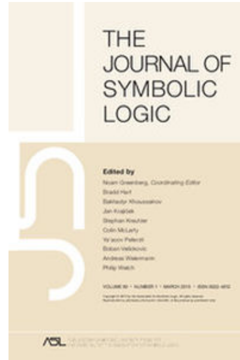

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Saharon Shelah and Mor Doron

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RELATIONAL STRUCTURES CONSTRUCTIBLE BY QUANTIFIER FREE DEFINABLE OPERATIONS

SAHARON SHELAH AND MOR DORON

Abstract. We consider the notion of bounded m-ary patch-width defined in [9], and its very close relative m-constructibility defined below. We show that the notions of m-constructibility all coincide for $m \geq 3$, while 1-constructibility is a weaker notion. The same holds for bounded m-ary patch-width. The case $m = 2$ is left open.

§1. Introduction.

1.1. Background. Our interest in this subject started from investigating spectra of monadic sentences, so let us begin with a short description of spectra. Let ϕ be a sentence in (a fragment of) second order logic (SOL). The spectrum of ϕ is the set $\{n \in \mathbb{N}\colon \phi$ has a model of size n$\}$. In 1952 Scholz defined the notion of spectrum and asked for a characterization of all spectra of first order (FO) sentences. In [1] Asser asked if the complement of a FO spectrum is itself a FO spectrum.

Definition 1.1. *A set $A \subseteq \mathbb{N}$ is eventually periodic if for some $n, p \in \mathbb{N}$, for all $m > n$, $m \in A$ if and only if $m + p \in A$.*

In [7] Durand, Fagin and Loescher showed that the spectrum of a FO sentence in a vocabulary with finitely many unary relation symbols and one function symbol is eventually periodic. In [10] Gurevich and Shelah generalized this for spectrum of monadic second order (MSO) sentence in the same vocabulary. Inspired by [10] Fisher and Makowsky in [9] showed that the spectrum of a CMSO sentence (a monadic sentence with counting quantifiers) is eventually periodic provided that all its models have bounded patch-width. A many sorted version for the context of graphs is the generalization of the Parikh's theorem proved by Courcelle in [6]. The notion of patch-width of structures (usually graphs) is a complexity measure on structures, generalizing clique-width. The proofs of [9] remains valid if we consider m-ary patch-width, i.e., we allow m-ary relations as auxiliary relations. Classes of bounded patch-width are of importance to the study of graphs. Two important example shown to be of bounded patch-width are graph languages generated by context free VR grammars and context free HR grammars. In [11] Shelah generalized the proof of [10] and showed eventual periodicity for a MSO sentence provided

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that all its models are constructible by recursion using operations that preserve monadic theory (see definitions below).

1.2. Summary of results. The above results on eventual periodicity led us to ask: What are the relations between the different notions for which we have eventual periodicity of MSO spectra? In other words do we have three different results, or are they all equivalent? In [4] Courcelle proved (using somewhat different notations) that a class of structures is constructible if and only if it is monadically interpretable in trees, thus implying that two of the results coincide. We give a proof of Courcelle's result more coherent with our definitions, which we use later on. We prove that the notions of bounded m-ary patch-width is very close to m-constructibility (constructibility where we allow m-ary relations as auxiliary relations) (see Lemmas 2.9 and 2.10). Next we show that for $m \geq 3$ a class of models is contained in a m-constructible class if and only if it is contained in a 3-constructible class (see Theorem 3.7). The same holds for classes of bounded m-ary patch-width. Finally we show that in the above theorem we cannot replace 3-constructible by 1-constructible. That is, there exists a 3-constructible class which is not contained in any 1-constructible class. We give a specific example (see 4.1). The case $m = 2$ is left open. We thank the referee for his or hers helpful comments.

§2. Preliminary definitions and previous results.

NOTATION 2.1. (1) *Let $\uplus$ denote the disjoint union operation.*

(2) *Let τ be a finite relational vocabulary.*

(3) *For $R \in \tau$ let $n(R)$ be the number of places of R. We say that R is $n(R)$-ary or $n(R)$ place. We allow $n(R) = 0$ i.e., the interpretation of R is in $\{\mathbb{T}, \mathbb{F}\}$. We call τ nice if $R \in \tau \Rightarrow n(R) > 0$.*

(4) *A τ-structure M has the form $\langle |M|, R^M : R \in \tau\rangle$ where M is a finite set called the domain (or universe) of M, and $R^M \subseteq {}^{n(R)}|M|$ for $R \in \tau$. (We denote by ${}^nA$ the Cartesien product of the set A with itself n times).*

(5) *For $k \in \mathbb{N}$, let τ_k be $\tau \uplus \{P_1, \dots, P_k\}$ with $P_1, \dots, P_k$ unary predicates.*

(6) *A k-colored τ-structure is a τ_k-structure in which the interpretation of the P_i's is a partition of the set of elements of the model (but some P_i's may be empty).*

(7) *A k-const τ-structure is a $\tau \uplus \{C_1, \dots, C_k\}$-structure where each C_i is an individual constant symbol. We denote such a structure by $(M, a_1, \dots, a_k)$ where M is a τ-structure and $a_1, \dots, a_k \in M$. We allow $a_i = a_j$ for $i \neq j$. The notation* const *stands constants. Our constants are referred to as sources in the hypergraph context, see* [4].

DEFINITION 2.2. (1) *A monadic second order (MSO) formula in vocabulary τ is a second order formula in which every second order quantifier quantifies a unary relation symbol. The notion of quantifier depth extends, naturally to MSO formulas.*

(2) *Let M be a τ-structure, and q a natural number. The monadic q-theory of M, $Th_q^{MSO}(M)$, is the set of all sentences of quantifier depth $\leq q$ that hold in M.*

(3) *Let M be a τ-structure, and n, q natural numbers. Let $\bar{a} = (a_1, \dots, a_n) \in {}^n|M|$. The q-type of $\bar{a}$ in M, $tp_q(\bar{a}, M)$, is the set of all τ formulas ϕ, of quantifier depth $\leq q$ in free variables $x_1, \dots, x_n$, such that: $M \models \phi[a_1, \dots, a_n]$. If $q = 0$ we sometimes write $tp_{qf}(\bar{a}, M)$.*

(4) *The notion of a q-type extends to MSO logic. We write* $tp_q^{MSO}(\bar{a}, M)$ *for the set of MSO formulas* ϕ, *of quantifier depth* $\leq q$ *in free variables* $x_1, \ldots, x_n$, *such that:* $M \models \phi[a_1, \ldots, a_n]$.
(5) *The set of all formally possible q-types in a vocabulary* τ *and in variables* $\langle x_1, \ldots, x_n \rangle$, *will be denoted by* $TP_q(\langle x_1, \ldots, x_n \rangle, \tau)$, *and similarly* $TP_q^{MSO}(\langle x_1, \ldots, x_n \rangle, \tau)$. *We may write* $TP_q^{MSO}(n, \tau)$ *instead of* $TP_q^{MSO}(\langle x_1, \ldots, x_n \rangle, \tau)$. *(By 'formally possible' we mean a set of formulas S such that* $\phi \in S \Leftrightarrow \neg\phi \notin S$, $\phi \wedge \psi \in S \Leftrightarrow \phi \in S$ *and* $\psi \in S$, *and we identify* ϕ *with* $\neg\neg\phi$. *We do not demand that S is realizable.)*

DEFINITION 2.3 (Patch-width, see [9]). (1) *Let* τ *be a nice vocabulary,* M *a* τ*-structure,* k *a natural number, and* $\mathfrak{P}$ *a finite set of* k*-colored* τ*-structures. We say that* M *has patch-width at most* k *(with respect to* $\mathfrak{P}$*) and denote* $pwd_{\mathfrak{P}}(M) \leq k$, *if* M *is the* τ*-reduct of a* k*-colored* τ*-structure which is in the closure of* $\mathfrak{P}$ *under the operations:*
 (i) *disjoint union* – $\uplus$,
 (ii) *recoloring* – $\rho_{i \to j}$ *(change all the elements with color* P_i *to color* P_j*) and*
 (iii) *modifications* – $\delta_{R,B}$ *(redefine the relation* $R \in \tau$ *by the quantifier free formula* B *in vocabulary* τ_k*).*

A class $\mathfrak{K}$ *of* τ*-structures is a* $PW(k)$*-class, if for some finite set of* k*-colored* τ*-structures* $\mathfrak{P}$ *the elements of* $\mathfrak{K}$ *are all the* τ*-reducts of structures of patch-width at most* k *with respect to* $\mathfrak{P}$. *We say* $\mathfrak{K}$ *is of bounded patch-width (BPW) if it is a* $PW(k)$*-class for some* $k \in \mathbb{N}$.
(2) *In the definition above we may instead of* k*-colored* τ*-structures, talk about* τ^+*-structures where* $\tau^+ \supseteq \tau$, $|\tau^+ \setminus \tau| = k$ *and every relation in* $\tau^+ \setminus \tau$ *is at most m-ary. We then talk about m-ary patch-width, where the rest of the definition remains unchanged. Note that the notions of patch-width and unary patch-width are close but not identical as in the former we demand that the sets of colors are disjoint.*

In [9] it is proved that:

THEOREM 2.4. *Let* ϕ *be a* $MSO(\tau)$ *sentence, and* $\mathfrak{K}$ *a class of* τ*-structures of bounded m-ary patch-width. Then the set* $\{\|M\| \colon M \in \mathfrak{K}, M \models \phi\}$ *is eventually periodic.*

We now define our set of operations. For $k, k_1, k_2 \in \mathbb{N}$, $\mathfrak{S}_{\tau,k,k_1,k_2}$ will be a set of binary operations that take as arguments one k_1-const τ-structure and one k_2-const τ-structure, and produce a k-const τ-structure. Each operation consists of quantifier free definitions of the relations in τ, and a (combinatorial) definition of the k constants of the resulting structure.

