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κ-bounded exponential-logarithmic power series fields

Salma Kuhlmann[a,*], Saharon Shelah[b]

[a] *Research Unit Algebra and Logic, University of Saskatchewan, Mc Lean Hall, 106 Wiggins Road, Saskatoon, SK S7N 5E6, Canada*

[b] *Department of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel*

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Abstract

In [F.-V. Kuhlmann, S. Kuhlmann, S. Shelah, Exponentiation in power series fields, Proc. Amer. Math. Soc. 125 (1997) 3177–3183] it was shown that fields of generalized power series cannot admit an exponential function. In this paper, we construct fields of generalized power series with *bounded support* which admit an exponential. We give a natural definition of an exponential, which makes these fields into models of real exponentiation. The method allows us to construct for every κ regular uncountable cardinal, 2^κ pairwise non-isomorphic models of real exponentiation (of cardinality κ), but all isomorphic as ordered fields. Indeed, the 2^κ exponentials constructed have pairwise distinct *growth rates*. This method relies on constructing lexicographic chains with many automorphisms.

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* Corresponding author.
E-mail addresses: skuhlman@math.usask.ca (S. Kuhlmann), shelah@math.huji.ac.il (S. Shelah).

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1. Introduction

In [9], Tarski proved his celebrated result that the elementary theory of the ordered field of real numbers admits elimination of quantifiers, and gave a recursive axiomatization of its class of models (the class of real closed fields). He asked whether analogous results hold for the elementary theory $T_{\exp}$ of $(\mathbb{R}, \exp)$ (the ordered field of real numbers with *exponentiation*). Addressing Tarski's problem, Wilkie [10] established that $T_{\exp}$ is model complete and o-minimal. Due to these results, the problem of constructing non-archimedean models of $T_{\exp}$ gained much interest.

Non-archimedean real closed fields are easy to construct; for example, any field of generalized power series (see Section 2) $\mathbb{R}((G))$ with exponents in a *divisible* ordered abelian group $G \neq 0$ is such a model. However, in [7] it was shown that fields of generalized power series cannot admit an exponential function, so different methods were needed to construct non-archimedean real closed exponential fields. In [3], van den Dries, Macintyre and Marker construct non-archimedean models (the logarithmic-exponential power series fields) of $T_{\exp}$ with many interesting properties. In [6], the exponential-logarithmic power series fields are constructed, providing yet another class of models. Although the two construction procedures are different (and produce different models, see [8]), both logarithmic-exponential or exponential-logarithmic series models are obtained as countable increasing unions of fields of generalized power series. In both cases, a partial exponential (logarithm) is constructed on every member of this union, and the exponential on the union is given by an inductive definition.

In this paper, we describe a different construction, which offers several advantages. The procedure is straightforward: we start with any non-empty chain Γ_0. For a given regular uncountable cardinal κ, we form the (uniquely determined) κ-th iterated lexicographic power $(\Gamma_\kappa, \iota_\kappa)$ of Γ_0 (see Section 4). We take G_κ and $\mathbb{R}((G_\kappa))_\kappa$ to be the corresponding κ-bounded Hahn group and κ-bounded power series field respectively (see Section 2). The logarithm on the positive elements of $\mathbb{R}((G_\kappa))_\kappa$ is now defined by a *uniform* formula (18). Under the additional hypothesis that $\kappa = \kappa^{<\kappa}$, $\mathbb{R}((G_\kappa))_\kappa$ is a model of cardinality κ.

As an application, we construct 2^κ pairwise non-isomorphic models of $T_{\exp}$ (of cardinality κ), but all isomorphic as real closed fields. This answers a question of D. Marker, and establishes an exponential analogue to the main result of [1].

The structure of the paper is as follows. In Section 2, we recall some preliminary notions and facts. In Section 3, we state and prove the Main Lemma: it provides sufficient conditions on a chain Γ, which allow a uniform definition of a logarithm on $\mathbb{R}((G_\kappa))_\kappa$. In Section 4, we give a canonical procedure to obtain chains satisfying the conditions of the Main Lemma. In Proposition 4, an additional sufficient condition, which allows us to obtain logarithms satisfying the *growth axiom scheme*, is given. In Section 5, we complete the construction of the model (Theorem 7). In Section 6, we introduce the *logarithmic rank*, which is an isomorphism invariant for the logarithm. Theorem 8 relates the logarithmic rank of our model to the orbital behaviour of automorphisms of our initial chain Γ_0. In Section 7, we construct chains with many automorphisms, which in turn allows the construction of models of $T_{\exp}$ with many logarithms (Theorem 9).

2. Preliminaries

We first need some definitions and general facts. Let Γ be a **chain** (that is, a totally ordered set). Let X, Y be subsets of Γ. We write $X < Y$ if $x < y$ for all $x \in X$ and $y \in Y$. A Dedekind cut in Γ is a pair (X, Y) of disjoint nonempty convex subsets of Γ whose union is Γ and $X < Y$. A Dedekind cut is a **gap** in Γ if X has no last element and Y has no first element. Γ is said to be Dedekind complete if there are no gaps in Γ. We denote by $\overline{\Gamma}$ the Dedekind completion of a chain Γ. We say that a point $\alpha \in \Gamma$ has **left character** $\aleph_0$ if $\{\alpha' \in \Gamma\,;\ \alpha' < \alpha\}$ has cofinality $\aleph_0$, and dually for right character. Similarly, the characters of a gap $\overline{s}$ in a chain Γ are those of $\overline{s}$ considered as a point in $\overline{\Gamma}$. If both characters are $\aleph_0$, we shall call it an $\aleph_0\aleph_0$-gap.

Given chains Γ and Γ', we denote by $\Gamma \vec{\amalg}\, \Gamma'$ the chain obtained by lexicographically ordering the Cartesian product $\Gamma \times \Gamma'$. In other words, we obtain the ordered sum of chains $\Gamma \vec{\amalg}\, \Gamma' \simeq \sum_{\gamma \in \Gamma} \Gamma'_\gamma$ (where Γ'_γ denotes the γ-th copy of Γ').

