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MANY PARTITION RELATIONS BELOW DENSITY

BY

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ABSTRACT

We force 2^λ to be large, and for many pairs in the interval $(\lambda, 2^\lambda)$ a strong version of the polarized partition relations holds. We apply this to problems in general topology. For example, consistently, every 2^λ is the successor of a singular and for every Hausdorff regular space X, $\mathrm{hd}(X) \leq s(X)^{+3}$, $\mathrm{hL}(X) \leq s(X)^{+3}$ and better when $s(X)$ is regular, via a half-graph partition relations. For the case $s(X) = \aleph_0$ we get $\mathrm{hd}(X)$, $\mathrm{hL}(X) \leq \aleph_2$.

Anotated Content

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0. Introduction

Out motivation is a problem in general topology and for this we get a consistency result in the partition calculus.

In Juhasz–Shelah [JuSh:899] we proved: if $(\forall \mu < \lambda)(\mu^{\aleph_0} < \lambda)$ then there is a c.c.c. forcing notion that adds a regular topological space, hereditarily Lindelof of density λ.

A natural question asked there ([JuSh:899]) is:

Problem 0.1: Assume $\aleph_1 < \lambda \leq 2^{\aleph_0}$. Does there exist (i.e., provably in ZFC) a hereditary Lindelof regular space of density λ?

On cardinal invariants in general topology, see [Juh80].

We prove the consistency of a negative answer, in fact of stronger results by proving the consistency of strong variants of polarized partition relations (the half-graphs, see below). They are strong enough to resolve the question about hereditary density (and hereditary Lindelof). Moreover, if $\lambda = \lambda^{<\lambda} < \mu = \mu^{<\mu}$ (and G.C.H. holds in $[\lambda, \mu)$), then there is a forcing extension making $2^\lambda \geq \mu$ neither adding new $(< \lambda)$-sequences nor collapsing cardinals such that, for many pairs $\lambda_* < \mu_*$ in the interval, we have the appropriate partition relations.

An earlier result is in the paper [Sh:276, Theorem 1.1, p. 357] and it states the following: if $\lambda > \kappa > \mu$ are regular cardinals, $\lambda > \kappa^{++}$, then there is a cardinal and cofinality preserving forcing that makes $2^\mu = \lambda$ and $\kappa^{++} \to (\kappa^{++}, (\kappa; \kappa)_\kappa)^2$ in addition to the main result there $2^\lambda \to [\lambda]^2_3$; see more in [Sh:289], [Sh:288], [Sh:481], [Sh:546]. The applied notion of forcing $(Q, \leq)$ is the following: $p \in Q$ if p is a function from a subset $\mathrm{Dom}(p) \in [\lambda]^{\leq \kappa}$

into $\mathrm{Add}(\mu,1)-\{\emptyset\}$, where $\mathrm{Add}(\mu,1)$ denotes the forcing adding a Cohen subset of μ; $p \le q$ if $\mathrm{Dom}(p) \supseteq \mathrm{Dom}(q), p(\alpha) \le q(\alpha)$ for $\alpha \in \mathrm{Dom}(q)$ and $|\{\alpha \in \mathrm{Dom}(q) : p(\alpha) \ne q(\alpha)\}| < \mu$.

For simultaneously many n-place polarized partition relation, Shelah–Stanley [ShSt:608] deals with it but there are problems there, so we do not rely on it.

Our main result in general topology is Theorem 3.10, and by it: consistently, G.C.H. fails badly (2^μ is a successor of a limit cardinal $> \mu$ except when μ is strong limit singular and then $2^\mu = \mu^+$) and $\mathrm{hd}(X)$,$\mathrm{hL}(X)$ are $\le s(X)^{+3}$ for every Hausdorff regular X and $|X| \le 2^{(\mathrm{hd}(X))^+}, w(X) \le 2^{(\mathrm{hL}(X))^+}$ for any Hausdorff X. (Usually $s(X)^{+2}$ suffices, so in particular "X is hereditary Lindelof $\Rightarrow X$ has density $\le \aleph_2$".)

Concerning partition relations we give a generalizaiton of the earlier result explained above, namely, the consistency of $2^{\aleph_0} = \lambda$ and $\mu^{++} \to (\mu,(\mu;\mu)_\mu)^2$ simultaneously holding for each regular cardinal μ such that $\mu^{++} \le \lambda$. This gives a model in which though GCH fails badly, we have strong enough partition relations implying that the hereditary density and the hereditary Lindelof numbers of a T_3 space X are bounded by $s(X)^{+3}$, where $s(X)$ stands for spread.

The notion of forcing $(P,\le)$ used for the argument is defined as follows. For each regular cardinal $\mu < \lambda$ define the following equivalence relation E_μ on λ: $xE_\mu y$ iff $x+\mu = y+\mu$. Let $[x]_\mu$ denote the equivalence class of x; $p \in P$ if p is a function from some set $\mathrm{Dom}(p) \subseteq \lambda$ into $\{0,1\}$ such that $|[x]_\mu \cap \mathrm{Dom}(p)| < \mu$ holds for every successor $\mu < \lambda, x < \lambda$; $p \le q$ if $p \supseteq q$ and for every successor $\mu < \lambda$ we have

$$|\{[x]_\mu : \emptyset \ne \mathrm{Dom}(q) \cap [x]_\mu \ne \mathrm{Dom}(p) \cap [x]_\mu\}| < \mu.$$

This notion of forcing $(P,\le)$, in a most remarkable way, imitates concurrently several different posets $(Q,\le)$ as defined above. Not surprisingly, in order to show that $(P,\le)$ is cardinal and cofinality preserving, the author uses ideas similar to those in [Sh:276].

In order to prove the main claim, that is, the partition relation, we use the following trick: we find a condition $\bar p$ such that the dense sets we are interested in are all dense below $\bar p$. It suffices, therefore, to show that forcing with the part below $\bar p$ gives the required result, and this reduces the problem to showing that a certain notion of forcing $(R,\le)$ forces the sought-for-partition relation, where $|R|$ is small (compared to μ). As $(R,<)$ is close to the poset $(Q,<)$ of [Sh:276], an elementary submodel argument similar to the one there applies.

The exposition of the method is axiomatic; the author formulates the most general situation where this method works, and then specifies it to the situation sketched above. This is not necessarily the optimal description for those who are only interested in the application given. There is, however, reason for the peculiar way of presenting this proof: we would like to include this method into the tool kit set, and simply quote it at possible later applications.

Recall (first appeared in Erdős-Hajnal [EH78], but probably raised by Galvin in letters in the mid-seventies):

Definition 0.2: (1) $\lambda \to (\mu;\mu)^2_\kappa$ means that: for every $\mathbf{c} : [\lambda]^2 \to \kappa$ there are ε and α_i, β_i for $i < \mu$ such that:
  - (a) $\varepsilon < \kappa$,
  - (b) if $i < j < \mu$ then $\alpha_i < \beta_i < \alpha_j < \lambda$,
  - (c) if $i \le j < \mu$ then $\mathbf{c}\{\alpha_i, \beta_j\} = \varepsilon$.

(2) We can replace μ by an ordinal and if $\kappa = 2$ we may omit it.

Definition 0.3: (1) Let $\lambda \to (\mu, (\mu;\mu)_\kappa)^2$ mean that: for every $\mathbf{c} : [\lambda]^2 \to 1 + \kappa$ there are ε and α_i, β_i for $i < \mu$ such that:
  - (a) $\varepsilon < \kappa$,
  - (b) $\alpha_i < \beta_i < \alpha_j < \lambda$ for $i < j < \mu$,
  - $(\mathrm{c})_0$ if $\varepsilon = 0$ then $i < j \Rightarrow \mathbf{c}\{\alpha_i, \alpha_j\} = \varepsilon$, so we can forget the β_i's,
  - $(\mathrm{c})_1$ if $\varepsilon \ge 1$ then $i \le j \Rightarrow \mathbf{c}\{\alpha_i, \beta_j\} = \varepsilon$.

(2) In part (1), if $\kappa = 1$ we may omit it. Above, replacing μ by "$< \mu$" means "for every $\xi < \mu$ we have ...".