DEFINITION 2.5 (Addition operations). (1) *Syntactic definition:*
For $k, k_1, k_2 \in \mathbb{N}$, *each* $\mathbf{s} \in \mathfrak{S}_{\tau,k,k_1,k_2}$ *consists of:*
 (i) *Sets* $A_l = A_l^{\mathbf{s}} \subseteq \{1, \ldots, k_l\}$ *for* $l \in \{1, 2\}$.
 (ii) *For* $l \in \{1, 2\}$, *a 1-1 function* $g_l = g_l^{\mathbf{s}}$ *from* A_l *to* $\{1, \ldots, k\}$ *such that:* $Im(g_1) \cup Im(g_2) = \{1, \ldots, k\}$.
 (iii) *For* $l \in \{1, 2\}$ *a set* $B_l \subseteq \{1, \ldots, k_l\}^2$, *and a set* $B \subseteq \{1, \ldots, k_1\} \times \{1, \ldots, k_2\}$.

(iv) *For each* $R \in \tau$ *with* $n(R) = n$ *and each* $w_l \subseteq \{1, \dots, n\}$ *for* $l \in \{1,2\}$, *a function* $f_{R,w_1,w_2} = f^{\mathbf{s}}_{R,w_1,w_2}$ *with range* $\{\mathbb{T}, \mathbb{F}\}$, *and domain a set of triples of the form* (p, q_1, q_2) *where*:
- $p \in TP_0(\langle x_1, \dots, x_n\rangle, \sigma)$ *where* σ *is a vocabulary with* $k_1 + k_2$ *individual constants and two unary predicates,*
- *For* $l \in \{1,2\}$, $q_l \in TP_0(\langle x_i : i \in w_l\rangle, \tau)$.

(2) *Semantic definition*:
Let $k, k_1, k_2 \in \mathbb{N}$ *and* $\mathbf{s} \in \mathfrak{S}_{\tau,k,k_1,k_2}$. *Let* $(M_l, a^l_1, \dots, a^l_{k_l})$ *be* k_l*-const* τ*-structure for* $l \in \{1,2\}$. *The addition* $(M_1, a^1_1, \dots, a^1_{k_1}) \circledast_{\mathbf{s}} (M_2, a^2_1, \dots, a^2_{k_2})$ *is defined whenever*:
- $(|M_1| \cap |M_2|) \subseteq (\{a^1_1, \dots, a^1_{k_1}\} \cap \{a^2_1, \dots, a^2_{k_2}\})$ *and*
- *For* $l \in \{1,2\}$: $a^l_i = a^l_j \Leftrightarrow (i,j) \in B_l$ *and* $a^1_i = a^2_j \Leftrightarrow (i,j) \in B$,

to be the k*-const* τ*-structure* $(M, b_1, \dots, b_k)$ *defined by*:
(i) $|M| = (|M_1| \setminus \{a^1_1, \dots, a^1_{k_1}\}) \cup (|M_2| \setminus \{a^2_1, \dots, a^2_{k_2}\}) \cup \{a^l_i : l \in \{1,2\}, i \in A_l\})$.
(ii) *For each* $l \in \{1,2\}$ *and* $i \in A_l$, $a^l_i = b_{g_l(i)}$.
(iii) *For all* $R \in \tau$ *with* $n(R) = n$ *and* $\bar{x} = (x_1, \dots, x_n) \in {}^n|M|$, *let* $w_l = \{i : x_i \in |M_l|\}$ *for* $l \in \{1,2\}$. *Let* p *be the quantifier free type of* $\bar{x}$ *in the model with* $\{a^1_1, \dots, a^1_{k_1}\} \cup \{a^2_1, \dots, a^2_{k_2}\}$ *as constants, and* $|M_1|, |M_2|$ *as unary predicates. For* $l \in \{1,2\}$ *let* $q_l = tp_{qf}(\langle x_i : i \in w_l\rangle, M_l)$. *Now the value of* $R^M(\bar{x})$ *is defined to be* $f^{\mathbf{s}}_{R,w_1,w_2}(p, q_1, q_2)$.

Note that we may have two different syntactic definitions (i.e., $\mathbf{s} \neq \mathbf{s}' \in \mathfrak{S}_{\tau,k,k_1,k_2}$*) that give rise to the same semantic operation (i.e.,* $\circledast_{\mathbf{s}}$ *and* $\circledast_{\mathbf{s}'}$ *are equal).*

(3) *For technical reasons we would like to allow empty structures. i.e., let* $\tau' := \{R \in \tau : n(R) = 0\}$, *and* $X \subseteq \tau'$. *Now* $Null_X$ *is the* τ*-structure with* $|Null_X| = \emptyset$ *and* $R^{Null_X} = True \Leftrightarrow R \in X$. *Then if* $\mathbf{s} \in \mathfrak{S}_{\tau,k,k_1,0}$, *and* M *is a* τ_{k_1}*- structure then* $M \circledast_{\mathbf{s}} Null_X$ *is a well defined* τ_k*-structure. Furthermore for any* τ*-structure* M, $M \uplus Null_\emptyset$ *is defined and equal to* M.

The important properties of the addition operations are the following:

THEOREM 2.6. *Let* $k, k_1, k_2 \in \mathbb{N}$. *Then*:
(1) $\mathfrak{S}_{\tau,k,k_1,k_2}$ *is finite.*
(2) The addition theorem:
Let M, M' *be* k_1*-const* τ*-structures such that* $Th^q_{MSO}(M) = Th^q_{MSO}(M')$, *and* N, N' *be* k_2*-const* τ*-structures such that* $Th^q_{MSO}(N) = Th^q_{MSO}(N')$, *and* $\mathbf{s} \in \mathfrak{S}_{\tau,k,k_1,k_2}$. *Assume that the additions* $M \circledast_{\mathbf{s}} N$ *and* $M' \circledast_{\mathbf{s}} N'$ *are defined. Then*

$$Th^q_{MSO}(M \circledast_{\mathbf{s}} N) = Th^q_{MSO}(M' \circledast_{\mathbf{s}} N').$$

PROOF. (1) Immediate from the definition as the number of quantifier free formulas in a finite relational vocabulary and a given set of variables is finite (up to logical equivalence).

(2) An easy proof can be given using Ehrenfeucht-Fraïssé games, it is a straight forward generalization of the proof given in [8] for the disjoint union operation. An other proof by Courcelle is Theorem 3.4 of [4]. ⊣

It follows from 2.6(2) that given $q \in \mathbb{N}$ and $\mathbf{s} \in \mathfrak{S}_{\tau,k,k_1,k_2}$, there exists a computable function $g_q^{\mathbf{s}}\colon TP_q^{MSO}(0,\tau_{k_1}) \times TP_q^{MSO}(0,\tau_{k_2}) \to TP_q^{MSO}(0,\tau_k)$, such that whenever $M \circledast_{\mathbf{s}} N$ is defined we have:

$$Th_{MSO}^q(M \circledast_{\mathbf{s}} N) = g_q^{\mathbf{s}}(Th_{MSO}^q(M), Th_{MSO}^q(N)).$$

DEFINITION 2.7 (Constructibility). *Let m^* and k^* be natural numbers. A class $\mathfrak{K}$ of τ-structures is (m^*,k^*)-constructible, if there exists: A finite relational vocabulary $\tau^+ \supseteq \tau$, a finite set of structures $\mathfrak{P}$, and a finite set of addition operations $\mathfrak{S}$ such that:*

(i) *Every relation in $\tau^+ \setminus \tau$ is at most m^*-ary.*
(ii) *Every structure in $\mathfrak{P}$ is a k-const τ^+-structure for some $k \leq k^*$.*
(iii) *Every operation in $\mathfrak{S}$ is in $\mathfrak{S}_{\tau^+,k,k_1,k_2}$ for some $k, k_1, k_2 \leq k^*$.*
(iv) *The elements of $\mathfrak{K}$ are all the τ-reducts of structures in the closure of $\mathfrak{P}$ under the operations in $\mathfrak{S}$.*

We say that $\mathfrak{K}$ is m^*-constructible if it is (m^*,k^*)-constructible for some k^*, and that it is constructible if it is m^*-constructible for some m^*.