Let G be a totally ordered abelian group. The archimedean equivalence relation on G is defined as follows:

$$\text{For } x, y \in G \setminus \{0\} :\ x \stackrel{+}{\sim} y \text{ if } \exists n \in \mathbb{N} \text{ s.t. } n|x| \geq |y| \text{ and } n|y| \geq |x|$$

where $|x| := \max\{x, -x\}$. We set $x \ll y$ if for all $n \in \mathbb{N}$, $n|x| < |y|$. We denote by $[x]$ is the archimedean equivalence class of x. We totally order the set of archimedean classes as follows: $[y] < [x]$ if $x \ll y$.

Let $(K, +, \cdot, 0, 1, <)$ be an ordered field. Using the archimedean equivalence relation on the ordered abelian group $(K, +, 0, <)$, we can endow K with the **natural valuation** v: for $x, y \in K$, $x, y \neq 0$ define $v(x) := [x]$ and $[x] + [y] := [xy]$. We call $v(K) := \{v(x) \mid x \in K, x \neq 0\}$ the **value group**, $R_v := \{x \mid x \in K \text{ and } v(x) \geq 0\}$ the **valuation ring**, $I_v := \{x \mid x \in K \text{ and } v(x) > 0\}$ the **valuation ideal** (the unique maximal ideal of R_v), $U_v^{>0} := \{x \mid x \in R_v, x > 0, v(x) = 0\}$ **the group of positive units** of R_v. The **residue field** is $\overline{K} := R_v/I_v$. For $x, y \in K^{>0} \setminus R_v$ we say that x and y are **multiplicatively-equivalent** and write $x \stackrel{\cdot}{\sim} y$ if: $\exists n \in \mathbb{N}$ s.t. $x^n \geq y$ and $y^n \geq x$. Note that

$$x \stackrel{\cdot}{\sim} y \text{ if and only if } v(x) \stackrel{+}{\sim} v(y). \tag{1}$$

An ordered field K is an **exponential field** if there exists a map

$$\exp : (K, +, 0, <) \longrightarrow (K^{>0}, \cdot, 1, <)$$

such that exp is an isomorphism of ordered groups. A map exp with these properties will be called an **exponential** on K. A **logarithm** on K is the compositional inverse $\log = \exp^{-1}$ of an exponential. Without loss of generality, we shall always require the exponentials (logarithms) under consideration to be v-**compatible**: $\exp(R_v) = U_v^{>0}$ or $\log(U_v^{>0}) = R_v$.

We are mainly interested in exponentials satisfying the **growth axiom** scheme:

$$(\mathbf{GA})\ \ x \geq n^2 \implies \exp(x) > x^n \qquad (n \geq 1).$$

Note that because of the hypothesis $x \geq n^2$, **(GA)** is only relevant for $v(x) \leq 0$. Let us consider the case $v(x) < 0$. In this case, "$x > n^2$" holds for all $n \in \mathbb{N}$ if x is positive. Restricted to $K \setminus R_v$, axiom scheme **(GA)** is thus equivalent to the assertion

$$\forall n \in \mathbb{N}: \ \exp(x) > x^n \qquad \text{for all } x \in K^{>0} \setminus R_v. \tag{2}$$

Applying the logarithm $\log = \exp^{-1}$ on both sides, we find that this is equivalent to

$$\forall n \in \mathbb{N}: \ x > \log(x^n) = n\log(x) \qquad \text{for all } x \in K^{>0} \setminus R_v. \tag{3}$$

Via the natural valuation v, this in turn is equivalent to

$$v(x) < v(\log(x)) \qquad \text{for all } x \in K^{>0} \setminus R_v. \tag{4}$$

A logarithm log will be called a **(GA)-logarithm** if it satisfies (4). For more details about ordered exponential fields and their natural valuations see [6].

In this paper, we will mainly work with ordered abelian groups and ordered fields of the following form: let Γ be any totally ordered set and R any ordered abelian group. Then R^Γ will denote the Hahn product with index set Γ and components R. Recall that this is the set of all maps g from Γ to R such that the **support** $\{\gamma \in \Gamma \mid g(\gamma) \neq 0\}$ of g is well-ordered in Γ. Endowed with the lexicographic order and pointwise addition, R^Γ is an ordered abelian group, called the **Hahn group**.

We want a convenient representation for the elements g of the Hahn groups. Fix a strictly positive element $1 \in R$ (if R is a field, we take 1 to be the neutral element for multiplication). For every $\gamma \in \Gamma$, we will denote by 1_γ the map which sends γ to 1 and every other element to 0 (1_γ is the characteristic function of the singleton $\{\gamma\}$). Hence, every $g \in R^\Gamma$ can be written in the form $\sum_{\gamma\in\Gamma} g_\gamma 1_\gamma$ (where $g_\gamma := g(\gamma) \in R$). Note that $g \overset{+}{\sim} g'$ if and only if $\min \operatorname{support} g = \min \operatorname{support} g'$.

For $G \neq 0$ an ordered abelian group, k an archimedean ordered field, $k((G))$ will denote the (generalized) **power series field** with coefficients in k and exponents in G. As an ordered abelian group, this is just the Hahn group k^G. When we work in $K = k((G))$, we will write t^g instead of 1_g . Hence, every series $s \in k((G))$ can be written in the form $\sum_{g\in G} s_g t^g$ with $s_g \in k$ and well-ordered support $\{g \in G \mid s_g \neq 0\}$. Multiplication is given by the usual formula for multiplying series.

The natural valuation on $k((G))$ is given by $v(s) = \min \operatorname{support} s$ for any series $s \in k((G))$. Clearly the value group is (isomorphic to) G and the residue field is (isomorphic to) k. The valuation ring $k((G^{\geq 0}))$ consists of the series with non-negative exponents, and the valuation ideal $k((G^{>0}))$ of the series with positive exponents. The **constant term** of a series s is the coefficient s_0. The units of $k((G^{\geq 0}))$ are the series in $k((G^{\geq 0}))$ with a non-zero constant term.

