Consistency of positive partition theorems for graphs and models

by

Saharon Shelah

Institute of Mathematics
The Hebrew University
Jerusalem, Israel

Department of Mathematics
Rutgers University
New Brunswick N.J. U.S.A.

Recently A. Hajnal, P. Komjath [1] have dealt with the partition relation $H \to (G)^2_\sigma$: if we colour the edges of a graph H by σ colours, there is an induced subgraph isomorphic to G which is monochromatic (i.e. all edges get the same colour). They prove (generalizing a proof from Shelah [2]) that it is consistent (with ZFC) that there is a graph G of cardinality $\aleph_1$ such that for no graph H : $H \to (G)^2_2$.

They ask whether the negation is consistent. We give here an affirmative answer (even for much stronger partition relations). We first prove it using a class of measurable cardinals (in §1, §2). In §3, §4 we eliminate this.

We can also generalize $M \to (N)^{<\beta(*)}_\theta$ to $M \to [N]^{<\beta(*)}_{\theta,\theta_1}$ (we get an isomorphism of N in which only $\le \theta_1$ colours occurs). Our positive independence result ([3],[4]) like $2^{\aleph_0} \to [\aleph_1]^2_{n,2}$ are generalized naturally. This will be discussed elsewhere. Later are given generalizations with finite conclusion, but infinite number of colours ; and we improve the bounds for $\to_{wsp}$.

This research was partially supported by the United States Israel Binational Science Foundation (Number 289). We thank Juris Steprans for handling the paper.

§ 1. The consistency of the partition theorem from a measurable cardinal.

1.1 Notation: K_σ^α (for $\alpha \le \omega$, σ a cardinal) is the class of triples $M = (A,<,F)$ where $<$ is a well ordering of the nonempty set A and $F : [A]^{<\alpha} \to \sigma$ such that $F(\emptyset) = 0$. We let $[A]^n = \{u : u \subseteq A, |u| = n\}$, $[A]^{<\alpha} = \bigcup\{A^{[n]} : n < \alpha\}$, and do not distinguish strictly between F and $\langle F^{[n]} : n < \alpha\rangle$. If A is a set of ordinals, $<$ will be the usual order and we omit it.

We write $M = (|M|, <^M, F^M)$, and use M for $|M|$ sometimes. Now $f : M \to N$ is an (K_σ^α)-embedding if $(\forall x,y \in M)[x <^M y \Leftrightarrow f(x) <^N f(y)]$ and $(\forall u \in [|M|]^{<\alpha})[F^M(u) = F^N(f''(u))]$ and we write $M \subseteq N$ (M a submodel of N) if the identity is an embedding.

1.1A Explanation: We are thinking of M as a model, $F^M(u)$ as the quantifier free type of u, more formally, if $u = \{a_0,\dots,a_{n-1}\}$, $a_0<\dots<a_{n-1}$ we call

$$\bigwedge_{v\subseteq u} F(\dots,x_i,\dots)_{i\in v} = F(\dots,a_i,\dots)_{i\in v} \wedge x_0<\dots<x_{n-1}$$

the quantifier free type tp_{qf} of u (this notation happens when we do not have a fixed model).

Below in 1.2, d is thought of as a coloring of M.

1.2 Definition: For $M,N \in K_\sigma^\alpha$, $\beta \le \omega$ and cardinal θ, $M \to (N)_\theta^{<\beta}$ if: for every function $d : [M]^{<\beta} \to \theta$, there is an embedding f of N into M such that for every non empty $u \in [N]^{<\beta}$, $d(f''(u))$ depend only on the quantifier free type of u in N.

The arrow $\to_{sp}$ is defined in 2.1. First we define the weaker notion $\to^{eh}$ where eh stands for end homogeneity.

1.3 Definition: For $M,N \in K_\sigma^\alpha$, $\beta < \omega$ and cardinal θ (eh stands for end homogeneous) $M \to^{eh} (N)_\theta^{<\beta}$ if: for every function $d : [M]^{<\beta} \to \theta$, there is an embedding f of N into M such that: if $m < \beta$, $\langle a_0,\dots,a_{m-1},a_m\rangle$ a

$<^N$-increasing sequence of members of N , and

$$tp_{qf}(\langle a_0,\ldots,a_{m-2},a_{m-1}\rangle,\emptyset,N) = tp_{qf}(\langle a_0,\ldots,a_{m-2},a_m\rangle,\emptyset,N)$$

then

$$d(f(a_0),\ldots,f(a_{m-2}) \ , \ f(a_{m-1})) = d(f(a_0),\ldots,f(a_{m-2}) \ , \ f(a_m)) \ .$$

1.3A Remark: There are obvious monotonicity properties. Here qf stands for quantifier free.

1.4 Fact: If $\lambda_0 \le \lambda_1 \le \cdots \le \lambda_{n-1}$ and L is a finite vocabulary with predicates only and if for $m < n$

$(*)_m$ for every model $N \in K^\alpha_\sigma$ of cardinality λ_m
for some model $M \in K^\alpha_\sigma$ of cardinality λ_{m+1}
$M \rightarrow^{eh} (N)^{<n}_M$ holds.

then for every model $N \in K^\alpha_\sigma$ of cardinality λ_0 there is an model $M \in K^\alpha_\sigma$ of cardinality λ_{n-1} such that $M \rightarrow (N)^{<n}_M$.

Proof: Trivial.

1.4A Remark: We can define canonization relations - say, in how many variables the coloring does not depend. See [EHMR]: [Sh95].

1.5 Lemma: Suppose $\mu = \mu^{<\mu}$, $\sigma < \mu \le \theta < \kappa \le \lambda$, $\alpha,\beta \le \omega$, and $\kappa \rightarrow_{sp} (\mu+1)^{<\mu,<\beta}_{\mu,\theta}$ (see Definition 2.1), let P be the forcing notion defined by: $p \in P$ iff p is a partial function from $[\lambda]^{<\alpha}$ to σ of cardinality $< \mu$ and $p(\emptyset) = 0$ and ordered by inclusion.

Then in V^P for some $M \in K^\alpha_\sigma$, $|M| = \kappa$, and for every $N \in K^\alpha_\sigma$ of order type $\le \mu+1$ we have: $M \rightarrow^{eh} (N)^{<\beta}_\theta$.

1.5A Remark: (1) P is really just adding λ Cohen subsets of μ . (2) If we let $|N| < \mu$, the proof is somewhat simplified.

Proof of Lemma 1.5 : Let $\underset{\sim}{M} = (\kappa, <, F^{\underset{\sim}{M}})$, where $F^{\underset{\sim}{M}}(u) = j$ iff for some $f \in G_P$ $f(u) = j$. Let $\underset{\sim}{d} : [\kappa]^{<\beta} \to \theta$, and $\underset{\sim}{N} \in K^{\alpha}_{\sigma}$ be P-names. Without loss of generality $|\underset{\sim}{N}|$ is an ordinal $\epsilon+1 \le \mu + 1$, and (as $\underset{\sim}{N}$ depends on at most μ of the Cohen subsets and doing P in two stages) without loss of generality $\underset{\sim}{N} = N$. Let χ be a large enough regular cardinal and let $<^*_\chi$ a well-ordering of $H(\chi)$. Let $p^* \in P$. By the hypothesis (see Definition 2.1 and 2.3) (for second phrase in (I) use for $i < \mu$ the function

$$F(x_0,\dots,x_j,\dots)_{j<i} \overset{\text{def}}{=} \langle x_j : j < i \rangle .$$

(*) There is $B \subseteq \kappa$, B of order type $\epsilon + 1$ and $\langle N_u : u \in [B]^{<\beta} \rangle$ and $H_{u,v}$, $(u,v \in [B]^{<\beta}, |u| = |v|)$ such that:

(I) $N_u \prec (H(\chi), \in, <^*_\chi)$, $(\forall a \in [N_u]^{<\mu})[a \in N_u]$ (hence $\mu \subseteq N_u$)

and

$$\{\langle u, \sigma, \kappa, \lambda, P, \underset{\sim}{d}, N \rangle\} \in N_u$$

(II) (b) - (h) of 2.1

(III) $p^* \in P \cap N_\emptyset$.

Let $B = \{\xi_i : i \le \epsilon\}$ and $[i < j \Rightarrow \xi_i < \xi_j]$. First assume $\epsilon < \mu$. Let g be the unique order preserving function from $\epsilon + 1$ onto B, i.e. $g(i) = \xi_i$.

Let for $i \le \epsilon$

$$I_i = [\{\xi_j : j < i \text{ or } j = \epsilon\}]^{<\beta} .$$

Next let for $u \in I_i$, $K(u) = \{(u,h) : h$ is a function from $[u]^{<\alpha}$ to $\sigma, h(\emptyset) = 0\}$

$$J_i = \{(u,h) : u \in I_i \text{ and } (u,h) \in K(u)\} .$$

Note that $[i < j \Rightarrow I_i \subseteq I_j]$, and $[i < j \Rightarrow J_i \subseteq J_j]$. We say $(u_1,h_1) \le (u_2,h_2)$ if $u_1 \subseteq u_2, h_1 \subseteq h_2$.

[**Explanation**: Note that we have already decided that the desired embedding will take i to $g(i)$, so the universe of the image of N will be $I \overset{\text{def}}{=} \{\xi_i : i \le \epsilon\}$. What we have to do is to find a condition in P forcing that the embedding is as required. Now ξ_ϵ is simultaneously a good "approximation" to ξ_j over $\{\xi_i : i < j\}$ and we define a condition for $\{\xi_j : j < i \text{ or } g(j) = \epsilon\}$ by induction on i, though in N, $j_1 \neq j_2$ may realize different quantifier free type over $\{i : i < j_1 \cap j_2\}$. We are saved by dealing simultaneously with conditions $p_{(u,h)}$ for $(u,h) \in J_i$.]

