ISRAEL JOURNAL OF MATHEMATICS, Vol. 28, Nos. 1-2, 1977

DECIDABILITY OF A PORTION OF THE PREDICATE CALCULUS

BY

SAHARON SHELAH[†]

ABSTRACT

We show decidability of the existence of a model (a finite model) for sentences with the string of quantifiers $\forall x(\exists y_1 \cdots y_n)$, for a language with equality, one one-place function, predicates and constants.

Since the predicate calculus was shown undecidable much work has been done on fragments of it, in particular, restricting ourselves to sentences in prenex normal form where the quantifier string belongs to a fixed set, and the formula belongs to a specific language. For instance, the problem of whether ψ has a model for

$$\psi \in \{(\forall xy)(\exists z_1 \cdots z_n)\varphi \colon \varphi \in L,\ \varphi \text{ quantifier free}\},$$

where L is without equality but with constants and some predicate symbols, was shown decidable by Gödel and Kalmar and Schutte. The answer to the same problem with equality is still open. For a survey see Gurevich [2] or Dreben and Goldfarb [1].

Gurevich conjectures decidability for $\pi = \{\forall x \exists y_1 \cdots y_n \varphi \colon \varphi \in L$ is quantifier free$\}$ for L containing equality, one one-place function symbol, constants and predicates; and apart from the problem mentioned above it remained the only open one. We affirm this conjecture (of course, the problem for $(\exists z_1, \cdots, z_k)(\forall x)(\exists y_1, \cdots, y_n)\varphi$ is the same).

There is also the problem of finite models. In most cases of decidability, the existence of a model implies the existence of a finite model (e.g., in Gödel's result). But

$$(\forall x)(\exists y)(\exists z)(f(y) = x \wedge f(z) = x \wedge y \neq z)$$

has no finite model.

[†] I would like to thank the Israel Academy of Science and Humanities for partially supporting this research by a grant.

Received August 27, 1975 and in revised form January 3, 1977.

We again affirm a conjecture of Gurevich that the problem is decidable. Our results were announced in [4].

Let us try to explain the proof. First suppose $\psi \in \pi$, $M \models \psi$, L has no constants and M contains no circles (any L-model M is considered as a graph, where a, b are connected iff $a = f(b)$ or $b = f(a)$, so circles and distance have their natural meaning). If M' is a disjoint copy of M, $M \cup M'$ is again a model of ψ, moreover, instances of relations involving elements from M and M' can be defined arbitrarily. We can also identify each $a \in A \subseteq M$ with its copy in M', for a set A closed under f, and the situation is similar. This leads us to define a weak notion of satisfiability, $\models^1$, which means that we are interested in the satisfiability of $R(a_1, \cdots, a_n)$ iff for some b, $a_i = f^{n(i)}(b)$, $n(i)$ not large compared to ψ. By repeating the above duplication, we can build from a $\models^1$-model of ψ, a real model of ψ. So it suffices to decide whether there is M, $M \models^1 \psi$, which can be expressed in a language with equality, f and one-place predicates, which is known to be decidable.

The existence of circles is a minor problem. Instead of demanding above that $n(i)$ is not too large, we should demand only that the distance of a from b is not large compared to ψ.

If individual constants are present, the situation is more complicated. The problem is that we cannot duplicate the almost constants ($= f^n(c)$, c a constant), but there may be infinitely many of them. So we introduce a finer weak satisfaction $M \models^2_C$, which means that we avoid almost constants when we can.

For finite models there is some more work (in fact, if in defining M^+ we were thinking of infinite models only, we would not "identify" $f^{l(1)}(x)$, $f^{l(2)}(x)$, but use a "periodic" component, which would simplify the proof; i.e., we replace $\langle a, v\rangle$ by $\langle a, v, k\rangle$, k an integer, and define $f(\langle f^{l(2)}(x), v, k\rangle) = \langle f^{l(1)+1}(x), v, k+1\rangle$).

M. Rubin asks what occurs if we restrict ourselves to models in which every circle has length $\leqq k_0$. For infinite models, we get decidability similarly (making the above-mentioned change in the M^+ definition). For finite models, by complicating our proofs we can prove decidability. (Hint: do not use the good v's to define M^+, but rather define $\models^3_C$ to say they exist, and then prove a parallel to 17 for finite models by replacing an interval, of almost constants, by a periodic interval with repetition big enough.) We can also prove that if $\psi \in \pi$ has a model, it has a model in which every component contains a circle (of length $\leqq h_{11}(\psi)$).†

† Detailed proof of those remarks, of Theorem 1 (somewhat simplified), as well as the functions h_i can be found in the M.Sc. thesis of Mrs. A. Sfard (in Hebrew).

Let L be a first order language, containing one one-place function f, and finitely many individual constants c_i and finitely many predicates R_i.

MAIN THEOREM 1. *Let* π_L *be the set of* $\psi \in L$ *which have the form* $(\forall x_0)(\exists x_1, \cdots, x_m)\psi'$, ψ' *is quantifier free. Then*:

(A) *We can effectively decide whether* $\psi \in \pi$ *has a model.*

(B) *We can effectively decide whether* $\psi \in \pi$ *has a finite model.*

(C) *The decision procedures are primitive recursive.*

(D) *In* (B) *there is a primitive recursive bound on the cardinality of the model.*

The proof is presented as a series of lemmas and claims. Let ψ be fixed. Usually there is no difference whether we ask for a model or for a finite model.