Given any series, we can truncate it at its constant term and write it as the sum of two series, one with strictly negative exponents, and the other with non-negative exponents. Thus a complement in $(k((G)), +)$ to the valuation ring is the Hahn group $k^{G^{<0}}$. We call it the **canonical complement to the valuation ring** and denote it by **Neg** $k((G))$ or by $k((G^{<0}))$. Note that Neg $k((G))$ is in fact a (non-unital) subring, and a k-algebra.

Given $s \in k((G))^{>0}$, we can factor out the monomial of smallest exponent $g \in G$ and write $s = t^g u$ with u a unit with a positive constant term. Thus a complement in

$(k((G))^{>0}, \cdot)$ to the subgroup $U_v^{>0}$ of positive units is the group consisting of the (monic) monomials t^g. We call it the **canonical complement to the positive units** and denote it by **Mon** $k((G))$.

Throughout this paper, **fix a regular uncountable cardinal** κ. We are particularly interested in the κ**-bounded Hahn group** $(R^\Gamma)_\kappa$, the subgroup of R^Γ consisting of all maps of which support has cardinality $< \kappa$. Similarly, we consider the κ**-bounded power series field** $k((G))_\kappa$, the subfield of $k((G))$ consisting of all series of which support has cardinality $< \kappa$. It is a valued subfield of $k((G))$. We denote by $k((G^{\geq 0}))_\kappa$ its valuation ring. A subfield F of $k((G))$ is said to be **truncation closed** if whenever $s \in F$, then all truncations (initial segments) of s belong to F as well. If F is truncation closed, then $\mathrm{Neg}(F) := \mathrm{Neg}\, k((G)) \cap F$ is a complement to the valuation ring of F. If F contains the subfield $k(t^g\ ;\ g \in G)$ generated by the monic monomials, then $\mathrm{Mon}(F) = \{t^g\ ;\ g \in G\}$ is a complement to the group of positive units in $(F^{>0}, \cdot)$. Note that $k((G))_\kappa$ is truncation closed and contains $k(t^g\ ;\ g \in G)$. We denote $\mathrm{Neg}\, k((G))_\kappa$ by $k((G^{<0}))_\kappa$.

Our goal is to define an exponential (logarithm) on $k((G))_\kappa$ (for appropriate choice of G). From the above discussion, we get the following useful result:

Proposition 1. *Set* $K = k((G))_\kappa$. *Then* $(K, +, 0, <)$ *decomposes lexicographically as the sum:*

$$(K, +, 0, <) = k((G^{<0}))_\kappa \oplus k((G^{\geq 0}))_\kappa. \tag{5}$$

Similarly, $(K^{>0}, \cdot, 1, <)$ *decomposes lexicographically as the product:*

$$(K^{>0}, \cdot, 1, <) = \mathrm{Mon}\,(K) \times U_v^{>0}. \tag{6}$$

Moreover, $\mathrm{Mon}\,(K)$ *is order isomorphic to* G *through the isomorphism* $(-v)(t^g) = -g$.

Proposition 1 allows us to achieve our goal in two main steps; by defining the logarithm first on $\mathrm{Mon}\,(K)$ (Lemma 2) and then on $U_v^{>0}$ (Proposition 6).

3. The Main Lemma

We are interested in developing a method to construct a **left logarithm** on $\mathbb{R}((G))_\kappa$, that is, an isomorphism of ordered groups from $\mathrm{Mon}\,\mathbb{R}((G))_\kappa$ onto $\mathrm{Neg}\,\mathbb{R}((G))_\kappa = \mathbb{R}((G^{<0}))_\kappa$. Moreover, we want a criterion to obtain a **(GA)-left logarithm**, that is, a left logarithm which satisfies $t^g > \log((t^g)^n) = n \log(t^g)$ for all $n \in \mathbb{N}$ and $g \in G^{<0}$.

Lemma 2. *Let* Γ *be a chain. Set*

$$G := (\mathbb{R}^\Gamma)_\kappa \text{ and } K := \mathbb{R}((G))_\kappa.$$

Every isomorphism of chains

$$\iota : \Gamma \to G^{<0}$$

lifts to an isomorphism of ordered groups

$$\hat{\iota} : (G, +) \to (\mathrm{Neg}\,(K), +)$$

given by

$$\hat{\iota}\left(\sum_{\gamma\in\Gamma} g_\gamma 1_\gamma\right) := \sum_{\gamma\in\Gamma} g_\gamma t^{\iota(\gamma)} \tag{7}$$

for $g = \sum_{\gamma\in\Gamma} g_\gamma 1_\gamma \in G$. *Furthermore, setting*

$$\log(t^g) := \hat{\iota}(-g) = \sum_{\gamma\in\Gamma} -g_\gamma t^{\iota(\gamma)} \tag{8}$$

defines a left logarithm on K, *which satisfies*

$$v(\log t^g) = \iota(\min \operatorname{support} g). \tag{9}$$

Moreover, log *is a (GA)-left logarithm if and only if*

$$\iota(\min \operatorname{support} g) > g \qquad \textit{for all } g \in G^{<0}. \tag{10}$$

Proof. The map $\hat{\iota}$ is well defined (because of the condition imposed simultaneously on the supports of elements of G and of K). It is straightforward to verify that $\hat{\iota}$ is an isomorphism of ordered groups and that (8) defines a left logarithm. Also (10) follows from (4). □

Remark 3. If ι is only an embedding, one would still obtain by (7) an embedding $\hat{\iota}$, and by (8) an embedding of Mon (K) into Neg (K) (a so-called left *pre*-logarithm). The maps $\hat{\iota}$ and log are surjective (isomorphisms) if and only if ι is surjective. This observation is used to construct pre-logarithms on Exponential-Logarithmic Power Series fields in [6]. In this paper, we will not make use of pre-logarithms.