We now define by induction on $i \leq \epsilon$, $\langle p^i_{(u,h)} : (u,h) \in J_i \rangle$ such that:

(a) $p^i_{(u,h)} \in P \cap N_u$, and $p^* \leq p^i_{(u,h)}$

(b) $h \subseteq p^i_{(u,h)}$

(c) if $j < i$, $(u,h) \in J_j$, then $p^j_{(u,h)} \leq p^i_{(u,h)}$.

(d) if $(u_1,h_1) \leq (u_2,h_2)$ then $p^i_{(u_1,h_1)} = p^i_{(u_2,h_2)} \restriction (\lambda \cup N_{u_1})$.

(e) $p^i_{(u,h)}$ forces a value to $\underset{\sim}{d}(v)$ for every $v \subseteq u$.

(f) $p^{i+1}_{u\cup\{\xi_i\},h_i} \geq H_{u\cup\{\epsilon\},u\cup\{\xi_i\}}(p^i_{(u\cup\{\epsilon\},h_\epsilon)})$ if $u \subseteq i$ and h_i, h_ϵ are functions such that $(u \cup \{\xi_i\},h_i) \in J_{i+1}$, $(u \cup \{\xi_i\},<,h_i) \cong (u \cup \{\epsilon\},<,h_\epsilon)$.

We shall carry out the definition in detail.

[**Explanation:** Condition (a) is in order to have control over the conditions and to utilize the indiscernibility.

Condition (b), note it is the role of $p^i_{(u,h)}$ to ensure our being able to deal with the case $h = (F^N \circ (g^{-1})) \restriction u$.

Condition (c) should be clear.

Condition (d) enables us to form the condition $\bigcup_{x \in X} p_{(u_x,h_x)}$ for suitable X .

Condition (e) as we want to form a condition forcing the "right" values of $\underset{\sim}{d}$, we certainly have to have approximations forcing some values for them.

Condition (f) This comes for the end-homogeneity, we want to say that ξ_i, ϵ are similar over $\{\xi_j : j < i\}$, of course the minute we want to force a value to $\underset{\sim}{d}(\{\xi_i,\xi_\epsilon\})$ this similarity cannot be maintained.]

The Induction.

Case A: $i = 0$.

So $I_i = \{\epsilon\}^{<\beta}$, so $I_i = \{\emptyset, \{\epsilon\}\}$ (except when $\beta \leq 1$ which is not so interesting).

For every $(u,h) \in J_i$ we have to define $p^i_{(u,h)}$. Let $u_0 = \emptyset, u_1 = \{\epsilon\}$.

Let us enumerate the functions $h : u_1 \to \sigma$, $h(\emptyset) = 0 : \{h_\gamma : \gamma < \gamma_0\}$, (so $\gamma_0 < \mu$). We define p_γ by induction on γ such that:

(i) $p_\gamma \in N_{\{\epsilon\}} \cap P$

(ii) $h_\gamma \subseteq p_\gamma$

(iii) for every $\beta < \gamma$, $p_\beta \restriction N_\emptyset \leq p_\gamma$

(iv) p_γ forces a value to $\underset{\sim}{d}(\{\epsilon\})$.

There is no problem in doing this by $\oplus_1, \oplus_2$ below. In the end let

$$p^i_{(u_0,\emptyset)} = \bigcup_{\gamma<\gamma_0} p_\gamma \restriction N_\emptyset$$

$$p^i_{(u_1,h_\gamma)} = p_\gamma \cup p^i_{(u_0,\emptyset)}$$

where

$\oplus_1$ if $q_\gamma \in P$ for $\gamma < \gamma(*) < \mu$, then $\bigcup_{\gamma<\gamma(*)} q_\gamma$ is in P and is the least upper bound of $\{q_\gamma : \gamma < \gamma(*)\}$ if and only if for any $\gamma_1, \gamma_2 < \gamma(*)$ the functions $q_{\gamma_1}, q_{\gamma_2}$ are compatible

$\oplus_2$ if $q_1 \in N_{u_1} \cap P$, $q_2 \in N_{u_2} \cap P$ then: q_1, q_2 are compatible if and only if $q_1 \restriction N_{u_1 \cap u_2}$, $q_2 \restriction N_{u_1 \cap u_2}$ are compatible [by 2.1 d].

Case B: i limit.

For any $(u,h) \in J_i$ there is $j_{(u,h)} < i$ such that $(u,h) \in J_{j_{(u,h)}}$

We let $p^i_{(u,h)} \overset{\text{def}}{=} \bigcup\{p^j_{(u,h)} : j < i \text{ and } (u,h) \in J_j\}$. There are no problems in checking the conditions (note: $p^i_{(u,h)} \in N_u$ because

$$[a \in N_u \wedge |a| < \mu \Rightarrow a \in N_u] \text{ by I) }.$$

Case C: $i = j + 1$.

Let us enumerate $J_i = \{(u_\gamma, h_\gamma) : \gamma < \gamma(*)\}$.
Note that $\gamma(*) < \mu$. We define by induction on $\gamma \le \gamma(*)$ a sequence $\langle q^\gamma_{(u,h)} : (u,h) \in J_i \rangle$ such that (compare with (a) - (f) above):

(a)′ $q^\gamma_{(u,h)} \in P \cap N_u$

(b)′ $h \subseteq q^\gamma_{(u,h)}$

(c)′ $\beta < \gamma$ implies $q^\beta_{(u,h)} \le q^\gamma_{(u,h)}$

(d)′ if $(u_1,h_1) \le (u_2,h_2)$ then

$$q^\gamma_{(u_1,h_1)} = q^\gamma_{(u_2,h_2)} \restriction N_{u_1}$$

(e)′ if $\beta < \gamma$ then $q^\gamma_{(u_\beta,h_\beta)}$ forces a value to $\underset{\sim}{d}(v)$ for every $v \subseteq u$.

(f)′ the parallel of (f) .

Subcase C (a): $\gamma = 0$. Define $q^\gamma_{(u,h)}$ as follows:

(α) it is $p^j_{(u,h)}$ if $u \in I_j$

(β) it is $H_{u_1,u}(p^j_{(u,h)})$ if $u \in I_i$, $\xi_i \in u$, $\varepsilon \notin u$, $v \overset{\text{def}}{=} u \setminus \{\xi_i\} \in I_j$; and we let:

$$u_1 \overset{\text{def}}{=} v \cup \{\varepsilon\} ,$$

(γ) it is $p^j_{(u_0,h \restriction u_0)} \cup H_{u_0,u_1}(p^j_{(u_0,h_0)})$ if $u = v \cup \{\xi_j,\varepsilon\}$, $\xi_j \notin v$, $\varepsilon \notin v$, and we let $u_1 \overset{\text{def}}{=} v \cup \{\xi_j\}$, $u_0 = v \cup \{\varepsilon\}$.

Subcase C (b): γ is limit.
Use unions $q^\gamma_{(u,h)} = \bigcup_{\beta \le \gamma} q^\beta_{(u,h)}$.

Subcase C (c): $\gamma = \beta + 1$.
Note that the only demand on γ in $\langle q^\gamma_{(u,h)} : (u,h) \in J_i \rangle$ which is not clearly satisfied by $\langle q^\beta_{(u,h)} : (u,h) \in J_i \rangle$ is (e)′ for β . We first choose $r = r^\gamma$, such that:

(i) $q^{\beta}_{(u_\beta,h_\beta)} \le r \in N_{u_\beta} \cap P$

(ii) r forces a value to $\underset{\sim}{d}(v)$ for every $v \subseteq u_\beta$.

Clearly such r exists. Now for every $(u,h) \in J_i$ let

$q^{\gamma}_{(u,h)} \overset{\text{def}}{=} q^{\beta}_{(u,h)} \cup\cup\{r \restriction N_v : v \subseteq u \cap u_\beta, h \restriction v = h_\beta \restriction v\}$ [note that it is possible that $v_1,v_2 \subseteq u \cap u_\beta$ and $h \restriction v_1 = h_\beta \restriction v_1$, $h \restriction v_2 = h_\beta \restriction v_2$ but $h \restriction (v_1 \cup v_2) \ne h_\beta \restriction (v_1 \cup v_2)$.]

Now $q^{\gamma}_{(u,h)} \in P$ by $\oplus_1,\oplus_2$ and as $q^{\beta}_{(u,h)} \restriction N_v = q^{\beta}_{(v,h\restriction v)} \le q^{\beta}_{(u_\beta,h_\beta)} \le r$ whenever $(v \subseteq u \cap u_\beta, h \restriction v = h_\beta \restriction v)$. It is easy to check that $\langle q^{\gamma}_{(u,h)} : (u,h) \in J_1\rangle$ is as required: i.e. conditions (a)′ – (f)′ are satisfied.

* * *

So we have defined $\langle q^{\gamma}_{(u,h)} : (u,h) \in J_\gamma\rangle$ for $\gamma \le \gamma(*)$ as required, and we can finish Case C: let $p^{i}_{(u,h)} \overset{\text{def}}{=} q^{\gamma(*)}_{(u,h)}$ for $(u,h) \in J_i$.

* * *

So we have finished the definition of $\langle p^{i}_{(u,h)} : (u,h) \in J_i\rangle$ for $i \le \varepsilon$.