We thank Shimoni Garti for many corrections and Istvan Juhász for questions and historical remarks; we may continue this research in [Sh:F884].

1. Strong polarized partition relations

We deal with sufficient conditions on a forcing notion for preserving such partition relations. For this, we use an expansion of a forcing notion. Instead of the usual pair $(Q, \le_{\mathbf{Q}})$, namely, the underlying set and the partial order, we use a quadruple of the form $\mathbf{Q} = (Q, \le_{\mathbf{Q}}, \le^{\mathrm{pr}}_{\mathbf{Q}}, \mathrm{ap}_{\mathbf{Q}})$.

The "pr" stands for pure, and the "ap" stands for apure. Both are included (as partial orders) in $\mathbf{Q}$.

Discussion 1.1: We define (below) the notion of "(λ, θ, ξ)-forcing" to give a sufficient condition for appropriate cases of the partition relations defined above to hold. We start with the quadruple $\mathbf{Q} = (Q, \leq_{\mathbf{Q}}, \leq^{\mathrm{pr}}_{\mathbf{Q}}, \mathrm{ap}_{\mathbf{Q}})$ such that $q \in Q \Rightarrow \mathrm{ap}_{\mathbf{Q}}(q) \subseteq Q$ and $\leq_{\mathbf{Q}}, \leq^{\mathrm{pr}}_{\mathbf{Q}}$ are quasi orders on Q. The idea is that if $r \in \mathrm{ap}_{\mathbf{Q}}(q)$, then r and q are compatible in $\mathbb{Q}$, close to "r is an a-pure extension of q".

Definition 1.2: (1) We say that $\mathbf{Q}$ is a (χ^+, θ, ξ)-forcing notion when χ^+, θ are regular uncountable cardinals, ξ an ordinal and $\circledast$ below holds; in writing $(\chi^+, \theta, < \zeta)$ we mean that $\circledast$ holds for every $\xi < \zeta$; also, we can replace χ^+ by λ:

$\circledast$ (a) $\mathbf{Q} = (Q, \leq_{\mathbf{Q}}, \leq^{\mathrm{pr}}_{\mathbf{Q}}, \mathrm{ap}_{\mathbf{Q}})$,

(b) $\mathbb{Q} = (Q, \leq_{\mathbf{Q}})$ is a forcing notion (i.e., a quasi order, so $\Vdash_{\mathbf{Q}}$ means $\Vdash_{\mathbb{Q}}$ and $p \in \mathbf{Q}$ means $p \in Q$ and $\mathbf{V}^{\mathbf{Q}}$ means $\mathbf{V}^{\mathbb{Q}}$ and $\underset{\sim}{\mathbf{G}}$ is the $\mathbb{Q}$-name of the generic set),

(c) $\leq^{\mathrm{pr}}_{\mathbf{Q}}$ is a quasi order on Q and $p \leq^{\mathrm{pr}}_{\mathbf{Q}} q$ implies $p \leq_{\mathbf{Q}} q$,

(d) (α) $\mathrm{ap}_{\mathbf{Q}}$ is a function with domain Q,

(β) for $q \in \mathbf{Q}$ we have[1] $q \in \mathrm{ap}_{\mathbf{Q}}(q) \subseteq Q$,

(γ) $r \in \mathrm{ap}_{\mathbf{Q}}(q) \Rightarrow r, q$ are compatible in $\mathbb{Q}$; moreover,

$(\gamma)^+$ if $r \in \mathrm{ap}_{\mathbf{Q}}(q) \wedge q \leq^{\mathrm{pr}}_{\mathbf{Q}} q^+$ then q^+, r are compatible in $\mathbb{Q}$; moreover, there is $r^+ \in \mathrm{ap}_{\mathbf{Q}}(q^+)$ such that $q^+ \Vdash_{\mathbf{Q}}$ "$r^+ \in \underset{\sim}{\mathbf{G}}_{\mathbf{Q}} \Rightarrow r \in \underset{\sim}{\mathbf{G}}_{\mathbf{Q}}$"[2],

(e) $(Q, \leq^{\mathrm{pr}}_{\mathbf{Q}})$ is $(< \theta)$-complete, i.e., any $\leq^{\mathrm{pr}}_{\mathbf{Q}}$-increasing sequence of length $< \theta$ has a $\leq^{\mathrm{pr}}_{\mathbf{Q}}$-upper bound in Q,

(f) $(Q, \leq^{\mathrm{pr}}_{\mathbf{Q}})$ satisfies the χ^+-c.c.,

(g) if $\bar{q} = \langle q_\varepsilon : \varepsilon < \theta \rangle$ is $\leq^{\mathrm{pr}}_{\mathbf{Q}}$-increasing then[3] for stationary many limit ordinals $\zeta < \theta$, the sequence $\bar{q} \restriction \zeta$ has an exact $\leq^{\mathrm{pr}}_{\mathbf{Q}}$-upper bound; see part (2) below,

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[1] It is natural to demand $q \in \mathrm{ap}_{\mathbf{Q}}(q)$, but not really necessary (if we do not demand it, this just complicates a little $\circledast$(c)(d)).

[2] No harm in asking that $r \leq^{\mathrm{pr}}_{\mathbf{Q}} s$ and $s \in \mathrm{ap}_{\mathbf{Q}}(q^+)$ and $q^+ \leq s$ for some s. Why does this not follow from our assumption? By the present demand r^+, q^+ have a common $\leq$-upper bound which is s, so $s \Vdash$ "$q^+, r^+ \in \underset{\sim}{\mathbf{G}}_{\mathbf{Q}}$ hence $r \in \underset{\sim}{\mathbf{G}}_{\mathbf{Q}}$", so without loss of generality $r \leq s$, but this does not say that $q \leq^{\mathrm{pr}}_{\mathbf{Q}} s$.

[3] Note that: we can restrict ourselves to the case $q_0 \in \mathcal{I}$, where $\mathcal{I}$ is a dense subset of $\mathbb{Q}$. Also, we can restrict ourselves to the set of $\bar{q}$ sequences which is the set of plays of a suitable game with one player using a fixed strategy, etc.

(h) if $\langle q_\varepsilon : \varepsilon < \theta \rangle$ is $\leq^{\rm pr}_{\mathbf{Q}}$-increasing and $p_\varepsilon \in \mathrm{ap}_{\mathbf{Q}}(q_\varepsilon)$ for $\varepsilon < \theta$ and $\xi < \theta$ then for some $\zeta < \theta$ we have $q_\zeta \Vdash_{\mathbf{Q}}$ "if $p_\zeta \in \underset{\sim}{\mathbf{G}}_{\mathbf{Q}}$ then $\xi \leq \mathrm{otp}\{\varepsilon < \zeta : p_\varepsilon \in \underset{\sim}{\mathbf{G}}_{\mathbf{Q}}\}$",

(i) if $q \in \mathbf{Q}$ then $\mathrm{ap}_{\mathbf{Q}}(q)$ has cardinality $< \theta$,

(j) if $q_* \leq r$ then there is a (q_*, r)-witness (q, p) which means
- $\bullet_1$ $q_* \leq^{\rm pr}_{\mathbf{Q}} q$,
- $\bullet_2$ $p \in \mathrm{ap}_{\mathbf{Q}}(q_*)$,
- $\bullet_3$ $q \Vdash_{\mathbb{Q}}$ "$p \in \underset{\sim}{\mathbf{G}} \Rightarrow r \in \underset{\sim}{\mathbf{G}}$".

(2) Assume $\mathbf{Q}$ satisfies clauses (a)–(e) of part (1).

Let $\bar{q} = \langle q_\varepsilon : \varepsilon < \delta \rangle$ be a $\leq^{\rm pr}_{\mathbf{Q}}$-increasing sequence of conditions, $\delta < \theta$ a limit ordinal. We say that q is an exact $\leq^{\rm pr}_{\mathbf{Q}}$-upper bound of $\bar{q}$ when $\varepsilon < \delta = \ell g(\bar{q}) \Rightarrow q_\varepsilon \leq^{\rm pr}_{\mathbf{Q}} q$ and:

$(*)_{\bar{q},q}$ if $p \in \mathrm{ap}_{\mathbf{Q}}(q)$ then for some $\varepsilon < \delta$ and $p' \in \mathrm{ap}_{\mathbf{Q}}(q_\varepsilon)$, we have $\Vdash_{\mathbf{Q}}$ "if $q, p' \in \underset{\sim}{\mathbf{G}}_{\mathbf{Q}}$ then $p \in \underset{\sim}{\mathbf{G}}_{\mathbf{Q}}$".