In [11] it is proved that:

THEOREM 2.8. *Let ϕ be a $MSO(\tau)$ sentence, and $\mathfrak{K}$ a constructible class of τ-structures. Then the set $\{\|M\|\colon M \in \mathfrak{K}, M \models \phi\}$ is eventually periodic.*

This is a generalization of 2.4 as we have:

LEMMA 2.9. *Let τ be a nice vocabulary, and $\mathfrak{K}$ be a m-ary $PW(k)$-class of τ-structures. Then $\mathfrak{K}$ is a $(m,0)$-constructible class.*

PROOF. First note that the disjoint union operation of τ^+-structures is in $\mathfrak{S}_{\tau^+,0,0,0}$. As for the recoloring and the modification operations, those are unary operations, so we look at the operation $\mathbf{s} \in \mathfrak{S}_{\tau^+,0,0,0}$ that acts as recoloring or modification on its left operand. So $M \circledast_{\mathbf{s}} Null_\emptyset$ is the desired recoloring or modification of M. ⊣

In the addition operations we allow omitting marked elements (i.e., we allow $|M \circledast_{\mathbf{s}} N| \subsetneqq |M| \cup |N|$, but we demand that the elements of the difference are constants of M or N). Moreover the universe of the the two operands is not necessarily disjoint, that is we allow that the intersection is a set of values of constants. This is not allowed in the operations of patch-width. It turns out though that these are the only essential differences between the two types of operations as suggested by the following:

LEMMA 2.10. *Let $\mathfrak{K}$ be a $(m,0)$-constructible class such that the vocabulary τ^+ associated with $\mathfrak{K}$ is nice. Then $\mathfrak{K}$ is of bounded m-ary patch-width.*

PROOF. $\mathfrak{K}$ is $(m,0)$-constructible so we have a vocabulary τ^+ and sets $\mathfrak{S}$ and $\mathfrak{P}$. Now the set of atomic structures for the patch-width definition will be the same $\mathfrak{P}$. The vocabulary of the patch-width definition will be: $\tau^+ \cup \{R'\colon R \in \tau^+\} \cup \{P_1, P_2\}$. P_1, P_2 are new unary relation symbols. We now have to show for each operation in $\mathfrak{S}$ how to simulate it by operations of patch-width. Let $\mathbf{s} \in \mathfrak{S}$ and let M_1, M_2 be τ^+-structures. Denote by M_1', M_2' the trivial extensions to the new vocabulary i.e., for $l \in \{1,2\}$ define $R^{M_l'} = \emptyset$ for $R \notin \tau^+$. We will now describe a series of patch-width operations on M_1', M_2' resulting in a structure M^* such that $M^*|_{\tau^+} \cong M_1 \circledast_{\mathbf{s}} M_2$, this will complete the proof. First color all the elements of M_l' by P_l for $l \in \{1,2\}$. Next for each $R \in \tau^+$ redefine R' to be the same as R, do this for both M_1', M_2'. Now take the disjoint union of the to resulting structures. Finally we have to redefine the

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relations of τ^+ of the disjoint union to be as in $M_1 \circledast_s M_2$. Let $R \in \tau^+$ be n-ary and let $w_1, w_2 \subseteq \{1, \dots, n\}$ satisfy $w_1 \cup w_2 = \{1, \dots, n\}$. Let p be the quantifier free type in the vocabulary with two unary relations S_1, S_2 "saying" that for $i \leq n$ and $l \in \{1, 2\}$, $x_i \in S_l$ if and only if $i \in w_l$. Now define:

$$\varphi_{R,w_1,w_2}(x_1, \dots x_n) := \bigwedge_{i \in w_1} P_1(x_i) \bigwedge_{i \in w_2} P_2(x_i) \wedge \Big[\bigvee_{\substack{q_l \in TP_0(\langle x_i :\, i \in w_l \rangle, \tau^+) \\ f^{\circledast}_{R,w_1,w_2}(p,q_1,q_2) = \mathbb{T}}} \wedge q_1' \wedge q_2' \Big].$$

Where $\wedge q_l'$ is the disjunction of all the formulas in q_l where we replace every relation $R \in \tau^+$ by R'. Now redefine the relation R using the modification $\delta_{R,B}$ for the formula:

$$B(x_1, \dots, x_n) := \bigwedge_{\substack{w_1, w_2 \subseteq \{1, \dots, n\} \\ w_1 \cup w_2 = \{1, \dots, n\}}} \varphi_{R,w_1,w_2}(x_1, \dots, x_n).$$

Do this for all $R \in \tau^+$ and we are done. ⊣

From the two lemmas above we see that if we do not use constant symbols, then the definitions of bounded patch-width and of constructibility coincide. The reason we prefer to use constructibility is that the binary operations suggests interpretations by binary trees (which we prefer), and the use of constants allows more general operations (which is useful in proofs). We could however used m-ary patch-width instead of m-constructibility without much difference.

NOTATION 2.11 (Trees). (1) *The vocabulary of trees, τ_{trees}, is $\{\leq, c_{rt}\}$.*

(2) *The vocabulary of k-colored-trees, $\tau_{k-trees}$, is $\{\leq, c_{rt}\} \cup \{P_1, \dots, P_k\}$ i.e., $(\tau_{trees})_k$.*

(3) *A tree $\mathfrak{T}$ is a τ_{trees}-structure in which:*
- *$\leq^{\mathfrak{T}}$ is a partial order on $|\mathfrak{T}|$.*
- *$c^{\mathfrak{T}}_{rt}$ is a singleton. We write $c^{\mathfrak{T}}_{rt}$ instead of $\{c^{\mathfrak{T}}_{rt}\}$.*
- *For every $t \in |\mathfrak{T}|$ the set $\{s \in |\mathfrak{T}| : s \leq^{\mathfrak{T}} t\}$ is linearly ordered by $\leq^{\mathfrak{T}}$.*
- *For all $x \in |\mathfrak{T}|$, $c^{\mathfrak{T}}_{rt} \leq^{\mathfrak{T}} x$.*

(4) *A k-colored-tree $\mathfrak{T}$ is a $\tau_{k-trees}$-structure, such that $\mathfrak{T} \restriction \tau_{trees}$ is a tree.*

(5) *A 2-colored-tree $\mathfrak{T}$ is directed binary (DB) if $(c^{\mathfrak{T}}_{rt}, P^{\mathfrak{T}}_1, P^{\mathfrak{T}}_2)$ is a partition of $|\mathfrak{T}|$, and each non-maximal element of $\mathfrak{T}$ has exactly two immediate successors one in $P^{\mathfrak{T}}_1$ and the other in $P^{\mathfrak{T}}_2$. For $k \geq 2$, a k-colored-tree $\mathfrak{T}$ is DB if $(|\mathfrak{T}|; \leq^{\mathfrak{T}}, c^{\mathfrak{T}}_{rt}, P^{\mathfrak{T}}_1, P^{\mathfrak{T}}_2)$ is.*

DEFINITION 2.12 (Monadic interpretation in trees). (1) *We call $\mathbf{c}$ a monadic k-interpretation scheme for a vocabulary τ if $\mathbf{c}$ consists of:*
- *Natural numbers $k_1 = k^{\mathbf{c}}_1$ and $k_1 = k^{\mathbf{c}}_2$ both less then or equal to k.*
- *For every $l \leq k_1$ a monadic $\tau_{k_2-trees}$-formula $\varphi^{\mathbf{c}}_{=,l}(x)$.*
- *For every n-place relation $R \in \tau$ and every $\eta \in {}^{\{1,\dots,n\}}\{0, \dots, k_1\}$ a monadic $\tau_{k_2-trees}$-formula: $\varphi = \varphi^{\mathbf{c}}_{R,\eta}(x_1, \dots, x_n)$.*

(2) *Let $\mathbf{c}$ be a monadic k-interpretation scheme for a vocabulary τ, and $\mathfrak{T}$ a $k^{\mathbf{c}}_2$-tree. The interpretation of $\mathfrak{T}$ by $\mathbf{c}$ denoted by $\mathfrak{T}^{[\mathbf{c}]}$ is the τ-structure M defined by:*
- *$|M| = \{(t, l) \in |\mathfrak{T}| \times \{0, \dots, k_1\} : \mathfrak{T} \models \varphi_{=,l}(t)\}$.*
- *For every $R \in \tau$ n-place relation:*

$$R^M = \{((t_i, l_i) : i \leq n) \in {}^n|M| : \mathfrak{T} \models \varphi_{R,(l_i :\, i \leq n)}(t_1, \dots, t_n)\}.$$

(3) *For* $\mathbf{c}$ *a monadic* k*-interpretation scheme for* τ *we denote by* $\mathfrak{K}_{\mathbf{c}}^{mo}$ *the class of all* τ*-structures* M *such that for some* $k_2^{\mathbf{c}}$*-tree,* $\mathfrak{T}$*, we have:* $\mathfrak{T}^{[\mathbf{c}]} \cong M$. $\mathfrak{K}_{\mathbf{c}}^{mo,db}$ *is the same as* $\mathfrak{K}_{\mathbf{c}}^{mo}$ *only we demand that* $\mathfrak{T}$ *is directed binary.*

(4) *We say that* $\mathbf{c}$ *has the leaf property if* $k_1^{\mathbf{c}} = 0$ *and for every* $k_2^{\mathbf{c}}$*-tree* $\mathfrak{T}$*, and every* $t \in |\mathfrak{T}|$: $\mathfrak{T} \models \varphi_{0,=}^{\mathbf{c}}[t]$ *implies that* t *is a maximal element in* $\mathfrak{T}$.

Our scheme $\mathbf{c}$ is a special case of a monadic second order definable transduction defined by Courcelle in [4, 5]. For comparison note that $\mathbf{c}$ is the (τ, τ_{trees}) $(k_1^{\mathbf{c}} + 1)$-copying with $k_2^{\mathbf{c}}$ parameters defined by: $(\phi_{trees}, \phi_{=,0}^{\mathbf{c}}, \dots, \phi_{=,k_1}^{\mathbf{c}}, (\phi_w^{\mathbf{c}})_{w \in \tau * k_1})$, (in the notations of [4, 5]) where ϕ_{trees} is the conjunctions of the tree axioms.

Without loss of generality we may assume that $k_1^{\mathbf{c}} = 0$. This is because of the following:

Lemma 2.13. *For every* $\mathbf{c}$ *a monadic* k*-interpretation scheme for a vocabulary* τ*, there exists* $\mathbf{c}'$ *a monadic* $(k+2)$*-interpretation scheme for* τ*, such that:*

- $k_1^{\mathbf{c}'} = 0$.
- $k_2^{\mathbf{c}'} = k_2^{\mathbf{c}} + 2$.
- *For every* $k_2^{\mathbf{c}}$*-tree* $\mathfrak{T}$*, there exists a* $k_2^{\mathbf{c}'}$*-tree* $\mathfrak{T}'$*, such that:* $\mathfrak{T}^{[\mathbf{c}]} \cong \mathfrak{T}'^{[\mathbf{c}']}$.