4. The κ-th iterated lexicographic power of a chain

Let $\Gamma_0 \neq \emptyset$ be a given chain. We shall construct canonically over Γ_0 a chain Γ_κ together with an isomorphism of ordered chains

$$\iota_\kappa : \Gamma_\kappa \to G_\kappa^{<0}$$

where $G_\kappa := (\mathbb{R}^{\Gamma_\kappa})_\kappa$. We call the pair $(\Gamma_\kappa, \iota_\kappa)$ the κ-th **iterated lexicographic power** of Γ_0.

We shall construct by transfinite induction on $\mu \leq \kappa$ a chain Γ_μ together with an embedding of ordered chains

$$\iota_\mu : \Gamma_\mu \to G_\mu^{<0}$$

where $G_\mu := (\mathbb{R}^{\Gamma_\mu})_\kappa$. We shall have $\Gamma_\nu \subset \Gamma_\mu$ and $\iota_\nu \subset \iota_\mu$ if $\nu < \mu$.

For $\mu = 0$, set $G_0 = (\mathbb{R}^{\Gamma_0})_\kappa$ and $\iota_0 : \Gamma_0 \to G_0^{<0}$ be defined by $\gamma \mapsto -1_\gamma$. Now assume that for all $\alpha < \mu$ we have already constructed Γ_α, $G_\alpha := (\mathbb{R}^{\Gamma_\alpha})_\kappa$, and the embedding

$$\iota_\alpha : \Gamma_\alpha \to G_\alpha^{<0}.$$

First assume that $\mu = \alpha + 1$ is a successor ordinal. Since Γ_α is isomorphic to a subchain of $G_\alpha^{<0}$ through ι_α, we can take $\Gamma_{\alpha+1}$ to be a chain containing Γ_α as a subchain and admitting an isomorphism $\iota_{\alpha+1}$ onto $G_\alpha^{<0}$ which extends ι_α. More precisely,

$$\Gamma_{\alpha+1} := \Gamma_\alpha \cup (G_\alpha^{<0} \setminus \iota_\alpha(\Gamma_\alpha)),$$

endowed with the **patch ordering**: if $\gamma_1, \gamma_2 \in \Gamma_{\alpha+1}$ both belong to Γ_α, compare them there; similarly if they both belong to $G_\alpha^{<0}$. If $\gamma_1 \in \Gamma_\alpha$ but $\gamma_2 \in G_\alpha^{<0}$ we set $\gamma_1 < \gamma_2$ if and only if $\iota_\alpha(\gamma_1) < \gamma_2$ in G_α. Then $\iota_{\alpha+1}$ is defined in the obvious way: $\iota_{\alpha+1}|_{\Gamma_\alpha} := \iota_\alpha$ and $\iota_{\alpha+1}|_{(G_\alpha^{<0} \setminus \iota_\alpha(\Gamma_\alpha))} :=$ the identity map. Note that

$$\iota_{\alpha+1}(\Gamma_{\alpha+1}) = G_\alpha^{<0}. \tag{11}$$

Thus $\iota_{\alpha+1}$ is an embedding of $\Gamma_{\alpha+1}$ into $G_{\alpha+1}^{<0}$.

If μ is a limit ordinal we set

$$\Gamma_\mu := \bigcup_{\alpha<\mu} \Gamma_\alpha, \quad \iota_\mu := \bigcup_{\alpha<\mu} \iota_\alpha \quad \text{and} \quad G_\mu := (\mathbb{R}^{\Gamma_\mu})_\kappa.$$

Note that by construction and (11)

$$\iota_\mu(\Gamma_\mu) = \bigcup_{\alpha<\mu} G_\alpha^{<0} \tag{12}$$

and $\bigcup_{\alpha<\mu} G_\alpha \subset G_\mu$.

This completes the construction of $\Gamma_\kappa := \bigcup_{\alpha<\kappa} \Gamma_\alpha$, $\iota_\kappa := \bigcup_{\alpha<\kappa} \iota_\alpha$ and $G_\kappa := (\mathbb{R}^{\Gamma_\kappa})_\kappa$. We now claim that

$$G_\kappa = \bigcup_{\alpha<\kappa} G_\alpha$$

(Once the claim is established, we conclude from (12) that $\iota_\kappa : \Gamma_\kappa \to G_\kappa^{<0}$ is an isomorphism, as required). Let $g \in G_\kappa$ and $\kappa > \delta := \text{card}\,(\text{support}\, g)$. Now support $g := \{\gamma_\mu\ ;\ \mu < \delta\} \subset \Gamma_\kappa$, so for every $\mu < \delta$ choose $\alpha_\mu < \kappa$ such that $\gamma_\mu \in \Gamma_{\alpha_\mu}$. Clearly card $(\{\alpha_\mu\ ;\ \mu < \delta\}) \le \delta < \kappa$ so $\{\alpha_\mu\ ;\ \mu < \delta\}$ cannot be cofinal in κ (since κ is regular), therefore it is bounded above by some $\alpha \in \kappa$. It follows that support $g \subset \Gamma_\alpha$, so $g \in G_\alpha$ as required.

Proposition 4. *Assume that $\sigma \in \text{Aut}\,(\Gamma_\kappa)$ is such that $\sigma|_{\Gamma_\mu} \in \text{Aut}\,(\Gamma_\mu)$ for all $\mu \in \kappa$ and $\sigma(\gamma) > \gamma$ for all $\gamma \in \Gamma_0$. Then the isomorphism*

$$l := \iota_\kappa \circ \sigma : \Gamma_\kappa \to G_\kappa^{<0}$$

satisfies (10).