Remark 1.3: Can we weaken clause (i) of $\circledast$ of 1.2(1) to "cardinality $\leq \theta$"?

(1) Here it mostly does not matter, but in one point of the proof of 1.4 it does: in proving $\circledast_4$ there, choosing $\zeta(*)$ such that it will be possible to choose $\varepsilon(*)$.

(2) There is a price for demanding a strict inequality. The price is (in 2.12(1)) that, recalling $\kappa = \kappa_{\mathbf{y}}$, instead of using $\mathrm{ap}_{\mathbf{y}}(q) = \{r : q \leq^{\rm ap}_{\kappa} r \in Q_{\mathbf{y}}\}$ we use $\mathrm{ap}_{\mathbf{y}}(q) = \{r : q \leq^{\rm ap}_{\kappa} r \in Q_{\mathbf{y}}$ and $\mathrm{supp}_\kappa(q, r) \subseteq \mathrm{supp}_\theta(p^{\mathbf{y}}_{\alpha_{\mathbf{y}}(q)}, q)\}$.

CLAIM 1.4: *If $\mathbf{Q}$ is a (χ^+, θ, ξ_*)-forcing notion, $\kappa < \theta = \mathrm{cf}(\theta)$ and $\chi = \chi^{<\theta}$ then $\chi^+ \to (\xi_*, (\xi_*; \xi_*)_\kappa)^2$ holds in $\mathbf{V}^{\mathbf{Q}}$.*

Remark 1.5: We can replace χ^+ by "regular χ' such that $\alpha < \chi' \Rightarrow |\alpha|^{<\theta} < \chi'$".

Proof. Let λ_* be large enough (so, in particular, $\mathbf{Q}, \theta, \ldots \in \mathcal{H}(\lambda_*^+)$). Choose a well ordering $<^*_{\lambda_*^+}$ of the set of $\mathcal{H}(\lambda_*^+)$. Recalling Definition 1.2, clearly $\theta > \aleph_0$, hence without loss of generality κ is infinite, so $1 + \kappa = \kappa$.

Toward a contradiction, assume $p^* \Vdash_{\mathbf{Q}}$ "$\underset{\sim}{\mathbf{c}}$ is a function from $[\chi^+]^2$ to κ" is a counterexample.

We now choose $\bar{M}$ such that

$\circledast_1$ (a) $\bar{M} = \langle M_\alpha : \alpha \leq \theta \rangle$,

(b) $M_\alpha \prec (\mathcal{H}(\lambda_*^+), \in)$,

(c) M_α has cardinality χ,

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(d) $[M_\alpha]^{<\theta} \subseteq M_\alpha$ if α is non-limit,
(e) M_α is $\prec$-increasing continuous,
(f) $\mathbf{Q}, p^*, \underset{\sim}{\mathbf{c}}$ belong to M_α and $\chi + 1 \subseteq M_\alpha$,
(g) $\bar{M} \restriction (\alpha + 1) \in M_{\alpha+1}$.

Note that $\chi = \chi^{<\theta}$ implies $\theta < \chi^+$, so let

$\circledast_2$ $\delta_* :=\ \min(\chi^+ \backslash M_\theta)$.

We shall now prove:

$\circledast_3$ if $q \in \mathbf{Q}$ and $\varphi(x, y) \in \mathbb{L}_{\theta,\theta}$ is a formula with parameters from M_θ such that $(\mathcal{H}(\lambda_*^+), \in, <^*_{\lambda_*^+}) \models \varphi[\delta_*, q]$ then for some pair $(\delta, q') \in M_\theta$ we have:
(a) $\delta < \delta_*$,
(b) $(\mathcal{H}(\lambda_*^+), \in, <^*_{\lambda_*^+}) \models \varphi[\delta, q']$,
(c) q', q has a common $\le^{\mathrm{pr}}_{\mathbf{Q}}$-upper bound.

Why does $\circledast_3$ hold? Let $\bar{r} = \langle r_\zeta : \zeta < \zeta^* \rangle$ list $\mathbf{Q}$, each member appearing χ^+ times, now without loss of generality $\bar{r} \in M_0$, so necessarily we can find $\zeta_1 \in \zeta^* \backslash M_\theta$ such that $q = r_{\zeta_1}$ and let $\zeta_2 =\ \min(M_\theta \cap (\zeta^* + 1) \backslash \zeta_1)$; of course, $\zeta^* \in M_\theta$ and $\zeta_2 \in M_\theta$ and $\zeta_1 < \zeta_2 \wedge \mathrm{cf}(\zeta_2) > \chi$.

Let

$$Y = \{q' \in \mathbf{Q} : (\mathcal{H}(\lambda_*^+), \in, <^*_{\lambda_*^+}) \models (\exists x)(\varphi(x, q') \wedge x \in \chi^+)\}.$$

Recall that $\chi^{<\theta} = \chi$, so

$\odot_{3.1}$ $Y \in M_\theta, Y \subseteq \mathbf{Q}$ and $q \in Y$.

Now we ask:

$\odot_{3.2}$ Is there $Z \subseteq Y$ of cardinality $\le \chi \in \chi^+$ such that, for every $q'' \in Y$ for at least one $q' \in Z$, the pair (q', q'') is $\le^{\mathrm{pr}}_{\mathbf{Q}}$-compatible?

Assume toward a contradiction that the answer is negative. Then, in particular, $|Y| > \chi$ and we can choose $r_\varepsilon \in Y$ by induction on $\varepsilon < \chi^+$ such that $\zeta < \varepsilon \Rightarrow$ the pair (r_ζ, r_ε) is $\le^{\mathrm{pr}}_{\mathbf{Q}}$-incompatible. Why? In stage ε try to use $Z := \{r_\zeta : \zeta < \varepsilon\}$, so $Z \subseteq Y$ has cardinality $\le |\varepsilon| \le \chi$, so some $r_\varepsilon \in Y$ can serve as q'' in $\odot_{3.2}$, by our assumption toward a contradiction. Hence $\langle r_\varepsilon : \varepsilon < \chi^+ \rangle$ contradicts clause (f) of Definition 1.2(1). So the answer to $\odot_{3.2}$ is yes, hence there is such $Z \in M_\theta$, but $\chi + 1 \subseteq M_\theta$, hence $Z \subseteq M_\theta$.

So apply the property of Z, with q standing for q'', so there is $q' \in Z \subseteq \mathbf{Q} \cap M_\theta$ such that the pair (q', q) is $\le^{\mathrm{pr}}_{\mathbf{Q}}$-compatible; but $Z \subseteq Y$, hence by the definition of Y there is $\delta \in \chi^+$ such that $(\mathcal{H}(\lambda_*^+), \in, <^*_{\lambda_*^+}) \models \varphi[\delta, q']$, and as $q' \in Z \subseteq M_\theta$ without loss of generality $\delta \in M_\theta$, hence $\delta \in \chi^+ \cap M_\theta$, so by the definition of δ_* we have $\delta < \delta_*$; so $\circledast_3$ holds indeed.