Lastly let

$$p^{**} = \cup\{p^{\varepsilon}_{(u,h)} : (u,h) \in J_\varepsilon\ ,\ u \subseteq \varepsilon\ , \text{ and } h(v) = F^N(g^{-1}(v)) \text{ for } v \in [u]^{<\beta}\} .$$

Clearly the union is well defined and forces what we need except when $\alpha > \beta$, then we have to add information to make g an embedding of N to M . So we have finished the case $\varepsilon < \mu$.

Secondly, we assume $\varepsilon = \mu$. We cannot use the definition above as the union will not be a condition (too large cardinality). But we can work in $V[G_P]$, and choose by induction on $i < \mu$, an ordinal α_i , such that $\sup_{j<i} \alpha_j < \alpha_i < \operatorname{Max} B$ and $p^{i}_{(u,h)}$ such that $u \subseteq \{\alpha_j : j < i\} \cup \{\varepsilon\}$ and $(u,h) \in J_\varepsilon$ satisfying the parallel of (a) – (f) above such that: if $u \subseteq \{\alpha_j : j < i\} \cup \{\varepsilon\}$, and $(u,h) \in J_\varepsilon$ and $h(v) = F^N(g^{-1}(v)$ for $v \in [u]^{<\omega}$ then $p^{i}_{(u,h)} \in \underset{\sim}{G}_P$. End of proof of lemma 1.5.

1.6 Conclusion: Assume that there is a class of measurable cardinals. Then in some generic extension

$\forall m,n < \omega \, \forall \theta \, \forall N \in K_\sigma^{<n} (\exists M)[M \in K_\sigma^{<n} \wedge |M| \leq \beth_{m+1}((|N| + \sigma + \theta)^+) \wedge M \to (M)_\theta^m]$.

Proof: Iterate the forcing (with e.g. Easton support) $Q_{\delta+n}$ (δ limit or zero) is adding $\kappa_{\delta+n+1}$ Cohen subsets to $\kappa_{\delta+n}$, where $\kappa_0 = \aleph_0$, for limit ordinal δ, $\kappa_\delta = \bigcup_{\alpha<\delta} \kappa_\alpha$, if κ_δ is singular $\kappa_{\delta+1} = \kappa_\delta^+$. In all other cases $\kappa_{\alpha+1}$ is the first measurable $> \kappa_\alpha$. By 1.5 we get enough instances of $\to^{eh}$. Iterating their use by 1.4 we get the desired conclusion.

§ 2. On $\to_{sp}$.

2.1 Definition: We define $\lambda \to_{sp} (\kappa)_{\mu,\theta}^{<\sigma,<n}$ where $\lambda,\kappa,\sigma,\theta$ are cardinals and $n \leq \omega$. It says that if N is an algebra with universe λ and with $\leq \mu$ operations each with $< \sigma$ places, then there is $A \in [\lambda]^\kappa$ and N_u for $u \in [A]^{<n}$ such that:

(a) N_u is a subalgebra of N

(b) N_u has cardinality μ

(c) $N_u \cap A = u$

(d) $N_u \cap N_v = N_{u\cap v}$ (the main point!)

(e) for $u,v \in [A]^{<n}$ of the same cardinality, $N_u \cong N_v$ the unique isomorphism from N_u onto N_v, order preserving, exists, we call it $H_{u,v}$.

(f) $H_{u,v}$ maps u to v

(g) $H_{u,u}$ = the identity, $H_{u_2,u_3} \circ H_{u_1,u_2} = H_{u_1,u_3}$ and for $u_1 \subseteq u$, $H_{u,v} \restriction N_{u_1} \subseteq H_{v_1,u_1}$ where $v_1 \subseteq v$ is such that $H_{u,v}$ maps u_1 onto v_1 (so equality holds),

(h) for $u \in [A]^{<n}$, $N_u \cap \theta \subseteq N_\emptyset$.

2.2 Definition: (1) We define $\lambda \to_{spn} (\kappa)_{\mu,\theta}^{<\sigma,<n}$ similarly adding

(i) if $v \subseteq u \in [A]^{<n}$, $(\exists\xi) v = u \cap \xi$ then $|N_v|$ is an initial segment

of $|N_u|$;

(2) We omit θ when $\theta = \aleph_0$, i.e. omit (h) in 2.1 .

2.3 Observation: (1) If $\lambda \to_{spn} (\kappa)^{<\sigma,<n}_{\mu,\aleph_0}$ and $\theta < \lambda$ then $\lambda \to_{sp} (\kappa)^{<\sigma,<n}_{\mu,\theta}$.

(2) Those arrows have obvious monotonicity properties: we can decrease κ,σ,n,θ . For $\to_{sp}$ we can increase λ .

(3) In 2.1, 2.2 we can use as N any algebra such that $\lambda \subsetneq |N|$. (See [Sh3] and § 3).

2.4 Fact: (1) If λ is measurable, $\kappa \le \lambda$, $\mu + \theta + \sigma < \lambda$, $n \le \omega$ then $\lambda \to_{spn} (\kappa)^{<\sigma,<n}_{\mu,\theta}$.

(2) If λ is minimal such that $\lambda \to (\kappa)^{<\omega}_{\theta}$, $\theta \ge \mu$ then $\lambda \to_{spn} (\kappa)^{<\omega,<\omega}_{\mu,\theta}$.

2.5 **Lemma**: If $\zeta \ge 3$, $\lambda \to_{sp} (\zeta)^{<\omega,<3}_{\omega,\mu}$ then $\lambda \to (\zeta)^{<\omega}_{\mu}$.

Proof: Let $\chi > 2^\lambda$ be a regular cardinal and let $<^*_\chi$ be a well order of $H(\chi)$. So we have $\langle M_u : u \in [\zeta]^{\le 2} \rangle$ such that:

(a) $M_u \prec (H(\chi),\in,<^*_\chi)$,

(b) $M_u \cap M_v = M_{u \cap v}$

(c) $\lambda \in M_u$

(d) if $|u| = |v|$, M_u,M_v are isomorphic and let $H_{u,v}$ denote the (unique) isomorphism

(e) if $u = \{i_1,i_2\}$, $v = \{j_1,j_2\}$, $i_1 < i_2$ and $j_1 < j_2$ then $H_{\{i_1\},\{j_1\}} \subseteq H_{u,v}$, $H_{\{i_2\},\{j_2\}} \subseteq H_{u,v}$, $id_{M_0} \subseteq H_{\{i_1\},\{j_1\}}$.

(f) $M_{\{i\}} \cap \lambda \ne M_\emptyset \cap \lambda$. (This follows from (c) of 2.1.)

Let $\alpha_i = \alpha(i) = \mathrm{Min}(M_{\{i\}} \cap \lambda - M_\emptyset)$. Clearly $H_{\{i\},\{j\}}(\alpha_i) = \alpha_j$ (use (d),(e)) and $\mu < \alpha_i$. Also $\alpha_i \ne \alpha_j$ (as $M_{\{i\}} \cap M_{\{j\}} = M_\emptyset$) for $i \ne j$.

So for all $i < j < \zeta$, "$\alpha_i < \alpha_j$" has the same truth value. Since

$\alpha_i \neq \alpha_j$ if $\zeta \geq \omega$, as the ordinals are well ordered:

(A) $\langle \alpha_i : i < \zeta \rangle$ is strictly increasing. If $\zeta < \omega$ we could inverse the indexing and also have (A).

Next we shall prove

(B) If $i < j$ and $\bar{c} \in M_{\{i\}}$, then $\bar{c}$ and $H_{\{i\},\{j\}}(\bar{c})$ realize the same type over $\{\gamma : \gamma < \alpha_i\}$ (in $(H(\chi), \in, <^*_\chi)$.)

[Proof: Let $\varphi(\bar{x},\bar{y})$ be a formula, $\lg(\bar{x}) = \lg(\bar{c}), \lg(\bar{y}) = n$. Let $<_{gd}$ be the following order (of Godel) on n-tuples of ordinals: $\bar{\beta} <_{gd} \bar{\gamma}$ if and only if $\mathrm{Max}(\bar{\beta}) < \mathrm{Max}(\bar{\gamma})$ or $\mathrm{Max}(\bar{\beta}) = \mathrm{Max}(\bar{\gamma})$ and $\bar{\beta}$ is smaller than $\bar{\gamma}$ in the lexicographic order.

Let $F_\varphi(\bar{x}_1,\bar{x}_2) =$ the $<_{gd}$-first sequence $\bar{y}$ (n-tuple of ordinals) such that: $\varphi(\bar{x}_1,\bar{y}) \equiv \neg\varphi(\bar{x}_2,\bar{y})$.

Clearly F_φ is definable in $(H(\chi),\in,<^*_\chi)$ hence each M_u is closed under F_φ.