CLAIM 2. We can effectively find $m(i)$, ψ_i such that:

(A) ψ, $(\forall x_0) \bigvee_i (\exists x_1 \cdots x_{m(i)})\psi_i(x_0, \cdots, x_{m(i)})$ are logically equivalent; so w.l.o.g. they are equal.

(B) $\psi_i = \bigwedge_j \psi_{i,j}$, and we shall write $\varphi \in \psi_i$ instead of $\varphi \in \{\psi_{i,j} : j\}$.

(C) $\psi_{i,j}$ is of the form $R(\cdots, \sigma_l, \cdots)$ or $\sigma_1 = \sigma_2$ or $\sigma_1 = f(\sigma_2)$ or is the negation of such formulas, where σ_l is a variable x_k $(k \leqq m(i))$ or an individual constant (i.e., terms like $f(f(x))$ do not appear).

(D) Let $TR(\psi_i)$ be the set of individual constants and variables appearing in ψ_i. If $\sigma_1, \sigma_2 \in TR(\psi_i)$ *then*: $\sigma_1 = \sigma_2 \in \psi_i$ or $\sigma_1 \neq \sigma_2 \in \psi_i$; and $\sigma_1 = f(\sigma_2)$ or $\sigma_1 \neq f(\sigma_2) \in \psi_i$; such a ψ_i is called complete.

So each ψ_i defines a natural division of $TR(\psi_i)$ to ψ_i-components; also if $M \models \psi_i[a_0, \cdots, a_{m(i)}]$, this division induces a natural partition of $\{a_0, \cdots, a_{m(i)}\}$ to what we call ψ_i-components.

REMARK. Note that though each ψ_i is complete, not all individual constants must appear in it.

DEFINITION 3. An element a in an L-model M is called an almost constant if for some constant c_k and natural number n, $M \models f^n(c_k) = a$. The almost constant nearest to $x \in M$ is $ac_M(x)$ (or $ac(x)$), if it exists.

NOTATION 4. h_k will denote a primitive recursive function from π_L to ω:

$$h_7(\psi),\ h_8(\psi) \ll h_{10}(\psi, 1) \ll h_{10}(\psi, 2) \ll \cdots \ll h_{10}(\psi, 5m^*) \ll h_1(\psi)$$

$$\ll h_3(\psi), h_2(\psi) \ll h_4(\psi),\ h_5(\psi)$$

$$(m^* = m^*(\psi) = \max m(i) + 1)$$

(where $\ll$ means sufficiently smaller than; the exact function can be determined by the reader, from the proof).

DEFINITION 5. *A simplification* of ψ is a sentence we get by replacing $(\exists x_1 \cdots x_{m(i)})\psi_i$ by $\bigvee_{j<j(0)}(\exists x_1 \cdots x_{m(i,j)})\psi_i^j$ where each ψ_i^j is complete and

(A) For each $j < j(0)$ there is a function $g_j: \{x_1, \cdots, x_{m(i)}\} \to TR(\psi_i^j)$ such that

$$\vdash \psi_i^j \to \operatorname{Sub}\begin{matrix} x_1, \cdots, x_{m(i)} \\ \\ g_j(x_1), \cdots g_j(x_{m(i)}) \end{matrix} \psi_i$$

(or, we can assume each conjunct in the second formula appears in the first).

(B) $j(0) \leqq h_3(\psi)$ and $\psi_i^j \in L'$, where we get L' from L by adding $\leqq h_2(\psi)$ individual constants.

(C) For each $j < j(0)$ at least one of the following conditions holds:

(α) There are *less* ψ_i^j-components without an individual constant *than* there are ψ_i-components without individual constants; and $m(i,j) < h_2(\psi)$.

(β) The number of ψ_i^j-components without individual constants is not greater than the numbers of ψ_i-components without individual constants; and $m(i,j) < m(i)$.

(γ) There is an individual constant c such that $x_0 = c \in \psi_i^j$, but for no individual constant c does $x_0 = c \in \psi_i$; and $m(i,j) < h_2(\psi)$.

REMARK. (1) Instead of demanding from ψ_i^j to be complete, we can somewhat strengthen the bound on $j(0)$.

(2) If $j(0) = 0$ clearly this is a simplification.

CLAIM 6. (A) The number of simplifications of ψ is $\leqq h_4(\psi)$.

(B) If ψ^{l+1} is a simplification of ψ^l for $l = 0, \cdots, n-1$, $\psi^0 = \psi$, then $n \leqq h_5(\psi)$.

(C) If ψ' is a simplification of ψ, $M \vDash \psi'$ then $M \vDash \psi$.

PROOF. Trivial.

CONCLUSION 7. In order to decide when $\psi \in \pi_L$ has a [finite] model, it suffices to decide statements of the form:

"suppose no simplification of ψ has a [finite] model, does ψ have a [finite] model?"

So we make

ASSUMPTION 8. No simplification of ψ has a [finite] model, and ψ_i "says" the variables are distinct and not equal to the constants appearing in ψ_i (except possibly on x_0); hence for each model M of ψ and i there are $a_0, a_1 \cdots$ such that $M \models \psi_i[a_0, a_1, \cdots]$.