Next (but its proof will take a while)

$\circledast_4$ if $q^0 \in \mathbf{Q}$ is above p^*, then for some triple (q^1, p, ι) we have:
- (a) $q^0 \le^{\rm pr}_{\mathbf{Q}} q^1$,
- (b) $\iota < \kappa$,
- (c) $p \in \operatorname{ap}_{\mathbf{Q}}(r)$ for some r satisfying $q^0 \le^{\rm pr}_{\mathbf{Q}} r \le^{\rm pr}_{\mathbf{Q}} q^1$,
- (d) if $\iota = 0$ then $p \le q^1$,
- (e) if q satisfies $q^1 \le^{\rm pr}_{\mathbf{Q}} q$ and $\varphi(x, y) \in \mathbb{L}_{\theta,\theta}$ is a formula with parameters from M_θ satisfied by the pair (δ_*, q) in the model $(\mathcal{H}(\lambda_*^+), \in, <^*_{\lambda_*^+})$, then we can find q', q'', δ such that the septuple $\mathbf{q} = (q, p, \iota, \varphi(x, y), q', q'', \delta)$ satisfies
  - $\boxtimes_{\mathbf{q}}$ $\bullet_1$ $\delta < \delta_*$ (hence $\delta \in M_\theta$),
    - $\bullet_2$ $(\mathcal{H}(\lambda_*^+), \in, <^*_{\lambda_*^+}) \models \varphi[\delta, q']$,
    - $\bullet_3$ if $\iota = 0$ then
      - (α) $q \le^{\rm pr}_{\mathbf{Q}} q''$,
      - (β) $q' \le^{\rm pr}_{\mathbf{Q}} q''$,
      - (γ) $q'' \Vdash$ "$\underset{\sim}{\mathbf{c}}\{\delta, \delta_*\} = 0$",
    - $\bullet_4$ if $\iota \in (0, \kappa)$, then $q \le^{\rm pr}_{\mathbf{Q}} q''$ and $q'' \Vdash$ "$p \in \underset{\sim}{\mathbf{G}}_{\mathbf{Q}} \Rightarrow \underset{\sim}{\mathbf{c}}\{\delta, \delta_*\} = \iota \wedge q' \in \underset{\sim}{\mathbf{G}}_{\mathbf{Q}}$".

Why? Assume toward a contradiction that $\circledast_4$ fails. We let $\langle S_\varepsilon : \varepsilon \le \theta\rangle$ be a $\subseteq$-increasing continuous sequence of subsets of θ with $S_\theta = \theta$, $|S_{\varepsilon+1} \backslash S_\varepsilon| = \theta$, $|S_0| = \theta$ and $\min(S_{\varepsilon+1} \backslash S_\varepsilon) \ge \varepsilon$. Now we try to choose $(q^*_\varepsilon, \mathbf{x}_\varepsilon, \varphi_\varepsilon)$ by induction on $\varepsilon < \theta$ (but φ_ε is chosen in the $(\varepsilon+1)$-th stage) such that:

$\odot_{4.1}$
- (α) $q^*_\varepsilon \in \mathbf{Q}$ and $\langle q^*_\zeta : \zeta \le \varepsilon\rangle$ is $\le^{\rm pr}_{\mathbf{Q}}$-increasing,
- (β) $q^*_0 = q^0$,
- (γ) if ε is a limit ordinal $(< \theta)$ and $\langle q^*_\zeta : \zeta < \varepsilon\rangle$ has an exact $\le^{\rm pr}_{\mathbf{Q}}$-upper bound (see part (2) of Definition 1.2) then q^*_ε is an exact $\le^{\rm pr}_{\mathbf{Q}}$-upper bound of it,
- (δ) $\mathbf{x}_\varepsilon = \langle (p^*_\xi, \iota_\xi) : \xi \in S_\varepsilon\rangle$ lists $\{(p, \iota) : \iota < \kappa$ and $p \in \operatorname{ap}_{\mathbf{Q}}(q^*_\zeta)$ for some ζ such that $\zeta = 0 \vee \zeta \le \varepsilon\}$; here we use clause (i) of 1.2(1) recalling $q^*_\zeta \in \operatorname{ap}_{\mathbf{Q}}(q^*_\zeta)$, by clause (d)$(\beta)$ of 1.2(1), so $1 \le |\operatorname{ap}_{\mathbf{Q}}(q^*_\zeta)| < \theta$,

(ε) for successor ordinal $\varepsilon = \zeta + 1$, let $(q^*_{\zeta+1}, \varphi_\zeta(x,y))$ exemplify that the triple $(q^*_\zeta, p^*_\zeta, \iota_\zeta)$ does not satisfy demand (e) on (q^1, p, ι) in $\circledast_4$, i.e.,

($*$) $q^*_\zeta \leq^{\rm pr}_{\mathbf{Q}} q^*_{\zeta+1}$ and $\varphi_\zeta(x,y) \in \mathbb{L}_{\theta,\theta}$ is a formula with parameters from M_θ which the pair $(\delta_*, q^*_{\zeta+1})$ satisfies in $(\mathcal{H}(\lambda^+_*), \in, <^*_{\lambda^+_*})$, but we cannot find q', q'', δ such that the septuple $\mathbf{q}_{\zeta+1} := (q^*_{\zeta+1}, p^*_\zeta, \iota_\zeta, \varphi_\zeta(x,y), q', q'', \delta)$ satisfies $\boxtimes_{\mathbf{q}_{\zeta+1}}$.

We show that the induction can be carried out. Assume we are stuck at ε. Now if $\varepsilon = 0$ we can satisfy clauses $(\alpha) + (\beta)$ and, recalling $1 \leq |\mathrm{ap}_{\mathbf{Q}}(q^0)| < \theta$, we can choose $\mathbf{x}_0$ to satisfy clause (δ), and since $(\gamma), (\varepsilon)$ are vacuous we are done.

Suppose $\varepsilon > 0$. For limit ε we can choose q^*_ε as required in clause (α) by clause (e) of Definition 1.2(1); also, clause (γ) is relevant but causes no problem; and lastly, we can choose $\mathbf{x}_\varepsilon$ and, since clause (ε) is vacuous for limit ordinals, we are done again. So ε is a successor; let $\varepsilon = \zeta + 1$, so q^*_ζ was defined. Now if we cannot choose $(q^*_{\zeta+1}, \varphi_\zeta(x,y)) = (q^*_\varepsilon, \varphi_\zeta(x,y))$, then the triple $(q^*_\zeta, p^*_\zeta, \iota_\zeta)$ is as required from the triple (q^1, p, ι) in $\circledast_4$. But this is impossible (by our assumption toward a contradiction), so we can find $(q^*_{\zeta+1}, \varphi_\zeta(x,y))$ as required; and again we can choose $\mathbf{x}_\varepsilon$ as for $\varepsilon = 0$.

So it is enough to get a contradiction from the assumption that we can carry out the induction. But by clause (g) of Definition 1.2(1) the set $S := \{\zeta < \theta : \zeta$ is a limit ordinal and the sequence $\langle q^*_\varepsilon : \varepsilon < \zeta\rangle$ has an exact $\leq^{\rm pr}_{\mathbf{Q}}$-upper bound$\}$ is stationary. As S is stationary, noting $\odot_{4.1}(\delta)$ and recalling clause (i) of Definition 1.2(1) which gives $|\mathrm{ap}_{\mathbf{Q}}(q^*_\varepsilon)| < \theta = \mathrm{cf}(\theta)$ for $\varepsilon < \theta$, clearly for some limit ordinal $\zeta(*) \in S$ we have: if $\iota < \kappa\,(< \theta)$ and $p \in \bigcup\{\mathrm{ap}_{\mathbf{Q}}(q^*_\varepsilon) : \varepsilon < \zeta(*)\}$ then for unboundedly many $\varepsilon < \zeta(*)$ we have $(p^*_\varepsilon, \iota_\varepsilon) = (p, \iota)$.

Let $\varphi(x,y) \in \mathbb{L}_{\theta,\theta}$ express all the properties that the pair $(\delta_*, q^*_{\zeta(*)})$ satisfies and are used below, i.e., $(\exists y_0, \ldots, y_{\zeta(*)})[x \in \chi^+ \wedge y = y_{\zeta(*)} \wedge \bigwedge_{\varepsilon<\zeta\leq\zeta(*)} y_\varepsilon \leq^{\rm pr}_{\mathbf{Q}} y_\zeta \wedge \bigwedge_{\varepsilon<\zeta(*)} \varphi_\varepsilon(x, y_{\varepsilon+1}) \wedge (y_{\zeta(*)}$ is an exact $\leq^{\rm pr}_{\mathbf{Q}}$-upper bound of $\langle y_i : i < \zeta(*)\rangle)]$.