Proof. Let $g \in G_\kappa^{<0}$ and $\gamma_\mu := \min \text{support}\, g \in \Gamma_\mu$ for the least such $\mu \in \kappa$. We prove that (10) holds by transfinite induction on μ. If $\mu = 0$, then $\gamma_0 \in \Gamma_0$ so

$$l(\gamma_0) = \iota_0 \circ \sigma(\gamma_0) = -1_{\sigma(\gamma_0)} > g.$$

Now assume that the assertion holds for all $\alpha < \mu$. Since

$$\iota_\kappa \circ \sigma(\Gamma_{\alpha+1}) = \iota_{\alpha+1}(\Gamma_{\alpha+1}) = G_\alpha^{<0},$$

by (11) and for μ limit

$$\iota_\kappa \circ \sigma(\Gamma_\mu) = \iota_\mu(\Gamma_\mu) = \bigcup_{\alpha<\mu} G_\alpha^{<0}$$

by (12), we have in any case that

$$l(\gamma_\mu) \in G_\alpha^{<0} \text{ for some } \alpha < \mu. \tag{13}$$

Set $l(\gamma_\mu) := g' \in G_\alpha^{<0}$. We have to show that $g < g'$; for this it is enough to show that $\min \operatorname{support} g < \min \operatorname{support} g'$, or equivalently that:

$$l(\min \operatorname{support} g) < l(\min \operatorname{support} g').$$

But the last inequality holds since by induction assumption we have that $g' < l(\min \operatorname{support} g')$. □

Proposition 5. *Let $\sigma_0 \in \operatorname{Aut}(\Gamma_0)$. Then σ_0 can be extended to $\sigma \in \operatorname{Aut}(\Gamma_\kappa)$ satisfying $\sigma|_{\Gamma_\mu} \in \operatorname{Aut}(\Gamma_\mu)$ for all $\mu \in \kappa$. In particular, if $\sigma_0 \in \operatorname{Aut}(\Gamma_0)$ satisfies $\sigma_0(\gamma) > \gamma$ for all $\gamma \in \Gamma_0$, then σ satisfies the hypothesis of Proposition 4.*

Proof. We first note that any $\sigma_\mu \in \operatorname{Aut}(\Gamma_\mu)$ lifts to $\hat{\sigma}_\mu \in \operatorname{Aut}(G_\mu)$ as follows. For $g = \sum_{\gamma\in\Gamma_\mu} g_\gamma 1_\gamma \in G_\mu$, set:

$$\hat{\sigma}_\mu\left(\sum_{\gamma\in\Gamma_\mu} g_\gamma 1_\gamma\right) := \sum_{\gamma\in\Gamma_\mu} g_\gamma 1_{\sigma_\mu(\gamma)}. \tag{14}$$

Observe that if $\alpha < \mu$ and $\sigma_\mu \in \operatorname{Aut}(\Gamma_\mu)$ extends $\sigma_\alpha \in \operatorname{Aut}(\Gamma_\alpha)$, then also $\hat{\sigma}_\mu$ extends $\hat{\sigma}_\alpha$. By induction on $\mu \le \kappa$, we now construct $\sigma_\mu \in \operatorname{Aut}(\Gamma_\mu)$ satisfying the following two properties:

$$\text{(i)}\ \hat{\sigma}_\mu \circ \iota_\mu = \iota_\mu \circ \sigma_\mu \quad\text{and}\quad \text{(ii)}\ \sigma_\mu \supset \sigma_\beta \quad \text{for all } \beta \le \mu. \tag{15}$$

Note that (15) part (i) implies that

$$\text{for all } g \in G_\mu^{<0}: \qquad \hat{\sigma}_\mu(g) \in \iota_\mu(\Gamma_\mu) \text{ if and only if } g \in \iota_\mu(\Gamma_\mu). \tag{16}$$

It is readily verified that σ_0 satisfies (15). Assume that for $\alpha < \mu$, σ_α has been constructed satisfying (15).

If $\mu = \alpha + 1$, define $\sigma_{\alpha+1}$ on $\Gamma_{\alpha+1} = \Gamma_\alpha \cup (G_\alpha^{<0} \setminus \iota_\alpha(\Gamma_\alpha))$ by setting: $\sigma_{\alpha+1}|_{\Gamma_\alpha} := \sigma_\alpha$ and $\sigma_{\alpha+1}|_{(G_\alpha^{<0}\setminus\iota_\alpha(\Gamma_\alpha))} := \hat{\sigma}_\alpha$. Since $\hat{\sigma}_\alpha$ satisfies (16), $\sigma_{\alpha+1}$ is well-defined. It easily follows from the definition of $\sigma_{\alpha+1}$ that $\sigma_{\alpha+1} \supset \sigma_\alpha$, and that $\sigma_{\alpha+1}$ is a bijection satisfying (15). It remains to verify that $\sigma_{\alpha+1}(\gamma_1) < \sigma_{\alpha+1}(\gamma_2)$ for $\gamma_1 < \gamma_2$, $\gamma_1, \gamma_2 \in \Gamma_{\alpha+1}$. We only verify this when $\gamma_1 \in \Gamma_\alpha$ and $\gamma_2 \in G_\alpha^{<0}$ (the verification in the other cases is straightforward). From $\iota_\alpha(\gamma_1) < \gamma_2$ in G_α it follows that $\hat{\sigma}_\alpha(\iota_\alpha(\gamma_1)) < \hat{\sigma}_\alpha(\gamma_2)$ in G_α. By (15), we therefore have $\iota_\alpha(\sigma_\alpha(\gamma_1)) < \hat{\sigma}_\alpha(\gamma_2)$ in G_α. That is, $\iota_\alpha(\sigma_{\alpha+1}(\gamma_1)) < \sigma_{\alpha+1}(\gamma_2)$ in G_α, or equivalently $\sigma_{\alpha+1}(\gamma_1) < \sigma_{\alpha+1}(\gamma_2)$ in $\Gamma_{\alpha+1}$ as required.