Hence $\mathfrak{K}_{\mathbf{c}}^{mo} \subseteq \mathfrak{K}_{\mathbf{c}'}^{mo}$.

Proof. Let s_1 and s_2 be the two "new" unary predicates, and let $\mathfrak{T}$ be a $k_2^{\mathbf{c}}$-tree. Define $\mathfrak{T}'$ as follows: $|\mathfrak{T}'| = |\mathfrak{T}| \cup (|\mathfrak{T}| \times \{0, \dots, k_1^{\mathbf{c}}\})$, $s_1^{\mathfrak{T}'} = |\mathfrak{T}|$, $s_2^{\mathfrak{T}'} = |\mathfrak{T}| \times \{0, \dots, k_1^{\mathbf{c}}\}$, and if t_1 is the immediate successor of t_2 in $\mathfrak{T}$ then define, $t_1 <^{\mathfrak{T}'} (t_1, 0) <^{\mathfrak{T}'} (t_1, 1) <^{\mathfrak{T}'} \cdots <^{\mathfrak{T}'} (t_1, k_1^{\mathbf{c}}) <^{\mathfrak{T}'} t_2$. Now define:

$$\varphi_{=,0}^{\mathbf{c}'}(x) := s_2(x) \wedge \bigwedge_{l < k_1^{\mathbf{c}}} (\forall y)[s_1(y) \wedge (\psi_l(x, y)] \to (\varphi_{=,l}^{\mathbf{c}}(y))^{s_1}.$$

Where $\psi_l(x, y)$ is a formula stating that there are exactly l elements between x and y and all of them are in s_2, and $(\varphi_{=,l}^{\mathbf{c}}(y))^{s_1}$ is the formula $\varphi_{=,l}^{\mathbf{c}}(y)$ relativized to s_1 i.e., we replace every quantifier of the form $\exists x$ or $\forall x$ by $\exists x s_1(x) \wedge$ or $\forall x s_1(x) \to$ respectively, and every quantifier of the form $\exists X$ or $\forall X$ by $\exists X (\forall x X(x) \to s_1(x)) \wedge$ or $\forall X (\forall x X(x) \to s_1(x)) \to$ respectively. It should be clear that $\mathfrak{T} \models \varphi_{=,l}^{\mathbf{c}}[t]$ if and only if $\mathfrak{T}' \models \varphi_{=,0}^{\mathbf{c}'}[(t, l)]$. The relations are dealt with in a similar way. $\dashv$

Lemma 2.14. *Let* $\mathfrak{K}$ *be a* (m^*, k^*)*-constructible class of* τ*-structures. Then there exists a natural number* k^{**} *and a monadic* k^{**}*-interpretation scheme* $\mathbf{c}$ *with the leaf property, such that* $\mathfrak{K} \subseteq \mathfrak{K}_{\mathbf{c}}^{mo,db}$.

We will not go into detail here. A similar result was proved by Courcelle in [4] Theorem 4.6. We do however give a sketch of the proof containing some definitions that will be useful later.

Sketch. Suppose $\mathfrak{P}$ and $\mathfrak{S}$ are the finite sets of structures and operations generating $\mathfrak{K}$, and τ^+ is the vocabulary associated with $\mathfrak{K}$ (see 2.7). Now with every $M \in \mathfrak{K}$ we can associate a DB tree which represents the construction of M from the structures in $\mathfrak{P}$. The leaves of this tree are structures in $\mathfrak{P}$, every non-maximal node of the tree is a τ^+-structure which is the result of its two immediate successors by an operation in $\mathfrak{S}$, and its root (restricted to τ) is M. Formally we define:

Definition 2.15. *We say that the pair* $(\mathfrak{T}, \mathfrak{M})$ *with* $\mathfrak{T} = \langle T; \leq^{\mathfrak{T}}, c_{rt}^{\mathfrak{T}}, S_1^{\mathfrak{T}}, S_2^{\mathfrak{T}} \rangle$ *a DB tree and* $\mathfrak{M} = \langle M_t : t \in T \rangle$*, is a full representation of* $M \in \mathfrak{K}$ *when:*

(1) *Every M_t is a k_t-const τ^+-structure for some $k_t \leq k^*$.*
(2) *For every $t \in T$ $\leq^{\mathfrak{T}}$-maximal, $M_t \in \mathfrak{P}$.*
(3) *The τ-reduct of $M_{c_{rt}^{\mathfrak{T}}}$ is M.*
(4) *For every t, a non-maximal element of T, let s_1, s_2 be its immediate successors with $s_l \in S_l^{\mathfrak{T}}$. Then $M_t = M_{s_1} \circledast_{\mathbf{s}} M_{s_2}$ for some $\mathbf{s} \in \mathfrak{S}_{\tau^+, k_{s_1}, k_{s_2}, k_t} \cap \mathfrak{S}$.*

We can encode the information necessary for the construction of M using labels (colors) for the nodes of the tree. Formally:

Definition 2.16. (1) *Let τ^* be the vocabulary $\tau_{k_2-trees}$ with the following unary predicates:*
(a) *S_1 and S_2.*
(b) *P_k for $k \leq k^*$.*
(c) *$Q_{\mathbf{s}}$ for $\mathbf{s} \in \mathfrak{S}$.*
(d) *R_N for $N \in \mathfrak{P}$.*
k_2 is the total number of unary predicates in τ^*, i.e., $k_2 = |\mathfrak{P}| + |\mathfrak{S}| + k^* + 2$.
(2) *A τ^*-structure $\mathfrak{T}$ is a representation of $M \in \mathfrak{K}$, if we can find $\mathfrak{M} = \langle M_t : t \in |\mathfrak{T}| \rangle$ such that:*
(a) *$((|\mathfrak{T}|, \leq^{\mathfrak{T}}, c_{rt}^{\mathfrak{T}}, S_1^{\mathfrak{T}}, S_2^{\mathfrak{T}}), \mathfrak{M})$ is a full representation of M.*
(b) *$\langle P_k^{\mathfrak{T}} : k \leq k^* \rangle$ is a partition of $|\mathfrak{T}|$. If $t \in P_k^{\mathfrak{T}}$, then $k_t = k$ i.e., M_t is a k-const τ^+-structure. We write $k^{\mathfrak{T}}(t) = k$ if and only if $t \in P_k^{\mathfrak{T}}$.*
(c) *$\langle Q_{\mathbf{s}}^{\mathfrak{T}} : \mathbf{s} \in \mathfrak{S} \rangle \cup \langle R_N^{\mathfrak{T}} : N \in \mathfrak{P} \rangle$ is a partition of $|\mathfrak{T}|$.*
(d) *For every $t \in |\mathfrak{T}|$ $\leq^{\mathfrak{T}}$-maximal, $t \in R_{M_t}^{\mathfrak{T}}$.*
(e) *For every $t \in |\mathfrak{T}|$ non-maximal, let s_1, s_2 be its immediate successors with $s_l \in S_l^{\mathfrak{T}}$. Suppose $M_t = M_{s_1} \circledast_{\mathbf{s}} M_{s_2}$ for some $\mathbf{s} \in \mathfrak{S}_{\tau^+, k_{s_1}, k_{s_2}, k_t} \cap \mathfrak{S}$. Then $t \in Q_{\mathbf{s}}^{\mathfrak{T}}$.*

Note that:

Observation 2.17. (1) *Every $M \in \mathfrak{K}$ has a full representation, and hence a representation.*
(2) *If $M_l \in \mathfrak{K}$ are represented by $\mathfrak{T}_l$ for $l \in \{1, 2\}$, and $\mathfrak{T}_1 \cong \mathfrak{T}_2$. Then $M_1 \cong M_2$.*

Now define: $k_1 = max\{|N| : N \in \mathfrak{P}\}$, k_2 is the number of unary predicates in τ^* (see 2.16(1)), and let $k^{**} = max\{k_1, k_2\}$. We can define a k^{**}-interpretation scheme $\mathbf{c}$ with $k_1^{\mathbf{c}} = k_1$ and $k_2^{\mathbf{c}} = k_2$ such that for all $M \in \mathfrak{K}$, and $\mathfrak{T}$ a representation of M we have $M \cong \mathfrak{T}^{[\mathbf{c}]}$. Note that indeed $\mathfrak{T}$ is a DB $k_2^{\mathbf{c}}$-colored-tree. We will not specify all the formulas of $\mathbf{c}$ as they tend to be very long and complicated, but do note that all the information about M can be decoded from the representation of M using monadic formulas. Finally by an argument very close to that of 2.13 we may assume that $\mathbf{c}$ has the leaf property. (See [3] Proposition 62 for more details). ⊣

§3. Equivalence of m-ary patch-width for $m \geq 3$.

We come now to the main part of our result. Basically what we do here is proving the reverse inclusion of 2.14. It turns out that in our constructible class we only need 3-ary relations as auxiliary relations, thus we can replace constructible by 3-constructible. It follows that a class $\mathfrak{K}$ is contained in a constructible class, if and only if it is contained in a 3-constructible class, and similarly for m-ary patch-width. We start with an investigation of directed binary trees that will be useful later. See [2] for a similar discussion including a proof of Lemma 3.2 formulated in the terms of tree automata.