So

($*$) $(\mathcal{H}(\lambda^+_*), \in, <^*_{\lambda^+_*}) \models \varphi[\delta_*, q^*_{\zeta(*)}]$.

By $\circledast_3$ we can find a pair (δ, q') such that:

$\odot_{4.2}$ (a) $\delta < \delta_*$, hence $\delta \in M_\theta$ and $q' \in M_\theta$,

(b) $(\mathcal{H}(\lambda^+_*), \in, <^*_{\lambda^+_*}) \models \varphi[\delta, q']$,

(c) $q', q^*_{\zeta(*)}$ are $\leq^{\rm pr}_{\mathbf{Q}}$-compatible.

Let q'' be such that

(d) $q' \leq^{\rm pr}_{\mathbf{Q}} q''$ and $q^*_{\zeta(*)} \leq^{\rm pr}_{\mathbf{Q}} q''$.

Let $\langle q'_\zeta : \zeta \leq \zeta(*)\rangle$ exemplify $\varphi[\delta, q']$ and, without loss of generality, $\{q'_\zeta : \zeta \leq \zeta(*)\} \subseteq M_\theta$; in particular, $\varepsilon \leq \zeta(*) \Rightarrow q'_\varepsilon \leq^{\rm pr}_{\mathbf{Q}} q'_{\zeta(*)} = q' \leq^{\rm pr}_{\mathbf{Q}} q''$ and, of course, $\varepsilon \leq \zeta(*) \Rightarrow q^*_\varepsilon \leq^{\rm pr}_{\mathbf{Q}} q^*_{\zeta(*)} \leq^{\rm pr}_{\mathbf{Q}} q''$.

Case 1: $q'' \Vdash_{\mathbf{Q}}$ "$\underset{\sim}{\mathbf{c}}\{\delta, \delta_*\} = 0$".

There is $\varepsilon < \zeta(*)$ such that $\iota_\varepsilon = 0$. We get a contradiction to the choice of the $(q^*_\varepsilon, \varphi_\varepsilon)$.

Why? Let us check that the septuple $\mathbf{q} = (q^*_{\varepsilon+1}, q^*_{\varepsilon+1}, 0, \varphi_\varepsilon(x, y), q'_{\varepsilon+1}, q'', \delta)$ is such that $\boxtimes_{\mathbf{q}}$ holds.

For $\bullet_1$: Recall $\odot_{4.2}$(a).

For $\bullet_2$: By $\odot_{4.1}(\varepsilon)(*)$ we have $(\mathcal{H}(\lambda^+_*), \in, <^*_{\lambda^+_*}) \models \varphi_\varepsilon(\delta_*, q^*_{\varepsilon+1})$, by the choice of $\varphi(x, y)$ and of $\langle q'_\zeta : \zeta \leq \zeta(*)\rangle$ we have $(\mathcal{H}(\lambda^+_*), \in, <^*_{\lambda^+_*}) \models \varphi_\varepsilon[\delta, q'_{\varepsilon+1}]$ as required.

For $\bullet_3(\alpha)$: It means $q^*_{\varepsilon+1} \leq^{\rm pr}_{\mathbf{Q}} q''$, which holds as $q^*_{\varepsilon+1} \leq^{\rm pr}_{\mathbf{Q}} q^*_{\zeta(*)}$ by $\odot_{4.1}(\alpha)$ and $q^*_{\zeta(*)} \leq^{\rm pr}_{\mathbf{Q}} q''$ by $\odot_{4.2}$(d).

For $\bullet_3(\beta)$: It means $q'_{\varepsilon+1} \leq^{\rm pr}_{\mathbf{Q}} q''$, which has been proved just before "case 1".

For $\bullet_3(\gamma)$: It means $q'' \Vdash$ "$\underset{\sim}{\mathbf{c}}\{\delta, \delta_*\} = 0$", which holds by the case assumption.

For $\bullet_4$: It is vacuous.

So indeed $\boxtimes_{\mathbf{q}}$ holds, contradicting the choice of $(q^*_{\varepsilon+1}, \varphi_\varepsilon)$ see $\odot_{4.1}(\epsilon)$.

Case 2: Not Case 1.

Choose (q^+, ι) such that $q^+ \in \mathbf{Q}, q^*_{\zeta(*)} \leq_{\mathbf{Q}} q'' \leq_{\mathbf{Q}} q^+$ and $q^+ \Vdash_{\mathbf{Q}}$ "$\underset{\sim}{\mathbf{c}}\{\delta, \delta_*\} = \iota$" where $\iota \in (0, \kappa)$; we use "not Case 1". By clause (j) of $\circledast$ of Definition 1.2 applied with $(q^*_{\zeta(*)}, q^+)$ here standing for (q_*, r) there, we can find a pair (s, p) such that

$\odot_{4.3}$ (a) $p \in \mathrm{ap}_{\mathbf{Q}}(q^*_{\zeta(*)})$,
(b) $q^*_{\zeta(*)} \leq^{\rm pr}_{\mathbf{Q}} s$,
(c) $s \Vdash_{\mathbf{Q}}$ "$p \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}} \Rightarrow q^+ \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}}$".

As $q^*_{\zeta(*)}$ is an exact $\leq^{\rm pr}_{\mathbf{Q}}$-upper bound of $\langle q^*_\varepsilon : \varepsilon < \zeta(*)\rangle$ because $\zeta(*) \in S$ and $p \in \mathrm{ap}_{\mathbf{Q}}(q^*_{\zeta(*)})$, see part (2) of Definition 1.2, there is a pair $(p', \varepsilon(*))$ such that:

$\odot_{4.4}$ (a) $\varepsilon(*) < \zeta(*)$,
(b) $p' \in \mathrm{ap}_{\mathbf{Q}}(q^*_{\varepsilon(*)})$,
(c) $\Vdash_{\mathbf{Q}}$ "if $q^*_{\zeta(*)}, p' \in \underset{\sim}{\mathbf{G}}_{\mathbf{Q}}$ then $p \in \underset{\sim}{\mathbf{G}}_{\mathbf{Q}}$".

So by the choice of $\zeta(*)$ for some $\zeta < \zeta(*)$ which is $> \varepsilon(*)$ we have $(p^*_\zeta, \iota_\zeta) = (p', \iota)$. Let $\mathbf{q} = (q^*_{\zeta+1}, p^*_\zeta, \iota_\zeta, \varphi_\zeta(x,y), q'_\zeta, s, \delta)$. This septuple satisfies $\boxtimes_{\mathbf{q}}$ because:

For $\bullet_1$: Recall $\odot_{4.2}$(a).

For $\bullet_2$: As in Case 1.

For $\bullet_3$: It is vacuous.

For $\bullet_4$: It means first $q^*_{\zeta+1} \leq^{\rm pr}_{\mathbf{Q}} s$, which holds as $q^*_{\zeta+1} \leq^{\rm pr}_{\mathbf{Q}} q^*_{\zeta(*)}$ by $\odot_{4.1}(\alpha)$ and $q^*_{\zeta(*)} \leq^{\rm pr}_{\mathbf{Q}} s$ by $\odot_{4.3}$(b). Second, $s \Vdash$ "$p^*_\zeta \in \underset{\sim}{\mathbf{G}}_{\mathbf{Q}} \Rightarrow \underset{\sim}{\mathbf{c}}\{\delta, \delta_*\} = \iota$", which holds as $p^*_\zeta = p'$, and assuming $\mathbf{G} \subseteq \mathbb{Q}$ is generic over $\mathbf{V}$ if $s, p' \in \mathbf{G}$ then by $\odot_{4.3}$(b) also $q^*_{\zeta(*)} \in \mathbf{G}$, hence by $\odot_{4.4}$(c) also $p \in \mathbf{G}$, hence by $\odot_{4.3}$(c) also $q^+ \in \mathbf{G}$, hence by the choice of q^+ in the beginning of the case we have $\mathbf{V}[\mathbf{G}]$ satisfies $\underset{\sim}{\mathbf{c}}[\mathbf{G}]\{\delta, \delta_*\} = \iota$.