Finally, if μ is a limit ordinal, set $\sigma_\mu := \bigcup_{\alpha<\mu} \sigma_\alpha$. Then $\sigma := \sigma_\kappa$ is the required $\sigma \in \operatorname{Aut}(\Gamma_\kappa)$. □

5. κ-bounded models

We now extend the definition of the logarithm to the positive units. Below, for $r \in \mathbb{R}$, $r > 0$ we denote by $\log r$ the natural logarithm of r.

Proposition 6. *Let G be any divisible ordered abelian group, and set $K := \mathbb{R}((G))_\kappa$. For $u \in U_v^{>0}$ write $u = r(1+\varepsilon)$ (with $r \in \mathbb{R}$, $r > 0$ and $\varepsilon \in I_v$ infinitesimal). Then*

$$\log(u) := \log r(1+\varepsilon) = \log r + \sum_{i=1}^{\infty}(-1)^{(i-1)}\frac{\varepsilon^i}{i} \tag{17}$$

defines an isomorphism of ordered groups from $U_v^{>0}$ onto R_v.

Proof. The formal sum given in (17), and more generally, any formal sum $\sum_{i=0}^{\infty} r_i\varepsilon^i$ (with $r_i \in \mathbb{R}$), is a well-defined element of $\mathbb{R}((G))$: it has well-ordered support, since $\operatorname{support}\varepsilon \subset G^{>0}$. Also, the map defined by (17) is a bijective, order preserving group homomorphism; cf. [4]. It remains to verify that

$$\operatorname{card}(\operatorname{support}\varepsilon) < \kappa \Longrightarrow \operatorname{card}\left(\operatorname{support}\sum_{i=0}^{\infty} r_i\varepsilon^i\right) < \kappa.$$

Note that

$$\operatorname{support} r_i\varepsilon^i \subset \oplus_i \operatorname{support}\varepsilon := \{g_1 + \cdots + g_i \mid g_j \in \operatorname{support}\varepsilon \text{ for all } j = 1,\ldots,i\},$$

and clearly, $\operatorname{card}(\oplus_i \operatorname{support}\varepsilon) < \kappa$ for all i, so $\operatorname{card}(\cup_i(\oplus_i \operatorname{support}\varepsilon)) < \kappa$. Now observe that $\operatorname{support}\sum_{i=0}^{\infty} r_i\varepsilon^i \subset \cup_i(\oplus_i \operatorname{support}\varepsilon)$. $\square$

We can now define the logarithm on the positive elements of $\mathbb{R}((G_\kappa))_\kappa$ making $\mathbb{R}((G_\kappa))_\kappa$ into a model of T_{exp} := **the elementary theory of the reals with exponentiation**. Below, T_{an} := the theory of the reals with restricted analytic functions and $T_{\text{an,exp}}$:= the theory of the reals with restricted analytic functions and exponentiation (see [2] for axiomatizations of these theories).

Theorem 7. *Let κ be a regular uncountable cardinal, Γ_0 a chain, Γ_κ the κ-th lexicographic iterated power of Γ_0, and $G_\kappa = (\mathbb{R}^{\Gamma_\kappa})_\kappa$. Let $\sigma \in \operatorname{Aut}(\Gamma_\kappa)$ and*

$$l : \Gamma_\kappa \to G_\kappa^{<0}$$

be as in Proposition 4. For positive $a \in \mathbb{R}((G_\kappa))_\kappa$, write $a = t^g r(1+\varepsilon)$, with $g = \sum_{\gamma\in\Gamma_\kappa} g_\gamma 1_\gamma \in G_\kappa$, $r \in \mathbb{R}^{>0}$, and ε infinitesimal. Then

$$\log(a) := \log(t^g r(1+\varepsilon)) = \sum_{\gamma\in\Gamma} -g_\gamma t^{l(\gamma)} + \log r + \sum_{i=1}^{\infty}(-1)^{(i-1)}\frac{\varepsilon^i}{i} \tag{18}$$

defines a logarithm on $\mathbb{R}((G_\kappa))_\kappa^{>0}$ making $\mathbb{R}((G_\kappa))_\kappa$ into a model of T_{exp}.

Proof. By Lemma 2, Proposition 4, and Proposition 6, (18) defines a (GA)-logarithm. Using the Taylor expansion of any analytic function, one can endow $\mathbb{R}((G_\kappa))_\kappa$ with a natural interpretation of the restricted analytic functions (as we did in Proposition 6 for the logarithm). This makes $\mathbb{R}((G_\kappa))_\kappa$ into a substructure of the T_{an} model $\mathbb{R}((G_\kappa))$ (cf. [2]). From the quantifier elimination results of [2], we get that $\mathbb{R}((G))_\kappa$ is a model of T_{an}. Since log is a (GA)-logarithm, it follows (from the axiomatization given in [2]) that $\mathbb{R}((G))_\kappa$ is a model of $T_{\mathrm{an,exp}}$. $\square$

6. Growth rates

Let Γ be a chain and $\sigma \in \mathrm{Aut}(\Gamma)$. Assume that

$$\sigma(\gamma) > \gamma \text{ for all } \gamma \in \Gamma. \tag{19}$$

An automorphism satisfying (19) will be called an increasing automorphism. By induction, we define the **n-th iterate** of σ: $\sigma^1(\gamma) := \sigma(\gamma)$ and $\sigma^{n+1}(\gamma) := \sigma(\sigma^n(\gamma))$. We define an equivalence relation on Γ as follows. For $\gamma, \gamma' \in \Gamma$, set

$$\gamma \sim_\sigma \gamma' \text{ if and only } \exists n \in \mathbb{N} \text{ such that } \sigma^n(\gamma) \geq \gamma' \text{ and } \sigma^n(\gamma') \geq \gamma. \tag{20}$$

The equivalence classes $[\gamma]_\sigma$ of $\sim_\sigma$ are convex and closed under application of σ. By the convexity, the order of Γ induces an order on $\Gamma/\sim_\sigma$ such that $[\gamma]_\sigma < [\gamma']_\sigma$ if $\gamma < \gamma'$. The order type of $\Gamma/\sim_\sigma$ is the **rank** of (Γ, σ).

