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## ON THE CARDINALITY OF ULTRAPRODUCT OF FINITE SETS

SAHARON SHELAH

**ABSTRACT.** We shall prove that if  $\mathcal{D}$  is an ultrafilter and  $\aleph_0 \leq \lambda = \prod n_i/\mathcal{D}$ ,  $\lambda^{\aleph_0} = \lambda$ . This affirms a conjecture of Keisler.

Let  $\mathcal{D}$  be an ultrafilter on  $I$ ,  $n_i, i \in I$ , be natural numbers. Keisler in [2, on the bottom of p. 49] conjectured the following:

**CONJECTURE.** If  $\lambda = \prod n_i/\mathcal{D} \geq \aleph_0$ , then  $\lambda^{\aleph_0} = \lambda$ .

We shall affirm this.

Let  $N$  be standard model of natural number, and let  $M = N^I/\mathcal{D}$ . In  $N$  the usual arithmetical functions are definable. Hence we shall use them freely. It is known that  $M$ , as an ultrapower of  $N$ , is elementarily equivalent to it, and  $M$  is  $\aleph_1$ -saturated (see, for example, [1] or [2]). Let  $a, b, c$  be elements of  $M$ , and  $\leq (<)$  be the ordering relation. We define

$$|b| = |\{a: a < b\}| = \text{the power of the set } \{a: a < b\}.$$

$m, n$  will be natural numbers.

It is easily seen that if  $b = \langle n_i: i \in I \rangle/\mathcal{D}$  then  $|b| = \prod n_i/\mathcal{D}$ . So it is sufficient to show that  $|b| \geq \aleph_0$  implies  $|b|^{\aleph_0} = |b|$ . Suppose  $|b| = \lambda \geq \aleph_0$ ,  $\lambda^{\aleph_0} > \lambda$ ; we shall get a contradiction, and so prove the conjecture.

It is easy to see that  $|ab| = |a| |b|$ . This is because  $xb + y$  for  $x = 0, \dots, a - 1$ ;  $y = 0, \dots, b - 1$  ranges the elements  $0, \dots, ab - 1$ , and every element  $< ab$  is obtained exactly once. Hence  $|a^4| = |a|^4$  and if  $|a| \geq \aleph_0$  then  $|a^4| = |a|$ . Let us define  $b_n$  for natural numbers  $n$ :  $b_0 = [\sqrt[4]{b}]$ ,  $b_{n+1} = [\sqrt[4]{b_n}]$  ( $[\sqrt[4]{x}]$  is the integral part of  $\sqrt[4]{x}$ ). It is easily seen that  $b_{n+1}^2 < b_n/2$ , and so  $\sum_{n_1 < n < n_2} b_n^2 < b_{n_1}$ , and also  $|b_n| = |b| = \lambda$ . Let  $C$  be the set of sequences  $\langle c_n: n < \omega \rangle$  such that  $c_n < b_n$  and  $\bar{c}$  denotes an element of  $C$ . It is clear that  $|C| = \prod_{n < \omega} |b_n| = \lambda^{\aleph_0} > \lambda$ . For every  $\bar{c} = \langle c_n: n < \omega \rangle \in C$  we define

$$p(\bar{c}) = \left\{ \sum_{n \leq m} c_n b_n < x < \sum_{n \leq m} c_n b_n + b_m : m < \omega \right\}.$$

As  $m_1 < m_2$  implies

$$\sum_{n \leq m_1} c_n b_n \leq \sum_{n \leq m_2} c_n b_n < \sum_{n \leq m_2} c_n b_n + b_{m_2} \leq \sum_{n \leq m_1} c_n b_n + b_{m_1}$$

every finite subset of  $p(\bar{c})$  is satisfiable in  $M$ , and so, as  $M$  is  $\aleph_1$ -saturated,  $p(\bar{c})$  is realized in  $M$  by  $a(\bar{c})$ .

Now  $a(\bar{c}) < \sum_{n \leq 0} c_n b_n + b_0 = (c_0 + 1)b_0 \leq b_0^2 < b$ . Suppose  $\bar{c}^1 \neq \bar{c}^2$ ,  $\bar{c}^1 = \langle c_n^1: n < \omega \rangle$ ,  $\bar{c}^2 = \langle c_n^2: n < \omega \rangle$ . There exist  $m < \omega$  such that  $c_n^1 = c_n^2$  for  $n < m$ , and  $c_m^1 \neq c_m^2$  and without loss of generality  $c_m^1 > c_m^2$ . So

$$a(\bar{c}^1) \geq \sum_{n \leq m} c_n^1 b_n = \sum_{n < m} c_n^1 b_n + c_m^1 b_m \geq \sum_{n < m} c_n^2 b_n + (c_m^2 + 1)b_m > a(\bar{c}^2).$$

Thus  $a(\bar{c}^1) \neq a(\bar{c}^2)$ .

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We can conclude that

$$|b| = |\{a: a < b\}| \geq |\{a(\bar{c}): \bar{c} \in C\}| = |C| = \lambda^{\aleph_0} > \lambda = |b|$$

a contradiction, and so we have proved the conjecture.

#### REFERENCES

- [1] T. FRAYNE, A. MOREL and D. SCOTT, *Reduced direct products*, *Fundamenta mathematicae*, vol. 51 (1962), pp. 195–248.  
[2] H. J. KEISLER, *Ultraproducts of finite sets*, this JOURNAL, vol. 32 (1967), pp. 47–57.

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