcup_{j<\lambda} B_j$, B_j increasing with j, $|B_j| < \lambda$. For each $\eta \in {}^{\mu}2$, (as $\mathrm{cf}(\lambda) \neq \mathrm{cf}(\mu)$) for some $j_\eta < \lambda$,

$$\mu = \sup\left\{\zeta < \mu: \eta \restriction \zeta \in B_{j_\eta}\right\}.$$

So as $\mathrm{cf}(\chi) > \mu$, for some ordinal $j^* < \lambda$ we have

$$\{\eta \in {}^{\mu}2: j_\eta \leq j^*\} \text{ has cardinality } \geq \chi.$$

As $\mathrm{cf}(\lambda) \neq \mathrm{cf}(\mu)$ and $\mu \leq \lambda$ (by its definition) clearly $\mu < \lambda$, hence $|B_{j^*}| \times \mu < \lambda$. Let

$$\mathcal{T} = \{\eta \restriction \epsilon: \epsilon < \ell g(\eta) \text{ and } \eta \in B_{j^*}\}.$$

It is as required.

SUBCASE 2B: *Not 2A so* $\mathrm{cf}(\lambda) = \mathrm{cf}(\mu)$. As $(\forall\sigma)[\theta \leq \sigma < \mu \Rightarrow \lambda = 2^\sigma \Rightarrow \mathrm{cf}(\lambda) = \mathrm{cf}(2^\sigma) > \sigma]$, clearly $\mathrm{cf}(\lambda) \geq \mu$ so μ is regular. If $\lambda = \mu$ we get $\lambda = \lambda^{<\lambda}$ contradicting an assumption.

So $\lambda > \mu$, so λ singular. So if $\alpha < \mu$, $\mu < \sigma_i = \mathrm{cf}(\sigma_i) < \lambda$ for $i < \alpha$ then (see [Sh-g, 345a, 1.3(10)]) $\max\mathrm{pcf}\{\sigma_i: i < \alpha\} \leq \prod_{i<\alpha} \sigma_i \leq \lambda^{|\alpha|} \leq \left(2^\theta\right)^{|\alpha|} \leq 2^{<\mu} = \lambda$, but as λ is singular and $\max\mathrm{pcf}\{\sigma_i: i < \alpha\}$ is regular (see [Sh345a, 1.9]), clearly the inequality is strict, i.e. $\max\mathrm{pcf}\{\sigma_i: i < \alpha\} < \lambda$. So let $\langle\sigma_i: i < \mu\rangle$ be a strictly increasing sequence of regulars in (μ, λ) with limit λ, and by [Sh355, 3.4] there is $T \subseteq \prod_{i<\mu} \sigma_i$, $|\{\nu \restriction i: \nu \in T\}| \leq \max\text{pcf}\{\lambda_j: j < i\} < \lambda$, and number of μ-branches $> \lambda$. In fact we can get any regular cardinal in $(\lambda, \text{pp}^+(\lambda))$ in the same way. Let $\lambda^* = \min\{\lambda': \mu < \lambda' \leq \lambda, \text{cf}(\lambda') = \mu$ and $pp(\lambda') > \lambda\}$, so (by [Sh355, 2.3]), also λ^* has those properties and $\text{pp}(\lambda^*) \geq \text{pp}(\lambda)$. So if $\text{pp}^+(\lambda^*) = (2^\mu)^+$ or $\text{pp}(\lambda^*) = 2^\mu$ is singular, we are done. So assume this fails.

If $\mu > \aleph_0$, then (as in 3.4) $\alpha < 2^\mu \Rightarrow \text{cov}(\alpha, \mu^+, \mu^+, \mu) < 2^\mu$ and we can finish as in subcase 2A (as in 3.4; actually $\text{cov}(2^{<\mu}, \mu^+, \mu^+, \mu) < 2^\mu$ suffices which holds by the previous sentence and [Sh355, 5.4]). If $\mu = \aleph_0$ all is easy. $\blacksquare_{6.3}$

6.4 Claim: *Assume $\mathfrak{b}_k \subseteq \mathfrak{b}_{k+1} \subseteq \cdots$ for $k < \omega$, $\mathfrak{a} = \bigcup_{k\leq\omega} \mathfrak{b}_k$ (and $|\mathfrak{a}| < \text{Min}\,\mathfrak{a}$) and $\lambda \in \text{pcf}\,\mathfrak{a} \setminus \bigcup_{k<\omega} \text{pcf}(\mathfrak{b}_k)$.*

(1) *Then we can find finite $\mathfrak{d}_k \subseteq \text{pcf}(\mathfrak{b}_k \setminus \mathfrak{b}_{k-1})$ (stipulating $\mathfrak{b}_{-1} = \emptyset$) such that $\lambda \in \text{pcf} \bigcup_{k<\omega} \mathfrak{d}_k$.*

(2) *Moreover, we can demand $\mathfrak{d}_k \subseteq (\text{pcf}\,\mathfrak{b}_k) \setminus (\text{pcf}(\mathfrak{b}_{k-1}))$.*

Proof: We start to repeat the proof of [Sh371, 1.5] for $\kappa = \omega$. But there we apply [Sh371, 1.4] to $\langle \mathfrak{b}_\zeta: \zeta < \kappa\rangle$ and get $\langle\langle \mathfrak{c}_{\zeta,\ell}: \ell \leq n_\zeta\rangle: \zeta < \kappa\rangle$ and let $\lambda_{\zeta,\ell} = \max\text{pcf}(\mathfrak{c}_{\zeta,\ell})$. Here we apply the same claim ([Sh371, 1.4]) to $\langle \mathfrak{b}_k \setminus \mathfrak{b}_{k-1}: k < \omega\rangle$ to get part (1). As for part (2), in the proof of [Sh371, 1.5] we let $\delta = |\mathfrak{a}|^+ + \aleph_2$ choose $\langle N_i: i < \delta\rangle$, but now we have to adapt the proof of [Sh371, 1.4] (applied to $\mathfrak{a}$, $\langle \mathfrak{b}_k: k < \omega\rangle$, $\langle N_i: i < \delta\rangle$); we have gotten there, toward the end, $\alpha < \delta$ such that $E_\alpha \subseteq E$. Let $E_\alpha = \{i_k: k < \omega\}$, $i_k < i_{k+1}$. But now instead of applying [Sh371, 1.3] to each $\mathfrak{b}_\ell$ separately, we try to choose $\langle \mathfrak{c}_{\zeta,\ell}: \ell \leq n(\zeta)\rangle$ by induction on $\zeta < \omega$. For $\zeta = 0$ we apply [Sh371, 1.3]. For $\zeta > 0$, we apply [Sh371, 1.3] to $\mathfrak{b}_\zeta$ but there defining by induction on ℓ $\mathfrak{c}_\ell = \mathfrak{c}_{\zeta,\ell} \subseteq \mathfrak{a}$ such that $\max(\text{pcf}(\mathfrak{a} \setminus \mathfrak{c}_{\zeta,0} \setminus \cdots \setminus \mathfrak{c}_{\zeta,\ell-1}) \cap \text{pcf}\,\mathfrak{b}_\zeta)$ is strictly decreasing with ℓ. We use:

6.4A Observation: If $|\mathfrak{a}_i| < \text{Min}(\mathfrak{a}_i)$ for $i < i^*$, then $\mathfrak{c} = \bigcap_{i<i^*} \text{pcf}(\mathfrak{a}_i)$ has a last element or is empty.

Proof: Wlog $\langle |\mathfrak{a}_i|: i < i^* \langle$ is nondecreasing. By [Sh345b, 1.12]

$$(*)_1 \qquad \mathfrak{d} \subseteq \mathfrak{c}\ \&\ |\mathfrak{d}| < \text{Min}\,\mathfrak{d} \Rightarrow \text{pcf}(\mathfrak{d}) \subseteq \mathfrak{c}.$$

By [Sh371, 2.6]

$$\text{if } \lambda \in \text{pcf}(\mathfrak{d}),\ \mathfrak{d} \subseteq \text{pcf}(\mathfrak{c}),\ |\mathfrak{d}| < \text{Min}(\mathfrak{d}) \text{ then}$$
$$\text{for some } \mathfrak{e} \subseteq \mathfrak{d} \text{ we have } |\mathfrak{e}| \leq \text{Min}\,|\mathfrak{a}_0|,\ \lambda \in \text{pcf}(\mathfrak{e}).$$

Now choose by induction on $\zeta < |\mathfrak{a}_0|^+$, $\theta_\zeta \in \mathfrak{c}$, satisfying $\theta_\zeta > \max\mathrm{pcf}\{\theta_\epsilon : \epsilon < \zeta\}$. If we are stuck in ζ, $\max\mathrm{pcf}\{\theta_\epsilon : \epsilon < \zeta\}$ is the desired maximum by $(*)_1$. If we succeed $\theta = \max\mathrm{pcf}\{\theta_\epsilon : \epsilon < |\mathfrak{a}_0|^+\}$ is in $\mathrm{pcf}\{\theta_\epsilon : \epsilon < \zeta\}$ for some $\zeta < |\mathfrak{a}_0|^+$ by $(*)_2$; easy contradiction. $\blacksquare_{6.4A}$

$\blacksquare_{6.4}$

6.5 Conclusion: Assume $\aleph_0 = \mathrm{cf}(\mu) \le \kappa \le \mu_0 < \mu$, $[\mu' \in (\mu_0, \mu) \,\&\, \mathrm{cf}(\mu') \le \kappa \Rightarrow \mathrm{pp}_\kappa(\mu') < \lambda]$ and $\mathrm{pp}_\kappa^+(\mu) > \lambda = \mathrm{cf}(\lambda) > \mu$. Then we can find λ_n for $n < \omega$, $\mu_0 < \lambda_n < \lambda_{n+1} < \mu$, $\mu = \bigcup_{n<\omega} \lambda_n$ and $\lambda = \mathrm{tcf} \prod_{n<\omega} \lambda_n / J$ for some ideal J on ω (extending J_ω^{bd}).

Proof: Let $\mathfrak{a} \subseteq (\mu, \mu) \cap \mathrm{Reg}$, $|\mathfrak{a}| \le \kappa$, $\lambda \in \mathrm{pcf}(\mathfrak{a})$. Without loss of generality $\lambda = \max\mathrm{pcf}\,\mathfrak{a}$, let $\mu = \bigcup_{n<\omega} \mu_n^0$, $\mu_0 \le \mu_n^0 < \mu_{n+1}^0 < \mu$, let $\mu_n^1 = \mu_n^0 + \sup\{\mathrm{pp}_\kappa(\mu') : \mu_0 < \mu' \le \mu_n^0 \text{ and } \mathrm{cf}(\mu') \le \kappa\}$, by [Sh355, 2.3] $\mu_n^1 < \mu$, $\mu_n^1 = \mu_n^0 + \sup\{\mathrm{pp}_\kappa(\mu') : \mu_0 < \mu' < \mu_n^1 \text{ and } \mathrm{cf}(\mu') \le \kappa\}$ and obviously $\mu_n^1 \le \mu_{n+1}^1$; by replacing by a subsequence without loss of generality $\mu_n^1 < \mu_{n+1}^1$. Now let $\mathfrak{b}_n = \mathfrak{a} \cap \mu_n^1$ and apply the previous claim: to $\mathfrak{b}_k =: \mathfrak{a} \cap (\mu_n^1)^+$, note:

$$\max\mathrm{pcf}(\mathfrak{b}_k) \le \mu_k^1 < \mathrm{Min}(\mathfrak{b}_{k+1} \backslash \mathfrak{b}_k). \qquad \blacksquare_{6.5}$$

6.6 CLAIM:

(1) *Assume $\aleph_0 < \mathrm{cf}(\mu) = \kappa < \mu_0 < \mu$, $2^\kappa < \mu$ and $[\mu_0 \le \mu' < \mu \,\&\, \mathrm{cf}(\mu') \le \kappa \Rightarrow \mathrm{pp}_\kappa \mu' < \mu]$. If $\mu < \lambda = \mathrm{cf}(\lambda) < \mathrm{pp}^+(\mu)$ then there is a tree $\mathcal{T}$ with κ levels, each level of cardinality $< \mu$, $\mathcal{T}$ has exactly λ κ-branches.*

(2) *Suppose $\langle \lambda_i : i < \kappa \rangle$ is a strictly increasing sequence of regular cardinals, $2^\kappa < \lambda_0$, $\mathfrak{a} =: \{\lambda_i : i < \kappa\}$, $\lambda = \max\mathrm{pcf}\,\mathfrak{a}$, $\lambda_j > \max\mathrm{pcf}\{\lambda_i : i < j\}$ for each $j < \kappa$ (or at least $\sum_{i<\kappa} \lambda_i > \max\mathrm{pcf}\{\lambda_i : i < j\}$) and $\mathfrak{a} \notin J$ where $J = \{\mathfrak{b} \subseteq \mathfrak{a} : \mathfrak{b}$ is the union of countably many members of $J_{<\lambda}[\mathfrak{a}]\}$ (so $J \supseteq J_{\mathfrak{a}}^{bd}$, $\mathrm{cf}\,\kappa > \aleph_0$). Then the conclusion of (1) holds with $\mu = \sum_{i<\kappa} \lambda_i$.*

Proof: (1) By (2) and [Sh371, §1] (or can use the conclusion of [Sh-g, AG 5.7]).