DEFINITION 9. For A, a set of elements of an L-model M, let $M \models^1_A \psi_i[a_0, a_1, \cdots]$ mean that $M \models \psi_{i,j}[a_{l_1}, \cdots]$ when (A) or (B), where

(A) $\psi_{i,j}$ is an equality or an inequality,

(B) there is an x_l ($l \leqq m(i)$) such that every x_k which appears in $\psi_{i,j}$ satisfies either (α) or (β):

(α) a_k is an individual constant or belongs to A,

(β) x_k is in the ψ_i-component of x_l;

and for some (natural number) q, $M \models f^q(a_l) = a_k$ or a_k is an almost constant.

REMARK 10. In (B) we can assume w.l.o.g. that x_l appears in $\psi_{i,j}$; notice that the satisfaction of (B) depends only on the variables appearing in $\psi_{i,j}$ and on the choice of the a_l's, and it holds iff it holds for any subset with two elements.

DEFINITION 11. (A) $M \models^1_A \psi$ if for each $a_0 \in |M|$ there are i and $a_1, \cdots, a_{m(i)} \in |M|$ such that

$$M \models^1_A \psi_i[a_0, a_1, \cdots, a_{m(i)}].$$

(B) In Definitions 9, 11(A) if we omit A and write $M \models^1 \psi$ we mean A is the set of almost constants.

DEFINITION 12. Let M be an L-model, $A \subseteq |M|, |A| \leqq m^* = \max_i m(i) + 1$. We define

(A) $$L^*_A = L \cup \{Q_1, Q_2, Q_3\} \cup \{a : a \in A\}$$

where Q_1, Q_2 are one-place predicates, and Q_3 a two-place predicate;

(B) M^*_A is an L^*_A-expansion of M, by interpreting (for $a \in A$) (the constant) a as (the element) a and letting:

$$Q_1 = \{b \in |M| : b \text{ lies in a circle, i.e. } (\exists n > 0)\ f^n(b) = b\},$$

$$Q_2 = \{b \in |M| : b \text{ is an almost constant}\},$$

$$Q_3 = \{\langle b, f^n(b)\rangle : b \in |M|,\ n > 0\};$$

(C) We define $C_i(M)$ for $i \leqq 5m^*$:

$C_0(M)$ = the set of individual constants of M,

$C_{i+1}(M) = \{ b \in M : b$ is an almost constant of M, defined in $M^*_\emptyset$ by a formula $\varphi(x) \in L^*_\emptyset$ of length $\leqq h_{10}(\psi, i)\}$;

(D) Let $\bar{C} = \bar{C}(M) = \langle C_i(M): i \leqq 5m^* \rangle$.

DEFINITION 13. The 5-tuple $v = \langle x, l_1, l_2, A, p \rangle$ is *good* for an L-model M, if

(A) $x \in |M|$, $A \subseteq |M|$, $|A| < m^*$, $p \leqq m^*$,

(B) $l_1 > h_7(\psi)$, $l_2 > l_1 + h_7(\psi)$,

(C) for each formula $\varphi(y) \in L^*_{A \cup \{x\}}$ of length $\leqq h_8(\psi)$

$$M^*_{A \cup \{x\}} \models \varphi[f^{l_1}(x)] \equiv \varphi[f^{l_2}(x)],$$

(D) x is on a circle iff $f^{l_2}(x)$ is on this circle, and l_2 is smaller than the length of the circle.

(E) for each constant c, $x \in \{f^m(c): m \geqq 0\}$ iff

$$f^{l_2}(x) \in \{f^m(c): m \geqq 0\}.$$

REMARK. As $\varphi \in L^*_{A \cup \{x\}}$, also $f^{l_1 \pm k}(x)$, $f^{l_2 \pm k}(x)$ are "similar" for small k. In fact A in the 5-tuple is not necessary.

DEFINITION 14. Let M be an L-model of ψ, and we shall define M^+: $|M^+| = |M| \cup \{\langle a, v \rangle: v = \langle x, l_1, l_2, A, p \rangle$ is good for M; and for some $n \geqq 0$, $f^n(a) = f^{l_2}(x)$, and when $f^{l_2}(x)$ is an almost constant or on a circle $(\forall m \leqq n)(f^m(a) \neq f^{l_1}(x))\}$. (So clearly $\langle a, v \rangle \in |M^+| - |M|$ implies a is not an individual constant of M (by Definition 13E).)

We define the relations such that M is a submodel of M^+, equality is defined naturally and when $v = \langle x, l_1, l_2, A, p \rangle$:

$$f(\langle a, v \rangle) = \begin{cases} \langle f(a), v \rangle & a \neq f^{l_2}(x), \\ \langle f^{l_1+1}(x), v \rangle & a = f^{l_2}(x). \end{cases}$$

As for relations, if b_i is a_i or $\langle a_i, v_i \rangle$ then

$$M^+ \models R[b_1, \cdots, b_k] \quad \text{iff} \quad M \models [a_1, \cdots, a_k]$$

except in the following cases:

(α) Suppose for some $v = \langle x, l_1, l_2, A, p \rangle$ for each i, $b_i = a_i$, $a_i \in A \cup C_0$ or $b_i = \langle a_i, v \rangle$, and $I \subseteq \{1, \cdots k\}$, $0 < |I| < k$; $i \notin I$ implies that the distance of a_i to $f^{l(2)}(x)$ is $\leqq m^*$, or $b_i \in A \cup C_0$ and for $i \in I$, $a_i = f^{l(1)+k(i)}(x)$, $k(i) \leqq m^*$. We define a'_i as a_i for $i \notin I$, and as $f^{l(2)+k(i)}(x)$ for $i \in I$; and

$$M^+ \models R[b_1, \cdots, b_k] \quad \text{iff} \quad M \models R[a'_1, \cdots, a'_k].$$

