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Author(s): Saharon Shelah
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ON THE CARDINALITY OF ULTRAPRODUCT OF FINITE SETS

SAHARON SHELAH

ABSTRACT. We shall prove that if $\mathscr{D}$ is an ultrafilter and $\aleph_0 \leq \lambda = \Pi n_i/\mathscr{D}$, $\lambda^{\aleph_0} = \lambda$. This affirms a conjecture of Keisler.

Let $\mathscr{D}$ be an ultrafilter on I, n_i, $i \in I$, be natural numbers. Keisler in [2, on the bottom of p. 49] conjectured the following:

CONJECTURE. If $\lambda = \Pi n_i/\mathscr{D} \geq \aleph_0$, then $\lambda^{\aleph_0} = \lambda$.

We shall affirm this.

Let N be standard model of natural number, and let $M = N^I/\mathscr{D}$. In N the usual arithmetical functions are definable. Hence we shall use them freely. It is known that M, as an ultrapower of N, is elementarily equivalent to it, and M is $\aleph_1$-saturated (see, for example, [1] or [2]). Let a, b, c be elements of M, and $\leq$ ($<$) be the ordering relation. We define

$$|b| = |\{a: a < b\}| = \text{the power of the set } \{a: a < b\}.$$

m, n will be natural numbers.

It is easily seen that if $b = \langle n_i: i \in I\rangle/\mathscr{D}$ then $|b| = \Pi n_i/\mathscr{D}$. So it is sufficient to show that $|b| \geq \aleph_0$ implies $|b|^{\aleph_0} = |b|$. Suppose $|b| = \lambda \geq \aleph_0$, $\lambda^{\aleph_0} > \lambda$; we shall get a contradiction, and so prove the conjecture.

It is easy to see that $|ab| = |a|\,|b|$. This is because $xb + y$ for $x = 0, \cdots, a-1$; $y = 0, \cdots, b-1$ ranges the elements $0, \cdots, ab-1$, and every element $<ab$ is obtained exactly once. Hence $|a^4| = |a|^4$ and if $|a| \geq \aleph_0$ then $|a^4| = |a|$. Let us define b_n for natural numbers n: $b_0 = [\sqrt[4]{b}]$, $b_{n+1} = [\sqrt[4]{b_n}]$ ($[\sqrt[4]{x}]$ is the integral part of $\sqrt[4]{x}$). It is easily seen that $b_{n+1}^2 < b_n/2$, and so $\sum_{n_1<n<n_2} b_n^2 < b_{n_1}$, and also $|b_n| = |b| = \lambda$. Let C be the set of sequences $\langle c_n: n < \omega\rangle$ such that $c_n < b_n$ and $\bar{c}$ denotes an element of C. It is clear that $|C| = \Pi_{n<\omega}|b_n| = \lambda^{\aleph_0} > \lambda$. For every $\bar{c} = \langle c_n: n < \omega\rangle \in C$ we define

$$p(\bar{c}) = \left\{\sum_{n\leq m} c_n b_n < x < \sum_{n \leq m} c_n b_n + b_m : m < \omega\right\}.$$

As $m_1 < m_2$ implies

$$\sum_{n\leq m_1} c_n b_n \leq \sum_{n\leq m_2} c_n b_n < \sum_{n\leq m_2} c_n b_n + b_{m_2} \leq \sum_{n\leq m_1} c_n b_n + b_{m_1}$$

every finite subset of $p(\bar{c})$ is satisfiable in M, and so, as M is $\aleph_1$-saturated, $p(\bar{c})$ is realized in M by $a(\bar{c})$.

Now $a(\bar{c}) < \sum_{n\leq 0} c_n b_n + b_0 = (c_0 + 1)b_0 \leq b_0^2 < b$. Suppose $\bar{c}^1 \neq \bar{c}^2$, $\bar{c}^1 = \langle c_n^1: n < \omega\rangle$, $\bar{c}^2 = \langle c_n^2: n < \omega\rangle$. There exist $m < \omega$ such that $c_n^1 = c_n^2$ for $n < m$, and $c_m^1 \neq c_m^2$ and without loss of generality $c_m^1 > c_m^2$. So

$$a(\bar{c}^1) \geq \sum_{n\leq m} c_n^1 b_n = \sum_{n<m} c_n^1 b_n + c_m^1 b_m \geq \sum_{n<m} c_n^2 b_n + (c_m^2 + 1)b_m > a(\bar{c}^2).$$

Thus $a(\bar{c}^1) \neq a(\bar{c}^2)$.

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We can conclude that

$$|b| = |\{a: a < b\}| \geq |\{a(\bar{c}): \bar{c} \in C\}| = |C| = \lambda^{\aleph_0} > \lambda = |b|$$

a contradiction, and so we have proved the conjecture.

REFERENCES

[1] T. FRAYNE, A. MOREL and D. SCOTT, *Reduced direct products*, ***Fundamenta mathematicae***, vol. 51 (1962), pp. 195–248.

[2] H. J. KEISLER, *Ultraproducts of finite sets*, this JOURNAL, vol. 32 (1967), pp. 47–57.

HEBREW UNIVERSITY OF JERUSALEM