(2) For each $\mathfrak{b} \subseteq \mathfrak{a}$ define the function $g_\mathfrak{b} : \kappa \to \mathrm{Reg}$ by

$$g_\mathfrak{b}(i) = \max\mathrm{pcf}[\mathfrak{b} \cap \{\lambda_j : j < i\}].$$

Clearly $[\mathfrak{b}_1 \subseteq \mathfrak{b}_2 \Rightarrow g_{\mathfrak{b}_1} \le g_{\mathfrak{b}_2}]$. As $\mathrm{cf}(\kappa) > \aleph_0$, J $\aleph_1$-complete, there is $\mathfrak{b} \subseteq \mathfrak{a}$, $\mathfrak{b} \notin J$ such that:

$$\mathfrak{c} \subseteq \mathfrak{b} \,\&\, \mathfrak{c} \notin J \Rightarrow \neg g_\mathfrak{c} <_J g_\mathfrak{b}.$$

Let $\lambda_i^* = \max \mathrm{pcf}(\mathfrak{b} \cap \{\lambda_j: j < i\})$. For each i let $\mathfrak{b}_i = \mathfrak{b} \cap \{\lambda_j: j < i\}$ and $\langle\langle f^{\mathfrak{b}}_{\lambda,\alpha}: \alpha < \lambda\rangle: \lambda \in \mathrm{pcf}\, \mathfrak{b}\rangle$ be as in [Sh371, §1]. Let

$$\mathcal{T}_i^0 = \left\{ \operatorname*{Max}_{\ell=1,n} f^{\mathfrak{b}}_{\lambda_\ell,\alpha_\ell} \restriction \mathfrak{b}_i : \lambda_\ell \in \mathrm{pcf}(\mathfrak{b}_i),\ \alpha_\ell < \lambda_\ell,\ n < \omega \right\}.$$

Let $\mathcal{T}_i = \{f \in \mathcal{T}_i^0$: for every $j < i, f \restriction \mathfrak{b}_j \in \mathcal{T}_j^0$ moreover for some $f' \in \prod_{j<\kappa} \lambda_j$, for every j, $f' \restriction j \in \mathcal{T}_i^0$ and $f \subseteq f'\}$, and $\mathcal{T} = \bigcup_{i<\kappa} \mathcal{T}_i$, clearly it is a tree, $\mathcal{T}_i$ its ith level (or empty), $|\mathcal{T}_i| \le \lambda_i^*$. By [Sh371, 1.3, 1.4] for every $g \in \prod \mathfrak{b}$ for some $f \in \prod \mathfrak{b}$, $\bigwedge_{i<\kappa} f \restriction \mathfrak{b}_i \in \mathcal{T}_i^0$ hence $\bigwedge_{i<\kappa} f \restriction \mathfrak{b}_i \in \mathcal{T}_i$. So $|\mathcal{T}_i| = \lambda_i^*$, and $\mathcal{T}$ has $\ge \lambda$ κ-branches. By the observation below we can finish (apply it essentially to $F = \{\eta$: for some $f \in \prod \mathfrak{b}$ for $i < \kappa$ we have $\eta(i) = f \restriction \mathfrak{b}_i$ and for every $i < \kappa$, $f \restriction \mathfrak{b}_i \in \mathcal{T}_i^0\})$, then find $A \subseteq \kappa$, $\kappa \setminus A \in J$ and $g^* \in \prod_{i<\kappa}(\lambda_i + 1)$ such that $Y' =: \{f \in F: f \restriction A < g^* \restriction A\}$ has cardinality λ and then the tree will be $\mathcal{T}'$ where $\mathcal{T}_i' =: \{f \restriction \mathfrak{b}_i: f \in Y'\}$ and $\mathcal{T}' = \bigcup_{i<\kappa} \mathcal{T}_i'$. (So actually this proves that if we have such a tree with $\ge \theta$ ($\mathrm{cf}(\theta) > 2^\kappa$) κ-branches then there is one with exactly θ κ-branches.)

6.6A Observation: (1) *If $F \subseteq \prod_{i<\kappa} \lambda_i$, J an $\aleph_1$-complete ideal on κ, and $[f \ne g \in F \Rightarrow f \ne_J g]$ and $|F| \ge \theta$, $\mathrm{cf}\,\theta > 2^\kappa$, then for some $g^* \in \prod_{i<\kappa}(\lambda_i + 1)$ we have:*

(a) *$Y = \{f \in F: f <_J g^*\}$ has cardinality θ,*

(b) *for $f' <_J g^*$, we have $|\{f \in F: f \le_J f'\}| < \theta$,*

(c) *there* are $f_\alpha \in Y$ for $\alpha < \theta$ such that: $f_\alpha <_J g^*$, $[\alpha < \beta < \theta \Rightarrow \neg f_\beta <_J f_\alpha]$.*

Proof: Let $Z =: \{g: g \in \prod_{i<\kappa}(\lambda_i + 1)$ and $Y_g =: \{f \in F: f \le_J g\}$ has cardinality $\ge \theta\}$. Clearly $\langle \lambda_i: i < \kappa\rangle \in Z$ so there is $g^* \in Z$ such that: $[g' \in Z \Rightarrow \neg g' <_J g^*]$; so (b) holds. Let $Y = \{f \in F: f <_J g^*\}$, easily $Y \subseteq Y_{g^*}$ and $|Y_{g^*} \setminus Y| \le 2^\kappa$ hence $|Y| \ge \theta$, also clearly $[f_1 \ne f_2 \in F\ \&\ f_1 \le_J f_2 \Rightarrow f_1 <_J f_2]$; if (a) fails, necessarily (by (b)) $|Y| > \theta$. For each $f \in Y$ let $Y_f = \{h \in Y: h \le_D f\}$, so $|Y_f| < \theta$ hence by the Hajnal free subset theorem for some $Z' \subseteq Z$, $|Z'| = \lambda^+$, and $f_1 \ne f_2 \in Z' \Rightarrow f_1 \notin Y_{f_2}$ so $[f_1 \ne f_2 \in Z' \Rightarrow \neg f_1 <_J f_2]$. But there is no such Z' of cardinality $> 2^\kappa$ ([Sh111, 2.2, p. 264]) so (a) holds. As for (c): choose $f_\alpha \in F$ by induction on α, such that $f_\alpha \in Y \setminus \bigcup_{\beta<\alpha} Y_{f_\beta}$; it exists by cardinality considerations and $\langle f_\alpha: \alpha < \theta\rangle$ is as required (in (c)). $\blacksquare_{6.6A}$

$\blacksquare_{6.6}$

* Or strightening clause (i) see the proof of 6.6B

6.6B OBSERVATION: *Let $\kappa < \lambda$ be regular uncountable, $2^\kappa < \mu_i < \lambda$ (for $i < \kappa$), μ_i increasing in i. The following are equivalent:*

(A) *there is $F \subseteq {}^\kappa\lambda$ such that:*
  (i) $|F| = \lambda$,
  (ii) $|\{f \restriction i : f \in F\}| \le \mu_i$,
  (iii) $[f \neq g \in F \Rightarrow f \neq_{J^{bd}_\kappa} g]$;

(B) *there be a sequence $\langle \lambda_i : i < \kappa \rangle$ such that:*
  (i) $2^\kappa < \lambda_i = \mathrm{cf}(\lambda_i) \le \mu_i$,
  (ii) $\max \mathrm{pcf}\{\lambda_i : i < \kappa\} = \lambda$,
  (iii) *for $j < \kappa$, $\mu_j \ge \max \mathrm{pcf}\{\lambda_i : i < j\}$;*

(C) *there is an increasing sequence $\langle \mathfrak{a}_i : i < \kappa \rangle$ such that $\lambda \in \mathrm{pcf} \bigcup_{i<\kappa} \mathfrak{a}_i$, $\mathrm{pcf}\, \mathfrak{a}_i \subseteq \mu_i$ (so $\mathrm{Min}(\bigcup_{i<\kappa} \mathfrak{a}_i) > |\bigcup_{i<\kappa} \mathfrak{a}_i|$).*

Proof:

(B)$\Rightarrow$(A): By [Sh355, 3.4].

(A)$\Rightarrow$(B): If $(\forall \theta)[\theta \ge 2^\kappa \Rightarrow \theta^\kappa \le \theta^+]$ we can directly prove (B) if for a club of $i < \kappa$, $\mu_i > \bigcup_{j<i} \mu_j$, and contradict (A) if this fails. Otherwise every normal filter D on κ is nice (see [Sh386, §1]). Let F exemplify (A).

Let $K = \{(D, g): D$ a normal filter on κ, $g \in {}^\kappa(\lambda + 1)$, $\lambda = |\{f \in F : f <_D g\}|\}$. Clearly K is not empty (let g be constantly λ) so by [Sh386] we can find $(D, g) \in K$ such that:

$(*)_1$ if $A \subseteq \kappa$, $A \neq \emptyset \bmod D$, $g_1 <_{D+A} g$ then $\lambda > |\{f \in F : f <_{D+A} g_1\}|$.

Let $F^* = \{f \in F : f <_D g\}$, so (as in the proof of 6.6) $|F^*| = \lambda$.

We claim:

$(*)_2$ if $h \in F^*$ then $\{f \in F^* : \neg h \le_D f\}$ has cardinality $< \lambda$.

[Why? Otherwise for some $h \in F^*$, $F' =: \{f \in F^* : \neg h \le_D f\}$ has cardinality λ, for $A \subseteq \kappa$ let $F'_A = \{f \in F^* : f \restriction A \le h \restriction A\}$ so $F' = \bigcup\{F'_A : A \subseteq \kappa, A \neq \emptyset \bmod D\}$, hence for some $A \subseteq \kappa$, $A \neq \emptyset \bmod D$ and $|F'_A| = \lambda$; now $(D + A, h)$ contradicts $(*)_1$].

By $(*)_2$ we can choose by induction on $\alpha < \lambda$, a function $f_\alpha \in F^*$ such that $\bigwedge_{\beta<\alpha} f_\beta <_D f_\alpha$. By [Sh355, 1.2A(3)] $\langle f_\alpha : \alpha < \lambda \rangle$ has an e.u.b. f^*. Let $\lambda_i = \mathrm{cf}(f^*(i))$, clearly $\{i < \kappa : \lambda_i \le 2^\kappa\} = \emptyset \bmod D$, so without loss of generality $\bigwedge_{i<\kappa} \mathrm{cf}(f^*(i)) > 2^\kappa$ so λ_i is regular $\in (2^\kappa, \lambda]$, and $\lambda = \mathrm{tcf}\left(\prod_{i<\kappa} \lambda_i / D\right)$. Let $J_i = \{A \subseteq i : \max \mathrm{pcf}\{\lambda_j : j < i\} \le \mu_i\}$; so (remembering (ii) of (A)) we can find $h_i \in \prod_{j<i} f^*(i)$ such that:

$(*)_3$ if $\{j: j < i\} \notin J_i$, then for every $f \in F$, $f \restriction i <_{J_i} h_i$.