(β) Suppose for some $\langle x, l_1, l_2, A, p \rangle$ for each i, $b_i = a_i$, $a_i \in A \cup C_0$, *or* $b_i = \langle a_i, v \rangle$, $v = \langle x, l_1, l_2, A, p \rangle$, $I \subseteq \{1, \cdots k\}$, $0 < |I| < k$; and $i \notin I$ implies the

Sh:58

distance of a_i to $f^{l(2)}x$ is $\leqq m^*$, or $b_i \in A \cup C_0$, and for $i \in I$, $a_i = f^{l(2)-k(i)}(x)$, $k(i) \leqq m^*$. We define a'_i as a_i for $i \notin I$, and as $f^{l(1)-k(i)}(x)$, for $i \in I$, and

$$M^+ \models R[b_1, \cdots, b_k] \quad \text{iff} \quad M \models R[a'_1, \cdots, a'_k].$$

Definition 15. $M \models^2_{\bar{C}}$ where $\bar{C} = \langle C_i \colon i \leqq 5m^* \rangle$, C_i increasing sequence of subsets of $|M|$ and each individual constant of M belongs to C_0, *if*

(A) For every $a_0 \in |M|$ there are i, $a_1, \cdots, a_{m(i)}$ such that $M \models^2_{\bar{C}} \psi_i[a_0, \cdots, a_{m(i)}]$, which means

(α) $M \models^1_{C_0} \psi_i[a_0, \cdots, a_{m(i)}]$.

(β) If among $\{a_l, f(a_l) \colon x_l$ is in the ψ_i-component of $x_0\}$ there is no almost constant *then*: if $a_l[f(a_l)]$ is not in C_0, then it is not an almost constant.

(γ) If among $\{a_l, f(a_l) \colon x_l$ is in the ψ_i-component of $x_0\}$ there is an element of $C_{k(1)}$, $k(1) \leqq 3m^*$ *then* there is k, $k(1) < k < k(1) + m^*$ such that

(i) $\{ac(a_l) \colon l \leqq m(i)\} \cap (C_{k+1} - C_k) = \emptyset$ (see Definition 3),

(ii) each $ac(a_l)$ which is not in C_k, is not an almost constant,

(iii) and if in the ψ_i-component of x_l there is no constant and in $\{ac(a_{l(1)}) \colon x_{l(1)} \in \psi_i$-component of $x_l\}$ there is no element of C_k then a_l lies in an M-component without constants,

(iv) $M \models^1_{C_k} \psi_i[a_0, \cdots]$.

(δ) If no element of $\{a_j, f(a_j) \colon x_j$ is in the ψ_i-component of $x_0\}$ is in C_{3m^*} but at least one is an almost constant and $\{ac(a_l) \colon l\} \cap (C_{k+1} - C_k) = \emptyset$, $1 < k < 3m^*$, and x_l is not in the ψ_i-component of x_0, and $ac(a_l) \notin C_k$ *then* $ac(a_l)$ is not an almost constant, and

$$M \models^1_{C_k} \psi_i[a_0, a_1, \cdots].$$

Remark. Note that up to now the only difference from $M \models^1 \psi$ is in (δ). Note that (β), (γ), (δ) are exclusive, and always one of them holds. The other parts of the definition are inessential.

(B) If $T = \{x_{l_1}, \cdots, x_{l_k}\}$ is a ψ_i-component with no individual constant, *then* there are distinct elements $a^p_{l_1}, \cdots$ $(p \leqq m^*)$ of M from an M-component with no individual constant such that $f(a^p_{l_{k(1)}}) \neq a^q_{l_{k(2)}}$, $M \models^1 \psi^T_i[a^p_{l_1}, \cdots, a^p_{l_k}]$, where $\psi^T_i(x_{l_1}, \cdots) = \psi_i \restriction T = \wedge \{\varphi \colon \varphi \in \psi_i$, and in φ there appear only $x_{l_1}, \cdots, x_{l_k}$ and individual constants$\}$. Also $\{a^p_{l_1}, \cdots, a^p_{l_k}\} \cap \{a^q_{l_1}, \cdots, a^q_{l_k}\} = \emptyset$ if $p \neq q$.

(C) If $x_{l_1}, \cdots, x_{l_k}$ are the variables of some ψ_i-component T, then there are distinct $a^p_{l_1}, \cdots$ $(p \leqq m^*)$ which are not almost constants, and $M \models^1 \psi^T_i[a^p_{l_1}, \cdots, a^p_{l_k}]$. Also $\{a^p_{l_1}, \cdots, a^p_{l_k}\} \cap \{a^q_{l_1}, \cdots, a^q_{l_k}\} = \emptyset$ if $p \neq q$.

(D) Each C_k consists of almost individual constants, and if c is an almost constant whose distance from C_k is $\leqq 2m^*$, then $c \in C_{k+1}$, and $|C_k| \leqq h_{10}(\psi, k)$.