Third, $s \Vdash$ "$p^*_\zeta \in \underset{\sim}{\mathbf{G}}_{\mathbf{Q}} \Rightarrow q'_\zeta \in \underset{\sim}{\mathbf{G}}_{\mathbf{Q}}$", which holds as $p^*_\zeta = p'$ and assuming $\mathbf{G} \subseteq \mathbb{Q}$ is generic over $\mathbf{V}$, if $s, p' \in \mathbf{G}$ then as above $q^+ \in \mathbf{G}$, hence by the choice of q^+ in the beginning of the case also $q'' \in \mathbf{G}$, hence by $\odot_{4.2}$(d) also $q' \in \mathbf{G}$, hence by the choice of φ and of $\langle q'_\zeta : \zeta \leq \zeta(*)\rangle$ we have $q'_\zeta \in \mathbf{G}$, as required.

Hence we get a contradiction to the choice of $(q^*_{\zeta+1}, \varphi_\zeta)$. So we are done proving $\circledast_4$.

Let the triple (q_*, p_*, ι_*) satisfy the demands on (q^1, p, ι) in $\circledast_4$ for $q^0 = p^*$ and let r_* be as guaranteed by clause (c) of $\circledast_4$, so

$\odot$ $p_* \leq^{\rm pr}_{\mathbf{Q}} r_* \leq^{\rm pr}_{\mathbf{Q}} q_*$ and $p \in \text{ ap}_{\mathbf{Q}}(r_*)$.

Now we choose $q_\zeta, q'_\zeta, q''_\zeta, q'''_\zeta, r_\zeta, p_\zeta, \alpha_\zeta, \beta_\zeta$ by induction on $\zeta < \theta$ such that:

$\circledast_5$
- (a) $q_\zeta \in \mathbf{Q}$,
- (b) $\langle q_\xi : \xi \leq \zeta\rangle$ is $\leq^{\rm pr}_{\mathbf{Q}}$-increasing,
- (c) $q_0 = q_*$,
- (d) $\alpha_\zeta < \beta_\zeta < \delta_*$ and $\varepsilon < \zeta \Rightarrow \beta_\varepsilon < \alpha_\zeta$,
- (e) $(q'_\zeta, q''_\zeta, \alpha_\zeta)$ is as (q', q'', δ) is guaranteed to be in clause (e) of $\circledast_4$, with q_ζ here standing for q there (and, of course, p_*, ι_* here stands for p, ι there) and a suitable φ, hence
  - (α) $\alpha_\xi, \beta_\xi < \alpha_\zeta < \delta_*$ for $\xi < \zeta$,
  - (β) $q_\zeta \leq^{\rm pr}_{\mathbf{Q}} q''_\zeta$,
  - (γ) the pair $(\alpha_\zeta, q'_\zeta) \in M_\theta$ is similar enough to (δ_*, q_ζ),
  - (δ) if $\iota_* > 0$ then $q''_\zeta \Vdash$ "if $p_* \in \underset{\sim}{G}_{\mathbf{Q}}$ then $\underset{\sim}{\mathbf{c}}\{\alpha_\zeta, \delta_*\} = \iota_*$ and $q'_\zeta \in \underset{\sim}{\mathbf{G}}_{\mathbf{Q}}$",

(ε) if $\iota_* = 0$ then $q'_\zeta \leq^{\rm pr}_{\mathbf{Q}} q''$ and $q''_\zeta \Vdash$ "$\underset{\sim}{\mathbf{c}}\{\alpha_\zeta, \delta_*\} = \iota_*$" and $q''_\zeta \Vdash$ "$\underset{\sim}{\mathbf{c}}\{\alpha_\varepsilon, \alpha_\zeta\} = \iota$" for $\varepsilon < \zeta$

(f) the quadruple $(\beta_\zeta, r_\zeta, p_\zeta, q'''_\zeta) \in M_\theta$ is similar enough to the quadruple $(\delta_*, r, p_*, q''_\zeta)$, i.e.,

(α) $\beta_\zeta \in (\alpha_\zeta, \delta_*)$,

(β) the pair (q'''_ζ, q''_ζ) is $\leq^{\rm pr}_{\mathbf{Q}}$-compatible,

(γ) $p_\zeta \in \mathrm{ap}_{\mathbf{Q}}(r_\zeta)$ and $r_\zeta \leq^{\rm pr}_{\mathbf{Q}} q'''_\zeta$,

(δ) $q'''_\zeta \Vdash_{\mathbf{Q}}$ "if $p_\zeta \in \underset{\sim}{\mathbf{G}}_{\mathbf{Q}}$ then $\underset{\sim}{\mathbf{c}}\{\alpha_\varepsilon, \beta_\zeta\} = \iota_*$ for $\varepsilon \leq \zeta$",

(g) $q''_\zeta \leq^{\rm pr}_{\mathbf{Q}} q_{\zeta+1}$ and $q'''_\zeta \leq^{\rm pr}_{\mathbf{Q}} q_{\zeta+1}$.

Why can we carry out the induction? Note that $q'_\zeta, \ldots, \beta_\zeta$ are chosen in the $(\zeta+1)$-th step.

For $\zeta = 0$ just let $q_0 = q_*$, so the only relevant clauses (a) and (c) are satisfied.

For ζ limit only clause (b) is relevant, and we can choose q_ζ by clause (e) of Definition 1.2.

We are left with ζ successor; let $\zeta = \xi + 1$.

We first choose $(q'_\xi, q''_\xi, \alpha_\xi)$ as required in clause (e) of $\circledast_5$ using appropriate φ and $\circledast_4(e)$ for our (q_*, p_*, ι_*). Clearly in $\circledast_5$, clause (e) holds as well as the second statement in clause (d). In particular, (e)(δ) comes from $\circledast_4$ (e) and (e)(ε) comes from φ i.e. as $\varepsilon < \zeta$.

Second, we choose $(\beta_\xi, r_\xi, p_\xi, q'''_\xi)$ as required in clause (f) of $\circledast_5$.

[Why? We can find $(\beta_\xi, r_\xi, p_\xi, q'''_\xi) \in M_\theta$ similar enough to $(\delta_*, r, p_*, q''_\xi)$. Using $\circledast_3$ with (δ_*, q''_ζ) here standing for (δ_*, q) there and q'''_ζ here standing for q' in the conclusion of $\circledast_3$ (and r_ξ, p_ξ are gotten by existential quantifier in choosing which φ holds as r_*, p_* witness). First note that $\alpha_\zeta < \delta_*$ holds as $\alpha_\zeta \in M_\theta$, hence $\beta_\xi < \delta_*$ but $\beta_\xi \in M_\theta$ so $\beta_\xi < \delta_*$, so clause (f)(α) holds. Second q'''_ζ, q''_ζ are $\leq^{\rm pr}_{\mathbf{Q}}$ compatible by $\circledast_3$(c) hence clause (f)(β) holds.

Third, the parallel of (f)(γ) holds for (p_*, r_*) by the choice of r_* and as $q_* = q_0 \leq^{\rm pr}_{\mathbf{Q}} q_\zeta \leq^{\rm pr}_{\mathbf{Q}} q''_\zeta$.

Fourth, the parallel of (f)(δ) holds for (q''_ζ, p_*) by (e)(δ).]

Third, as q''_ξ, q'''_ξ are $\leq^{\rm pr}_{\mathbf{Q}}$-compatible, there is $q_\zeta = q_{\xi+1}$ as required in clause (g).

So we can satisfy $\circledast_5$.

Now we apply clause (h) of Definition 1.2(1) to the sequence $\langle (q_\varepsilon, p_\varepsilon) : \varepsilon < \theta \rangle$, hence there is $\zeta < \theta$ as there, so as $p_\varepsilon \in \mathrm{ap}_{\mathbf{Q}}(q_\varepsilon)$ the conditions $p_\varepsilon, q_\varepsilon$ are compatible in $\mathbb{Q}$, hence they have a common upper bound $r \in \mathbf{Q}$, hence by the choice of $\langle (p_\varepsilon, q_\varepsilon) : \varepsilon < \theta \rangle$ above, $r \Vdash_{\mathbf{Q}}$ "$\xi_* \leq$ otp$\{\varepsilon < \zeta : q_\varepsilon, p_\varepsilon \in \underset{\sim}{\mathbf{G}}_{\mathbf{Q}}\}$".