Let $h \in \prod_{i<\kappa} f^*(i)$ be defined by: $h(i) = \sup\{h_j(i): j \in (i, \kappa)$ and $\{j: j < i\} \notin J_i\}$. As $\bigwedge_i \mathrm{cf}[f^*(i)] > 2^\kappa$, clearly $h < f^*$ hence by the choice of f^* for some $\alpha(*) < \lambda$ we have: $h <_D f_{\alpha(*)}$ and let $A =: \{i < \kappa: h(i) < f_{\alpha(*)}\}$, so $A \in D$. Define λ'_i as follows: λ'_i is λ_i if $i \in A$, and is $(2^\kappa)^+$ if $i \in \kappa \backslash A$. Now $\langle \lambda'_i: i < \kappa\rangle$ is as required in (B).

(B)$\Rightarrow$(C): Straightforward.

(C)$\Rightarrow$(B): By [Sh371, §1]. $\blacksquare_{6.6B}$

6.6C CLAIM: *If $F \subseteq {}^\kappa\mathrm{Ord}$, $2^\kappa < \theta = \mathrm{cf}(\theta) \le |F|$ then we can find $g^* \in {}^\kappa\mathrm{Ord}$ and a proper ideal I on κ and $A \subseteq \kappa$, $A \in I$ such that:*

(a) *$\prod_{i<\kappa} g^*(i)/I$ has true cofinality θ, and for each $i \in \kappa \setminus A$ we have $\mathrm{cf}[g^*(i)] > 2^\kappa$,*

(b) *for every $g \in {}^\kappa\mathrm{Ord}$ satisfying $g \restriction A = g^* \restriction A$, $g \restriction (\kappa\backslash A) < g^* \restriction (\kappa\backslash A)$ we can find $f \in F$ such that: $f \restriction A = g^* \restriction A$, $g \restriction (\kappa\backslash A) < f \restriction (\kappa\backslash A) < g^* \restriction (\kappa\backslash A)$.*

Proof: As in [Sh410, 3.7 proof of (A)$\Rightarrow$(B)]. (In short let $f_\alpha \in F$ for $\alpha < \theta$ be distinct, χ large enough, $\langle N_i: i < (2^\kappa)^+\rangle$ as there, $\delta_i =: \sup(\theta \cap N_i)$, $g_i \in {}^\kappa\mathrm{Ord}$, $g_i(\zeta) =: \mathrm{Min}\,[N \cap \mathrm{Ord} \backslash f_{\delta_i}(\zeta)]$, $A \subseteq \kappa$ and $S \subseteq \{i < (2^\kappa)^+: \mathrm{cf}(i) = \kappa^+\}$ stationary, $[i \in S \Rightarrow g_i = g^*]$, $[\zeta < \alpha \,\&\, i \in S \Rightarrow [f_{\delta_i}(\zeta) = g^*(\zeta) \equiv \zeta \in A]]$ and for some $i(*) < (2^\kappa)^+$, $g^* \in N_{i(*)}$, so $[\zeta \in \kappa \setminus A \Rightarrow \mathrm{cf}\, g^*(\zeta) > 2^\kappa]$.) $\blacksquare_{6.6C}$

6.6D CLAIM: *Suppose D is a filter on $\theta = \mathrm{cf}(\theta)$, σ-complete, $\theta > |\alpha|^\kappa$ for $\alpha < \sigma$, and for each $\alpha < \theta$, $\bar\beta = \langle \beta^\alpha_\epsilon: \epsilon < \kappa\rangle$ is a sequence of ordinals. Then for every $X \subseteq \theta$, $X \ne \emptyset \bmod D$ there is $\langle \beta^*_\epsilon: \epsilon < \kappa\rangle$ (a sequence of ordinals) and $w \subseteq \kappa$ such that:*

(a) *$\epsilon \in \kappa\backslash w \Rightarrow \sigma \le \mathrm{cf}(\beta^*_\epsilon) \le \theta$,*

(b) *if $\beta'_\epsilon \le \beta^*_\epsilon$ and $[\epsilon \in w \equiv \beta'_\epsilon = \beta^*_\epsilon]$, then $\{\alpha \in X$: for every $\epsilon < \kappa$ we have $\beta'_\epsilon \le \beta^\alpha_\epsilon \le \beta^*_\epsilon$ and $[\epsilon \in w \equiv \beta^\alpha_\epsilon = \beta^*_\epsilon]\} \ne \emptyset \bmod D$.*

Proof: Essentially by the same proof as 6.6C (replacing δ_i by $\mathrm{Min}\{\alpha \in X$: for every $Y \in N_i \cap D$ we have $\alpha \in Y\}$). See more [Sh513, §6]. $\blacksquare_{6.6D}$

6.6E Remark: We can rephrase the conclusion as:

(a) $B =: \{\alpha \in X$: if $\epsilon \in w$ then $\beta^\alpha_\epsilon = \beta^*_\epsilon$, and: if $\epsilon \in \kappa \setminus w$ then β^α_ϵ is $< \beta^*_\epsilon$ but $> \sup\{\beta^*_\zeta: \zeta < \epsilon, \beta^\alpha_\zeta < \beta^*_\epsilon\}\}$ is $\ne \emptyset \bmod D$.

(b) If $\beta'_\epsilon < \beta_\epsilon$ for $\epsilon \in \kappa \setminus w$ then $\{\alpha \in B$: if $\epsilon \in \kappa \setminus w$ then $\beta^\alpha_\epsilon > \beta'_\epsilon\} \ne \emptyset \bmod D$.

(c) $\epsilon \in \kappa \setminus w \Rightarrow \mathrm{cf}(\beta'_\epsilon)$ is $\leq \theta$ but $\geq \sigma$.

6.6F Remark: (1) If $|\mathfrak{a}| < \min(\mathfrak{a})$, $F \subseteq \Pi\mathfrak{a}$, $|F| = \theta = \mathrm{cf}\,\theta \notin \mathrm{pcf}(\mathfrak{a})$ and even $\theta > \sigma = \sup(\theta^+ \cap \mathrm{pcf}(\mathfrak{a}))$ then for some $g \in \Pi\mathfrak{a}$, the set $\{f \in F\colon f < g\}$ is unbounded in θ (or use a σ-complete D as in 6.6E). (This is as $\Pi\mathfrak{a}/J_{<\theta}[\mathfrak{a}]$ is $\min(\mathrm{pcf}(\mathfrak{a}) \setminus \theta)$-directed as the ideal $J_{<\theta}[\mathfrak{a}]$ is generated by $\leq \sigma$ sets; this is discussed in [Sh513, §6].)

6.6G Remark: It is useful to note that 6.6D is useful to use [Sh462, §4, 5.14]: e.g. for if $n < \omega$, $\theta_0 < \theta_1 < \cdots < \theta_n$, satisfying $(*)$ below, for any $\beta'_\epsilon \leq \beta^*_\epsilon$ satisfying $[\epsilon \in w \equiv \beta'_\epsilon < \beta^*_\epsilon]$ we can find $\alpha < \gamma$ in X such that:

$$i \in w \equiv \beta^\alpha_\epsilon = \beta^*_\epsilon,$$

$$\{\epsilon, \zeta\} \subseteq \kappa \setminus w \ \&\ \{\mathrm{cf}(\beta^*_\epsilon), \mathrm{cf}(\beta^*_\zeta)\} \subseteq [\theta_l, \theta_{l+1})) \ \&\ l \text{ even } \Rightarrow \beta^\alpha_\epsilon < \beta^\gamma_\zeta,$$

$$\{\epsilon, \zeta\} \subseteq \kappa \setminus w \ \&\ \{\mathrm{cf}(\beta^*_\zeta), \mathrm{cf}(\beta^*_\zeta)\} \subseteq [\theta_l, \theta_{l+1}) \ \&\ l \text{ odd } \Rightarrow \beta^\gamma_\epsilon < \beta^\alpha_\zeta$$

where

$(*)$ (a) $\epsilon \in \kappa \setminus w \Rightarrow \mathrm{cf}(\beta^*_\epsilon) \in [\theta_0, \theta_n)$, and

(b) $\max\mathrm{pcf}[\{\mathrm{cf}(\beta^*_\epsilon)\colon \epsilon \in \kappa \setminus w\} \cap \theta_l] \leq \theta_l$ (which holds if $\theta_l = \sigma_l^+$, $\sigma_l^\kappa = \sigma_l$ for $l \in \{1, \ldots, n\}$).

6.7 Claim: *For any $\mathfrak{a}$, $|\mathfrak{a}| < \mathrm{Min}(\mathfrak{a})$, we can find $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\lambda\colon \lambda \in \mathfrak{a}\rangle$ such that:*

(α) *$\bar{\mathfrak{b}}$ is a generating sequence, i.e.*

$$\lambda \in \mathfrak{a} \Rightarrow J_{\leq\lambda}[\mathfrak{a}] = J_{<\lambda}[\mathfrak{a}] + \mathfrak{b}_\lambda,$$

(β) *$\bar{\mathfrak{b}}$ is smooth, i.e. for $\theta < \lambda$ in $\mathfrak{a}$,*

$$\theta \in \mathfrak{b}_\lambda \Rightarrow \mathfrak{b}_\theta \subseteq \mathfrak{b}_\lambda,$$

(γ) *$\bar{\mathfrak{b}}$ is closed, i.e. for $\lambda \in \mathrm{pcf}(\mathfrak{a})$ we have $\mathfrak{b}_\lambda = \mathfrak{a} \cap \mathrm{pcf}(\mathfrak{b}_\lambda)$.*

Proof: Let $\langle \mathfrak{b}_\theta[\mathfrak{a}]\colon \theta \in \mathrm{pcf}\,\mathfrak{a}\rangle$ be as in [Sh371, 2.6]. For $\lambda \in \mathfrak{a}$, let $\bar{f}^{\mathfrak{a},\lambda} = \langle f^{\mathfrak{a},\lambda}_\alpha\colon \alpha < \mathfrak{a}\rangle$ be a $<_{J_\lambda[\mathfrak{a}]}$-increasing cofinal sequence of members of $\prod \mathfrak{a}$, satisfying:

$(*)_1$ if $\delta < \lambda$, $|\mathfrak{a}| < \mathrm{cf}(\delta) < \mathrm{Min}\,\mathfrak{a}$ and $\theta \in \mathfrak{a}$ then:

$$f^{\mathfrak{a},\lambda}_\delta(\theta) = \mathrm{Min}\left\{\bigcup_{\alpha \in C} f^{\mathfrak{a},\lambda}_\alpha(\theta)\colon C \text{ a club of } \delta\right\}$$

[exists by [Sh345a, Def. 3.3(2)^b + Fact 3.4(1)]].

Let $\chi = \beth_\omega(\sup \mathfrak{a})^+$, $|\mathfrak{a}| < \kappa = \mathrm{cf}\,\kappa < \mathrm{Min}\,\mathfrak{a}$ (without loss of generality there is such κ) and $\bar{N} = \langle N_i \colon i < \kappa \rangle$ be an increasing continuous sequence of elementary submodels of $(H(\chi), \in, <^*_\chi)$, $N_i \cap \kappa$ an ordinal, $\bar{N} \restriction (i+1) \in N_{i+1}$, $\|N_i\| < \kappa$, and $\mathfrak{a}$, $\langle \bar{f}^{\mathfrak{a},\lambda} \colon \lambda \in \mathfrak{a} \rangle$ belong to N_0. Let $N_\kappa = \bigcup_{i<\kappa} N_i$. For every $\lambda \in \mathfrak{a}$, for some club E_λ of κ,

$(*)$ $\theta \in \mathfrak{a} \Rightarrow f^{\mathfrak{a},\lambda}_{\sup(N_\kappa \cap \lambda)}(\theta) = \bigcup_{\alpha \in E_\lambda} f^{\mathfrak{a},\lambda}_{\sup(N_\alpha \cap \lambda)}(\theta)$.