THEOREM 16. *If* $M \models \psi$ *then* $M^+ \models^2_{\bar{C}} \psi$ *($\bar{C}$ as defined in Definition* 12C).

PROOF. We check the parts of Definition 15.

Part (B). Let $T = \{x_{l_1}, \cdots, x_{l_k}\}$ be the ψ_i-component without constants.

Case (i). There are in M $a^p_{l_1}, \cdots, a^p_{l_k}$ $(p \leqq m^*)$ satisfying ψ_i^T which are from a component without constants. Trivially M^+ satisfies it.

Case (ii). Not (i) but there are $a_{l_1}, \cdots \in m$ satisfying ψ_i^T whose distance from almost constants is $\geqq h_1(\psi)$, and in their M-component there is a constant. Then we can find a $v_p = \langle x, l_1, l_2, \emptyset, p\rangle$ $(p \leqq m^*)$ good for M, such that $\{a_{l_1}, \cdots\} \subseteq \{b \colon (\exists n) f^n(b) = f^{l_1}(x)\}$ and $f^{l_2}(x)$ is not an almost constant nor on a circle. It is easy to check that $\langle a_{l_1}, v_p\rangle, \cdots, \langle a_{l_k}, v_p\rangle$ are as required.

Case (iii). Neither (i) nor (ii), but there is an individual constant c, and $0 < q(0) < q(1) < \cdots < q(h_1(\psi))$, and for each $r < h_1(\psi)$ there are $a^r_{l_1}, \cdots, a^r_{l_k}$ satisfying ψ_i^T such that

$$\{a^r_{l_1}, \cdots, a^r_{l_k}\} \subseteq \{b \colon (\exists n)[f^n(b) = f^{q(r)}(c) \wedge f^{n-1}(b) \neq f^{q(r)-1}(c)]\}.$$

It is easy to find a $v_p = \langle x, l_1, l_2, \emptyset, p\rangle$ $(p \leqq m^*)$ good for M, and $0 \leqq r_1 < r_2 < h_1(\psi)$ such that $f^{l_1}(x) = f^{q(r_1)-1}(c)$, $f^{l_2}(x) = f^{q(r_2)-1}(x)$. Hence $\langle a_{l_1}, v_p\rangle, \cdots, \langle a_{l_k}, v_p\rangle \in M^+$ will prove the assertion.

Case (iv). None of the above. By not (i), there are d_p $(p \leqq m(i)^2)$ such that $M \models \psi_i^T[a_{l_1}, \cdots]$ implies $\{a_{l_1}, \cdots\}$ and $\{d_p \colon p < m(i)^2\}$ are not disjoint.

Suppose $M \models \psi_i^T[a_{l_1}, \cdots]$ and a_{l_1} is in a component of M with an almost constant, and let $d = d(a_{l_1}, \cdots, a_{l_k})$ be the nearest one. So (by not (ii)) the distance of d to $a_{l_1}, \cdots$ is $\leqq h_1(\psi)$. If (by not (iii)) the number of possible $d(a_{l_1} \cdots)$ is $\leqq h_1(\psi)|L|$ we can find a simplification of ψ with a model: we replace $(\exists x_1 \cdots)\psi_i$ by

$$(\exists x_1 \cdots x_{m(i)})(\psi_i \wedge \vee \{f^n(x_{l_1}) = d_p \colon n \leqq h_1(\psi) + m(i),\ p \leqq h_1(\psi)|L| + m(i)^2\}).$$

So there are $\geqq h_1(\psi)$ possible d's. For each d there is an individual constant $c = c(d)$ such that $d \in \{f^n(c) \colon n \geqq 0\}$, so for some c we have $h_1(\psi)$ d's, and this contradicts not (iii).

Part (C). Similar to (B) with cases (i) and (iv) (with simpler proof) only.

Part (D). Trivial.

Part (A). Let $a_0 \in |M^+|$, and we should find suitable i, $a_1, \cdots$.

Case (i). $a_0 = \langle a^0, v\rangle$, $a^0 \in |M|$. So, as $M \models \psi$, there are i, $a^1, \cdots \in M$ such that $M \models \psi_i[a^0, \cdots, a^{m(i)}]$. Let $T(0), T(1), \cdots, T(r)$ be the ψ_i-components, $x_0 \in T_0$. Let $\psi_i^{T(q)} = \psi_i^{T(q)}(x_{l(1,q)}, \cdots)$ and if $T(q)$ $(q > 0)$ has no constant, by part B there are $a_{l(1,q)}, \cdots$ satisfying $\psi_i^{T(q)}$ from a component without constants. If $T(q)$ $(q > 0)$ has constants we can find by part (C) $a_{l(1,q)} \cdots$ satisfying $\psi_i^{T(q)}$, which are not almost constant. As in B, C we have m^* such possible disjoint choices, we can ensure there will be no accidental indentification.

For $T(0)$, we usually choose $a_{l(\alpha,0)} = \langle a^{l(\alpha,0)}, v\rangle$ and clearly if this definition is possible, the conclusion of part A holds. In the remaining case the distance of a_0 to $f^{l_1}(x)$ or to $f^{l_2}(x)$ is $\leqq m^*$; this we leave to the reader (look carefully at the exceptions in Definitions 14).

Case (ii). $a_0 \in |M|$, and in $\{a_l, f(a_l)$: x_l is in the ψ_i-component of $x_0\}$ there is no almost constant. (β) of Definition 15, part A applies to this case, and the proof is like the previous one.