NOTATION 3.1. *Let $\mathfrak{T}$ be a DB k-colored-tree. Let $n \in \mathbb{N}$ and $x_1, \dots, x_n \in T$ be fixed maximal elements of $\mathfrak{T}$.*

(1) *For $x, y \in T$ denote by $x \wedge y$ the maximal element z with $z \leq x, y$.*
(2) *For $x, y \in T$ with $x \leq y$ denote $[x, y] := \{z \in T : x \leq z \leq y\}$ and $(x, y) := \{z \in T : x < z < y\}$.*
(3) *Define $Y := \{x_1, \dots, x_n\} \cup \{x_i \wedge x_j : i, j \leq n\} \cup \{c_{rt}^{\mathfrak{T}}\}$, and fix $(y_1, \dots, y_m)$ an enumeration of Y. Note that $m \leq 2n$.*
(4) *For any non-maximal $x \in T$ let $F_R(x) \in T$ (resp. $F_L(x) \in T$) be the unique immediate successor of x which is in $P_1^{\mathfrak{T}}$ (resp. $P_2^{\mathfrak{T}}$).*
(5) *Let $R_R(y, y')$ and $R_L(y, y')$ be binary relations meaning $F_R(y) \leq y'$ and $F_L(y) \leq y'$ respectively.*
(6) *The branching structure of $x_1, \dots, x_2$ is the structure $\langle Y, \leq^{\mathfrak{T}}, R_R, R_L, x_1, \dots, x_n \rangle$. The elements of Y will be referred to as branching points. Note that there are (up to isomorphism) finitely many possible branching structures for a fixed n.*
(7) *For $y, y' \in Y$ with $y < y'$ define, $T_{y,y'} = \{x \in T : x \geq y \wedge x \not\geq y'\}$.*
(8) *Let $T_R = \{c_{rt}^{\mathfrak{T}}\} \cup \{t \in T : R_R(c_{rt}^{\mathfrak{T}}, t)\}$, and $T_L = \{c_{rt}^{\mathfrak{T}}\} \cup \{t \in T : R_L(c_{rt}^{\mathfrak{T}}, t)\}$.*

LEMMA 3.2. *Let $q \in \mathbb{N}$. The type $tp_q^{MSO}((y_1, \dots, y_m), \mathfrak{T})$ is computable from the branching structure of $x_1, \dots, x_n$, the types $tp_q^{MSO}(c_{rt}^{\mathfrak{T}}, \mathfrak{T}|_{T_L})$, $tp_q^{MSO}(c_{rt}^{\mathfrak{T}}, \mathfrak{T}|_{T_R})$, and the types $tp_q^{MSO}((y, y'), \mathfrak{T}|_{T_{y,y'}})$ for y, y' adjacent branching points.*

PROOF. Let $\tau = \tau_{k-trees} \cup \{R_R, R_L\}$. We proceed by induction on n.

For $n = 0$: We can define an operation $\mathbf{s} \in \mathfrak{S}_{\tau,1,1,1}$ such that for every DB tree $\mathfrak{T}$ we have: $(\mathfrak{T}, c_{rt}^{\mathfrak{T}}) = (\mathfrak{T}|_{T_L}, c_{rt}^{\mathfrak{T}}) \circledast_{\mathbf{s}} (\mathfrak{T}|_{T_R}, c_{rt}^{\mathfrak{T}})$. The result now follows from the addition Theorem (2.6(2)).

For $n = 1$: Similarly to the previous case we can define an operation $\mathbf{s_L} \in \mathfrak{S}_{\tau,2,2,1}$ such that for every DB tree $\mathfrak{T}$ and every $x \in T_L$ we have: $(\mathfrak{T}, c_{rt}^{\mathfrak{T}}, x) = (\mathfrak{T}|_{T_{c_{rt}^{\mathfrak{T}},x}}, c_{rt}^{\mathfrak{T}}, x) \circledast_{\mathbf{s_L}} (\mathfrak{T}|_{T_R}, c_{rt}^{\mathfrak{T}})$. Symmetrically we can define $\mathbf{s_R} \in \mathfrak{S}_{\tau,2,1,2}$ such that for every DB tree $\mathfrak{T}$ and every $x \in T_R$ we have: $(\mathfrak{T}, c_{rt}^{\mathfrak{T}}, x) = (\mathfrak{T}|_{T_L}, c_{rt}^{\mathfrak{T}}) \circledast_{\mathbf{s_R}} (\mathfrak{T}|_{T_{c_{rt}^{\mathfrak{T}},x}}, c_{rt}^{\mathfrak{T}}, x)$. Now from the branching structure of x we can compute if $x \in T_R$ or $x \in T_L$ holds and use $\mathbf{s_R}$ or $\mathbf{s_L}$ accordingly.

For $n+1$: Assume we have proven the lemma for $x_1, \dots, x_n \in T$ and let $x_{n+1} \in T$. We make use of the following fact: Let y be a branching point of $x_1, \dots, x_n$, and let:

$$T_{\geq y} = \{x \in T : x \geq y\},$$
$$T_{\geq y}^R = \{x \in T : R_R(y, x)\} \cup \{y\},$$
$$T_{\geq y}^L = \{x \in T : R_L(y, x)\} \cup \{y\}.$$

Let $\overline{y}_{\geq y}$, $\overline{y}_{\geq y}^R$ and $\overline{y}_{\geq y}^L$ be the restrictions of $(y_1, \dots, y_m)$ to $T_{\geq y}$, $T_{\geq y}^R$ and $T_{\geq y}^L$ respectively. Then we can compute the types:

$$tp_q^{MSO}(\overline{y}_{\geq y}, \mathfrak{T}|_{T_{\geq y}}),\ tp_q^{MSO}(\overline{y}_{\geq y}^R, \mathfrak{T}|_{T_{\geq y}^R}) \text{ and } tp_q^{MSO}(\overline{y}_{\geq y}^L, \mathfrak{T}|_{T_{\geq y}^L}).$$

Why? from the induction hypothesis we can compute $Tp_q^{MSO}(y_1, \dots, y_m, \mathfrak{T})$, so all we have to do is to restrict the quantifiers of our formulas appropriately.

We return to the proof of the case $n+1$. Let us first deal with the following case:

$$(*) \qquad \begin{array}{lll} x_1,\dots,x_n \in T_L & \text{and} & x_{n+1} \in T_R \quad \text{or} \\ x_1,\dots,x_n \in T_R & \text{and} & x_{n+1} \in T_L. \end{array}$$

(which can be computed from the branching structure of $x_1,\dots,x_{n+1}$). Similarly to the case $n=1$ we can define $\mathbf{s}'_{\mathrm{L}} \in \mathfrak{S}_{\tau,m+1,m,2}$ such that for every DB k-colored-tree $\mathfrak{T}$ and every $x_1,\dots,x_n \in T_L$ and $x_{n+1} \in T_R$ we have:

$$(\mathfrak{T}, y_1,\dots,y_m, x_{n+1}) = (\mathfrak{T}|_{T_L}, y_1,\dots,y_m) \circledast_{\mathbf{s}'_{\mathrm{L}}} (\mathfrak{T}|_{T_{c^{\mathfrak{T}}_{rt},x_{n+1}}}, c^{\mathfrak{T}}_{rt}, x_{n+1}).$$

Symmetrically we can define $\mathbf{s}'_{\mathbf{R}} \in \mathfrak{S}_{\tau,m+1,2,m}$ such that for every DB k-colored-tree $\mathfrak{T}$ and every $x_1,\dots,x_n \in T_R$ and $x_{n+1} \in T_L$ we have: $(\mathfrak{T}, y_1,\dots,y_m,x_{n+1}) = (\mathfrak{T}|_{T_{c^{\mathfrak{T}}_{rt},x_{n+1}}}, c^{\mathfrak{T}}_{rt}, x_{n+1}) \circledast_{\mathbf{s}'_{\mathrm{L}}} (\mathfrak{T}|_{T_R}, y_1,\dots,y_m)$. In both cases we can compute the MSO theory of both operands, one is given and the other from the induction hypothesis. So by the addition theorem we are done.

Lastly assume that $(*)$ does not hold. Hence we have a unique $y \neq x_{n+1}$ which is a branching point of $x_1,\dots,x_{n+1}$ and not of $x_1,\dots,x_n$, and let y', y'' be the adjacent branching points of $x_1,\dots,x_n$ such that $y' < y < y''$. There are four possibilities for the branching structure of y, y', y'' which can be dealt with symmetrically. Without loss of generality let us assume that the possibility $R_R(y', y)$ and $R_L(y, y'')$ holds. Now from the types $tp^{MSO}_q(\overline{y}_{\geq y''}, \mathfrak{T}|_{T_{\geq y''}})$ (given by the fact above) and $tp^{MSO}_q((y, y''), \mathfrak{T}|_{T_{y,y''}})$ (given by the assumptions of the lemma) we can compute $tp^{MSO}_q((\overline{y}_{\geq y''}, y), \mathfrak{T}|_{T^L_{\geq y}})$. From this and the type $tp^{MSO}_q((y, x_{n+1}), \mathfrak{T}|_{T_{y,x_{n+1}}})$ (again given by the assumptions of the lemma) we can compute $tp^{MSO}_q((\overline{y}_{\geq y''}, y, x_{n+1})\mathfrak{T}|_{T_{\geq y}})$. From this and the type $tp^{MSO}_q((y', y), \mathfrak{T}|_{T_{y',y}})$ we can compute $tp^{MSO}_q((\overline{y}^R_{\geq y'}, y, x_{n+1}), \mathfrak{T}|_{T^R_{\geq y'}})$. From this and the type $tp^{MSO}_q(\overline{y}^L_{\geq y'}, \mathfrak{T}|_{T^L_{\geq y'}})$ (again given by the fact above) we can compute $tp^{MSO}_q((\overline{y}^R_{\geq y'}, y, x_{n+1}), \mathfrak{T}|_{T_{\geq y'}})$. If $y' = c^{\mathfrak{T}}_{rt}$ then $T_{\geq y'} = T$ and $\overline{y}^R_{\geq y'} = (y_1,\dots y_m)$ so we are done. Else assume (without loss of generality) that $R_R(c^{\mathfrak{T}}_{rt}, y')$ holds, and let $\overline{y'}$ be the restriction of $(y_1,\dots,y_m)$ to $T_{c^{\mathfrak{T}}_{rt},y'}$. We can compute the types $tp^{MSO}_q(\overline{y}^L_{\geq c^{\mathfrak{T}}_{rt}}, \mathfrak{T}|_{T^L_{\geq c^{\mathfrak{T}}_{rt}}})$ and $tp^{MSO}_q(\overline{y'}, y', c^{\mathfrak{T}}_{rt}, \mathfrak{T}|_{T_{c^{\mathfrak{T}}_{rt},y'}})$ which together with the above gives us the desired $tp^{MSO}_q(y_1,\dots,y_m,y,x_{n+1},\mathfrak{T})$. $\dashv$