So $r \Vdash_{\mathbf{Q}}$ "the sequence $\langle (\alpha_\varepsilon, \beta_\varepsilon) : \varepsilon < \zeta$ and $q_\varepsilon, p_\varepsilon \in \underset{\sim}{\mathbf{G}}_{\mathbf{Q}} \rangle$ is as required".

So we are done. $\blacksquare_{1.4}$

Noting that:

- if $\iota_* > 0$, then $q_{\zeta+1} \Vdash$ "$\underset{\sim}{\mathbf{c}}\{\alpha_\varepsilon, \beta_\zeta\}$"$= \iota_*$ for $\varepsilon \leq \zeta$,
- if $\iota_* = 0$, then $q_{\zeta+1} \Vdash$ "$\underset{\sim}{\mathbf{c}}\{\alpha_\varepsilon, d_\zeta\}$"$= \iota_*$ for $\varepsilon \leq \zeta$.

2. Many strong polarized partition relations

We can say more below on strongly inaccessible $\theta \in \Theta$.

Hypothesis 2.1: Let $\mathbf{p} = (\lambda, \mu, \Theta, \bar{\partial})$ satisfy:

(a) $\lambda = \lambda^{<\lambda} < \mu = \mu^{<\mu}$,

(b) $\Theta \subseteq [\lambda, \mu]$ is a set of regular cardinals with $\lambda, \mu \in \Theta$,

(c) $\bar{\partial} = \langle \partial_\theta : \theta \in \Theta \rangle$ is an increasing sequence of cardinals such that:

  (α) $\partial_\theta =$ cf(∂_θ),

  (β) $\partial_\theta = (\partial_\theta)^{<\partial_\theta}$,

  (γ) $\partial_\theta \leq \theta$, and if $\theta < \kappa$ are from Θ then $\partial_\theta < \partial_\kappa$,

  (δ) $\partial_\theta \geq \kappa$ if $\kappa \in (\Theta \cap \theta)$,

  (ε) if $\theta = \lambda$ then $\partial_\theta = \lambda$.

The reader may concentrate on (see 3.4):

Example 2.2: Assume

(a) $\mathbf{V}$ satisfies G.C.H. from λ to μ, i.e., $\partial \in [\lambda, \mu) \Rightarrow 2^\partial = \partial^+$,

(b) $\lambda = \lambda^{<\lambda} < \mu = \mu^{<\mu}$,

(c) $\Theta := \{\theta^+ : \lambda \leq \theta < \mu\} \cup \{\lambda, \mu\}$, and

(d) $\partial_\theta = \theta$ for every $\theta \in \Theta$, so in 2.3(5) below we have $\partial^\theta =$ min$\{\theta^+, \mu\}$.

For the rest of this section $\mathbf{p}$, i.e., $\lambda, \mu, \Theta, \bar{\partial}$, are fixed.

Definition 2.3: (1) For $\kappa \in \Theta$, let E_κ be the equivalence relation on μ defined by

($*$) $iE_\kappa j$ iff $i + \kappa = j + \kappa$.

(2) For any cardinal $\kappa \in [\lambda, \mu]$, define $E_{<\kappa}$ as Eq$_\lambda \cup \bigcup\{E_\theta : \theta \in \Theta \cap \kappa\}$. For such κ, if $\kappa \notin \Theta$, let $E_\kappa = E_{<\kappa}$.

(3) For $i < \mu$ and $\kappa \in \Theta$, let $[i]_\kappa = i/E_\kappa =$ the E_κ-equivalence class of i, and for $A \subseteq \mu$, let $A/E_\kappa = \{i/E_\kappa : i \in A\}$. For $i < \mu, A \subseteq \mu$ we say that i/E_κ is **represented in** A iff $A \cap (i/E_\kappa) \neq \emptyset$. If $A \subseteq B \subseteq \mu$, we say that i/E_κ **grows from** A to B iff $\emptyset \neq A \cap (i/E_\kappa) \neq B \cap (i/E_\kappa)$. If we write functions p, q instead of A, B, we mean $\mathrm{Dom}(p)$, $\mathrm{Dom}(q)$, respectively.

(4) Note that for all $i, j < \mu$ we have $iE_\mu j$. Thus, the following definition makes sense: if i, j are $< \mu$, we let $\kappa(i,j)$ be the minimal $\kappa \in \Theta$ such that $iE_\kappa j$.

(5) Suppose $\kappa \in \Theta$; let

$$\partial^\kappa = \min\{\partial_\theta : \kappa < \theta \in \Theta\} \text{ if } \kappa < \mu \text{ and } \partial^\kappa = \mu \text{ if } \kappa = \mu.$$

(Notice that κ is just an index in ∂^κ, and this is not cardinal exponentiation.)

Thus, in particular,

Observation 2.4: (1) *For $i, j < \mu$ we have: $\kappa(i,j)$ is well defined and for $i, j < \mu, \theta \in [\lambda, \mu)$ we have $iE_\theta j \Leftrightarrow \theta \geq \kappa(i,j)$ as*

(∗) *if $\theta < \kappa$ are both from Θ, then E_θ refines E_κ and, in fact, each E_κ-equivalence class is the union of κ many E_θ-equivalence classes.*

(2a) *If $\kappa < \theta$ are from Θ then $\partial^\kappa \leq \partial_\theta$; used in 2.8(1).*

(2b) *$\partial_\theta < \partial^\theta$ except possibly for $\theta = \mu$ (still $\partial_\mu \leq \mu = \partial^\mu$); recall 2.1(c)($\gamma$).*

(2c) *$\sup(\Theta \cap \kappa) \leq \partial_\kappa$ for $\kappa \in \Theta$; recall 2.1(c)(δ).*

(2d) *$\partial^\theta = (\partial^\theta)^{<\partial^\theta}$ for $\theta \in \Theta$.*

(2e) *If $\kappa \in \Theta$ then each $E_{<\kappa}$-equivalence class has cardinality $\leq \partial_\kappa$ (by (2c)); used in the proof of 2.8(3)).*

(3a) *$\partial_\lambda = \lambda$.*

(3b) *If $\theta < \kappa$ are successive elements of Θ then $\partial^\theta = \partial_\kappa$.*

(3c) *If $\kappa \in \Theta$ and $\bigcup(\Theta \cap \kappa)$ is a singular cardinal, then $\partial_\kappa \geq (\bigcup(\Theta \cap \kappa))^+$.*

Definition 2.5: (1) The forcing notion $\mathbb{Q}_{\mathbf{p}} = (Q_{\mathbf{p}}, \leq_{\mathbb{Q}_{\mathbf{p}}})$ (but we may omit $\mathbf{p}$ when it is clear from the context) is defined by:

(A) $q \in Q$ iff
  - (a) q is a (partial) function from μ to $\{0, 1\}$,
  - (b) if $i < \mu$ and $\kappa \in \Theta$, then the cardinality of $(i/E_\kappa) \cap \mathrm{Dom}(q)$ is $< \partial_\kappa$ (note: taking $\kappa = \mu$, the cardinality of $\mathrm{Dom}(q)$ is $< \partial_\mu \leq \mu$).

(B) $p \leq_{\mathbb{Q}} q$ iff
  - (a) $p \subseteq q$, i.e., $\mathrm{Dom}(p) \subseteq \mathrm{Dom}(q)$ and $\alpha \in \mathrm{Dom}(p) \Rightarrow p(\alpha) = q(\alpha)$,

(b) for every $\theta \in \Theta$ the set $\{A \in \mu/E_\theta : A$ grows from p to $q\}$ has cardinality $< \partial_\theta$.

(2) For $\kappa \in \Theta\backslash\{\mu\}$ and $p, q \in Q$, let:

(A) $p \leq^{\rm pr}_{\mathbf{p},\kappa} q$ or $p \leq^{\rm pr}_\kappa q$ iff

(a) $p \leq q$, and

(b) no E_κ-equivalence class grows from p to q.

(B) $p \leq^{\rm ap}_{\mathbf{p},\kappa} q$ or $p \leq^{\rm ap}_\kappa q$ iff

(a) $p \leq q$,

(b) $\mathrm{Dom}(q)/E_\kappa = \mathrm{Dom}(p)/E_\kappa$.