Case (iii). $a_0 \in M$ and among $\{a_l, f(a_l)$: x_l is in the ψ_i-component of $x_0\}$ there is an element of $C_{k(1)}$, $k(1) \leqq 3m^*$.

In this case there is minimal k, $k(1) < k < k(1) + 2m^*$, such that $\{ac(a_l)$: $l \leqq m(i)\} \cap (C_{k+1} - C_k) = \emptyset$. Let $T(0), T(1), \cdots, T(r)$ be the ψ_i-components, and if $C_k \cap \{ac(a_l)$: $x_l \in T_q\} \neq \emptyset$ we define $a'_l = a_l$. Let $A = \{a_0, \cdots, a_{m(i)}\} \cap C_k$; and if $\{ac(a_l)$: $x_l \in T(q)\} \cap C_k = \emptyset$, we find a good $v = \langle x, l_1, l_2, A, q\rangle$, such that for $x_l \in T(q)$, $\langle a_l, v\rangle \in M^+$, and the distance of a_l from $f^{l(1)}(x)$, and from $f^{l(2)}(x)$ is at least $2m^*$. Then we let $a'_l = \langle a_l, v\rangle$.

Notice that if $T(q)$ contains a constant $\{a_l$: $x_l \in T_q$, a_l an almost constant$\} \subset C_1$ hence (γ) of Definition 15A holds.

Case (iv). None of the above. This parallels (δ) of Definition 15A, and no new idea is needed.

THEOREM 17. *If there is an* M, $M \models^2_C \psi$ then *there is an* M, $M \models \psi$.

We prove some claims.

CLAIM 18. (A) The classes $\{M$: $M \models^1 \psi\}$ and $\{M$: $M \models^2_C \psi\}$ are closed under increasing unions.

(B) If $M \subseteq N$, $a_0 \in M$, $M \models^{(1)}_{(A)} (\exists x_1, \cdots, x_{m(i)}) \psi_i(a_0, x_1, \cdots)$ then $N \models^{(1)}_{(A)} (\exists x_1 \cdots) \psi(a_0, x_1, \cdots)$.

(C) If $M \models^2_C \psi$, $M \subseteq N$, then $N \models^2_C \psi$ iff it satisfies part A of Definition 15.

PROOF. Trivial.

CLAIM 19. Suppose $M \models^2_C \psi$, and N is defined such that

(i) $|N| = \{\langle a, p\rangle$: $p \leqq p_0\}$, (p_0 an ordinal) we identify $\langle a, 0\rangle$ with a, and if $a \in B_3 \cup C_{2m^*}$ we identify $\langle a, p\rangle$ with a, where

$$B_1 = \{a \in M\text{: in the } M\text{-component of } a \text{ there are no constants}\},$$

$$B_2 = \{a \in M\text{: } a \notin B_1,\ ac(a) \in C_{2m^*}\},$$

$$B_3 = |M| - B_1 \cup B_2;$$

(ii) M is a submodel of N;

(iii) $f(\langle a,p\rangle)=\langle f(a),p\rangle$ (this is compatible with the definition of equality);

(iv) Let $a_1,\cdots,a_n\in M$, $R(x_1,\cdots,x_n)\in L$, $p\leqq p_0$, $A=\{a_1,\cdots,a_n\}-C_{3m^*}$ lies in one component which contains no constant, then

$$M\models R[a_1,\cdots,a_n] \quad \text{iff} \quad N\models R[\langle a_1,p\rangle,\cdots\langle a_n,p\rangle];$$

(v) if $a_1,\cdots,a_n\in B_2\cup C_{3m^*}$, $R(x_1,\cdots,x_n)\in L$, $p\leqq p_0$ then

$$M\models R[a_1,\cdots,a_n] \quad \text{iff} \quad N\models R[\langle a_1,p\rangle,\cdots,\langle a_n,p\rangle];$$

(vi) the instances of relations not defined previously are absolutely arbitrary. *Then* $N\models^2_C\psi$.

PROOF. By claim 18C it suffices to check part A of Definition 15.

Case I. $a_0\in M$. Trivial by (ii).

Case II. $a_0=\langle a^0,p\rangle$, $a^0\in B_1$, $p>0$. Clearly there are i, $a^1,\cdots,a^{m(i)}\in M$ such that Definition 15A (α)–(δ) are satisfied. As a^0 lies in a component without an individual constant, no a^i is an almost constant and for every i if $ac(a^i)$ is an almost constant, $ac(a^i)\in C_{2m^*}$ (by (β) of Definition 15A) hence no a^i is in B_3. By (α) $M\models^1_\varnothing\psi_i[a_0,\cdots,a_{m(i)}]$ hence by (iv) $N\models^1_\varnothing\psi_i[\langle a^0,p\rangle,\langle a^1,p\rangle,\cdots]$, and clearly Definition 15A holds.

Case III. $a_0=\langle a^0,p\rangle$, $a^0\in B_2$, $p>0$. So there are i, $a^1,\cdots,a^{m(i)}$ satisfying Definition 15A (α)–(δ). If $A=\{a_l, f(a_l)$: x_l belongs to the ψ_i-component of $x_0\}$ contains no almost constant, we have finished. In other cases, as $a^0\in B_2$, in A there is an element of C_{2m^*}. So by Definition 15A (γ) there is a proper $k<3m^*$. Define a_l: if x_l lies in a ψ_i-component without constants, and $ac(a_l)\notin C_k$ then $a_l=a^l$. Otherwise $a_l=\langle a^l,p\rangle$.