LEMMA 3.3. *Let k^* be a natural number, and $\mathbf{c}$ a monadic k^*-interpretation scheme with the leaf property for a vocabulary τ. Then there exists a natural number k^{**}, and a $(3, k^{**})$-constructible class of τ structures, $\mathfrak{K}$, such that: $\mathfrak{K}^{mo,db}_{\mathbf{c}} \subseteq \mathfrak{K}$.*

PROOF. First we introduce some notation. Let q^* be the maximal quantifier rank of the formulas $\{\varphi_{Q,0} : Q \in \tau\}$. Define the vocabulary τ^+ to consist of:

- τ.
- $\tau_{k_2-trees}$ where $k_2 = k^{\mathbf{c}}_2$.
- Two 3-place relations R_R and R_L.
- For each $\mathbf{t} \in TP^{MSO}_{q^*}(2, \tau_{k_2-trees})$, a 3-place relation $R^3_{\mathbf{t}}$.
- For each $\mathbf{t} \in TP^{MSO}_{q^*}(2, \tau_{k_2-trees})$, a 2-place relation $R^2_{\mathbf{t}}$.
- For each $\mathbf{t} \in TP^{MSO}_{q^*}(1, \tau_{k_2-trees})$ two 0-place relations $R^R_{\mathbf{t}}$ and $R^L_{\mathbf{t}}$.

Given a k_2-colored-tree we use the new relations of τ^+ to encode the logical types we are interested in. We call the structure "correct" if indeed the new relations are interpreted as we want.

DEFINITION 3.4. *A τ^+-structure, $\mathfrak{T}$, is called a correct k_2-colored-tree if:*

- *For each $Q \in \tau$, $Q^{\mathfrak{T}} = \emptyset$.*
- *$\mathfrak{T} \restriction \tau_{k_2-trees}$ is a DB k_2-colored-tree.*
- *For each x_1, x_2, x_3 maximal elements of $|\mathfrak{T}|$, let $y = x_1 \wedge x_2$ and $y' = y \wedge x_3$. Then we have,*

$$R_R^{\mathfrak{T}}(x_1, x_2, x_3) \Leftrightarrow F_R(y) \leq y',$$

and similarly for R_L.

- *For each $\mathbf{t} \in TP_{q^*}^{MSO}(2, \tau_{k_2-trees})$, and x_1, x_2, x_3 maximal elements of $|\mathfrak{T}|$, let $y = x_1 \wedge x_2$ and $y' = y \wedge x_3$ then we have,*

$$(R_{\mathbf{t}}^3)^{\mathfrak{T}}(x_1, x_2, x_3) \Leftrightarrow tp_{q^*}^{MSO}((y, y'), \mathfrak{T}|_{T_{y,y'}}) = \mathbf{t}.$$

- *For each $\mathbf{t} \in TP_{q^*}^{MSO}(2, \tau_{k_2-trees})$, and x_1, x_2 maximal elements of $|\mathfrak{T}|$, let $y = x_1 \wedge x_2$ then we have,*

$$(R_{\mathbf{t}}^2)^{\mathfrak{T}}(x_1, x_2) \Leftrightarrow tp_{q^*}^{MSO}((c_{rt}^{\mathfrak{T}}, y), \mathfrak{T}|_{T_{c_{rt}^{\mathfrak{T}}, y}}) = \mathbf{t}.$$

- *For each $\mathbf{t} \in TP_{q^*}^{MSO}(1, \tau_{k_2-trees})$, $(R_{\mathbf{t}}^R)^{\mathfrak{T}} = \mathbb{T}$ if and only if*

$$tp_{q^*}^{MSO}(c_{rt}^{\mathfrak{T}}, \mathfrak{T}|_{T_R}) = \mathbf{t},$$

and similarly for $R_{\mathbf{t}}^L$.

Note that every DB k_2-colored-tree can be uniquely extended to a correct DB k_2-colored-tree. Now define $\mathfrak{P}$ to consist of all singleton correct models (models with one element) of the vocabulary τ^+, plus all the null τ^+-structures (see Definition 2.5(5)).

We now turn to the definition of the operations in $\mathfrak{S}$. Let u be a possible "color" of a singleton k_2-colored-tree. Formally $u \subseteq \{P_3, \dots, P_{k_2}\}$. We define the operation $\oplus_u$ on DB k_2-colored-trees as the addition of two trees with root of color u. Formally Let $\mathfrak{T}_1, \mathfrak{T}_2$ be DB k_2-colored-trees define $\mathfrak{T} = \mathfrak{T}_1 \oplus_u \mathfrak{T}_2$ by:

- $|\mathfrak{T}| = |\mathfrak{T}_1| \cup |\mathfrak{T}_2| \cup \{c\}$.
- c is the root of $\mathfrak{T}$ i.e., $c_{rt}^{\mathfrak{T}} = \{c\}$ and for all $t \in |\mathfrak{T}|, c \leq^{\mathfrak{T}} t$.
- c has color u i.e., for all $i \geq 3$, $c \in P_i^{\mathfrak{T}}$ if and only if $i \in u$.
- $c_{rt}^{\mathfrak{T}_1} \in P_1^{\mathfrak{T}}$ and $c_{rt}^{\mathfrak{T}_2} \in P_2^{\mathfrak{T}}$.
- The rest of the relations on $\mathfrak{T}_1$ and $\mathfrak{T}_2$ remain unchanged.

Note that indeed $\mathfrak{T}_1 \oplus_u \mathfrak{T}_2$ is a DB k_2-colored-tree whenever $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are, and hence $\oplus_u$ extends uniquely to an operation on correct k_2-trees.

Now for $l \in \{1, 2\}$ let $\mathfrak{A}_l$ be a τ^+-structure such that there exists a correct k_2-colored-tree with $|\mathfrak{A}_l| \subseteq |\mathfrak{T}_l|$, $\mathfrak{T}_l|_{|\mathfrak{A}_l|} = \mathfrak{A}_l$, and every element of $\mathfrak{A}_l$ is maximal in $\mathfrak{T}_l$. Define an operation $\mathbf{s}_u$ on such structures by: $\mathfrak{A}_1 \circledast_{\mathbf{s}_u} \mathfrak{A}_2 = (\mathfrak{T}_1 \oplus_u \mathfrak{T}_2)|_{|\mathfrak{A}_1| \cup |\mathfrak{A}_2|}$. It is easy to verify that $\circledast_{\mathbf{s}_u}$ is well defined and indeed belongs to $\mathfrak{S}_{\tau^+, 0,0,0}$. We now have:

LEMMA 3.5. *For every correct k_2-colored-tree, $\mathfrak{T}$ and every set $A \subseteq |\mathfrak{T}|$ of maximal elements, the restriction $\mathfrak{T}|_A$ is in the closure of $\mathfrak{P}$ under the operations $\{\mathbf{s}_u : u \subseteq \{3, \dots, k_2\}\}$.*

PROOF. First it is obvious that we can construct $\mathfrak{T}$ from $\mathfrak{P}$ using the operations $\{\oplus_u : u \subseteq \{3, \dots, k_2\}\}$. Now use the same construction only replace in each step the operation $\mathfrak{T}_1 \oplus_u \mathfrak{T}_2$ by the operation $\mathfrak{T}_1|_A \circledast_{\mathbf{s}_u} \mathfrak{T}_2|_A$. ⊣

The last thing we need now is to "decode" the relations in the correct structure into the relations in our vocabulary τ. For this we use:

LEMMA 3.6. *There exist $\mathbf{s}^* \in \mathfrak{S}_{\tau^+,0,0,0}$ such that For every correct k_2-tree, $\mathfrak{T}$ and every set $A \subseteq |\mathfrak{T}|$ of maximal elements, the structure $\mathfrak{A}' = \mathfrak{T}|_A \circledast_{\mathbf{s}^*} Null_\emptyset$ satisfies for each $Q \in \tau$ with $n(Q) = n$,*

$$(*) \qquad Q^{\mathfrak{A}'} = \{(x_1, \dots, x_n) \in {}^nA : \mathfrak{T} \models \varphi_{Q,0}(x_1, \dots, x_n)\}.$$