Let $\bar{c}_j = H_{\{i\},\{j\}}(\bar{c})$ for each $j < \zeta$ and assume that $F(\bar{c}_{j_1},\bar{c}_{j_2})$ is defined for some ($\equiv$ all) $j_1 < j_2 < 3$: otherwise (B) is immediate. So $F(\bar{c}_{j_1},\bar{c}_{j_2}) \in M_{\{j_1,j_2\}}$. However by a classical trick, if $j_1 < j_2 < j_3$ then the set $\{F_\varphi(\bar{c}_{j_1},\bar{c}_{j_2}), F_\varphi(\bar{c}_{j_1},\bar{c}_{j_3}), F_\varphi(\bar{c}_{j_2},\bar{c}_{j_3})\}$ has only two members. Assume e.g. that the first two are equal, so

$F_\varphi(\bar{c}_{j_1},\bar{c}_{j_2}) = F_\varphi(\bar{c}_{j_1},\bar{c}_{j_3}) \in M_{\{j_1,j_2\}} \cap M_{\{j_1,j_3\}} = M_{\{j_1\}}$. Generally (according to which of the three possible equalities holds) $F_\varphi(\bar{c}_{j_1},\bar{c}_{j_2})$ belongs to $M_{\{j_1\}}$ or to $M_{\{j_2\}}$. As clearly (for $\ell = 1,2$, as $\bar{c}_{j_m} = F_{\{i\},\{j_2\}}(\bar{c})$)

$F_\varphi(\bar{c}_{j_1},\bar{c}_{j_2}) \geq \sup(\alpha_{j_1} \cap \alpha_{j_2} \cap M_\emptyset) = \sup(\alpha_{j_\ell} \cap M_{\{j_\ell\}})$ we can deduce

$F_\varphi(\bar{c}_{j_1},\bar{c}_{j_2}) \geq \mathrm{Min}\{\alpha_{j_1},\alpha_{j_2}\}$. As φ was any formula, we have finished the proof of (B)].

(C) α_i is strongly inaccessible.

[Proof: Note that all α_i realize the same type in $(H(\chi),\in,<^*_\chi)$.

If each α_i is singular, there is $f_1 \in M_\emptyset$ such that for $\delta < \lambda$ singular, $f_1(\delta)$ is a club of δ of order type $cf(\delta)$. As $cf(\alpha_i) < \alpha_i$ and $cf(\alpha_i) \in M_{\{i\}}$ clearly $cf(\alpha_i) \in M_\emptyset$ hence for some $\theta \in M_\emptyset$, $(\forall i < \zeta)[cf(\alpha_i) = \theta]$. Let $f_2 \in M_\emptyset$ be such that for $\delta < \lambda$ of cofinality $\theta, f_2(\delta)$ is a one-to-one function from θ onto $f_1(\delta)$. So easily $f_1(\alpha_i) \cap M_{\{i\}} = f_1(\alpha_i) \cap M_\emptyset = \{f_2(\alpha_i)(\gamma) : \gamma \in \theta \cap M_\emptyset\}$; w.l.o.g. f_1, f_2 are definable over $\emptyset$ (in the model $(H(\chi),\in,<^*_\chi))$. Now if $i_1 < i_2$, we get a contradiction to (B).

Next if α_i are not strong limit, then there is $\mu < \alpha_i$, $\mu \in M_{\{i\}}$, $2^\mu \geq \alpha_i$. So $\mu \in M_\emptyset$, and by the $H_{\{i\},\{j\}}$'s , $2^\mu \geq \alpha_j$ for each j , so in $M_\emptyset$ there is a (definable from $\emptyset$) one-to-one function from 2^μ to $\mathcal{P}(\mu)$ and we get contradiction to (B).]

(D) W.l.o.g. $M_{\{i,j\}}$ is the Skolem hull of $M_{\{i\}} \cup M_{\{j\}}$.

(E) For $i < j < \zeta$ the intersection of the Skolem hull of $M_{\{j\}} \cup (\alpha_j \cap M_{\{i\}})$ with α_j is included in $M_{\{i\}}$.

[Proof; If not, there are $\bar{c} \in \alpha_j \cap M_{\{i\}}$, $\bar{d} \in M_{\{j\}}$, $y = G(\bar{c},\bar{d})$, G definable, $y \in \alpha_j \setminus M_{\{i\}}$.

Let w.l.o.g. $j < j_1 < \zeta$ (we use that w.l.o.g. $i = 0$, $j = 1$ and remember $\zeta \geq 3$. As $\bar{d}$ and $\bar{d}' \stackrel{\text{def}}{=} H_{\{j\},\{j_1\}}(\bar{d})$ realize the same type over $\{\gamma : \gamma < \alpha_j\}$, clearly $y = G(\bar{c},\bar{d}')$ too, so

$$y \in M_{\{i,j\}} \cap M_{\{i,j_1\}} = M_{\{i\}} \text{ .]}$$

We shall code, for each formula $\varphi(x_0,\ldots,x_{n-1},y)$, φ-types of n-tuples over $\{\gamma : \gamma < \alpha_i\}$ by an ordinal $< 2^{\alpha_i}$.

(F) For each φ , and $n \geq 1$, there is $\bar{c}_{\varphi,n,i} \in M_{\{i\}}$ such that:

(1) $\bar{c}_{\varphi,n,i}$ codes the φ-type of $\langle \alpha_i, \alpha_{i_1}, \ldots, \alpha_{i_{n-1}} \rangle$ over $\{\gamma : \gamma < \alpha_i\}$ whenever $i < i_1 < \ldots < i_{n-1} < \zeta$.

(2) $H_{\{i\},\{j\}}(\bar{c}_{\varphi,n,i}) = \bar{c}_{\varphi,n,j}$.

[**Proof:** For $n = 1$ this is easy.

For $n + 1$ if $i < i_1$, $\bar{c}_{\varphi,n,i}$ can be computed from α_i, $\bar{c}_{\varphi,n,i_1}$ (just think of the meaning) so $\bar{c}_{\varphi,n+1,i} = G(\alpha_i, \bar{c}_{\varphi,n,i_1})$ where G a definable function (over $\emptyset$).

However, by coding such things naturally $\bar{c}_{\varphi,n+1,i}$ is an ordinal $< 2^{\alpha_i}$, hence $< \alpha_1$ (by (C)). So it necessarily belongs to $M_{\{i\}}$ by (E), so (0),(1) holds.

By the way $\bar{c}_{\varphi,n+1,i}$ was defined, also (2) holds.]

If there is $F : [\lambda]^{<\omega} \to \mu$ which is a counterexample to the desired conclusion of 2.5, then such F belongs to $M_\emptyset$ and is definable over $\emptyset$ (in $(H(\chi), \in, <^*_\chi)$!), and $\langle \alpha_i : i < \zeta \rangle$ contradict its choice (by (F) above), so that the lemma 2.5 follows.

§ 3 Refining the combinatorics

3.1. Definition:

(1) For $x \in \{sp, spn\}$ we define

$$\lambda \underset{ex(k)}{\longrightarrow} (\kappa)^{<\sigma,n}_{\mu,\theta} \text{ like } \lambda \underset{x}{\longrightarrow} (\kappa)^{<\sigma,n}_{\mu,\theta}$$

(see definitions 2.1 and 2.2) except that we replace (e), (f), (g) by

$(e)^e$ if $u, v \in [A]^{<n}$ and $u \sim_k v$ (which means that for some w, w is an initial segment of u and v and $|u \setminus w| = |v \setminus w| \le k$) then $N_u \cong N_v$ and let $H_{u,v}$ be the unique isomorphism

$(f)^e$ $H_{u,v}$ when defined maps u onto v; $H_{u,u} = id$

$(g)^e$ if $u_1 \sim_k u_2 \sim_k u_3$, $u_i \in [A]^{<n(*)}$

then $H_{u_2,u_3} \circ H_{u_1,u_2} = H_{u_1,h_3}$ and for any $u_1' \subset u_1$ if $u_2' = H_{u_1,u_2}(u_1')$ then $H_{u_1,u_2} \restriction N_{u_1'} \subseteq H_{u'_1,u'_2}$ (so equality holds).

(2) If $k = 1$ we omit it.

(3) For $x \in \{sp, spn, esp, espn\}$ we define

$$\lambda \underset{wx}{\longrightarrow} (\kappa)^{<\sigma,<n}_{\mu,\theta} \text{ just like } \lambda \underset{x}{\longrightarrow} (\kappa)^{<\sigma,<n}_{\mu,\theta}$$

replacing (d) by $(d)^w$ $N_u \cap N_v \subset N_{u\cap v}$ and if $\alpha < \beta$ are from β, $u \in [\beta]^{<n}$ and $(\alpha,\beta) \cap u = \emptyset$ then $(\alpha,\beta) \cap M_u = \emptyset$.

(Note that now in (g) equality does not follow.)

(4) For any of the x for which $\underset{x}{\longrightarrow}$ was defined $\lambda \underset{xv}{\longrightarrow} (\kappa)^{<\sigma,<n}_{\mu,\theta}$ is defined as above except that also $d : [\lambda]^{<n(*)} \to \theta$ is given and (h) is replaced by:

$(h)^v$ For each ℓ, $d \restriction [A]^{\ell}$ is constant when e does not appear in x ; and $h(u)$ ($u \in [A]^{<n}$) does not depend on $\max(u)$ when it appears and, more generally, exv(k) does not depend on the last k members of u (i.e. if $u_1,u_2 \in [A^{\ell}]$ and w is a common initial segment of u_1,u_2 $|u_\ell - w| \le k$ then $h(u_1) = h(u_2)$).

3.2. Observation:

(1) We have $\lambda \underset{x}{\longrightarrow} (\kappa)^{<\sigma,<n}_{\mu,\theta} \Rightarrow \lambda \underset{y}{\longrightarrow} (\kappa)^{<\sigma,<n}_{\mu,\theta}$ where: y is x when we omit n or x is y when we omit e or w or v.