Similarly, let K be a real closed field and log a **(GA)**-logarithm on $K^{>0}$. Define an equivalence relation on $K^{>0} \setminus R_v$:

$$a \sim_{log} a' \text{ if and only if } \exists n \in \mathbb{N} \text{ such that } \log_n(a) \leq (a') \text{ and } \log_n(a') \leq a \tag{21}$$

(where $\log_n$ is the n-th iterate of the log). Again, the log-equivalence classes are convex and closed under application of log. The order type of the chain of equivalence classes is the **logarithmic rank** of $(K^{>0}, \log)$. Note that if x and y are archimedean-equivalent or multiplicatively-equivalent (cf. (1)), then they are a fortiori log-equivalent.

We now compute the logarithmic rank of the models described in Theorem 7. Below, set $\sigma_0 := \sigma|_{\Gamma_0}$.

Theorem 8. *The logarithmic rank of* $(\mathbb{R}((G_\kappa))_\kappa^{>0}, \log)$ *is equal to the rank of* (Γ_0, σ_0).

Proof. Let $a \in K^{>0} \setminus R_v$, write $a = t^g u$ (with u a unit, $g \in G_\kappa^{<0}$). Since a is archimedean-equivalent to t^g, it is log-equivalent to it. So it is enough to consider monomials t^g with $g = \sum_{\gamma \in \Gamma_\kappa} g_\gamma 1_\gamma \in G_\kappa^{<0}$. Set $\gamma_\mu := \min \operatorname{support} g \in \Gamma_\mu$ for the least such $\mu \in \kappa$. We show by transfinite induction on μ that there exists $g_0 \in G_\kappa^{<0}$ such that $\gamma_0 := \min \operatorname{support} g_0 \in \Gamma_0$ and t^g is log-equivalent to t^{g_0}.

If $\mu = 0$ there is nothing to prove. Assume that the assertion holds for all $\alpha < \mu$. Now

$$\log(t^g) = \sum_{\gamma \in \Gamma} -g_\gamma t^{l(\gamma)} \tag{22}$$

is archimedean-equivalent (cf. (9)), so log-equivalent to $t^{l(\gamma_\mu)}$. By (13) and induction hypothesis, the assertion holds for $t^{l(\gamma_\mu)}$, and thus for t^g by transitivity.

Now we determine the logarithmic equivalence class of t^g for $g \in G_\kappa^{<0}$ such that $\gamma_0 := \min \operatorname{support} g \in \Gamma_0$. Now t^g is multiplicatively-equivalent, so log-equivalent to $t^{-1_{\gamma_0}}$, so it is enough to consider monomials of the form t^{-1_γ} with $\gamma \in \Gamma_0$. We claim that

$$\text{for all } \gamma, \gamma' \in \Gamma_0 : t^{-1_\gamma} \sim_{log} t^{-1_{\gamma'}} \text{ if and only if } \gamma \sim_\sigma \gamma'.$$

We first find a formula for $\log_n(t^{-1_\gamma})$. Using (22) we compute: $\log(t^{-1_\gamma}) = t^{l(\gamma)} = t^{\iota_0 \circ \sigma(\gamma)} = t^{\iota_0(\sigma(\gamma))} = t^{-1_{\sigma(\gamma)}}$ (since $\sigma(\gamma) \in \Gamma_0$). By induction, we see that for all $n \in \mathbb{N}$:

$$\log_n(t^{-1_\gamma}) = t^{-1_{\sigma^n(\gamma)}}.$$

We conclude: $\gamma \sim_\sigma \gamma' \iff \exists n \in \mathbb{N}$ such that $\sigma^n(\gamma) \geq \gamma'$ and $\sigma^n(\gamma') \geq \gamma \iff 1_{\sigma^n(\gamma)} \leq 1_{\gamma'}$ and $1_{\sigma^n(\gamma')} \leq 1_\gamma \iff -1_{\gamma'} \leq -1_{\sigma^n(\gamma)}$ and $-1_\gamma \leq -1_{\sigma^n(\gamma')} \iff$

$$t^{-1_{\gamma'}} \geq t^{-1_{\sigma^n(\gamma)}} = \log_n(t^{-1_\gamma}) \text{ and } t^{-1_\gamma} \geq t^{-1_{\sigma^n(\gamma')}} = \log_n(t^{-1_{\gamma'}}),$$

if and only if $t^{-1_\gamma} \sim_{log} t^{-1_{\gamma'}}$ as required. □

Theorem 9. *Let κ be a regular uncountable cardinal with $\kappa = \kappa^{<\kappa}$. Let Γ_0 be any chain of cardinality κ which admits a family $\mathcal{A} = \{\sigma_0^\alpha \mid \alpha \in 2^\kappa\} \subset \operatorname{Aut}(\Gamma_0)$ of increasing automorphisms of pairwise distinct ranks. Let Γ_κ be the κ-th iterated lexicographic power of Γ_0, $G_\kappa := (\mathbb{R}^{\Gamma_\kappa})_\kappa$ the corresponding κ-bounded Hahn group, and $K = \mathbb{R}((G_\kappa))_\kappa$ the corresponding κ-bounded power series field of cardinality κ. Then K admits a family $\{\exp^\alpha \mid \alpha \in 2^\kappa\}$ of 2^κ exponentials. For every $\alpha \in 2^\kappa$, $(K, \exp^\alpha)$ is a model of real exponentiation. The 2^κ exponentials are of pairwise distinct exponential rank, but all agree on the valuation ring of K.*

Proof. For every σ_0^α, let $\sigma^{(\alpha)} \in \operatorname{Aut}(\Gamma_\kappa)$ be the corresponding extension (Proposition 5). Set $l^\alpha := \iota_\kappa \circ \sigma^{(\alpha)}$, and let $\log^\alpha$ be the corresponding logarithm (obtained by replacing in l by l^α in Eq. (18)). Now apply Theorem 8. □

In the next section, we give an explicit construction of chains satisfying the hypothesis of this theorem.