Let $E = \bigcap_{\lambda \in \mathfrak{a}} E_\lambda$, so E is a club of κ. For any $i < j < \kappa$ let

$$\mathfrak{b}^{i,j}_\lambda = \left\{ \theta \in \mathfrak{a} \colon \sup(N_i \cap \theta) < f^{\mathfrak{a},\lambda}_{\sup(N_j \cap \lambda)}(\theta) \right\}.$$

As in the proof of [Sh371, 1.3], possibly shrinking E, we have:

$(*)_2$ for $i < j$ from* E and $\lambda \in \mathfrak{a}$, we have:

- (α) $J_{\leq\lambda}[\mathfrak{a}] = J_{<\lambda}[\mathfrak{a}] + \mathfrak{b}^{i,j}_\lambda$ (hence $\mathfrak{b}^{i,j}_\lambda = \mathfrak{b}_\lambda[\mathfrak{a}] \bmod J_{<\lambda}[\mathfrak{a}]$),
- (β) $\mathfrak{b}^{i,j}_\lambda \subseteq \lambda^+ \cap \mathfrak{a}$,
- (γ) $\langle \mathfrak{b}^{i,j}_\lambda \colon \lambda \in \mathfrak{a} \rangle \in N_{j+1}$,
- (δ) $f^{\mathfrak{a},\lambda}_{\sup(N_\kappa \cap \lambda)} \restriction \mathfrak{b}^{i,j}_\lambda = \langle (\theta, \sup(N_\kappa \cap \theta)) \colon \theta \in \mathfrak{b}^{i,j}_\lambda \rangle$,
- (ϵ) $f^{\mathfrak{a},\lambda}_{\sup(N_\kappa \cap \lambda)} \leq \langle (\theta, \sup(N_\kappa \cap \theta)) \colon \theta \in \mathfrak{a} \rangle$.

We now define by induction on $\epsilon < |\mathfrak{a}|^+$, for $\lambda \in \mathfrak{a}$ (and $i < j < \kappa$), the set $\mathfrak{b}^{i,j,\epsilon}_\lambda$:

$$\mathfrak{b}^{i,j,0}_\lambda = \mathfrak{b}^{i,j}_\lambda$$

$$\mathfrak{b}^{i,j,\epsilon+1}_j = \mathfrak{b}^{i,j,\epsilon}_\lambda \cup \bigcup \left\{ \mathfrak{b}^{i,j,\epsilon}_\theta \colon \theta \in \mathfrak{b}^{i,j,\epsilon}_\lambda \right\} \cup \left\{ \theta \in \mathfrak{a} \colon \theta \in \mathrm{pcf}\, \mathfrak{b}^{i,j,\epsilon} \right\},$$

$$\mathfrak{b}^{i,j,\epsilon}_\lambda = \bigcup_{\zeta<\epsilon} \mathfrak{b}^{i,j,\zeta}_\lambda \quad \text{for } \epsilon < |\mathfrak{a}|^+ \text{ limit.}$$

Clearly for $\lambda \in \mathfrak{a}$, $\langle \mathfrak{b}^{i,j,\epsilon}_\lambda \colon \epsilon < |\mathfrak{a}|^+ \rangle$ belongs to N_{j+1} and is a non-decreasing sequence of subsets of $\mathfrak{a}$, hence for some $\epsilon(i,j,\lambda) < |\mathfrak{a}|^+$,

$$\left[\epsilon \in (\epsilon(i,j,\lambda), |\mathfrak{a}|^+) \Rightarrow \mathfrak{b}^{i,j,\epsilon}_\lambda = \mathfrak{b}^{i,j,\epsilon(i,j,\lambda)}_\lambda \right].$$

So letting $\epsilon(i,j) = \sup_{\lambda \in \mathfrak{a}} \epsilon(i,j,\lambda) < |\mathfrak{a}|^+$ we have:

$(*)_3$ $\epsilon(i,j) \leq \epsilon < |\mathfrak{a}|^+ \Rightarrow \bigwedge_{\lambda \in \mathfrak{a}} \mathfrak{b}^{i,j,\epsilon(i,j)}_\lambda = \mathfrak{b}^{i,j,\epsilon}_\lambda$.

Which of the properties required from $\langle \mathfrak{b}_\lambda \colon \lambda \in \mathfrak{a} \rangle$ are satisfied by $\langle \mathfrak{b}^{i,j,\epsilon(i,j)}_\lambda \colon \lambda \in \mathfrak{a} \rangle$? Note ($\beta$), ($\gamma$) hold by the inductive definition of $\mathfrak{b}^{i,j,\epsilon}_\lambda$ (and the choice of $\epsilon(i,j)$), as for property (α), one half, $J_{\leq\lambda}[\mathfrak{a}] \subseteq J_{<\lambda}[\mathfrak{a}] + \mathfrak{b}^{i,j,\epsilon(i,j)}_\lambda$ hold by $(*)_2(\alpha)$ (and $\mathfrak{b}^{i,j}_\lambda = \mathfrak{b}^{i,j,0}_\lambda \subseteq \mathfrak{b}^{i,j,\epsilon(i,j)}_\lambda$), so it is enough to prove (for $\lambda \in \mathfrak{a}$) :

* Actually for any $i < j < \kappa$ clauses (β), (γ), (δ) hold.

$(*)_4$ $\mathfrak{b}_\lambda^{i,j,\epsilon(i,j)} \in J_{\leq\lambda}[\mathfrak{a}]$.

For this end we define by induction on $\epsilon < |\mathfrak{a}|^+$ functions $f_\alpha^{\mathfrak{a},\lambda,\epsilon}$ with domain $\mathfrak{b}_\lambda^{i,j,\epsilon}$ for every $\alpha < \lambda \in \mathfrak{a}$, such that $\zeta < \epsilon \Rightarrow f_\alpha^{\mathfrak{a},\lambda,\zeta} \subseteq f_\alpha^{\mathfrak{a},\lambda,\epsilon}$, so the domain increases with ϵ.

We let $f_\alpha^{\mathfrak{a},\lambda,0} = f_\alpha^{\mathfrak{a},\lambda} \restriction \mathfrak{b}_\lambda^{i,j}$, $f_\alpha^{\mathfrak{a},\lambda,\zeta} = \bigcup_{\zeta<\epsilon} f_\alpha^{\mathfrak{a},\lambda,\zeta}$ for $\epsilon < |\mathfrak{a}|^+$ limit, and $f_\alpha^{\mathfrak{a},\lambda,\epsilon+1}$ is defined by defining each $f_\alpha^{\mathfrak{a},\lambda,\epsilon+1}(\theta)$ as follows:

Case 1: If $\theta \in \mathfrak{b}_\lambda^{i,j,\epsilon}$ then $f_\alpha^{\mathfrak{a},\lambda,\epsilon}(\theta)$.

Case 2: If $\mu \in \mathfrak{b}_\lambda^{i,j,\epsilon}$, $\theta \in \mathfrak{b}_\mu^{i,j,\epsilon}$ and not Case 1 and μ minimal under those conditions, then $f_\beta^{\mathfrak{a},\mu,\epsilon}(\theta)$ where we choose $\beta = f_\alpha^{\mathfrak{a},\lambda,\epsilon}(\mu)$.

Case 3: If $\theta \in \mathfrak{a} \cap \mathrm{pcf}(\mathfrak{b}_\lambda^{i,j,\epsilon})$ and not Case 1 or 2, then

$$\mathrm{Min}\left\{\gamma < \theta\colon f_\alpha^{\mathfrak{a},\lambda,\epsilon} \restriction \mathfrak{b}_\theta[\mathfrak{a}] \leq_{J_{<\theta}[\mathfrak{a}]} f_\gamma^{\mathfrak{a},\theta,\epsilon}\right\}.$$

Now $\langle\langle \mathfrak{b}_\lambda^{i,j,\epsilon}\colon \lambda \in \mathfrak{a}\rangle\colon \epsilon < |\mathfrak{a}|^+\rangle$ can be computed from $\mathfrak{a}$ and $\langle \mathfrak{b}_\lambda^{i,j}\colon \lambda \in \mathfrak{a}\rangle$. But the latter belong* to N_{j+1}, so the former belongs to N_{j+1}, so as also $\langle\langle f_\alpha^{\mathfrak{a},\lambda}\colon \alpha < \lambda\rangle\colon \lambda \in \mathrm{pcf}\,\mathfrak{a}\rangle$ belongs to N_{j+1} we clearly get that

$$\langle\langle\langle f_\alpha^{\mathfrak{a},\lambda,\epsilon}\colon \epsilon < |\mathfrak{a}|^+\rangle\colon \alpha < \lambda\rangle\colon \lambda \in \mathfrak{a}\rangle$$

belongs to N_{j+1}. Next we prove by induction on ϵ that, for $\lambda \in \mathfrak{a}$, we have:

$$\otimes_1 \qquad \theta \in \mathfrak{b}_\lambda^{i,j,\epsilon} \;\&\; \lambda \in \mathfrak{a} \Rightarrow f^{\mathfrak{a},\lambda,\epsilon}_{\sup(N_\kappa\cap\theta)}(\theta) = \sup(N_\kappa \cap \theta).$$

For $\epsilon = 0$ this is by $(*)_2(\delta)$. For ϵ limit, by the induction hypothesis and the definition of $f_\alpha^{\mathfrak{a},\lambda,\epsilon}$. For $\epsilon + 1$, we check $f^{\mathfrak{a},\lambda,\epsilon+1}_{\sup(N_\kappa\cap\lambda)}(\theta)$ according to the case in its definition; for Case 1 use the induction hypothesis applied to $f^{\mathfrak{a},\lambda,\epsilon}_{\sup(N_\kappa\cap\lambda)}$. For Case 2 (with μ), by the induction hypothesis applied to $f^{\mathfrak{a},\mu,\epsilon}_{\sup(N_\kappa\cap\mu)}$. Lastly, for Case 3 (with θ) we should note:

(i) $\mathfrak{b}_\lambda^{i,j,\epsilon} \cap \mathfrak{b}_\theta[\mathfrak{a}] \notin J_{<\theta}[\mathfrak{a}]$ (by the case's assumption and $(*)_2(\alpha)$ above),

(ii) $f^{\mathfrak{a},\lambda,\epsilon}_{\sup(N_\kappa\cap\lambda)} \restriction (\mathfrak{b}_\lambda^{i,j,\epsilon} \cap \mathfrak{b}_\theta^{i,j,\epsilon}) \subseteq f^{\mathfrak{a},\theta,\epsilon}_{\sup(N_\kappa\cap\theta)}$ (by the induction hypothesis for ϵ, used concerning λ and θ) hence (by the definition in case 3 and (i) + (ii)),

(iii) $f^{\mathfrak{a},\lambda,\epsilon+1}_{\sup(N_\kappa\cap\lambda)}(\theta) \leq \sup(N_\kappa \cap \theta)$.

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* As $\langle \mathfrak{b}_\lambda^{i,j,\epsilon} : \lambda \in \mathfrak{a}\rangle : \epsilon|\mathfrak{a}|^+\rangle$ is eventually constant, also each member of the sequence belongs to N_{j+1}.

Now if $\gamma < \sup(N_\kappa \cap \theta)$ then for some $\gamma(1)$, $\gamma < \gamma(1) \in N_\kappa \cap \theta$, so letting $\mathfrak{b} =: \mathfrak{b}^{i,j,\epsilon}_\lambda \cap \mathfrak{b}_\theta[\mathfrak{a}] \cap \mathfrak{b}^{i,j,\epsilon}_\theta$, it belongs to $J_{\leq\theta}[\mathfrak{a}] \setminus J_{<\theta}[\mathfrak{a}]$, we have

$$f^{\mathfrak{a},\theta}_\gamma \restriction \mathfrak{b} <_{J_{<\theta}[\mathfrak{a}]} f^{\mathfrak{a},\theta}_{\gamma(1)} \restriction \mathfrak{b} \leq f^{\mathfrak{a},\theta,\epsilon}_{\sup(N_\kappa\cap\theta)}$$

hence $f^{\mathfrak{a},\lambda,\epsilon+1}_{\sup(N_\kappa\cap\lambda)}(\theta) > \gamma$; as this holds for every $\gamma < \sup(N_\kappa \cap \theta)$ we have obtained

(iv) $f^{\mathfrak{a},\lambda,\epsilon+1}_{\sup(N_\kappa\cap\lambda)}(\theta) \geq \sup(N_\kappa \cap \theta)$;

together we have finished proving the inductive step for $\epsilon + 1$, hence we have proved $\otimes_1$.