(2) If (e appears in x and) $\lambda_2 \underset{x(k)}{\longrightarrow} (\lambda_1)^{<\sigma,<n}_{\mu,\theta}$ and $\lambda_1 \underset{x(\ell)}{\longrightarrow} (\lambda_0)^{<\sigma,<n}_{\mu,\theta}$ then $\lambda_2 \underset{x(k+\ell)}{\longrightarrow} (\lambda_0)^{<\sigma,<n}_{\mu,\theta}$.

(3) If $\lambda \underset{x(k)}{\longrightarrow} (\kappa)^{<\sigma,<n}_{\mu,\theta}$ and $k \ge n-1$ then $\lambda \underset{y}{\longrightarrow} (\kappa)^{<\sigma,<n}_{\mu,\theta}$ where y is x with e omitted.

(4) If $\lambda_2 \underset{x(k)}{\longrightarrow} (\lambda_1)^{<\sigma,<n}_{\mu,\theta}$, and $\ell = n-1-k$, y is x with e omitted and $\lambda_1 \to (\lambda_0)^{\ell}_{2^M}$ then $\lambda_2 \underset{y}{\longrightarrow} (\lambda_0)^{<\sigma,<n}_{\mu,\theta}$.

3.2A Remark. By 3.2(4) even $\lambda \underset{x(0)}{\longrightarrow} (\kappa)^{<\sigma,<n}_{\mu,\theta}$, when $n \ge 3$, is quite strong (when w does not appear in x).

3.3. Definition. $\lambda \longrightarrow^{+} (\kappa)^{<\omega}_{\mu}$ means: for each club $C \subseteq \lambda$ and for $n < \omega$, $i < \mu$, $F_{n,i} : [\lambda]^n \to \lambda$ there is $A \in [C]^{\kappa}$ such that if $\alpha_0 < \dots < \alpha_{n-1}$ belongs to A and $m < n$, $i < \mu$ and $F_{n,i}(\{\alpha_0, \dots, \alpha_{n-1}\}) < \alpha_m$ then it does not depend on $\alpha_m, \dots, \alpha_{n-1}$.

3.3.A. Remark: Replacing "a club $C \subseteq \lambda$" by "a final segment $C \subseteq \lambda$" does not change anything except that in the later version, if $\lambda = \bigcup_{i<l} \lambda_i$, each λ_i satisfying the first definition the λ satisfies the second definition.

3.4. Fact. If $\mu \le \theta < \lambda$, $\lambda \overset{+}{\longrightarrow} (\kappa)^{<\omega}_{\mu}$ then $\lambda \underset{\text{spn}}{\longrightarrow} (\kappa)^{<\omega,<\omega}_{\mu,\theta}$ (κ can be, in fact, any limit ordinal, $\omega\kappa = \kappa$).

3.5. Lemma. If $\lambda \underset{\text{spn}}{\longrightarrow} (\xi)^{<\omega,<3}_{\omega,\mu}$, $\xi \ge 3$ then $\lambda \overset{+}{\longrightarrow} (\xi)^{<\omega}_{\mu}$.

Proof. Similar to the proof of 2.5 but by the definition of $\underset{\text{spn}}{\longrightarrow}$ we know $\sup(N_\emptyset \cap \lambda) < \alpha_0 = \lambda$. In the end, if there is a sequence $\langle C, \langle F_{n,i} : n < \omega, i < \mu \rangle \rangle$ contradicting the conclusion, wlog it is definable over $\emptyset$ as $C \in M_\emptyset$ and easily $\alpha_i \in C$ and continue as before.

3.6. Lemma. (1) For every $n < \omega$, there is $k = k^1_n < \omega$ (e.g. $k = (2n-1)^2$) such that : if $\kappa^{<\sigma} = \kappa$ then $\beth_k(\kappa)^+ \underset{\text{wsp}}{\longrightarrow} (\kappa^+)^{<\sigma,<n}_{\kappa,\kappa}$.

(2) $\forall n < \omega\ \exists\ k = k^2_n < \omega$ such that : if $\sigma, \mu, \kappa < \lambda$ and λ is $(\alpha+k)$-Mahlo strongly inaccessible cardinal then $\lambda \underset{\text{wspn}}{\longrightarrow} (\kappa)^{<\sigma,<n}_{\mu,\mu}$.

Remark. 1. Using part 1 for 4.1 note that $\underset{\text{wsp}}{\longrightarrow}$ is stronger than $\underset{\text{wesp}}{\longrightarrow}$.

2. Part 2 is used for e.g. consistency of $2^{\mu} \to [\mu^+]^2_{0,3}$.

3. We do not try here to get the best bound (but see 3.8 and see [4]).

Proof. (1) Let $\lambda_0 = \kappa$, $\lambda_{\ell} + 1 = \beth_{2n+1}(\lambda_\ell)$. Suppose N^* is an algebra with universe λ_n and at most κ functions each with $< \sigma$ places. We define, by induction on $m \le n$, a set A_m and N^m_u ($u \in [A_m]^{<n}$) :

I (i) $A_0 = \lambda_n$.

(ii) $A_{m+1} \subset A_m$.

(iii) $|A_m| = \lambda_{n-m}$.

II (i) N^m_u (for $u \in [A_m]^{<n}$) is a submodel of N^* of cardinality $\le \kappa$.

(ii) The answer to "is the γ_1-th element of N^m_v equal to the γ_2-th element of N^m_u ?" where $u,v \in [A_{m+1}]^{<n}$ depends just on γ_1,γ_2 and the isomorphism type of $(u \cup v, u, v, < \restriction (u \cup v))$.

(iii) If $u,v \in [A_m]^{<n}$ then

$$|N^m_u| \cap |N^m_v| \subseteq N^{m+1}_{u \cap v} .$$

(iv) If $u \in [A_{m+1}]^{<n}$, $|u| \ge n - 1 - m$ then $N^m_u = N^{m+1}_u$.

For $m = 0$ let $A_0 = \lambda_n$, N^m_u is the subalgebra of N^* generated by the set $u \cup \kappa$. If $m < n - 1$, choose $A_{m+1} \subset A_m$ $|A_{m+1}| = \lambda_{n-m-1}$ such that II.(ii) holds (using $\lambda_{n-m} \to (\lambda_{n-m-1})^{2(n-1)}_{2^\kappa})$. Now for $u \in [A_{m+1}]^{<n}$, let N^{m+1}_u be: if $|u| < n - 1 - m$, the Skolem hull of $\bigcup\{N^m_w \cap N^m_v : w,v \in [A_{m+1}]^n$ and $u = w \cap v\}$ and if $|u| \ge n - 1 - m$, N^m_u . The cardinality of N^{m+1}_u is $\le \kappa$ by II (ii) i.e. if $X \in N^m_w \cap N^m_v$ and $w_1,v_1 \in [A_{m+1}]^{<n}$,

$$|w_1| = |w|, |v_1| = |v|, w_1 \cap v_1 = w \cap v$$

and $(\forall\alpha \in w\cup v)(\exists\beta \in w_1 \cup v_1)\ [|w\cap\alpha| = |w_1\cap\beta| \wedge |v\cap\alpha| = |v_1\cap\beta|]$ then $X \in N^m_{w_1} \cap N^m_{v_1}$.

(2) We need 3.7 below instead of Erdos-Rado and then the proof is similar to that of part 1.

3.7. Lemma: If λ is $(\alpha+n)$-Mahlo and strongly inaccessible and N^* is an algebra with universe λ and $< \lambda$ operations each with arity $< \lambda$ and A_0 is unbounded in λ <u>then</u> for every $\mu < \lambda$ there is $\kappa : \mu < \kappa < \lambda$, κ is α-Mahlo and strongly inaccessible and there is $A \subset A_0 \cap \kappa$ unbounded in κ :

(*) If for $\ell = 1,2$ $\beta < \alpha^\ell_0 < \dots \alpha^\ell_{n-1}$ are from A then

$$\langle\alpha^1_0, \dots \alpha^1_{n-1}\rangle , \langle\alpha^2_0, \dots, \alpha^2_{n-1}\rangle$$

realize the same type over $\{\gamma : \gamma < \beta\}$ is N^* .

3.7A Historical Remark: We proved 3.7 in 1968 as part of some research on transfer theorems in model theory. As Schmerl was doing parallel research, it appeared in [ScSh20] but somehow this version does not appear - only the version with a finite conclusion. Subsequently Schmerl found a better lower bound for λ (how Mahlo it should be) and proved that it was exact. Hajnal independently proved 3.7 and the author wrongly told him it had appeared in [ScSh20].

Proof: We prove it by induction on n.
For $n = 0$ there is nothing to prove. For $n > 1$ use the induction hypothesis to find $\kappa < \lambda$ which is $(\alpha+1)$-Mahlo and $A_0 \subset \kappa$ as there for $n - 1$. Expand $M \restriction \kappa$ by a predicate for A_0 and (as $n > 1$) apply the induction hypothesis for $n = 1$. For $n = 1$, let $C = \{\kappa < \lambda : \kappa$ is a strong limit and for each $\mu < \kappa$, there is $(N,A \restriction N) \prec_{L_{\mu,\mu}} (M,A)$ such that $\alpha \subsetneq |N| \subsetneq \kappa\}$. Clearly C is a club of λ, so there is $\kappa \in C$ which is α-Mahlo.

Choose $\gamma \in A - \kappa$, define a function $f : \kappa \to \kappa$ by $f(\alpha) = \min\{\gamma' \in A \cap \kappa : \gamma'$ realizes the type of γ over $\{i : i \leq \alpha\}\}$. Let $C' = \{\beta < \kappa : (\forall\alpha < \beta) f(\alpha) < \beta\}$. Clearly C' is a club of κ and $A_0 = \{f(\beta) : \beta \in C'\}$ is as required.