CLAIM 20. If $M\models^2_C\psi$, then there is an N, $M\subseteq N$, $N\models^1\psi$.

PROOF. Let B_1, B_2, B_3 be as in Claim 19, $B_3=\{a^\alpha_0: 0<\alpha\leqq p(0)\}$. For each α choose $k(\alpha)$, $i(\alpha)$, $a^\alpha_1,\cdots,a^\alpha_{i(\alpha)}$ as in Definition 15A. Define N as in Claim 19 and correct it so that $N\models^1\psi_{i(\alpha)}[a^\alpha_0,\langle a^\alpha_1,\alpha\rangle,\cdots,\langle a^\alpha_{i(\alpha)},\alpha\rangle]$. Changes are made only in case (δ) of Definition 19A, and we can choose $k(\alpha)<2m^*$.

Clearly the correction does not contradict (i)–(iv), so still $N\models^2_C\psi$. However, as remarked in Definition 19A, the only case there in which we do not get $M\models^1\psi_i[a_0,\cdots]$ is (δ), $|N|-|M|$ is disjoint to $B_3(N)$ (defined naturally) so we have just taken care of all such possibilities.

CLAIM 21. Let $M\models^1\psi$, $b\in M$, α an ordinal. We define M^α_b:

(i) $M^\alpha_b = \{\langle a, p\rangle: a \in M, p < \alpha\}$, but we indentify $\langle a, 0\rangle$ with a, and if a is an almost-constant we indentify $\langle a, p\rangle$ with a, and if $a \in \{f^n(b): n \geqq 0\}$ we identify $\langle a, 1\rangle$ with a;

(ii) $M \subseteq M^\alpha_b$;

(iii) $f(\langle a, p\rangle) = \langle f(a), p\rangle$;

(iv) if $p < \alpha$, $a_1, \cdots, a_n \in M$, $R(x_1 \cdots x_n) \in L$ and there is a_k such that each a_l satisfies

(α) a_l is an almost constant, or

(β) $a_l \in \{f^n(a_k): n \geqq 0\}$ and the distance between a_k and a_l is $< m^*$, *then* $M^\alpha_b \models R[\langle a_1, p\rangle, \cdots, \langle a_n, p\rangle]$ iff $M \models R[a_1, \cdots, a_n]$;

(v) the other instances of relations are arbitrary.

Then $M^\alpha_b \models^1 \psi$.

PROOF. By checking.

CLAIM 22. If $M \models^1 \psi$, $M \models^1 \psi_i[a_0, a_1, \cdots]$, then there is N, $M \subseteq N \models^1 \psi$, and $a'_1, \cdots, a'_{m(i)}$ such that $N \models \psi_i[a_0, a'_1, \cdots, a'_{m(i)}]$.

PROOF. Let $M' = M^{m^*}_{a_0}$ be as in Claim 21, let $T_0, T_1, \cdots, T_r$ be the ψ_i-components $x_0 \in T_0$. We define the a'_l: if $x_l \in T_k$, $a'_l = \langle a_l, k+1\rangle$. Now change M' to N just to make $\psi_i[a_0, a'_1, \cdots, a'_{m(i)}]$ hold; and it is easy to check that the conditions of Claim 21 still hold.

PROOF OF THEOREM 17. By Claim 20 there is $M_1 \models^1 \psi$; by repeating Claim 22, and remembering Claim 18A we get $M_2 \models \psi$.

THEOREM 23. *For each ψ, we can effectively find a sentence φ_ψ such that*

(1) *φ_ψ is a monadic sentence in L_0, L_0 contains $=$, f, the constants of L, and the following one-place predicates: Q_W, when $W = \langle R(x_{i_0}, \cdots, x_{i_l}, \cdots), x_{i_l}, T\rangle$ where T is a ψ_i-component for some i; Q_{cr} (= elements of circles) and for each $c \in C_0$, $Q_c = \{f^n(c): n \geqq 0\}$;*

(2) *there is an M, $M \models^2_C \psi$ iff there is a model for φ_ψ;*

(3) *there is a finite model M, $M \models^2_C \psi$ iff there is a finite model of φ_ψ.*

PROOF. Easy, left to the reader.

PROOF OF MAIN THEOREM 1 PART A. We can make Assumption 8. By Theorems 16 and 17, there is an M such that $M \models \psi$ iff there is an M, $M \models^2_C \psi$; so by Theorem 23 this is equivalent to the existence of a model for φ_ψ. As the monadic theory of one function symbol with one-place f and one-place predicates is decidable (see [3] for countable models, which is the case we need), we finish.

PROOF OF MAIN THEOREM 1 PART B. As for every finite M, M^+ is finite (check Definition 14), and as also the theory of finite models of one-place function symbols with one-place predicates is decidable, what is missing is only the parallel to Theorem 17 for finite models. It is easy to check that also in Claims 19, 20 we can preserve finiteness.