This is enough for proving $\mathfrak{b}^{i,j,\epsilon}_\lambda \in J_{\leq\lambda}[\mathfrak{a}]$: Why? If it fails, as $\mathfrak{b}^{i,j,\epsilon}_\lambda \in N_{j+1}$ and $\langle f^{\mathfrak{a},\lambda,\epsilon}_\alpha : \alpha < \lambda\rangle$ belongs to N_{j+1}, there is $g \in \prod \mathfrak{b}^{i,j,\epsilon}_\lambda$ s.t.

$$(*) \qquad \alpha < \lambda \Rightarrow f^{\mathfrak{a},\lambda,\epsilon}_\alpha \restriction \mathfrak{b}^{i,j,\epsilon} < g \bmod J_{\leq\lambda}[\mathfrak{a}].$$

Wlog $g \in N_{j+1}$; by $(*)$, $f^{\mathfrak{a},\lambda,\epsilon}_{\sup(N_\kappa\cap\lambda)} < g \bmod J_{\leq\lambda}[\mathfrak{a}]$. But $g < \langle \sup(N_\kappa \cap \theta) : \theta \in \mathfrak{b}^{i,j,\epsilon}_\lambda\rangle$. Together this contradicts $\oplus_1$!

This ends the proof of 6.7. $\blacksquare_{6.7}$

6.7A Claim: *Assume $|\mathfrak{a}| < \kappa = \mathrm{cf}(\kappa) < \mathrm{Min}(\mathfrak{a})$, σ an infinite ordinal, $|\sigma|^+ < \kappa$. Let $\bar{f}$, $\bar{N} = \langle N_i : i < \kappa\rangle$, N_κ be as in the proof of 6.7. Then we can find $\bar{i} = \langle i_\alpha : \alpha \leq \sigma\rangle$, $\bar{\mathfrak{a}} = \langle \mathfrak{a}_\alpha : \alpha < \sigma\rangle$ and $\langle\langle \mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}] : \lambda \in \mathfrak{a}_\beta\rangle : \beta < \sigma\rangle$ such that:*

(a) *$\bar{i}$ is a strictly increasing continuous sequence of ordinals $< \kappa$,*

(b) *for $\beta < \sigma$ we have $\langle i_\alpha : \alpha \leq \beta\rangle \in N_{i_{\beta+1}}$ (hence* $\langle N_{i_\alpha} : \alpha \leq \beta\rangle \in N_{i_{\beta+1}}$) and $\langle \mathfrak{b}^\gamma_\lambda[\bar{\mathfrak{a}}] : \lambda \in \mathfrak{a}_\gamma$ and $\gamma \leq \beta\rangle \in N_{i_{\beta+1}}$,*

(c) *$\mathfrak{a}_\beta = N_{i_\beta} \cap \mathrm{pcf}(\mathfrak{a})$, so $\mathfrak{a}_\beta$ is increasing continuous in β, $\mathfrak{a} \subseteq \mathfrak{a}_\beta \subseteq \mathrm{pcf}\,\mathfrak{a}$, $|\mathfrak{a}_\beta| < \kappa$,*

(d) *$\mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}] \subseteq \mathfrak{a}_\beta$ (for $\lambda \in \mathfrak{a}_\beta$),*

(e) *$J_{\leq\lambda}[\mathfrak{a}_\beta] = J_{<\lambda}[\mathfrak{a}_\beta] + \mathfrak{b}^\beta_\lambda[\mathfrak{a}]$ (so $\lambda \in \mathfrak{b}_\lambda[\mathfrak{a}]$ and $\mathfrak{b}_\lambda[\mathfrak{a}] \subseteq \lambda^+$),*

(f) *if $\mu < \lambda$ are in $\mathfrak{a}_\beta$ and $\mu \in \mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}]$ then $\mathfrak{b}^\beta_\mu[\bar{\mathfrak{a}}] \subseteq \mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}]$ (i.e. smoothness),*

(g) *$\mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}] = \mathfrak{a}_\beta \cap \mathrm{pcf}\,\mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}]$ (i.e. closedness),*

(h) *if $\mathfrak{c} \subseteq \mathfrak{a}_\beta$, $\beta < \sigma$, $\mathfrak{c} \in N_{i_{\beta+1}}$ then for some finite $\mathfrak{d} \subseteq \mathfrak{a}_{\beta+1} \cap \mathrm{pcf}(\mathfrak{c})$, we have $\mathfrak{c} \subseteq \bigcup_{\mu\in\mathfrak{d}} \mathfrak{b}^{\beta+1}_\mu[\bar{\mathfrak{a}}]$; more generally,***

(h)$^+$ *if $\mathfrak{c} \subseteq \mathfrak{a}_\beta$, $\beta < \sigma$, $\mathfrak{c} \in N_{i_{\beta+1}}$, $\theta = \mathrm{cf}(\theta) \in N_{i_{\beta+1}}$, then for some $\mathfrak{d} \in N_{i_{\beta+1}}$, $\mathfrak{d} \subseteq \mathfrak{a}_{\beta+1} \cap \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{c})$ we have $\mathfrak{c} \subseteq \bigcup_{\mu\in\mathfrak{d}} \mathfrak{b}^{\beta+1}_\mu[\bar{\mathfrak{a}}]$ and $|\mathfrak{d}| < \theta$,*

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* We can get $\bar{i} \restriction (\beta + 1) \in N_{i_\beta+1}$ if κ succesor of regular and $\bar{C}$ a square later.

** If in (h)$^+$, $\theta = \aleph_0$, we get (h).

(i) $\mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}]$ *increases with* β.

This will be proved below.

6.7B Claim: *In 6.7A we can also have:*

(1) *if we let* $\mathfrak{b}_\lambda[\bar{\mathfrak{a}}] = \mathfrak{b}^\sigma_\lambda[\mathfrak{a}] = \bigcup_{\beta<\sigma} \mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}]$, $\mathfrak{a}_\sigma = \bigcup_{\beta<\sigma} \mathfrak{a}_\beta$ *then also for* $\beta = \sigma$ *we have* (b) *(use* $N_{i_\beta+1}$*)*, (c), (d), (f), (i).

(2) *If* $\sigma = \mathrm{cf}(\sigma) > |\mathfrak{a}|$ *then for* $\beta = \sigma$ *also* (e), (g).

(3) *If* $\mathrm{cf}(\sigma) > |\mathfrak{a}|$, $\mathfrak{c} \in N_{i_\sigma}$, $\mathfrak{c} \subseteq \mathfrak{a}_\sigma$ *(hence* $|\mathfrak{c}| < \mathrm{Min}(\mathfrak{c})$ *and* $\mathfrak{c} \subseteq \mathfrak{a}_\sigma$*), then for some finite* $\mathfrak{d} \subseteq (\mathrm{pcf}\, \mathfrak{c}) \cap \mathfrak{a}_\sigma$ *we have* $\mathfrak{c} \subseteq \bigcup_{\mu\in\mathfrak{d}} \mathfrak{b}_\mu[\bar{\mathfrak{a}}]$. *Similarly for* θ*-complete,* $\theta < \mathrm{cf}(\sigma)$ *(i.e. we have clauses* (h), (h)$^+$ *for* $\beta = \sigma$*).*

(4) *We can have continuity in* $\delta \le \sigma$ *when* $\mathrm{cf}(\delta) > |\mathfrak{a}|$, *i.e.* $\mathfrak{b}^\delta_\lambda = \bigcup_{\beta<\delta} \mathfrak{b}^\beta_\lambda$.

6.7C Remark:

(1) If we want to use length κ, use $\bar{N}$ as produced in [Sh420, 2.6] so $\sigma = \kappa$.

(2) Concerning 6.7B, in 6.7C(1) for a club E of $\sigma = \kappa$, we have $\alpha \in E \Rightarrow \mathfrak{b}^\alpha_\lambda[\bar{\mathfrak{a}}] = \mathfrak{b}_\lambda[\bar{\mathfrak{a}}] \cap \mathfrak{a}_\alpha$.

(3) We can also use 6.7 (6.7A, 6.7B) to give an alternative proof of part of the localization theorems similar to the one given in the Spring '89 lectures. For example:

(3A) If $|\mathfrak{a}| < \theta = \mathrm{cf}\,\theta < \mathrm{Min}(\mathfrak{a})$, for no $\lambda_i \in \mathrm{pcf}\,\mathfrak{a}$ $(i < \theta)$ $\alpha < \theta$, do we have $\bigwedge_{\alpha<\theta}[\lambda_\alpha > \max\mathrm{pcf}\{\lambda_i : i < \alpha\}]$.

(3B) if $|\mathfrak{a}| < \mathrm{Min}(\mathfrak{a})$, $|\mathfrak{b}| < \mathrm{Min}\,\mathfrak{b}$, $\mathfrak{b} \subseteq \mathrm{pcf}(\mathfrak{a})$, $\lambda \in \mathrm{pcf}(\mathfrak{a})$, then for some $\mathfrak{c} \subseteq \mathfrak{b}$ we have $|\mathfrak{c}| \le |\mathfrak{a}|$ and $\lambda \in \mathrm{pcf}(\mathfrak{c})$.

Proof of (3A) from 6.7C(3): Without loss of generality $\mathrm{Min}\,\mathfrak{a} > \theta^{+3}$, let $\kappa = \theta^{+2}$, let $\bar{N}$, N_κ, $\bar{\mathfrak{a}}$, $\mathfrak{b}$ (as a function), $\langle i_\alpha : \alpha \le \sigma =: |\mathfrak{a}|^+\rangle$ be as in 6.7A but also $\langle \lambda_i : i < \theta\rangle \in N_0$. So for $j < \theta$, $\mathfrak{c}_j =: \{\lambda_i : i < j\} \in N_0$ (and $\mathfrak{c}_j \subseteq \mathfrak{a}_0$) hence (by clause (h) of 6.7A), for some finite $\mathfrak{d}_j \subseteq \mathfrak{a}_1 \cap \mathrm{pcf}\,\mathfrak{c}_j = N_{i_1} \cap \mathrm{pcf}\,\mathfrak{a} \cap \mathrm{pcf}\,\mathfrak{c}_j$ we have $\mathfrak{c}_j \subseteq \bigcup_{\lambda\in\mathfrak{d}_j} \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}]$. Assume $j(1) < j(2) < \theta$. Now if $\mu \in \mathfrak{a} \cap \bigcup_{\lambda\in\mathfrak{d}_{j(1)}} \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}]$ then for some $\mu_0 \in \mathfrak{d}_{j(1)}$ we have $\mu \in \mathfrak{b}^1_{\mu_0}[\bar{\mathfrak{a}}]$; now $\mu_0 \in \mathfrak{d}_{j(1)} \subseteq \mathrm{pcf}(\mathfrak{c}_{j(1)}) \subseteq \mathrm{pcf}(\mathfrak{c}_{j(2)}) \subseteq \mathrm{pcf}\left(\bigcup_{\lambda\in\mathfrak{d}_{j(2)}} \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}]\right) = \bigcup_{\lambda\in\mathfrak{d}_{j(2)}} \mathrm{pcf}(\mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}])$ hence (by clause (g) of 6.7A as $\mu_0 \in \mathfrak{d}_{j(0)} \subseteq N_1$) for some $\mu_1 \in \mathfrak{d}_{j(2)}$, $\mu_0 \in \mathfrak{b}^1_{\mu_1}[\bar{\mathfrak{a}}]$. So by clause (f) of 6.7A we have $\mathfrak{b}^1_{\mu_0}[\bar{\mathfrak{a}}] \subseteq \mathfrak{b}^1_{\mu_1}[\bar{\mathfrak{a}}]$ so remembering $\mu \in \mathfrak{b}^1_{\mu_0}[\bar{\mathfrak{a}}]$, we have $\mu \in \mathfrak{b}^1_{\mu_1}[\bar{\mathfrak{a}}]$. Remembering μ was any member of $\mathfrak{a} \cap \bigcup_{\lambda\in\mathfrak{d}_{j(1)}} \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}]$, we have $\mathfrak{a} \cap \bigcup_{\lambda\in\mathfrak{d}j(1)} \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}] \subseteq \mathfrak{a} \cap \bigcup_{\lambda\in\mathfrak{d}j(2)} \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}]$ (holds without "$\mathfrak{a}\cap$" but not used). So $\langle \mathfrak{a} \cap \bigcup_{\lambda\in\mathfrak{d}_j} \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}] : j < \theta\rangle$ is a non-decreasing sequence of subsets of $\mathfrak{a}$, but $\mathrm{cf}(\theta) > |\mathfrak{a}|$, so the sequence is eventually constant, say for $j \geq j(*)$. But

$$\begin{aligned}\max\mathrm{pcf}\left(\mathfrak{a}\cap\bigcup_{\lambda\in\mathfrak{d}_j}\mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}]\right) &\leq \max\mathrm{pcf}\left(\bigcup_{\lambda\in\mathfrak{d}_j}\mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}]\right)\\ &= \max_{\lambda\in\mathfrak{d}_j}\left(\max\mathrm{pcf}(\mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}])\right)\\ &= \max_{\lambda\in\mathfrak{d}_j}\lambda \leq \max\mathrm{pcf}\{\lambda_i: i<j\} < \lambda_j\\ &= \max\mathrm{pcf}\left(\mathfrak{a}\cap\bigcup_{\lambda\in\mathfrak{d}_{j+1}}\mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}]\right)\end{aligned}$$