3.8. Lemma

Suppose $\lambda = \theta^+, \theta^\kappa = \theta$, $\kappa^\mu = \kappa$, $\mu^{<\sigma} = \mu$, τ is a vocabulary such that $|\tau| \leq \mu$ and each member of τ has arity $< \sigma$. *If* M is a τ-model with universe λ *then* we can find δ, α, B, $\langle M_s : s \in [B]^{\leq 2} \rangle$, $\langle M^-_{\{i\}} : i \in B \rangle$ $\langle H_{s,t} : |s| = |t| ; s,t \in [B]^{\leq 2} \rangle$ and W such that:

(a) $\delta < \lambda$, $\mathrm{cf}\delta = \mu^+$.

(b) B is a subset of δ of order type μ^+ (we could get $\mu^+ + 1$, actually but then $M_{\{\max B\}}$ is not defined).

(c) $M_s \prec_{L_{\mu,\sigma}} M$ for $s \in [B]^2$ and $M^-_{\{i\}} \prec_{L_{\mu,\sigma}} M_{\{i\}} \prec_{L_{\mu,\sigma}} M$ for $i \in B$ and

$M_\emptyset \prec_{L_{\mu,\sigma}} M$.

(d) $M_s \cap B = s$, $M_{\{i\}} \cap B = M^-_{\{i\}} \cap B = \{i\}$, $M_\emptyset \cap B = \emptyset$.

(e) For $s,t \in [B]^{\le 2}$, $|s| = |t|$: $H_{s,t}$ is an isomorphism from M_s onto M_t (and $H_{s,s} = id_{M_s}$, $H_{s,t} = H^{-1}_{t,s}$ and $H_{s_0,s_2} = H_{s_1,s_2} \circ H_{s_0,s_1}$) and $H_{\{i\},\{j\}}$ maps $M^-_{\{i\}}$ onto $M^-_{\{j\}}$.

(f) All $H_{s,t}$ are compatible; $H_{s,t}$ maps s onto t .

(g) $M_s \cap M_t \subseteq M_{s\cap t}$.

(h) $\forall i < j < k$ from B

(α) $M_{\{i,j\}} \cap M_{\{i,k\}} = M_{\{i\}}$,

(β) $M_{\{i,j\}} \cap M_{\{j,k\}} = M^-_{\{j\}}$,

(γ) $M_{\{i,k\}} \cap M_{\{j,k\}} = M^-_{\{k\}}$.

(i) For $i < j$ from B , j is the first element of $M_{\{i,j\}} \setminus M_{\{i\}}$.

(j) $M_\emptyset \subset M^-_{\{k\}}$, $M_\emptyset \subset M_s$ for $k \in B$, $s \in [B]^{\le 2}$

(k) (α) $W \subset \lambda$, $\forall \xi \in W$ $cf\xi = \mu$

(β) $\delta = \max W$

(γ) If $H_{s,t}(\beta) = \gamma$ and $k \in W$ then $\beta < k \equiv \gamma < k$

(δ) If $\beta \in M_{\{i\}} - M_\emptyset, i \in B$, $\kappa = \min\{\xi : \xi \in W , \beta < \xi\}$ then $\xi \neq \beta$ and $\langle H_{\{i\},\{j\}}(\beta) : j \in B\rangle$ is increasing converging to δ .

(ϵ) If $\beta_1,\beta_2 \in M_{\{i\}} - M_\emptyset$, $\xi_e = \min\{\xi \in W : \beta_e \le \xi\}$, $\xi_1 \neq \xi_2$, and $i < j \in B$ then $H_{\{i\},\{j\}}(\beta_1) > \beta_2$.

3.8A **Remark:** (1) We can instead " $\lambda = \theta^+$ " assume λ is inaccessible $\forall\alpha < \lambda$ $[|\alpha|^\kappa < \lambda]$. Similarly for μ .

(2) For simplicity, θ,κ,μ,σ are regular and $<$ is a relation of M .

(3) We can replace $L_{\mu,\sigma}$ by any fragment of $L_{\mu^+,\sigma}$ of cardinality μ .

Proof: Let $M_0 \prec_{L_{\theta,\kappa^+}} M$ where $|M_0|$ is an ordinal $\delta_a < \lambda$ of cofinality θ , (or at least κ^+) so $\|M_0\| = \theta$. Let $N_a \prec_{L_{\kappa,\mu^+}} M$, $\delta_a \in N_a$, $|N_a| = \kappa$. By the choice of M_0 there is a model $N_b \prec_{L_{\kappa,\mu^+}} M_0$ and an isomorphism f from

N_a onto N_b over $N = M \restriction (|N_a| \cap |M_0|)$. Let $\delta_b = f(\delta_a)$. Let $N^* \prec_{L_{\mu,\sigma}} M$, $|N^*| = \mu$ be such that $\delta_a \in N^*$ and $\{N, N_a, N_b, f\} \in N^*$ in some coding. We let $M^-_{\{\delta_a\}} = N^* \restriction |N_a|$ and $M^-_{\{\delta_b\}} = N^* \restriction |N_b|$; $M_{\{\delta_b\}} = N^* \restriction |M_0|$, $M_{\{\delta_b,\delta_a\}}$ = the Skolem hull in M of $M^-_{\{\delta_a\}} \cup M_{\{\delta_b\}}$. Let

$h : |M_{\{\delta_b\}}| \to |N_a| : h(\beta) = \min\{\gamma : \gamma \in |N_a| , \beta < \gamma\}$. Let

$W = \text{range } (h) - \{\delta_a\}$. Let α be minimal element of N such that

$(\forall\beta)[\beta \in M_{\{\delta_b\}} \wedge (\exists\gamma \in N)\ \beta < \gamma \Rightarrow \beta < \alpha]$ i.e. $\alpha = \sup W$.

Now we define by induction on $\zeta < \mu^+$, δ_j , $M_{\{\delta_\zeta\}}$, $M^-_{\{\delta_\zeta\}}$, $M_{\{\delta_\xi,\delta_\zeta\}}$ for $(\xi < \zeta)$, $M_{\{\delta_\zeta,\delta_a\}}$ and $H_{\{\xi\},\{\zeta\}}$, $H_{\{\xi,\delta_a\},\{\zeta,\delta_a\}}$ for $\xi < \zeta$ (understand δ_ζ to be the ζ-th member of B) such that: the relevant cases of the desired conclusion holds, and $M_{\{\delta_\zeta\}} \subset N$, for $\xi < \zeta$, $M_{\{\delta_\xi,\delta_\zeta\}} \subset N$, $M_{\{\delta_\zeta,\delta_a\}} \subset N_a$, etc. and lemma 3.8 is proved.

3.9. Lemma: Suppose GCH for simplicity $\mu = \mu^{<\sigma}$, $\kappa = \kappa^\mu$, $\lambda \geq \kappa^{++}$, $\sigma < \mu < \kappa < \kappa^+ < \lambda$ are regular. There is a forcing notion P such that:

I A. P is strategically κ^+-complete.

B. P preserves cardinalities and cofinalities.

C. $|P| = \lambda$.

II (In V^P)

(*) There are $S^* \subset S \subset \lambda$, $\{C_\delta : \delta \in S\}$ and for $\delta \in S^*$, τ_δ, $\langle M^*_{\delta,s} : s \in [B^*_\delta]^{\leq 2}\rangle$, $\langle M^-_{\delta,s} : s \in [B_\delta]^1\rangle$, $H^\alpha = \langle H^\delta_{s,t} : s,t \in [B_\delta]^{\leq 2}$, $|s| = |t|\rangle$, W_δ, ξ_δ, ζ_δ, τ .

(A) The relevant conclusion of 1.1 holds for each $\delta \in S^*$ with B_δ an unbounded subset of $\xi_\delta < \kappa^{++}$, $\xi_0 = \min W < \kappa^+$.

(B) If $\xi_{\delta(1)} = \zeta_{\delta(2)}$ then there is a function $H_{\delta(n),\delta(2)}$ from $\bigcup_s M_{\delta(1),s} \cup W_{\delta(1)} \cup C_{\delta(1)}$ onto $\bigcup_s M_{\delta(2),s} \cup W_{\delta(2)} \cup C_{\delta(2)}$ which is order-preserving and preserves all relevant properties and the domain and range are disjoint.

(C) $\delta \in S^* \Rightarrow \mathrm{cf}\,\delta = \mu$ (follows from (A)) and for $\delta \in S$ C_δ is a club of δ of cardinality $\le \mu$ and

(D) if $\delta \in S^*$, α is an accumulation point of C_δ then $\alpha \in S \wedge C_\alpha = C_\delta \cap \alpha$ (follows from (B)).

(E) For $\delta \in S^*$, $W_\delta \subset C_\delta$.

Proof. If $\lambda = \kappa^{++}$ we shall force by approximations of cardinality κ . If we succeed to force for λ , we can force for λ^+ by approximations of cardinality κ^+ . For $\lambda = \kappa^{+n}$, we iterate this, for $\lambda > \kappa^{+\omega}$ we have to take care of the singular case.

§ 4. Eliminating the Measurables

4.1. Lemma.

(1) Suppose $\mu = \mu^{<\mu}$, $\mu \le \theta < \kappa \le \lambda$, $\alpha \le \omega$, $\beta(*) < \omega$ and $\kappa \underset{\text{wesp}}{\longrightarrow} (\mu+1)^{\mu,<\beta(*)}_{\mu,\theta}$ (see def. 3.1).