PROOF. Let $Q \in \tau$ be an n-place relation symbol, and $w_1, w_2 \subseteq \{1, \dots, n\}$. We should define $f^{\mathbf{s}^*}_{Q,w_1,w_2}$ in such a way that $(*)$ will hold. As we have $k_1^{\mathbf{s}^*} = k_2^{\mathbf{s}^*} = k^{\mathbf{s}^*} = 0$ and we are only interested in $Null_\emptyset$ as the right operand, the only relevant case is $w_1 = \{1, \dots, n\}$ and $w_2 = \emptyset$. In order to have $(*)$ We need to define a function:

$$f^{\mathbf{s}^*}_Q : \{p : \text{p is a quantifier free type of n variables in vocabulary } \tau^+\} \to \{\mathbb{T}, \mathbb{F}\}$$

such that for all $(x_1, \dots, x_n) \in {}^nA$, $f_Q(tp_{qf}((x_1, \dots, x_n), \mathfrak{A}')) = \mathbb{T}$ if and only if $\mathfrak{T} \models \varphi_{Q,0}(x_1, \dots, x_n)$. Recall that by Lemma 3.2 the value of $\mathfrak{T} \models \varphi_{Q,0}(x_1, \dots, x_n)$, is determined by the branching structure of $x_1, \dots, x_n$, the types $tp_q^{MSO}((y, y'), \mathfrak{T}|_{T^3_{y,y'}})$ for y, y' adjacent branching points of $x_1, \dots, x_n$, and the types $tp_q^{MSO}(c^{\mathfrak{T}}_{rt}, \mathfrak{T}|_{T_L})$ and $tp_q^{MSO}(c^{\mathfrak{T}}_{rt}, \mathfrak{T}|_{T_R})$ (see 3.2 and Notation 3.1). But as $\mathfrak{T}$ is correct these all are determined by p so we are done. ⊣

We can now conclude the proof of Lemma 3.3. Define

$$\mathfrak{S} = \{\mathbf{s}_u : u \subseteq \{3, \dots, k_2\}\} \cup \{\mathbf{s}^*\},$$

and let $\mathfrak{K}$ be the constructible class of τ-structures defined by $\mathfrak{P}$ and $\mathfrak{S}$. Let M be a τ-structure in $\mathfrak{K}^{mo,db}_{\mathbf{c}}$. So we have $M \cong \mathfrak{T}_1^{[\mathbf{c}]}$ for some DB k_2-colored-tree $\mathfrak{T}_1$. Let $\mathfrak{T}_2$ be the correct extension of $\mathfrak{T}_1$. Let $A = \{x \in |\mathfrak{T}_2| : \mathfrak{T}_1 \models \varphi_{=,0}(x)\}$, and $\mathfrak{A} = \mathfrak{T}_2|_A \circledast_{\mathbf{s}^*} Null_\emptyset$. From Lemma 3.5 we have that $\mathfrak{T}_2|_A$ is in the closure of $\mathfrak{P}$ under the operations in $\mathfrak{S}$ and hence so is $\mathfrak{A}$. From Lemma 3.6 and the definition of $\mathfrak{T}_1^{[\mathbf{c}]}$, we have that $\mathfrak{A}|_\tau = \mathfrak{T}_1^{[\mathbf{c}]} \cong M$, so $M \in \mathfrak{K}$ as desired. ⊣

From Lemmas 3.3 and 2.14 we conclude our main:

THEOREM 3.7. *Let $\mathfrak{K}$ be a class of τ-structures. Then $\mathfrak{K}$ is contained in a m-constructible class for some $m \in \mathbb{N}$ if and only if $\mathfrak{K}$ is contained in a 3-constructible class.*

The same holds for patch-width:

COROLLARY 3.8. *Let τ be a nice vocabulary and $\mathfrak{K}$ a class of τ-structures. Then $\mathfrak{K}$ is contained in a class of bounded m-ary patch-width for some $m \in \mathbb{N}$ if and only if $\mathfrak{K}$ is contained in a class of bounded 3-ary patch-width.*

PROOF. Assume $\mathfrak{K} \subseteq \mathfrak{K}'$ for some $\mathfrak{K}'$ of bounded m-ary patch-width. By Lemma 2.9 $\mathfrak{K}'$ is $(m, 0)$-constructible. By Theorem 3.7 $\mathfrak{K}'$ is contained in some 3-constructible $\mathfrak{K}''$. Notice that that the set $\mathfrak{S}$ defined in the proof of 3.3 satisfies that $\mathfrak{S} \subseteq \mathfrak{S}_{\tau^+,0,0,0}$ so $\mathfrak{K}''$ is in fact $(3, 0)$-constructible. Notice further that in the proof of 3.3 we do not need null structures in the construction, hence we may replace

τ^+ by a nice vocabulary. So by Lemma 2.10 $\mathfrak{K}''$ is a bounded 3-ary patch-width class as desired. $\dashv$

A close result to this corollary is Proposition 63 in [3] by Blumensath and Courcelle. They proved that if a class of τ-structures is monadically interpretable in trees then it can be constructed from singleton τ^+-structures, where $max\{n(R) : R \in \tau^+ \setminus \tau\} < max\{n(R) \colon R \in \tau\}$. The operations used for the construction though are more limited then ours so our Corollary 3.8 is not a generalization of their result.

§4. The cases $m = 1$ and $m = 2$. Can we improve Theorem 3.7 by replacing 3-constructible by 2-constructible? Consider a unary operation on k-const τ-structures, $\mathbf{s}_i$ for $i \leq k$ defined by: $\mathbf{s}_i(M)$ is the restriction of M to the set $\{x \in M \colon x \notin P_i^M\}$. If we would allow these operations in the definition of constructibility then the answer would be yes. Why? In the proof of Lemma 3.3 we could construct the full tree rather then restrict to the leaves (i.e., we could use the operations $\oplus_u$ instead of $\circledast_{\mathbf{s}_u}$). In this case we could encode the necessary logical types by binary relations. If we do not change the definition of constructibility then the question remains open.

Can we improve Theorem 3.7 by replacing 3-constructible by 1-constructible? In this case we know the answer to be negative. This follows from the following:

THEOREM 4.1. *There exists a nice vocabulary τ, and a class of τ-structures $\mathfrak{K}$, contained in some 3-constructible class, that is not contained in any 1-constructible class.*

The idea behind the proof is the following: In the complete binary tree of depth n, we will define a 4-ary relation $R(x_1, x_2, x_3, x_4)$ on leaves of the tree, which is some relation q on the distances (in the binary tree) between x_1 and x_2 and between x_3 and x_4 where the distances are taken mod p. The relation R will be defined in terms of of general p and q, and particular p and q are chosen in the end of the proof. A counting argument will prove that R cannot be constructed using unary auxiliary relations only. We give the proof in detail:

PROOF. Let $\tau = \{R\}$ with $n(R) = 4$. Set $p \in \mathbb{N}$ be large enough (to be defined later). Let $\mathfrak{T}$ be a tree. A set $X \subseteq T$ is convex in $\mathfrak{T}$ if $z, z' \in X$ and $z < z'' < z'$ implies $z'' \in X$. For $x, y \in T$ Define:

- $x \wedge^{\mathfrak{T}} y = x \wedge y =$ the $<^{\mathfrak{T}}$ maximal $z \in T$ such that $x \geq^{\mathfrak{T}} z$ and $y \geq^{\mathfrak{T}} z$.
- $d^{\mathfrak{T}}(x, y) = d(x, y) = min\{|S| \colon S \subseteq T, x, x \wedge y \in S, \text{ S is convex in } (T, <^{\mathfrak{T}})\}$.
- $d_p^{\mathfrak{T}}(x, y) = d_p(x, y) = d(x, y) \pmod p$.

Let $q \colon \{0, \dots, p-1\}^2 \to \{0, 1\}$ be some function that will be defined later. We now define $\mathbf{c}$ a 0-interpretation scheme for τ:

- $k_1^{\mathbf{c}} = k_2^{\mathbf{c}} = 0$.
- $\varphi_{=,0}^{\mathbf{c}}(x) = \neg\exists y y > x$ i.e., the elements of the interpreted structure are the leaves of the tree.
- $\varphi_{R,0}^{\mathbf{c}}(x_1, x_2, x_3, x_4) =$ "$q(d_p(x_1, x_2), d_p(x_3, x_4)) = 0$".

We have to show that $\varphi_{R,0}$ is indeed a monadic formula in τ_{trees}. Note that there exists a monadic formula $\varphi_{d_p=0}(x, y)$ such that for any tree $\mathfrak{T}$, $\mathfrak{T} \models \varphi_{d_p=0}(x, y)$ if and only if $d_p^{\mathfrak{T}}(x, y) = 0$. $\varphi_{d_p=0}(x, y)$ will "say" that there exists a set X such that:

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- $x, x \wedge y \in X$,
- the set X is convex in $\mathfrak{T}$,
- if z' is the immediate successor in X of $z \in X$, then there exist exactly $p-1$ elements (of T) between them.

similarly we have formulas $\varphi_{d_p=i}(x,y)$ for $0 < i < p$. Now define:

$$\varphi_{R,0}(x_1,x_2,x_3,x_4) = \bigvee_{\substack{n_1,n_2\in\{0,\dots,p-1\}\\ n_1\equiv n_2 \pmod p}} \varphi_{d_p=n_1}(x_1,x_2) \wedge \varphi_{d_p=n_2}(x_3,x_4).$$