(last equality as $\mathfrak{b}_{\lambda_j}[\mathfrak{a}] \subseteq \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}] \bmod J_{<\lambda}[\mathfrak{a}_1]$). Contradiction. $\blacksquare_{6.7C}$

Proof of 6.7C(3B) (like [Sh371, §3]): Included for completeness. If this fails choose a counterexample $(\mathfrak{a},\mathfrak{b},\lambda)$ with $|\mathfrak{b}|$ minimal, and among those with $\max\mathrm{pcf}(\mathfrak{b})$ minimal and among those with $\bigcup\{\mu^+: \mu\in\lambda\cap\mathrm{pcf}(\mathfrak{b})\}$ minimal. So $\max\mathrm{pcf}(\mathfrak{b})=\lambda$, and $\mu=\sup[\lambda\cap\mathrm{pcf}(\mathfrak{a})]$ is not in $\mathrm{pcf}(\mathfrak{b})$ or $\mu=\lambda$. Try to choose by induction on $i<|\mathfrak{a}|^+$, $\lambda_i\in\lambda\cap\mathrm{pcf}(\mathfrak{b})$, $\lambda_i>\max\mathrm{pcf}\{\lambda_j: j<i\}$, by 6.7C(3A), we will be stuck at some i, and by the previous sentence (and choice of $(\mathfrak{a},\mathfrak{b},\lambda)$, i is limit, so $\mathrm{pcf}(\{\lambda_j: j<i\})\not\subseteq\lambda$ but it is $\subseteq\mathrm{pcf}(\mathfrak{b})\subseteq\lambda^+$, so $\lambda=\max\mathrm{pcf}\{\lambda_j: j<i\}$. For each j, by the minimality condition for some $\mathfrak{b}_j\subseteq\mathfrak{b}$, we have $|\mathfrak{b}_j|\leq|\mathfrak{a}|$, $\lambda_j\in\mathrm{pcf}(\mathfrak{b}_j)$. So $\lambda\in\mathrm{pcf}\{\lambda_j: j<i\}\subseteq\mathrm{pcf}(\bigcup_{j<i}\mathfrak{b}_j)$ but $\bigcup_{j<i}\mathfrak{b}_j$ is a subset of $\mathfrak{b}$ of cardinality $\leq|i|\times|\mathfrak{a}|=|\mathfrak{a}|$.

6.7D Proof of 6.7A: Let $\langle\langle f^{\mathfrak{a},\lambda}_\alpha: \alpha<\lambda\rangle: \lambda\in\mathrm{pcf}\,\mathfrak{a}\rangle$ be chosen as in the proof of 6.7. For $\zeta<\kappa$ we define $\mathfrak{a}^\zeta =: N_\zeta\cap\mathrm{pcf}\,\mathfrak{a}$; we also define ${}^\zeta\bar{f}$ as $\langle\langle f^{\mathfrak{a}^\zeta,\lambda}_\alpha: \alpha<\lambda\rangle: \lambda\in\mathrm{pcf}\,\mathfrak{a}\rangle$ where $f^{\mathfrak{a}^\zeta,\lambda}_\alpha\in\prod\mathfrak{a}^\zeta$ is defined as follows:

(a) if $\theta\in\mathfrak{a}$, $f^{\mathfrak{a}^\zeta,\lambda}_\alpha(\theta)=f^{\mathfrak{a},\lambda}_\alpha(\theta)$,

(b) if $\theta\in\mathfrak{a}^\zeta\setminus\mathfrak{a}$ and $\mathrm{cf}(\alpha)\notin(|\mathfrak{a}^\zeta|,\mathrm{Min}\,\mathfrak{a})$, then

$$f^{\mathfrak{a}^\zeta,\lambda}_\alpha(\theta)=\mathrm{Min}\left\{\gamma<\theta: f^{\mathfrak{a},\lambda}_\alpha\restriction\mathfrak{b}_\theta[\mathfrak{a}]\leq_{J_{<\theta}[\mathfrak{b}_\theta[\mathfrak{a}]]}f^{\mathfrak{a},\theta}_\gamma\restriction\mathfrak{b}_\theta[\mathfrak{a}]\right\},$$

(c) if $\theta\in\mathfrak{a}^\zeta\setminus\mathfrak{a}$ and $\mathrm{cf}(\alpha)\in(|\mathfrak{a}^\zeta|,\mathrm{Min}\,\mathfrak{a})$, define $f^{\mathfrak{a}^\zeta,\lambda}_\alpha(\theta)$ so as to satisfy $(*)_1$ in the proof of 6.7.

Now ${}^\zeta\bar{f}$ is legitimate except that we have only

$$\beta<\gamma<\lambda\in\mathrm{pcf}\,\mathfrak{a}\Rightarrow f^{\mathfrak{a}^\zeta,\lambda}_\beta\leq f^{\mathfrak{a}^\zeta,\lambda}_\gamma \bmod J_{<\lambda}[\mathfrak{a}^\zeta]$$

(instead of strict inequality) and $\bigwedge_{\beta<\lambda} \bigvee_{\gamma<\lambda} \left[f_\beta^{\mathfrak{a}^\zeta,\lambda} < f_\gamma^{\mathfrak{a}^\zeta,\lambda} \bmod J_{<\lambda}[\mathfrak{a}^\zeta] \right]$, but this suffices. (The first statement is actually proved in [Sh371, 3.2A], the second in [Sh371, 3.2B]; by it also ${}^\zeta\bar{f}$ is cofinal in the required sense.)

For every $\zeta < \kappa$ we can apply 6.7 with $(N_\zeta \cap \mathrm{pcf}\,\mathfrak{a})$, ${}^\zeta\bar{f}$ and $\langle N_{\zeta+1+i}: i < \kappa\rangle$ here standing for $\mathfrak{a}$, $\bar{f}$, $\bar{N}$ there. In the proof of 6.7 get a club E_ζ of κ (so any $i < j$ from E_ζ are O.K.). Now we can define for $\zeta < \kappa$ and $i < j$ in E_ζ, ${}^\zeta\mathfrak{b}_\lambda^{i,j}$ and $\langle {}^\zeta\mathfrak{b}_\lambda^{i,j,\epsilon}: \epsilon < |\mathfrak{a}^\zeta|^+\rangle$, $\langle \epsilon^\zeta(i,j,\lambda): \lambda \in \mathfrak{a}^\zeta\rangle$, $\epsilon^\zeta(i,j)$, as well as in the proof of 6.7. Let:

$$E = \{i < \kappa: i \quad \text{is a limit ordinal } (\forall j < i)(j+j < i \,\&\, j\times j < i) \text{ and } \bigwedge_{j<i} i \in E_j\ \}.$$

So by [Sh420, §1] we can find $\bar{C} = \langle C_\delta: \delta \in S\rangle$, $S \subseteq \{\delta < \kappa: \mathrm{cf}\,\delta = \mathrm{cf}\,\sigma\}$ stationary, C_δ a club of δ, $\mathrm{otp}\, C_\delta = \omega^2\sigma$ such that:

(1) for each $\alpha < \lambda$, $\{C_\delta \cap \alpha: \alpha \in \mathrm{nacc}(C_\delta)\}$ has cardinality $< \kappa$,* and

(2) for every club E' of θ for stationarily many $\delta \in S$, $C_\delta \subseteq E'$.

Without loss of generality $\bar{C} \in N_0$. For some δ^*, $C_{\delta^*} \subseteq E$, and let $\{j_\zeta: \zeta \leq \omega^2\sigma\}$ enumerate $C_{\delta^*} \cup \{\delta^*\}$. So $\langle j_\zeta: \zeta \leq \omega^2\sigma\rangle$ is a strictly increasing continuous sequence of ordinals from $E \subseteq \kappa$ such that $\langle j_\epsilon: \epsilon \leq \zeta\rangle \in N_{j_{\zeta+1}}$. Let $j(\zeta) = j_\zeta$, $i(\zeta) = i_\zeta =: j_{\omega^2(1+\zeta)}$, $\mathfrak{a}_\zeta = N_{i_\zeta} \cap \mathrm{pcf}\,\mathfrak{a}$, and $\bar{\mathfrak{a}} =: \langle \mathfrak{a}_\zeta: \zeta < \sigma\rangle$, $\mathfrak{b}_\lambda^\zeta[\bar{\mathfrak{a}}] =: {}^{i(\zeta)}\mathfrak{b}_\lambda^{j(\omega^2\zeta+1),j(\omega^2\zeta+2),\epsilon^\zeta(j(\omega^2\zeta+1),j(\omega^2\zeta+2))}$. Most of the requirements follow immediately, as

(*) for each $\zeta < \sigma$, we have $\mathfrak{a}_\zeta$, $\langle \mathfrak{b}_\lambda^\zeta[\bar{\mathfrak{a}}]: \lambda \in \mathfrak{a}_\zeta\rangle$ are as in 6.7 and belong to $N_{i_\beta+3} \subseteq N_{i_{\beta+1}}$.

We are left (for proving 6.7A) with proving (h)$^+$ and (i) (remember (h) is a special case of (h)$^+$ choosing $\theta = \aleph_0$).

For proving clause (i) note that for $\zeta < \xi < \kappa$, $f_\alpha^{\mathfrak{a}^\zeta,\lambda} \subseteq f_\alpha^{\mathfrak{a}^\xi,\lambda}$ hence ${}^\zeta\mathfrak{b}_\lambda^{i,j} \subseteq {}^\xi\mathfrak{b}_\lambda^{i,j}$. Now we can prove by induction on ϵ that ${}^\zeta\mathfrak{b}_\lambda^{i,j,\epsilon} \subseteq {}^\xi\mathfrak{b}_\lambda^{i,j,\epsilon}$ for every $\lambda \in \mathfrak{a}_\zeta$ (check the definition after $(*)_2$ in the proof of 6.7) and the conclusion follows.

Instead of proving (h)$^+$ we prove an apparently weaker version (h)$'$ below, and then note that $\bar{i}' = \langle i_{\omega^2\zeta}: \zeta < \sigma\rangle$, $\bar{\mathfrak{a}}' = \langle \mathfrak{a}_{\omega^2\zeta}: \zeta < \sigma\rangle$, $\langle N_{i(\omega^2\zeta)}: \zeta < \sigma\rangle$, $\langle \mathfrak{b}_\lambda^{\omega^2\zeta}[\bar{\mathfrak{a}}']: \zeta < \sigma, \lambda \in \mathfrak{a}'_\zeta = \mathfrak{a}_{\omega^2\zeta}\rangle$ will exemplify the conclusion** where

(h)$'$ if $\mathfrak{c} \subseteq \mathfrak{a}_\beta$, $\beta < \sigma$, $\mathfrak{c} \in N_{i_{\beta+1}}$, $\theta = \mathrm{cf}(\theta) \in N_{i_{\beta+1}}$ *then* for some $\mathfrak{d} \in N_{i_{\beta+\omega+1}+1}$, $\mathfrak{d} \subseteq \mathfrak{a}_{\beta+\omega} \cap \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{c})$ we have $\mathfrak{c} \subseteq \bigcup_{\mu\in\mathfrak{d}} \mathfrak{b}_\mu^{\beta+\omega}[\bar{\mathfrak{a}}]$ and $|\mathfrak{d}| < \theta$.

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* If κ is successor of regular, then we can get $[\gamma \in C_\alpha \cap C_\beta \Rightarrow C_\alpha \cap \gamma = C_\beta \cap \gamma]$.