Let P be the forcing action as in 1.5. Then in V^P for some $M \in K^\alpha_\sigma$ of cardinality κ for every $N \in K^\alpha_v$ of power $\le \mu$, $M \overset{\text{eh}}{\rightarrow} (N)^{<\beta(*)}_\theta$.

(2) In applying $\underset{\text{wesp}}{\longrightarrow}$ we can weaken it replacing (d) in 3.1 (3) by

(d)$^-$: if $u \cup v \cup \{\alpha,\beta\} \subset A$, $(\forall i \in u \cup v)[i < \alpha \wedge i < \beta] \wedge [|u|,|v| < n-1]$ then $N_{u\cup\{\alpha\}} \cap N_{v\cup\{\beta\}} \cap \lambda \subset N_u \cap N_v$. However we still need $N_u \cap A = u$! but $\lambda = {}_{\beta(*)-2}(\kappa^{++})$ suffices for κ^+ .

Proof: We indicate the changes in the proof of 1.5. Of course, we replace "(b) to (h) of 2.1" by the appropriate variants from definition 3.1 (3). Defining $p^i_{(u,h)}$ for $(u,h) \in J_i$ by induction on i we change (d) to:

(d) *If* (u_1,h_1), (u_2,h_2) (both from J), are compatible (i.e. $h_1 \restriction (u_1 \cap u_2) = h_2 \restriction (u_1 \cap u_2)$) *then*

$$p^i_{(u_1,h_1)} \restriction N_{u_1} \cap N_{u_2} \cap \lambda = p^i_{(u_2,h_2)} \restriction N_{u_1} \cap N_{u_2} \cap \lambda$$

and in case (C) we change (d)′ to

(d)′ If $(u_1,h_1),(u_2,h_2)$ are compatible

$$q^{\gamma}_{(u_1,h_1)} \restriction (N_{u_1} \cap N_{u_2} \cap \lambda) = q^{\gamma}_{(u_2,h_2)} \restriction (N_{u_1} \cap N_{u_2} \cap \lambda) .$$

In subcase (C)(a), we use (d) above (this influence (γ) there) and in the proof of subcase (C)(c), we let, for $(u,h) \in J_i$; $q^{\gamma}_{(u,h)}(w)$ is defined iff $w \in \mathrm{dom}\, q^{\beta}_{(u,h)}$ or $r(w)$ is defined and $w \in [N_u \cap \kappa]^{<\alpha(*)}$. The value of $q^{\gamma}_{(u,h)}(w)$ is $q^{\beta}_{(u,h)}(w)$ when defined and $r(w)$ otherwise. Let us check (a)′ - (f)′ .

(a)′ Trivially $q^{\gamma}_{(u,h)} \in P$ as $q^{\gamma}_{(u,h)} \subset N_u$ (as a set of pairs, by its definition) clearly $q^{\gamma}_{(u,h)} \in N_u$ by the demand $((\forall a \in [N_u]^{<\mu})[a \in N_u]$ from I in the beginning of the proof of 1.5).

(b)′ as $h \subset q^{\beta}_{(u,h)} \subset q^{\gamma}_{(u,h)}$

(c)′ $\beta' < \gamma$ implies $\beta' < \beta$ or $\beta' = \beta$ and check

(d)′ assuming $(u_1,h_1),(u_2,h_2)$ are compatible we have

$$q^{\beta}_{(u_1,h_1)} \restriction (N_{u_1} \cap N_{u_2} \cap \lambda) = q^{\beta}_{(u_2,h_2)}(N_{u_1} \cap N_{u_2} \cap \lambda) .$$

As clearly, $\mathrm{dom}\, q^{\gamma}_{(u_e,h_e)} = \mathrm{dom}\, q^{\beta}_{(u_e,h_e)} \cup (\mathrm{Dom}\,(r) \cap N_{u_e})$ the equality of the domains is easy, similarly check equalities of values.

(e)′ (f)′ immediate.

4.2. Conclusion: Assume, for simplicity only, that V satisfies GCH . Then in some generic extension, not collapsing cardinals nor changing cofinalities,

(a) $2^{\aleph_\alpha} < \aleph_{\alpha+\omega}$ for every α

(b) for every $n < \omega$ and model $N \in K^{<n}_{\sigma}$ and $m < \omega$ and θ for some $k < \omega$ and model M , $|M| < \ _k(\|N\| + \sigma + \theta)$ and $M \to (N)^m_\theta$. (By 3.6 (1) (see remark) and 4.1.)

Proof: Like 1.6 using 4.1 instead of 2.5.

§ 5 $K_4 \subseteq G \to (3)^2_{\aleph_0}$

The question we address is an old one of Erdos and Hajnal. K_n is the complete graph with n vertices.

Question: Is there a graph G which embeds no K_4 such that $G \to (3)^2_{\aleph_0}$?

We get here the consistency of a slightly stronger statement. We still deal with graphs although the proof says something more general. More on the case we are interested in (forbidden infinite subgraphs) will appear later.

5.1. **Lemma**: Suppose $\mu < \lambda < \kappa$, κ is measurable (or just $\kappa > l(\lambda)$ or $\lambda \to_{wsp} (2k(*))^{\omega,<3}_{\mu,\mu}$) , $2 \le m < \omega_1$ and $\lambda = \lambda^{<\lambda}$. For some λ^+-c.c. λ-complete forcing notion P of power κ , $\Vdash_P$" $2^\lambda = \kappa$ and for some graph G of power κ ,

(i) $G \to (K_{k(*)})^2_\mu$

(ii) G embeds no $K_{k(*)+1}$."

Proof: The forcing P introduces just the graph G . Let $|G|$, the set of vertices of G be

$$[\kappa]^m_{inc} = \{(\alpha_0,\dots,\alpha_{m-1}) : \alpha_0<\dots<\alpha_{m-1} < \kappa\} .$$

We say $\eta = (\alpha_0,\dots,\alpha_{m-1})$, $\nu = (\beta_0,\dots,\beta_{m-1})$ from $[\kappa]^m_{inc}$ are potentially connected if $\alpha_0 < \beta_0 < \alpha_1 < \beta_1 <\dots<\alpha_{m-1} < \beta_{m-1}$ (or interchange them). Let $P = \{G : K_{k(*)+1}$ is not embeddable into G and G is a graph as above on $[\mathrm{dom}(G)]^m_{inc}$ where $\mathrm{dom}\, G$ is a subset of κ of power $< \lambda\}$. We say $G_1 < G_2$ if and only if $G_1 = G_2 \restriction [\mathrm{dom}\, G_1]^m_{inc}$. Clearly P is λ-complete, $P \vDash \lambda$-c.c., $\Vdash_P$ "$2^\lambda = \kappa$" and $|P| = \kappa$. Let $\underset{\sim}{Gr}$ be the P-name of $\bigcup\{L : L \in \underset{\sim}{G}_P\}$. It is a graph of the right form. Let $\underset{\sim}{d}$ be a P-name of a function from the set of edges of $\underset{\sim}{Gr}$ to μ and $p \in P$. Let χ be large enough. By the choice of κ and the partition theorem, we can find $U \subset \kappa$ such that $|U| = \lambda$ (U is really larger but this does not help). Let $I^\alpha = \{s \subset U : |s| \le 2^m\}$ and let $\{M_s : s \in I^\alpha\}$ be such that $U \cap M_s = s$,

$(\forall\alpha,\beta)[\alpha < \beta \wedge \alpha \in U \wedge \beta \in U \Rightarrow (\alpha,\beta) \cap M_\emptyset = \emptyset]$; $M_s \cap M_t = M_{s\cap t}$ (or just $M_s \cap M_t \subset M_{s\cap t}$) $(\forall a \subset M_s)(|a| < \lambda \Rightarrow a \in M_s)$ and $\|M_s\| = \lambda$ and for $s,t \in I^\alpha$, $|s| = |t|$ we have

$H_{s,t} : M_s \to M_t$, an isomorphism onto, so that $H_{s,t}(s) = t$ all the diagrams commute and $\bigwedge_s [p,P,\lambda,\mu,\kappa,\underset{\sim}{G}_r,\underset{\sim}{d} \in M_s]$.

Now we want to find $p \leq q \in P$ such that q forces a monochromatic $K_{k(*)}$. Let $\eta_e = (\alpha_0^e,\alpha_1^e,\ldots,\alpha_{m-1}^e) \in [U]^m$ for $e < k(*)$ such that

$$\alpha_0^0 < \alpha_0^1 < \alpha_0^2 < \ldots \alpha_0^{k(*)-1} < \alpha_1^0 < \alpha_1^1 < \ldots \alpha_1^{k(*)-1} < \ldots < \alpha_i^e \in U ,$$

$$t_e = \text{range } \eta_e .$$

We shall find a condition $q \geq p$. If $q \in P \cap M_{t_0}$, $p \leq q$ then we can find $r \in M_{t_0 \cup t_1} \cap P$ and $\xi < \mu$ such that 1. – 6. below holds, where

1. $r \Vdash (\eta_0,\eta_1) \in$ edges of $\underset{\sim}{Gr}$
2. $r \restriction M_{t_0} \geq q$
3. $r \restriction M_{t_1} \geq h_{t_0,t_1}(q)$
4. $r \Vdash \underset{\sim}{d}(\eta_0,\eta_1) = \xi$
5. $r \Vdash \forall x,y \in$ vertices (q), if $x,y \notin M_\emptyset$

$$[\{x,y\} \neq \{\eta_0\} \Rightarrow \langle x,h_{t_0,t_1}(y)\rangle \notin \text{edges of } \underset{\sim}{Gr}].$$

6. if $r \restriction M_{t_0} \leq q' \in P \cap M_{t_0}$ and $q' \restriction M_\emptyset = q'' \restriction M_\emptyset$ and $r \restriction M_{t_1} \leq q'' \in P \cap M_{t_1}$ then we can find r' such that $q',q'' \leq r' \in M_{t_0 \cup t_1}$ and r' satisfies 1 – 4 and: $[x \in$ vertices $q' - M_\emptyset$ and $y \in$ vertices $q'' - M_\emptyset$ and $(x,y) \in$ edges $r' \Rightarrow (xy) = (\eta_0,\eta_1)$ and $r' \Vdash$ "$\underset{\sim}{d}(\eta_0,\eta_1) = \xi$" .

