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NOTE

Regressive Ramsey Numbers Are Ackermannian*

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We give an elementary proof of the fact that regressive Ramsey numbers are Ackermannian. This fact was first proved by Kanamori and McAloon with mathematical logic techniques. © 1999 Academic Press

Nous vivons encore sous le règne de la logique, voilà, bien entendu, à quoi je voulais en venir. Mais les procédés logiques, de nos jours, ne s'appliquent plus qu'à la résolution de problèmes d'intérêt secondaire (André Breton, *Manifeste du surréalisme*).

1. INTRODUCTION

DEFINITION 1. 1. Let A be a set of natural numbers. A coloring c: $[A]^e \to \mathbb{N}$ of unordened e-tuples from A is *regressive* if $c(x) < \min x$ for all $x \in [A]^e$.

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2. A subset $B \subseteq A$ is *min-homogeneous* for a coloring c of $[A]^e$ if for all $x \in [A]^e$ the color $c(x)$ depends only on $\min x$.

THEOREM 2 (Kanamori and McAloon). 1. *For every k and e there exists N such that for every regressive coloring of e-tuples from $\{1, 2, \cdots, N\}$ there exists a* min-*homogeneous subset of size k.*

2. *The statement in* (1) *cannot be proved from the axioms of Peano Arithmetic* (*although it can be* phrased *in the language of PA*)

3. *Let $v(k)$ be the least N which satisfies* 1 *for $e=2$. The function v eventually dominates every primitive recursive function.*

Part (3) of Kanamori and McAloon's result [2] was proved with methods from mathematical logic. We present below an elementary proof of 3.

2. THE LOWER BOUND

For every function $f: \mathbb{N} \to \mathbb{N}$ and n, $f^{(n)}$ is defined by $f^{(0)}(x)=x$ and $f^{(n+1)}(x)=f(f^{(n)}(x))$ for all $x \in \mathbb{N}$.

Define a sequence of (strictly increasing) integer functions $f_i: \mathbb{N} \to \mathbb{N}$ for $i \geqslant 1$ as follows:

$$f_1(n)=n+1, \tag{1}$$

$$f_{i+1}(n)=f_i^{(\lfloor \sqrt{n}/2 \rfloor)}(n). \tag{2}$$

Fix an integer $k>2$. Define a sequence of semi-metrics $\langle d_i: i \in \mathbb{N} \rangle$ on $\{n: n \geqslant 4k^2\}$ by putting, for $m, n \geqslant 4k^2$,

$$d_i(m,n)=|\{l \in \mathbb{N}: m<f_i^{(l)}(4k^2) \leqslant n\}|. \tag{3}$$

For $n>m \geqslant 4k^2$ let $I(m, n)$ be the greatest i for which $d_i(m, n)$ is positive, and $d(m, n)=d_{I(m,n)}(m, n)$.

CLAIM 3. *For all $n \geqslant m \geqslant 4k^2$, $d(m, n) \leqslant \sqrt{m}/2$.*

Proof. Let $i=I(m, n)$. Since $d_{i+1}(m, n)=0$, there exists t and l such that $t=f_{i+1}^{(l)}(4k^2) \leqslant m \leqslant n < f_{i+1}^{(l+1)}(4k^2)=f_{i+1}(t)$. But $f_{i+1}(t)=f_i^{(\lfloor \sqrt{t}/2 \rfloor)}(t)$ and therefore $\sqrt{t}/2 \geqslant d_i(t, f_{i+1}(t)) \geqslant d(m, n)$. ∎

Let us fix the following (standard) pairing function Pr on $\mathbb{N}^2$

$$\Pr(m, n)=\binom{m+n+1}{2}+n.$$

Pr is a bijection between $\mathbb{N}^2$ and $\mathbb{N}$ and is monotone in each variable. Observe that if $m, n \leqslant l$ then $\Pr(m, n) < 4l^2$ for all $l > 2$.

Define a pair coloring c on $\{n: n \geqslant 4k^2\}$ as follows:

$$c(\{m, n\}) = \Pr(I(m, n), d(m, n)). \tag{4}$$

CLAIM 4. *For every $i \in \mathbb{N}$, every sequence $x_0 < x_1 < \cdots < x_i$ that satisfies $d_i(x_0, x_i) = 0$ is not min-homogeneous for c.*

Proof. The claim is proved by induction on i. If $i = 1$ then there are no $x_0 < x_1$ with $d_1(x_0, x_1) = 0$ at all. Suppose to the contrary that $i > 1$, that $x_0 < x_1 < \cdots < x_i$ form a min-homogeneous sequence with respect to c and that $d_i(x_0, x_i) = 0$. Necessarily, $I(x_0, x_i) = j < i$. By min-homogeneity, $I(x_0, x_1) = j$ as well, and $d_j(x_0, x_i) = d_j(x_0, x_1)$. Hence, $\{x_1, x_2, \cdots x_i\}$ is min-homogeneous with $d_j(x_1, x_i) = 0$—contrary to the induction hypothesis. ∎