7. Chains with 2^κ automorphisms of distinct ranks

Lemma 10. *Let β be an ordinal, and consider the chain $\Gamma_0 := \beta \vec{\amalg} \mathbb{Q}$. For every $\alpha \in \beta$, let $\mathbb{Q}_\alpha$ be the α-th-copy of $\mathbb{Q}$. Fix τ_α and $\tau'_\alpha \in \operatorname{Aut}(\mathbb{Q}_\alpha)$ increasing automorphisms of rank 1 and $\mathbb{Z}$ respectively. For every $S \subset \beta$ define τ_S as follows:*

$$\tau_S|_{\mathbb{Q}_\alpha} := \begin{cases} \tau_\alpha & \text{if } \alpha \in S \\ \tau'_\alpha & \text{otherwise.} \end{cases}$$

Then the rank of $\tau_S = \sum_{\alpha\in\beta} \delta_S(\alpha)$, where

$$\delta_S(\alpha) := \begin{cases} 1 & \text{if } \alpha \in S \\ \mathbb{Z} & \text{otherwise.} \end{cases}$$

Lemma 10 is a consequence of the following more general observation:

Proposition 11. *Let I be a chain, and $\{(\Gamma_i, \tau_i) \mid i \in I\}$ a collection of chains Γ_i endowed with an increasing automorphism τ_i. Set*

$$\Gamma := \sum_{i \in I} \Gamma_i \text{ and } \tau := \sum_{i \in I} \tau_i,$$

(that is, $\tau|_{\Gamma_i} = \tau_i$). Then the rank of (Γ, τ) is equal to $\sum_{i \in I}$ rank (Γ_i, τ_i).

The proof is straightforward and we omit it.

Remark 12. (i) In [5], other arithmetic operations on chains are studied; it may be interesting, for future work, to study the behaviour of automorphism ranks with respect to these operations.

(ii) Automorphisms τ_α and $\tau'_\alpha \in \text{Aut}(\mathbb{Q}_\alpha)$ such as in Lemma 10 exist: for example, set $\tau(q) := q + 1$, $\tau \in \text{Aut}(\mathbb{Q})$ is of rank 1. To produce $\tau' \in \text{Aut}(\mathbb{Q})$ of rank $\mathbb{Z}$, note that by Cantor's Theorem $\mathbb{Q} \simeq \mathbb{Z} \vec{\amalg} \mathbb{Q}$. Define τ' piecewise as follows: for $z \in \mathbb{Z}$ we let $\tau'|_{\mathbb{Q}_z} \in \text{Aut}(\mathbb{Q}_z)$ be the translation automorphism $\tau'(q) = q + 1$ for $q \in \mathbb{Q}_z$, then τ' is defined by patching, and has clearly rank $\mathbb{Z}$ as required.

(iii) If β is an infinite cardinal, then card $(\beta \vec{\amalg} \mathbb{Q}) = \beta$.

We now state and prove the main result of this section. Below, we keep the notation of Lemma 10.

Proposition 13. *Let β be an ordinal and $s \subset \beta$. Set*

$$\Delta_S := \sum_{\alpha \in \beta} \delta_S(\alpha).$$

Then

$$\Delta_S \simeq \Delta_{S'} \text{ if and only if } S = S'.$$

Proof. Fix an isomorphism $\varphi : \Delta_S \simeq \Delta_{S'}$. We show by induction on $\alpha \in \beta$ that

$$\varphi(\delta_S(\alpha)) = \delta_{S'}(\alpha). \tag{23}$$

(The proposition is proved once (23) is established: it follows from (23) that $\delta_S(\alpha) = 1$ if and only if $\delta'_S(\alpha) = 1$, i.e. $S = S'$.) Let $\alpha = 0$. Assume that $\delta_S(0) = 1$. Then necessarily $\delta_{S'}(0) = 1$ and (23) holds (since φ has to map the least element of Δ_S to the least element of $\Delta_{S'}$). Assume now that $\delta_S(0) = \mathbb{Z}$, then necessarily $\delta_{S'}(0) = \mathbb{Z}$. We claim that (23) holds in this case too. Clearly, since $\delta_S(0)$ is an initial segment of Δ_S, $\varphi(\delta_S(0))$ is an initial segment of $\Delta_{S'}$. It thus suffices to show that $\varphi(\delta_S(0)) \subset \delta_{S'}(0)$. Assume for a contradiction that $\varphi(\delta_S(0)) \cap \delta_{S'}(1) \neq \emptyset$. There are two cases to consider. If $\delta_{S'}(1) = 1$, then 1 has left character $\aleph_0$. This is impossible since no such element exists in $\delta_S(0)$. If $\delta_{S'}(1) = \mathbb{Z}$, then $\varphi(\delta_S(0))$ has an $\aleph_0\aleph_0$-gap. This is impossible since no such gap exists in $\mathbb{Z}$. The claim is established.

Now assume that (23) holds for all $\alpha < \mu < \beta$, we show it holds for μ. From induction hypothesis we deduce that

$$\varphi\left(\sum_{\alpha < \mu} \delta_S(\alpha)\right) = \sum_{\alpha < \mu} \delta_{S'}(\alpha), \tag{24}$$

therefore

$$\varphi\left(\sum_{\nu\geq\mu}\delta_S(\nu)\right)=\sum_{\nu\geq\mu}\delta_{S'}(\nu). \tag{25}$$

With the help of (24) and (25), the same argument as the one used for the induction begin (with μ and $\mu+1$ instead of 0 and 1) applies now to establish (23) for μ. □

Corollary 14. *The chain $\Gamma_0 = \kappa \,\vec{\amalg}\, \mathbb{Q}$ admits of family of 2^κ increasing automorphisms, of pairwise distinct ranks.*

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