As P is λ-complete also $P \cap M_{t_0}$ is μ^+-complete so there are q_0,ξ_0 , such that $p \leq q_0 \in P \cap M_{t_0}$ and: $\forall q : q_0 \leq q \in P \cap M_{t_0}$ we can find r as above for $\xi = \xi_0$. Note $\Vdash$ "the distance in $\underset{\sim}{Gr}$ of η_0 from vertices in $\underset{\sim}{Gr} \cap M_\emptyset \cap \alpha_0$ is $\geq m$".

Now we can find $\xi_0 < \mu$ and $\langle q_\ell^0 : \ell < k(*)\rangle$ such that

(i) $q^0_\ell \in M_{t_\ell}$

(ii) $q^0_\ell \restriction M_\emptyset = q^0_0 \restriction M_\emptyset$

(iii) $h_{t_0,t_i}(q) \le q^0_\ell$

(iv) for $\ell_1 < \ell_2 < k(*)$ we have: if $q^0_{\ell_1} \le q' \in M_{t_{\ell_0}}$ and $q^0_{\ell_2} \le q'' \in M_{t_{\ell_1}}$ and $q' \restriction M_\emptyset = q'' \restriction M_\emptyset$ then we can find r as above.

[Why? We define, by induction on $i < k(*)$, $\langle q^{0,i}_\ell : \ell \le i\rangle$ such that $\langle q^{0,i}_\ell : \ell \le i\rangle$ satisfies (i),(ii),(iii),(iv) above with the natural restrictions. For $i = 0$, $q^{0,0}_0 = q_0$. For $i = j+1$ apply the assertion above (before 1. - 6.) so with $h_{t_\ell,t_0}(q^{o,j}_j)$ here standing for q there; get there r and let

$$q^{0,i}_i = h_{t_0 \cup t_1, t_j \cup t_i}(r \restriction M_{t_1})$$
$$q^{0,i}_j = h_{t_0 \cup t_1, t_j \cup t_i}(r \restriction M_{t_0}) ,$$

and for $\ell < j$, $q^{0,i}_\ell = q^{0,j}_\ell$.

In the end let $q^0_\ell = q^{o,k(*)-1}_\ell$.

Let $\{(\beta_\ell,\gamma_\ell) : \ell \le \binom{k(*)}{2} = m\}$ list the increasing pairs. Now we define by induction on $\ell \le \binom{k(*)}{2}$

$$\{q^\ell_\beta : \beta \le k(*)\} , \quad r_{\beta_\ell,\gamma_\ell} \text{ such that:}$$

1. $q^{\ell_1}_\beta \le q^{\ell_2}_\beta$ for $\ell_1 \le \ell_2$
2. $q^\ell_\beta \in M_{t_\beta}$
3. $q^\ell_{\beta_1} \restriction M_\emptyset = q^\ell_{\beta_2} \restriction M_\emptyset$
4. $r_{\beta_\ell,\gamma_\ell} \restriction M_{t_{\beta_\ell}} \le q^{\ell+1}_{\beta_\ell}$
5. $r_{\beta,\gamma} \restriction M_{t_{\gamma_\ell}} \le q^\ell_{\gamma_\ell}$
6. $r_{\beta,\gamma} \Vdash \underset{\sim}{d}(\eta_\beta,\eta_\gamma) = \xi_0$

7. If e_{ℓ_i} is an edge of $r_{\beta_{\ell_i},\gamma_{\ell_i}}$ not in

$(M_{t_{\beta_{\ell_i}}} \times M_{t_{\beta_{\ell_i}}}) \cup (M_{t_{\gamma_{\ell_i}}} \times M_{t_{\gamma_{\ell_i}}}) \cup \{(\eta_{\beta_{\ell_i}},\eta_{\gamma_{\ell_i}})\}$ for $i \in 0,1$ and $\ell_0 \neq \ell_1$

then e_0,e_1 have no vertex in common.

8. If $\gamma \notin \{\alpha_\ell,\beta_\ell\}$ then $\text{edges}(q^{\ell+1}_\gamma) = \text{edges}(q^\ell_\gamma) \cup \text{edges}(q^{\ell+1}_{\alpha_\ell} \restriction M_\emptyset)$.

There is no problem in this - q_0 is tailor-made for this.

Now we define q :

$$\text{dom } q = \bigcup_{\beta,\ell} \text{dom } q^\ell_\beta \cup \bigcup_{\ell \le \binom{k(*)}{2}} \text{dom } r_{\alpha_\ell,\beta_\ell}$$

edges of q = union of the set of edges of q^ℓ_β , $r_{\alpha_\ell,\beta_\ell}$.

(Note that any node in $\text{dom } r_{\beta_i,\gamma_i} \setminus (M_{t_{\beta_i}} \cup M_{y_{\gamma_i}})$ is connected.)

(Note that the q^ℓ_β , $r_{\alpha_\ell,\beta_\ell}$ are pairwise compatible.)

The least trivial is to show $K_{k(*)+1}$ is not embeddable into q . Let Ξ be a set of $k(*)+1$ vertices.

Assume that Ξ is a complete graph (in q) and we shall derive a contradiction.

If we omit the edges $\{(\eta_i,\eta_j):\ i < j < k(*)\}$ from q , the resulting graph is obtained by successive edgeless amalgamation (look at the restriction to $\bigcup_{\ell \le i} \text{Dom } q^i_\ell \cup_{j<i} \text{Dom } r_{\beta_j,\gamma_j}$, for $i \le k(*)$). Hence it has no subgraph isomorphic to $K_{k(*)+1}$. So necessarily for some $i(1)$, $\eta_{i(1)} \in \Xi$. Now by the definition of "potential edge" and as ($m \ge 2$ and) the interval $(\eta_{i(1)}(0), \eta_{i(1)}(1))$ is disjoint to $M_\emptyset$, we have: $\eta_{i(1)}$ is not connected to any vertex from $M_\emptyset$. So $\Xi \cap M_\emptyset = \emptyset$. Now consider the sequence

$$\left\langle \bigcup_{\ell<k(*)} \text{Dom } q^i_\ell \cup \bigcup_{j<i} \text{Dom } r_{\beta_0,\gamma_j} \setminus M_\emptyset : i \le k(*) \right\rangle$$

and the restrictions of the graph q to them. Easily the first is in P , and in each step we use edgeless amalgamation (we could have started with this argument) so we finish.

Concluding Remarks:

5.2 Easy variants: We can have $G \to (H)^2_\mu$ such that the family of finite subgraphs of G is S (up to isomorphism) where for some n :

1. $S \neq \emptyset$
2. S closed under edgeless amalgamation
3. If $L_1,\dots,L_{|H|} \in S$; $i \neq j \Rightarrow L_i \cap L_j = L$;

   $x_i \in L_i$ and the distance of x_i from L in L_i is $\geq n$ then

   $L^* \in S$ where: vertices $(L^*) = \bigcup_{i=1}^{n}$ vertices (L_i)

   $\text{edges}(L^*) = \bigcup_{i=1}^{n} \text{edges}(L_i) \cup \text{edges}(L^* \restriction \{x_i : i = 1,\dots,|H|\})$

   $L^* \restriction \{x_1,\dots,x_{|H|}\} \cong H$.

5.3 Easy Remark: Instead of graphs we can have a model where relations are a partition of the singleton and of the pairs.

5.4 Note that the proof of 5.1 tells us that in 4.2 for $n = 3$ (i.e. coloring of singletons and pairs) we do not need 1.4 but can directly prove hence lowering the required cardinal.

5.5 On generalizing 5.2 to relation and colorings with more places see later works.

References

[1] A. Hajnal and P. Komjath, Embedding graphs and colored graphs.

[2] S. Shelah, Notes on combinatorial set theory. *Israel J. Math.* 14 (1973) 262-277.

[3] S. Shelah, Was Sierpinski right ? I. *Israel J. Math.* 62 (1988) 355-380.

[4] S. Shelah, Was Sierpinski right ? II. Preprint.

[5] J. Nesestril and V. Rodl, Partition (Ramsey) theory, a survey. Colloq. Math. Soc. Janos Bolyai, Vol. 18, North Holland, Amsterdam, (1978), 75-192.

[Sh95] S. Shelah, Canonization theorems and applications, J. of Symb. Logic 40 (1981) 345-353.

[EHMR] P. Erdos, A. Hajnal, A. Mate and R. Rado, Combinatorial Set Theory: Partition relations for cardinals. Disquisitiones Mathematicae Hungaricae 13, Akademiai Kiado Budapest, 1984, North Holland.

[Sc Sh20] J. Schmerl and S. Shelah, On power-like models of hyperinaccessible cardinals, J. of Symb. Logic 37 (1972) 531-537.