CLAIM 5. *The coloring c is regressive on the interval $[4k^2, f_k(4k^2))$.*

Proof. Clearly, $d_{k+1}(m, n) = 0$ for $4k^2 \leqslant m < n < f_k(4k^2)$ and therefore $I(m, n) < k \leqslant \sqrt{m}/2$. From Claim 3 we know that $d(m, n) \leqslant \sqrt{m}/2$. Thus, $c(\{m, n\}) \leqslant \Pr(\lfloor \sqrt{m}/2 \rfloor, \lfloor \sqrt{m}/2 \rfloor)$, which is $< m$, since $\sqrt{m} > 2$. ∎

We show that $f_k(4k^2)$ grows eventually faster than every primitive recursive function by comparing the functions f_i with the usual approximations of Ackermann's function. It is well known that every primitive recursive function is dominated by some approximation of Ackermann's function (see, e.g., [1]).

Let $A_i(n)$ be defined as follows:

$$A_1(n) = n + 1, \tag{5}$$

$$A_{i+1}(n) = A_i^{(n)}(n). \tag{6}$$

The A_i-s are the usual approximations to Ackermann's function, which is defined by $Ack(n) = A_n(n)$.

CLAIM 6. 1. *for all $n \geqslant 16$ and $i \geqslant 7$,*

(a) $16n^2 \leqslant f_i(n)$;

(b) $f_i(16n^2) \leqslant f_i^{(2)}(n)$.

2. $A_i(n) \leqslant f_{i+6}^{(2)}(n)$ *for all $i \geqslant 1$ and $n \geqslant 16$.*

Proof. Inequality (a) is verified directly.

Inequality (b) follows from (a) by substituting $f_i(n)$ for $16n^2$ in $f_i(16n^2)$, since f_i is increasing.

We prove 2 by induction on i. For $i=1$ it holds that $n+1 < f_7^{(2)}(n)$ for all $n \geqslant 16$ by (a).

Suppose the inequality holds for i and all $n \geqslant 16$, and let $n \geqslant 16$ be given. Since $A_i(n) \leqslant f_{i+6}^{(2)}(n)$ for all $n \geqslant 16$, it follows by monotonicity of A_i that $A_i^{(n)}(n) \leqslant f_{i+6}^{(2n)}(n)$. The latter term is smaller than $f_{i+6}^{(2n)}(16n^2)$ by monotonicity, which equals $f_{i+7}(16n^2)$ by (2). Inequality (b) implies that $A_i^{(n)}(n) \leqslant f_{i+7}^{(2)}(n)$. Finally, $A_i^{(n)}(n) = A_{i+1}(n)$ by (6). ∎

Claim 7. *For all $i \geqslant 7$ and $n \geqslant 16$ it holds that $A_i(n) \leqslant f_{i+7}(n)$.*

Proof. By 2 in the previous claim, $A_i(n) \leqslant f_{i+6}^{(2)}(n)$ for $n \geqslant 16$. If $n \geqslant 16$, then $\sqrt{n}/2 \geqslant 2$ and hence, by (2), $f_{i+7}(n) \stackrel{\text{def}}{=} f_{i+6}^{\lfloor(\sqrt{n}/2\rfloor)} \geqslant f_{i+6}^{(2)}(n)$. ∎

Corollary 8. *The function $v(k)$ eventually dominates every primitive recursive function.*

3. DISCUSSION

3.1. *Other Ramsey Numbers*

Paris and Harrington [7] published in 1976 the first finite Ramsey-type statement that was shown to be independent over Peano Arithmetic. Soon after the discovery of the Paris–Harrington result, Erdős and Mills studied the Ramsey–Paris–Harrington numbers in [6]. Denoting by $R_c^e(k)$ the Ramsey–Paris–Harrington number for exponent e and c many colors, Erdős and Mills showed that $R_2^2(k)$ is double exponential in k and that $R_c^2(k)$ is Ackermannian as a function of k and c. In the same paper, several small Ramsey–Paris–Harrington numbers were computed. Later Mills tightened the double exponential upper bound for $R_2^2(k)$ in [4].

Canonical Ramsey numbers for pair colorings were treated in [3] and were also found to be double exponential.

The second author showed that van der Waerden numbers are primitive recursive, refuting the conjecture that they were Ackermannian, in [8] (see also [5]).

We remark that an upper bound for regressive Ramsey numbers for pairs is $R_2^3(k)$—the Ramsey–Paris–Harrington number for *triples*. Let N be large enough and suppose that c is regressive on $\{1, 2, ..., N-1\}$. Color a triple $x<y<z$ red if $c(x, y)=c(x, z)$ and blue otherwise. Find a homogeneous set A of size at least k and so that $|A| > \min A+1$. The homogeneous color on A cannot be blue for $k>5$, and therefore A is min-homogeneous for c.

3.2. *Problems*

The following two problems about regressive Ramsey numbers remain open:

Problem 9. 1. Find a concrete upper bound for regressive Ramsey numbers.

2. Compute small regressive Ramsey numbers.

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