** Assuming $\sigma > \aleph_0$ hence, $\omega^2\sigma = \sigma$ for notational simplicity.

Proof of (h)′: So let θ, β, $\mathfrak{c}$ be given; let $\langle \mathfrak{b}_\mu[\mathfrak{a}]: \mu \in \mathrm{pcf}\,\mathfrak{c}\rangle (\in N_{i_{\beta+1}})$ be a generating sequence. We define by induction on $n < \omega$, A_n, $\langle \mathfrak{c}_\eta, \lambda_\eta: \eta \in A_n\rangle$ such that:

(a) $A_0 = \{\langle\rangle\}$, $\mathfrak{c}_{\langle\rangle} = \mathfrak{c}$, $\lambda_{\langle\rangle} = \max\mathrm{pcf}\,\mathfrak{c}$,

(b) $A_n \subseteq {}^n\theta$, $|A_n| < \theta$,

(c) if $\eta \in A_{n+1}$ then $\eta \restriction n \in A_n$, $\mathfrak{c}_\eta \subseteq \mathfrak{c}_{\eta\restriction n}$, $\lambda_\eta < \lambda_{\eta\restriction n}$ and $\lambda_\eta = \max\mathrm{pcf}(\mathfrak{c}_\eta)$,

(d) A_n, $\langle \mathfrak{c}_\eta, \lambda_\eta: \eta \in A_n\rangle$ belongs to $N_{i_{\beta+1+n}}$ hence $\lambda_\eta \in N_{i_{\beta+1+n}}$,

(e) if $\eta \in A_n$ and $\lambda_\eta \in \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{c}_\eta)$ and $\mathfrak{c}_\eta \not\subseteq \mathfrak{b}^{\beta+1+n}_{\lambda_\eta}[\bar{\mathfrak{a}}]$ then $(\forall\nu)[\nu \in A_{n+1} \,\&\, \eta \subseteq \nu \Leftrightarrow \nu = \eta\hat{}\langle 0\rangle]$ and $\mathfrak{c}_{\eta\hat{}\langle 0\rangle} = \mathfrak{c}_\eta \setminus \mathfrak{b}^{\beta+1+n}_{\lambda_\eta}[\bar{\mathfrak{a}}]$ (so $\lambda_{\eta\hat{}\langle 0\rangle} = \max\mathrm{pcf}\,\mathfrak{c}_{\eta\hat{}\langle 0\rangle} < \lambda_\eta = \max\mathrm{pcf}\,\mathfrak{c}_\eta$),

(f) if $\eta \in A_n$ and $\lambda_\eta \notin \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{c}_\eta)$ then

$$\mathfrak{c}_\eta = \bigcup\left\{\mathfrak{b}_{\lambda_{\eta\hat{}\langle i\rangle}}[\mathfrak{c}]: i < i_n < \theta, \eta\hat{}\langle i\rangle \in A_{n+1}\right\},$$

and if $\nu = \eta\hat{}\langle i\rangle \in A_{n+1}$ then $\mathfrak{c}_\nu = \mathfrak{b}_{\lambda_\nu}[\mathfrak{c}]$,

(g) if $\eta \in A_n$, and $\lambda_\eta \in \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{c}_\eta)$ but $\mathfrak{c}_\eta \subseteq \mathfrak{b}^{\beta+1-n}_{\lambda_n}[\bar{\mathfrak{a}}]$, then $\neg(\exists\nu)[\eta \triangleleft \nu \in A_{n+1}]$.

There is no problem to carry the definition (we use 6.7F(1) below*, the point is that $\mathfrak{c} \in N_{i_{\beta+1+n}}$ implies $\langle \mathfrak{b}_\lambda[\mathfrak{c}]: \lambda \in \mathrm{pcf}_\theta[\mathfrak{c}]\rangle \in N_{i_{\beta+1+n}}$ and as there is $\mathfrak{d}$ as in 6.7F(1), there is one in $N_{i_{\beta+1+n+1}}$ so $\mathfrak{d} \subseteq \mathfrak{a}_{\beta+1+n+1}$). Now let

$$\mathfrak{d}_n =: \left\{\lambda_\eta: \eta \in A_n \text{ and } \lambda_\eta \in \underset{\theta-\mathrm{complete}}{\mathrm{pcf}}(\mathfrak{c}_\eta) \text{ and } \mathfrak{c}_\eta \subseteq \mathfrak{b}^{\beta+1+n}_{\lambda_\eta}[\mathfrak{a}]\right\}$$

and $\mathfrak{d} =: \bigcup_{n<\omega} \mathfrak{d}_n$; we shall show that it is as required.

The main point is $\mathfrak{c} \subseteq \bigcup_{\lambda\in\mathfrak{d}} \mathfrak{b}^{\beta+\omega}_\lambda[\bar{\mathfrak{a}}]$; note that

$$\left[\lambda_\eta \in \mathfrak{d}, \eta \in A_n \Rightarrow \mathfrak{b}^{\beta+1+n}_{\lambda_\eta}[\bar{\mathfrak{a}}] \subseteq \mathfrak{b}^{\beta+\omega}_{\lambda_\eta}[\bar{\mathfrak{a}}]\right]$$

hence it suffices to show $\mathfrak{c} \subseteq \bigcup_{n<\omega}\bigcup_{\lambda\in\mathfrak{d}_n} \mathfrak{b}^{\beta+1+n}_\lambda[\bar{\mathfrak{a}}]$, so assume $\theta \in \mathfrak{c}\setminus\bigcup_{n<\omega}\bigcup_{\lambda\in\mathfrak{d}_n} \mathfrak{b}^{\beta+1+n}_\lambda[\bar{\mathfrak{a}}]$, and we choose by induction on n, $\eta_n \in A_n$ such that $\eta_0 = \langle\rangle$, $\eta_{n+1} \restriction n = \eta_n$ and $\theta \in \mathfrak{c}_\eta$; by clauses (e) + (f) above this is possible and $\langle \max\mathrm{pcf}\,\mathfrak{c}_{\eta_n}: n < \omega\rangle$ is strictly decreasing, contradiction.

The minor point is $|\mathfrak{d}| < \theta$; if $\theta > \aleph_0$ note that $\bigwedge_n |A_n| < \theta$ and $\theta = \mathrm{cf}(\theta)$ so $|\mathfrak{d}| \leq |\bigcup_n A_n| < \theta + \aleph_1 = \theta$.

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* No vicious circle; 6.7F(1) does not depend on 6.7B.

If $\theta = \aleph_0$ (i.e. clause (h)) we should have $\bigcup_n A_n$ finite; the proof is as above noting the clause (f) is vacuous now. So $\bigwedge_n |A_n| = 1$ and $\bigvee_n A_n = \emptyset$, so $\bigcup_n A_n$ is finite. Another minor point is $\mathfrak{d} \in N_{i_{\beta+\omega+1}}$; this holds as the construction is unique from $\langle N_j: j < i_{\beta+\omega}\rangle$, $\langle i_j: j \le \beta+\omega\rangle$, $\langle(\mathfrak{a}_{i(\zeta)}, \langle \mathfrak{b}^{\zeta}_\lambda: \lambda \in \mathfrak{a}_{i(\zeta)}\rangle): \zeta \le \beta+\omega\rangle$; no "outside" information is used so $\langle(A_n, \langle(c_\eta, \lambda_\eta): \eta \in A_n\rangle): n < \omega\rangle \in N_{i_{\beta+\omega+1}}$, so (using a choice function) really $\mathfrak{d} \in N_{i_{\beta+\omega+1}}$. $\blacksquare_{6.7A}$

6.7E Proof of 6.7B: Let $\mathfrak{b}_\lambda[\bar{\mathfrak{a}}] = \mathfrak{b}^\sigma_\lambda = \bigcup_{\beta<\sigma} \mathfrak{b}^\beta_\lambda[\mathfrak{a}_\beta]$ and $\mathfrak{a}_\sigma = \bigcup_{\zeta<\sigma} \mathfrak{a}_\zeta$. Part (1) is straightforward. For part (2), for clause (g), for $\beta = \sigma$, the inclusion "$\subseteq$" is straightforward; so assume $\mu \in \mathfrak{a}_\beta \cap \mathrm{pcf}\, \mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}]$. Then by 6.7A(c) for some $\beta_0 < \beta$, we have $\mu \in \mathfrak{a}_{\beta_0}$, and by 6.7C(3B) (which depends on 6.7A only) for some $\beta_1 < \beta$, $\mu \in \mathrm{pcf}\, \mathfrak{b}^{\beta_1}_\lambda[\bar{\mathfrak{a}}]$; by monotonicity wlog $\beta_0 = \beta_1$, by clause (g) of 6.7A applied to β_0, $\mu \in \mathfrak{b}^{\beta_0}_\lambda[\bar{\mathfrak{a}}]$. Hence by clause (i) of 6.7A, $\mu \in \mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}]$, thus proving the other inclusion.

The proof of clause (e) (for 6.7B(2)) is similar, and also 6.7B(3). For 6.7(B)(4) for $\delta < \sigma$, $\mathrm{cf}(\delta) > |\mathfrak{a}|$ redefine $\mathfrak{b}^\delta_\lambda[\bar{\mathfrak{a}}]$ as $\bigcup_{\beta<\delta} \mathfrak{b}^{\beta+1}_\lambda[\mathfrak{a}]$. $\blacksquare_{6.7B}$

6.7F CLAIM: *Let θ be regular.*

(0) *If $\alpha < \theta$, $\mathrm{pcf}_{\theta-\mathrm{complete}}\left(\bigcup_{i<\alpha} \mathfrak{a}_i\right) = \bigcup_{i<\alpha} \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{a}_i)$.*

(1) *If $\langle \mathfrak{b}_\theta[\mathfrak{a}]: \theta \in \mathrm{pcf}\,\mathfrak{a}\rangle$ is a generating sequence for $\mathfrak{a}$, $\mathfrak{c} \subseteq \mathfrak{a}$, then for some $\mathfrak{d} \subseteq \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{c})$ we have: $|\mathfrak{d}| < \theta$ and $\mathfrak{c} \subseteq \bigcup_{\theta\in\mathfrak{a}} \mathfrak{b}_\theta[\mathfrak{a}]$.*

(2) *If $|\mathfrak{a} \cup \mathfrak{c}| < \mathrm{Min}\,\mathfrak{a}$, $\mathfrak{c} \subseteq \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{a})$, $\lambda \in \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{c})$ then $\lambda \in \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{a})$.*

(3) *In (2) we can weaken $|\mathfrak{a} \cup \mathfrak{c}| < \mathrm{Min}\,\mathfrak{a}$ to $|\mathfrak{a}| < \mathrm{Min}\,\mathfrak{a}$, $|\mathfrak{c}| < \mathrm{Min}\,\mathfrak{c}$.*

(4) *We cannot find $\lambda_\alpha \in \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{a})$ for $\alpha < |\mathfrak{a}|^+$ such that $\lambda_i > \sup \mathrm{pcf}_{\theta-\mathrm{complete}}(\{\lambda_j: j < i\})$.*

(5) *Assume $\theta \le |\mathfrak{a}|$, $\mathfrak{c} \subseteq \mathrm{pcf}_{\theta-\mathrm{complete}}\,\mathfrak{a}$ (and $|\mathfrak{c}| < \mathrm{Min}\ \mathfrak{c}$; of course $|\mathfrak{a}| < \mathrm{Min}\,\mathfrak{a}$). If $\lambda \in \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{c})$ then for some $\mathfrak{d} \subseteq \mathfrak{c}$ we have $|\mathfrak{d}| \le |\mathfrak{a}|$ and $\lambda \in \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{d})$.*

Proof: (0) and (1): Check.

(2) See [Sh345b, 1.10-1.12].

(3) Similarly.