This gives us $\mathbf{c}$ as desired. Define $\mathfrak{K} = \mathfrak{K}_{\mathbf{c}}^{mo.db}$. By 3.3 $\mathfrak{K}$ is contained in a 3-constructible class (in fact in a 3-ary BPW class). For each $n \in \mathbb{N}$ let $M_n = ({}^{n\geq}2, \lhd)$ i.e., M_n is the complete binary tree of depth n, and $N_n = M_n^{[\mathbf{c}]}$ (see Definition 2.12(2)) so N_n is just the relation R. Let $\mathfrak{K}'$ be a constructible class of τ-structures, so $\tau^+ = \tau_k$ for some $k \in \mathbb{N}$. Towards contradiction assume that $N_n \in \mathfrak{K}'$ for all $n \in \mathbb{N}$. Let $\mathfrak{P}$ be the set of "atomic" structures associates with $\mathfrak{K}'$. Without loss of generality we may assume that $\mathfrak{P}$ consists of singleton structures only. Otherwise increase k by $max\{|M|: M \in \mathfrak{P}\}$ and construct each $M \in \mathfrak{P}$ from singletons of distinct colors. Now let $K \in \mathfrak{K}'$, and let $(\mathfrak{T}, \mathfrak{M})$ be a full representation of K (see 2.15). Assume $K \cong N_n$ for some n. So we have a 1-1 function f, from ${}^n2$ to the leaves of $\mathfrak{T}$, as every $\eta \in {}^n2$ corresponds to a unique element $a \in K$ under the isomorphisms, and for every element of $a \in K$ there exist a unique t a leaf of $\mathfrak{T}$ such that $a = |M_t|$. Define $f(\eta) = t$. Note that f is not onto, as some of the leaves of $\mathfrak{T}$ may be omitted during the creation process. For each $t \in T$ let $A_t = \{f^{-1}(s): s \leq^{\mathfrak{T}} \wedge s \in range(f)\}$. So $A_t \subseteq {}^n2$. For each $\eta \in A_t$ let $a = a_\eta = |M_{f(\eta)}|$. a_η is an element of M_t, so A_t is divided into 2^k parts according to the color of a_η in M_t, (more formally according to the type $tp_{qf}^{\tau'\setminus\tau}(a_\eta, M_t)$). We therefore have $B_t \subseteq A_t$ such that $|B_t| \geq \frac{|A_t|}{2^k}$, and all the elements of $f^{-1}(B_t)$ have the same color. Now define:

$$C_t = \{d_p^{N_n}(\eta, \eta \wedge \nu): \eta, \nu \in B_t\} \subseteq \{0, \dots, p-1\}.$$

We have $\frac{|A_t|}{2^k} \leq |B_t| \leq 2^{|C_t|}$. For the right-hand inequality use induction on $|C_t|$. Hence we conclude

$$|A_t| \leq |B_t| \cdot 2^k \leq 2^{|C_t|+k}.$$

Now note that if $C_t \neq \{0, \dots, p-1\}$, then $|C_t| \leq n - \lfloor \frac{n}{p} \rfloor$ and hence $|A_t| \leq 2^{|C_t|+k} \leq 2^{n-\lfloor \frac{n}{p} \rfloor+k}$. We now consider two cases:

CASE 1. There exist $s \in T$ with two immediate successors $t_1, t_2 \in T$ such that: $|A_{t_1}|, |A_{t_2}| > 2^{n-\lfloor \frac{n}{p} \rfloor+k}$.

According to what we saw above we have $C_1 = C_2 = \{0, \dots, p-1\}$. So for $l \in \{1,2\}$ we have $\langle(\rho_{t_l,i}, \nu_{t_l,i}: i \in \{0, \dots, p-1\}\rangle$ such that:

(α) $\{\rho_{t_l,i}, \nu_{t_l,i}: i \in \{0, \dots, p-1\}\}$ all have the same color in M_{t_l}.
(β) $d_p^{N_n}(\rho_{t_l,i}, \nu_{t_l,i}) = i$ for all $i < p$.

Denote by m the number of quantifier free types of couples in the vocabulary τ (actually in our case $m = 2^{(2^4)}$). Note that m does not depend on p. So for each $l \in \{1,2\}$, $\{0, \dots, p-1\}$ is divided into m parts according to the type

$tp_{qf}((\rho_{t_l,i}, v_{t_l,i}), M_{t_l})$. We claim that we can (a priori) choose p (large enough) and q in such a way that we can find i_1, i_2, j_1, j_2 such that for each $l \in \{1,2\}$: $(\rho_{t_l,i_l}, v_{t_l,i_l})$ and $(\rho_{t_l,j_l}, v_{t_l,j_l})$ have the same quantifier free type in vocabulary τ in M_{t_l}, and on the other hand: $q(i_1, j_1) \neq q(i_2, j_2)$. This is of course a contradiction as the quantifier free type of $(\rho_{t_l,i_l}, v_{t_l,i_l})$ and $(\rho_{t_l,j_l}, v_{t_l,j_l})$ in vocabulary τ_k in M_{t_l} determines the value of $R(\rho_{t_l,i_l}, v_{t_l,i_l}, \rho_{t_l,j_l}, v_{t_l,j_l})$ in M_s and hence in $M_{c_{rt}^{\mathfrak{T}}}$. But this value is true if and only if $q(i_l, j_l) = 0$ in contradiction with $q(i_1, j_1) \neq q(i_2, j_2)$. Why can we choose p and q as desired? Let $f\colon \{0, \dots, p-1\}^2 \to \{0,1\}$. By a partition of $\{0, \dots, p-1\}^2$ into m parts we will mean a pair (e_1, e_2) of equivalence relations on $\{0, \dots p-1\}$ both with at most m equivalence classes. We say that f respects the partition (e_1, e_2) if for each $i_1, i_2, j_1, j_2 \in \{0, \dots, p-1\}$ we have $\bigwedge_{l\in\{1,2\}}(i_l, j_l) \in e_l \Rightarrow \bigwedge_{l\in\{1,2\}} f(i_l) = f(j_l)$. Now for a given p the number of partitions of $\{0, \dots, p-1\}^2$ into m parts is $\leq m^p \cdot m^p$, and the number of functions from $\{0, \dots, p-1\}^2$ to $\{0,1\}$ that respect a given partition is $\leq 2^{m\cdot m}$. Hence the number of functions that we cannot choose (i.e., functions that respects some partition of $\{0, \dots, p-1\}$ into m parts) is $\leq 2^{2p\log(m)+m^2}$. But the total number of functions is 2^{p^2}. So if we choose (a priori) p such that $p^2 > 2p\log(m) + m^2$ we can choose a function q as desired.

Assume now that the assumption of CASE 1 does not hold. Assume also that we have chosen n large enough such that $2^{\lfloor \frac{n}{p} \rfloor - k} > 4$. In this case we can find $t_0, t_1, \dots, t_d \in T$ such that:

- $d \geq 5$.
- $t_0 = c_{rt}^{\mathfrak{T}}$.
- The element t_d is maximal in $\mathfrak{T}$.
- For $0 \leq i < d$, t_{i+1} is an immediate successor in $\mathfrak{T}$, of t_1.
- For $0 \leq i < d$, denote by s_{i+1} the immediate successor of t_i different from t_{i+1}, then $|A_{s_{i+1}}| \leq 2^{\lfloor \frac{n}{p} \rfloor - k}$.

Note that for any $0 < i \leq d$: $\bigcup_{0<j\leq i} A_{s_j}$ and A_{t_i} is a partition of $A_{c_{rt}^{\mathfrak{T}}}$, and that $|A_{c_{rt}^{\mathfrak{T}}}| = 2^n$. So we can find $0 < i^* \leq d$ such that $|\bigcup_{0<j\leq i^*} A_{s_j}|, |A_{t_{i^*}}| > 2^{\lfloor \frac{n}{p} \rfloor - k}$. We proceed similarly to CASE 1. As there we can find $\langle(\rho_{t_{i^*},i}, v_{t_{i^*},i} \in A_{t_{i^*}}$: $i \in \{0, \dots, p-1\}\rangle$ that satisfy (α) and (β) above, and the same for $\langle(\rho_i, v_i : i \in \{0, \dots, p-1\}\rangle$ where $\rho_i, v_i \in \bigcup_{0<j\leq i^*} A_{s_j}$. Again let m denote the number of quantifier free types of couples in the vocabulary τ. This time we want to choose p and q in such a way that we can find: i, j_1, j_2 such that: $(\rho_{t_{i^*},j_1}, v_{t_{i^*},j_1})$ and $(\rho_{t_{i^*},j_2}, v_{t_{i^*},j_2})$ have the same quantifier free type in vocabulary τ in $M_{t_{i^*}}$, and on the other hand: $q(i, j_1) \neq q(i, j_2)$. Again this is a contradiction as the quantifier free type of $(\rho_{t_{i^*},j_l}, v_{t_{i^*},j_l})$ for $l \in \{1,2\}$ determines the value of $R(\rho_{t_{i^*},j_l}, v_{t_{i^*},j_l}, \rho_i, v_i)$ in $M_{c_{rt}^{\mathfrak{T}}}$. Again this value is true if and only if $q(i, j_l) = 0$ in contradiction with $q(i, j_1) \neq q(i, j_2)$. Why can we choose p and q as desired? For a given p the number of functions from $\{0, \dots, p-1\}^2$ to $\{0,1\}$ such that we cannot choose as above is the number of partitions m^p, times the number of functions that respect that partition $2^{m\cdot p}$, or $2^{p\log(m)+m\cdot p}$. So if we choose p such that $p^2 > p\log(m) + m\cdot p$ we can choose a function q as desired. Note that the function we used for the second case will also work for the first case so we can use one definition of q. In both cases we get a contradiction and the proof is complete. $\dashv$

§5. **Conclusion.** We considered a set of quantifier free definable operations on relational structures, and the classes of structures constructible by these operations when we allow auxiliary relations. Our main result, Theorem 3.7 shows that we can restrict to 3-ary auxiliary relations without losing constructible classes. A parallel result for patch-width is Corollary 3.8. An example given in 4.1 shows that if we restrict to unary auxiliary relations we do lose constructible classes. The question regarding binary relations is left open.

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THE HEBREW UNIVERSITY OF JERUSALEM
EINSTEIN INSTITUTE OF MATHEMATICS
EDMOND SAFRA CAMPUS
GIVAT RAM
JERUSALEM 91904, ISRAEL
E-mail: shelah@math.huji.ac.il
E-mail: mord@math.huji.ac.il