DEFINITION 24. (A) For $a \in M$, $h(a, M)$ is the distance of a from the set of almost constant and elements of circles,

(B) $h(M) = \max\{h(a, M) + 1 \colon a \in M\}$,

(C) $H_i(M) = \{a \colon h(a, M) \leqq i\}$,

(D) $M \vDash_i^1 \psi$ if $M \vDash_A^1 \psi$ where $A = H_i(M)$.

LEMMA 25. *If $M \vDash^1 \psi$ is finite, then there is N, $M \subseteq N$, N finite, $N \vDash_0^1 \psi$, $h(N) = h(M)$.*

PROOF. We shall define $p(0)$ later, and let us define M_1 just as in Claim 21, but $|M_1^*| = \{\langle a, p\rangle \colon a \in M,\ p \leqq p(0)\}$, and $\langle a, p\rangle$ is identified with a whenever $a \notin B_1$, i.e. in the M-component of a there is a constant.

Let $|M| = \{a^\alpha \colon \alpha < \|M\|\}$, and for each α there are $i = i(\alpha)$ and $a_1^\alpha, \cdots, a_{m(i)}^\alpha$ such that $M \vDash^1 \psi_i[a^\alpha, a_1^\alpha, \cdots]$. Let $T_0^i, \cdots, T_{r(i)}^i$ be the ψ_i-components of ψ_i, $x_0 \in T_0$. $T_1, \cdots, T_{q(i)}$ have constants, $T_{q(i)+1}, \cdots, T_{r(i)}$ do not have constants. We shall define functions $g(\alpha, p, l)$ ($\alpha < \|M\|$, $p \leqq p(0)$, $l \leqq m^*$) into $\{p \colon p \leqq p(0)\}$, and change M_1 to N just so that for each α, p

$$N \vDash_0^1 \psi_{i(\alpha)}[\langle a^\alpha, p\rangle, \langle a_1^\alpha, g(\alpha, p, 1)\rangle, \langle a_2^\alpha, g(\alpha, p, 2)\rangle, \cdots].$$

Sufficient demands on g are

(α) if $a_l^\alpha \notin B_1$, $i = i(\alpha)$ then $g(\alpha, p, l) = 0$,

(β) if $a_l^\alpha \in B_1$, $i = i(\alpha)$ then $g(\alpha, p, l) \neq 0$,

(γ) if $l \in T_0^i$, $i = i(\alpha)$, then $g(\alpha, p, l) = p$; and $x_{l(1)}, x_{l(2)} \in T_q^i$ iff $g(\alpha, p, l(1)) = g(\alpha, p, l(2))$,

(δ) in the other cases $g(\alpha, p, l)$ is not defined,

(ε) if $\{g(\alpha_1, p_1, l) \colon l \leqq m^*\} \cap \{g(\alpha_2, p_2, l) \colon l \leqq m^*\} - \{0\}$ has at least two elements, then $\alpha_1 = \alpha_2$, $p_1 = p_2$.

Such choice is known.

LEMMA 26. *Let M be finite, $B \subseteq A = H_i(M)$, $C = A \cup \{c \in |M| \colon f(c) \in B\}$ and let $a \in A - B$ and suppose $M \vDash_C^1 \psi$. Then there is a finite model N, $M \subseteq N$, such that*

(A) $H_i(M) = H_i(N)$,

(B) *if* $b \in B$, $\{c \in N \colon f(c) = b\} = \{c \in M \colon f(c) = b\}$,

(C) $N \vDash_{C(1)}^1 \psi$ *where* $C(1) = C \cup \{c \in N \colon f(c) = a\}$.

PROOF. Similar to Lemma 25.

CONCLUSION 27. If there is a finite $M \vDash^2_C \psi$ then ψ has a finite model.

PROOF. By Claim 22, Theorem 23 and Lemma 25 we can assume there is a finite M_0, $M_0 \vDash^1_0 \psi$.

We now prove by induction on $k \leqq h(M_0)$ that there is a finite M_k, $M_{k-1} \subseteq M_k$, $h(M_k) = h(M_0)$, $M_k \vDash^1_k \psi$. For a fixed k let $H_k(M_k) = \{a_\alpha : \alpha < \alpha(0)\}$ and we define by induction on $\alpha \leqq \alpha(0)$ a finite M^α_k, such that $M^0_k = M_k$, $M^\alpha_k \subseteq M^{\alpha+1}_k$, $H_k(M^\alpha_k) = H_k(M_k)$ and $M^\alpha_k \vDash^1_{A(k,\alpha)} \psi$ where

$$A(k,\alpha) = H_k(M_k) \cup \{c : f(c) \in \{a_\beta : \beta < \alpha\}\}.$$

The induction step is by Lemma 26, and $M_{k+1} = M^{\alpha(0)}_k$. So we finish Conclusion 27, hence Main Theorem 1B.

PROOF OF MAIN THEOREM PARTS C, D. Left to the reader.

REFERENCES

1. B. Dreben and D. Goldfarb, a book in preparation.

2. Y. Gurevich, *The decision problem for standard classes*, J. Symbolic Logic **41** (1976), 460–464.

3. M. O. Rabin, *Decidability of second-order theories and automata on infinite trees*, Trans. Amer. Math. Soc. **141** (1969), 1–35.

4. S. Shelah, *Various results in mathematical logic*, Notices Amer. Math. Soc. **22** (1975), A-23(75T-E4).

INSTITUTE OF MATHEMATICS
THE HEBREW UNIVERSITY OF JERUSALEM
JERUSALEM, ISRAEL