(4) If $\theta = \aleph_0$ we already know it (e.g. 6.7C(3A)), so assume $\theta > \aleph_0$ and, without loss of generality, θ is regular $\le |\mathfrak{a}|$. We use 6.7A with $\{\theta, \langle \lambda_i: i < |\mathfrak{a}|^+\rangle\} \in N_0$, $\sigma = |\mathfrak{a}|^+$, $\kappa = |\mathfrak{a}|^{+3}$ where, without loss of generality, $\kappa < \mathrm{Min}(\mathfrak{a})$. For each $\alpha < |\mathfrak{a}|^+$ by (h)$^+$ of 6.7A there is $\mathfrak{d}_\alpha \in N_{i_1}$, $\mathfrak{d}_\alpha \subseteq \mathrm{pcf}_{\theta-\mathrm{complete}}(\{\lambda_i: i < \alpha\})$, $|\mathfrak{d}_\alpha| < \theta$

such that $\{\lambda_i: i < \alpha\} \subseteq \bigcup_{\theta\in\mathfrak{d}_\alpha} \mathfrak{b}^1_\theta[\bar{\mathfrak{a}}]$; hence by clause (g) of 6.7A and 6.7F(0) we have $\mathfrak{a}_1 \cap \mathrm{pcf}_{\theta-\mathrm{complete}}(\{\lambda_i: i < \alpha\}) \subseteq \bigcup_{\theta\in\mathfrak{d}_\alpha} \mathfrak{b}^1_\theta[\bar{\mathfrak{a}}]$. So for $\alpha < \beta < |\mathfrak{a}|^+$, $\mathfrak{d}_\alpha \subseteq \mathfrak{a}_1 \cap \mathrm{pcf}_{\theta-\mathrm{complete}}\{\lambda_i: i < \alpha\} \subseteq \mathfrak{a}_1 \cap \mathrm{pcf}_{\theta-\mathrm{complete}}\{\lambda_i: i < \beta\} \subseteq \bigcup_{\theta\in\mathfrak{d}_\beta} \mathfrak{b}^1_\theta[\bar{\mathfrak{a}}]$. As the sequence is smooth (i.e. clause (f) of 6.7A) clearly $\alpha < \beta \Rightarrow \bigcup_{\mu\in\mathfrak{d}_\alpha} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \subseteq \bigcup_{\mu\in\mathfrak{d}_\beta} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}]$.

So $\langle \bigcup_{\mu\in\mathfrak{d}_\alpha} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \cap \mathfrak{a}: \alpha < |\mathfrak{a}|^+\rangle$ is a non-decreasing sequence of subsets of $\mathfrak{a}$ of length $|\mathfrak{a}|^+$, hence for some $\alpha(*) < |\mathfrak{a}|^+$ we have:

$(*)_1$ $\alpha(*) \le \alpha < |\mathfrak{a}|^+ \Rightarrow \bigcup_{\mu\in\mathfrak{d}_\alpha} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \cap \mathfrak{a} = \bigcup_{\mu\in\mathfrak{d}_{\alpha(*)}} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \cap \mathfrak{a}$.

If $\tau \in \mathfrak{a}_1 \cap \mathrm{pcf}_{\theta-\mathrm{complete}}(\{\lambda_i: i < \alpha\})$ then $\tau \in \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{a})$ (by 6.7F(2),(3)), and $\tau \in \mathfrak{b}^1_{\mu_\tau}[\bar{\mathfrak{a}}]$ for some $\mu_\tau \in \mathfrak{d}_\alpha$ so $\mathfrak{b}^1_\tau[\bar{\mathfrak{a}}] \subseteq \mathfrak{b}^1_{\mu_\tau}[\bar{\mathfrak{a}}]$, also $\tau \in \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{b}^1_\tau[\bar{\mathfrak{a}}] \cap \mathfrak{a})$ (by clause (e) of 6.7A), hence

$$\begin{aligned}\tau \in \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{b}^1_\tau[\bar{\mathfrak{a}}] \cap \mathfrak{a}) &\subseteq \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{b}^1_{\mu_\tau}[\bar{\mathfrak{a}}] \cap \mathfrak{a})\\ &\subseteq \mathrm{pcf}_{\theta-\mathrm{complete}}\left(\bigcup_{\mu\in\mathfrak{d}_\alpha} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \cap \mathfrak{a}\right).\end{aligned}$$

So $\mathfrak{a}_1 \cap \mathrm{pcf}_{\theta-\mathrm{complete}}(\{\lambda_i: i < \alpha\}) \subseteq \mathrm{pcf}_{\theta-\mathrm{complete}}\left(\bigcup_{\mu\in\mathfrak{d}_\alpha} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \cap \mathfrak{a}\right)$. But for each $\alpha < |\mathfrak{a}|^+$ we have $\lambda_\alpha > \sup \mathrm{pcf}_{\theta-\mathrm{complete}}(\{\lambda_i: i < \alpha\})$, whereas $\mathfrak{d}_\alpha \subseteq \mathrm{pcf}_{\sigma-\mathrm{complete}}\{\lambda_i: i < \alpha\}$, hence $\lambda_\alpha > \sup \mathfrak{d}_\alpha$ hence

$(*)_2$ $\lambda_\alpha > \sup_{\mu\in\mathfrak{d}_\alpha} \max\mathrm{pcf}\, \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \ge \sup \mathrm{pcf}_{\theta-\mathrm{complete}}\left(\bigcup_{\mu\in\mathfrak{d}_\alpha} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \cap \mathfrak{a}\right)$.

On the other hand,

$(*)_3$ $\lambda_\alpha \in \mathrm{pcf}_{\theta-\mathrm{complete}}\{\lambda_i: i < \alpha+1\} \subseteq \mathrm{pcf}_{\theta-\mathrm{complete}}\left(\bigcup_{\mu\in\mathfrak{d}_{\alpha+1}} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \cap \mathfrak{a}\right)$.

For $\alpha = \alpha(*)$ we get contradiction by $(*)_1 + (*)_2 + (*)_3$.

(5) Assume $\mathfrak{a}$, $\mathfrak{c}$, λ form a counterexample with λ minimal. Without loss of generality $|\mathfrak{a}|^{+3} < \mathrm{Min}(\mathfrak{a})$ and $\lambda = \max\mathrm{pcf}\,\mathfrak{a}$ and $\lambda = \max\mathrm{pcf}\,\mathfrak{c}$ (just let $\mathfrak{a}' =: \mathfrak{b}_\lambda[\mathfrak{a}]$, $\mathfrak{c}' =: \mathfrak{c} \cap \mathrm{pcf}_\theta[\mathfrak{a}']$; if $\lambda \notin \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{c}')$ then necessarily $\lambda \in \mathrm{pcf}(\mathfrak{c}\backslash\mathfrak{c}')$ (by 6.7F(0)) and similarly $\mathfrak{c}\backslash\mathfrak{c}' \subseteq \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{a}\backslash\mathfrak{a}')$ hence by 6.7F(2),(3) $\lambda \in \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{a}\backslash\mathfrak{a}')$,
contradiction).

Also without loss of generality $\lambda \notin \mathfrak{c}$. Let κ, σ, $\bar{N}$, $\langle i_\alpha = i(\alpha): \alpha \le \sigma\rangle$, $\bar{\mathfrak{a}} = \langle \mathfrak{a}_i: i \le \sigma\rangle$ be as in 6.7A with $\mathfrak{a} \in N_0$, $\mathfrak{c} \in N_0$, $\lambda \in N_0$, $\sigma = |\mathfrak{a}|^+$, $\kappa = |\mathfrak{a}|^{+3} < \mathrm{Min}\,\mathfrak{a}$. We choose by induction on $\epsilon < |\mathfrak{a}|^+$, λ_ϵ, $\mathfrak{d}_\epsilon$ such that:

(a) $\lambda_\epsilon \in \mathfrak{a}_{\omega^2\epsilon+\omega+3}$, $\mathfrak{d}_\epsilon \in N_{i(\omega^2\epsilon+\omega+1)}$,

(b) $\lambda_\epsilon \in \mathfrak{c}$,

(c) $\mathfrak{d}_\epsilon \subseteq \mathfrak{a}_{\omega^2\epsilon+\omega+1} \cap \mathrm{pcf}_{\theta-\mathrm{complete}}(\{\lambda_\zeta: \zeta < \epsilon\})$,

(d) $|\mathfrak{d}_\epsilon| < \theta$,

(e) $\{\lambda_\zeta \colon \zeta < \epsilon\} \subseteq \bigcup_{\theta \in \mathfrak{d}_\epsilon} \mathfrak{b}_\theta^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]$,

(f) $\lambda_\epsilon \notin \mathrm{pcf}_{\theta-\mathrm{complete}} \left(\bigcup_{\theta \in \mathfrak{d}_\epsilon} \mathfrak{b}_\theta^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}] \right)$.

For every $\epsilon < |\mathfrak{a}|^+$ we first choose $\mathfrak{d}_\epsilon$ as the $<^*_\chi$-first element satisfying (c) + (d) + (e) and then if possible λ_ϵ as the $<^*_\chi$-first element satisfying (b) + (f). It is easy to check the requirements and in fact $\langle \lambda_\zeta \colon \zeta < \epsilon \rangle \in N_{\omega^2\epsilon+1}$, $\langle \mathfrak{d}_\zeta \colon \zeta < \epsilon \rangle \in N_{\omega^2\epsilon+1}$ (so clause (a) will hold). But why can we choose at all? Now $\lambda \notin \mathrm{pcf}_{\theta-\mathrm{complete}}\{\lambda_\zeta \colon \zeta < \epsilon\}$ as $\mathfrak{a}$, $\mathfrak{c}$, λ form a counterexample with λ minimal and $\epsilon < |\mathfrak{a}|^+$ (by 6.7F(3)). As $\lambda = \max\mathrm{pcf}\,\mathfrak{a}$ necessarily $\mathrm{pcf}_{\theta-\mathrm{complete}}(\{\lambda_\zeta \colon \zeta < \epsilon\}) \subseteq \lambda$ hence $\mathfrak{d}_\epsilon \subseteq \lambda$ (by clause (c)). By part (0) of the claim (and clause (a)) we know:

$$\mathrm{pcf}_{\theta-\mathrm{complete}} \left[\bigcup_{\mu \in \mathfrak{d}_\epsilon} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}] \right] = \bigcup_{\mu \in \mathfrak{d}_\epsilon} \mathrm{pcf}_{\theta-\mathrm{complete}} \left[\mathfrak{b}_\mu^{\omega^2+\omega+1}[\bar{\mathfrak{a}}] \right]$$

$$\subseteq \bigcup_{\mu \in \mathfrak{d}_\epsilon} (\mu + 1) \subseteq \lambda$$

(note $\mu = \max\mathrm{pcf}\,\mathfrak{b}_\mu^\beta[\bar{\mathfrak{a}}]$). So $\lambda \notin \mathrm{pcf}_{\theta-\mathrm{complete}} \left(\bigcup_{\mu \in \mathfrak{d}_\epsilon} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}] \right)$ hence by part (0) of the claim $\mathfrak{c} \not\subseteq \bigcup_{\mu \in \mathfrak{d}_\epsilon} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]$ so λ_ϵ exists. Now $\mathfrak{d}_\epsilon$ exists by 6.7A clause (h)$^+$.

Now clearly $\left\langle \mathfrak{a} \cap \bigcup_{\mu \in \mathfrak{d}_\epsilon} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}] \colon \epsilon < |\mathfrak{a}|^+ \right\rangle$ is non-decreasing (as in the earlier proof) hence eventually constant, say for $\epsilon \geq \epsilon(*)$ (where $\epsilon(*) < |\mathfrak{a}|^+$). But

(α) $\lambda_\epsilon \in \bigcup_{\mu \in \mathfrak{d}_{\epsilon+1}} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]$ [clause (e) in the choice of $\lambda_\epsilon, \mathfrak{d}_\epsilon$],

(β) $\mathfrak{b}_{\lambda_\epsilon}^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}] \subseteq \bigcup_{\mu \in \mathfrak{d}_{\epsilon+1}} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]$ [by clause (f) of 6.7A and (α) alone],

(γ) $\lambda_\epsilon \in \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{a})$ [as $\lambda_\epsilon \in \mathfrak{c}$ and a hypothesis],

(δ) $\lambda_\epsilon \in \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{b}_{\lambda_\epsilon}^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}])$ [by (γ) above and clause (e) of 6.7A],

(ϵ) $\lambda_\epsilon \notin \mathrm{pcf}(\mathfrak{a} \setminus \mathfrak{b}_{\lambda_\epsilon}^{\omega^2\epsilon+\omega+1})$,

(ζ) $\lambda_\epsilon \in \mathrm{pcf}_{\theta-\mathrm{complete}} \left(\mathfrak{a} \cap \bigcup_{\mu \in \mathfrak{d}_{\epsilon+1}} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}] \right)$ [by (δ) + (ϵ) + (β)].

But for $\epsilon = \epsilon(*)$, the statement (ζ) contradicts the choice of $\epsilon(*)$ and clause (f) above. $\blacksquare_{6.7F}$

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