(3) For $\kappa = \mu$ and $p, q \in Q$, let:

(A) $p \leq^{\rm pr}_\mu q$ iff $p = q$,

(B) $p \leq^{\rm ap}_\mu q$ iff $p \leq q$.

(4) Let $\mathbf{Q}_\kappa = \mathbf{Q}_{\mathbf{p},\kappa} = (Q, \leq_{\mathbb{Q}}, \leq^{\rm pr}_\kappa, \mathrm{ap}_\kappa)$, where $\mathrm{ap}_\kappa = \mathrm{ap}_{\mathbf{p},\kappa}$ is the function with domain Q such that $\mathrm{ap}_\kappa(q) = \{q' : q \leq^{\rm ap}_\kappa q'\}$; so $\mathbf{Q}_\kappa$ as a forcing notion is $\mathbb{Q}$.

(5) Let $\leq^{\rm us}_{\mathbf{p},\kappa} = \leq^{\rm us}_\kappa = \leq_{\mathbf{p}}$ be $\leq_{\mathbb{Q}_{\mathbf{p}}}$ for $\kappa \in \Theta$.

Remark 2.6: Clearly $\mathbf{Q}_\kappa$ is related to §1, and if κ is the last member of $\Theta \cap \mu$ we can use it (enough if $\Theta = \{\lambda, \mu\}$, but not in general, so we shall use a variant).

Claim 2.7: *Concerning Definition 2.5:*

(a) (α) *if $\kappa \in \Theta$, then $\leq, \leq^{\rm pr}_\kappa, \leq^{\rm ap}_\kappa$ are partial orderings of Q,*

(β) *$p \leq^{\rm pr}_\kappa q \Rightarrow p \leq q$ and $p \leq^{\rm ap}_\kappa q \Rightarrow p \leq q$,*

(γ) *if $\kappa = \mu$ then $\leq^{\rm ap}_\kappa = \leq$,*

(δ) *if $\kappa = \mu$ then $\leq^{\rm pr}_\kappa$ is the equality;*

(b) (α) *if $p_1, p_2 \in Q$ and they are compatible as functions, then $p_1 \cup p_2 \in Q$,*

(β) *moreover, letting $q = p_1 \cup p_2$, if clause (b) of 2.5(1)(B) holds between p_k and q, for $k = 1, 2$, then q is the lub, in $\mathbb{Q}$, of p_1 and p_2;*

(c) *if $p \leq q$ and $\kappa \in \Theta$, then there are $r, s \in Q$ such that:*

(α) *$p \leq^{\rm pr}_\kappa r \leq^{\rm ap}_\kappa q$,*

(β) *$p \leq^{\rm ap}_\kappa s \leq^{\rm pr}_\kappa q$,*

(γ) *$q = r \cup s$,*

(δ) *q is the $\leq$-lub of r, s;*

(d) *if $q \in Q$ then:*

(α) *$\emptyset \leq q$ (and $\emptyset$, the empty function, $\in Q_{\mathbf{p}}$),*

(β) $(\forall r)(q \le r \equiv q \le^{\rm ap}_\mu r)$,

(γ) $\kappa \in \Theta\backslash\{\mu\} \Rightarrow \emptyset \le^{\rm pr}_\kappa q$,

(δ) $\emptyset \ne q \Rightarrow \emptyset \nleq^{\rm ap}_\kappa q$ *for any* $\kappa \in \Theta\backslash\{\mu\}$;

(e) *if* $\kappa_1 \le \kappa_2$ *are both from* Θ, *then:*
$\le^{\rm pr}_{\kappa_2} \subseteq \le^{\rm pr}_{\kappa_1}$ *and* $\le^{\rm ap}_{\kappa_1} \subseteq \le^{\rm ap}_{\kappa_2}$;

(f) *if* $\kappa \in \Theta$ *and* $p \le^{\rm ap}_\kappa q$ *and* $p \le^{\rm pr}_\kappa r$, *then:*

(α) $q \cup r$ *is a well-defined function* $\in Q$,

(β) $p \le (q \cup r)$,

(γ) $q \le^{\rm pr}_\kappa (q \cup r)$,

(δ) $r \le^{\rm ap}_\kappa (q \cup r)$,

(ε) $q \cup r$ *is a* $\le$*-lub of* q, r *in* $\mathbb{Q}_{\mathbf{p}}$;

(g) *if* $\kappa \in \Theta, p \le^{\rm pr}_\kappa q_i\, (i = 1, 2)$ *and* $q_1,\ q_2$ *are compatible in* $\mathbf{Q}$ *(even just as functions), then* $p \le^{\rm pr}_\kappa (q_1 \cup q_2)$;

(h) *if* $p \le^{\rm ap}_\kappa q_k$ *for* $k = 1, 2$, *and* q_1, q_2 *are compatible in* $\mathbf{Q}$ *(even just as functions), then* $q_k \le^{\rm ap}_\kappa q_1 \cup q_2$ *for* $k = 1, 2$;

(i) (α) *if* $\{p_\varepsilon : \varepsilon < \zeta\}$ *has an* $\le$*-upper bound then* $\cup\{p_\varepsilon : \varepsilon < \zeta\}$ *is an upper bound,*

(β) *similarly for* $\le^{\rm pr}_\kappa, \le^{\rm ap}_\kappa$,

(γ) *assume* $p_\varepsilon \in Q$ *for every* $\varepsilon < \zeta$, *and* p_ε, p_ξ *has a common* $\le^{\rm x}_\kappa$*-upper bound for any* $\varepsilon, \xi < \zeta$; *then the union of* $\{p_\varepsilon : \varepsilon < \zeta\}$ *is a* $\le^{\rm x}_\kappa$*-lub, when* $x =$ us,ap *and* $\zeta < \lambda$;

(δ) *if* $\{p_\varepsilon : \varepsilon < \zeta\} \subseteq Q$ *has a common* $\le^{\rm pr}_\kappa$*-upper bound and* $\zeta < \partial^\kappa$, *then* $\{p_\varepsilon : \varepsilon < \zeta\}$ *has a* $\le^{\rm pr}_\kappa$*-lub—the union;*

(j) *if* $p \le^{\rm ap}_\kappa q$ *then* $\mathrm{Dom}(q)\backslash\mathrm{Dom}(p)$ *has cardinality* $< \partial_\kappa$;

(k) *if* $p_1 \le^{\rm ap}_\kappa p_3$ *and* $p_1 \le p_2 \le p_3$ *then* $p_1 \le^{\rm ap}_\kappa p_2$ *and* $p_2 \le^{\rm ap}_\kappa p_3$;

(l) *if* $p_1 \le^{\rm pr}_\kappa p_2, p_\ell \le^{\rm ap}_\kappa q_\ell$ *for* $\ell = 1, 2$ *and* $q_1 \cup q_2$ *is a function, then* $q := q_1 \cup q_2$ *is a* $\le$*-lub of* q_1, q_2 *and* $q_2 \le^{\rm ap}_\kappa q, q_1 \le q$;

(m) *assume* p_1, p_2 *are compatible in* $\mathbb{Q}$; *then there is a pair* (q, t) *such that:*

$\bullet_1$ $p_1 \le^{\rm pr}_\kappa q$,

$\bullet_2$ $p_2 \le^{\rm ap}_\kappa t$,

$\bullet_3$ $q \Vdash$ "$t \in \underset{\sim}{\mathbf{G}} \Rightarrow p_1 \in \underset{\sim}{\mathbf{G}}$",

$\bullet_4$ q, t *are compatible and we say* (q, t) *is a witness for* (p_1, p_2);

(n) *if* $\langle p^\ell_\alpha : \alpha < \delta\rangle$ *is* $\le^{\rm pr}_\kappa$*-increasing for* $\ell = 1, 2$, δ *a limit ordinal of cofinality* $< \partial_\kappa$ *and* $\alpha < \delta \Rightarrow p^1_\alpha \le^{\rm ap}_\kappa p^2_\alpha$, *then* $\bigcup\limits_{\alpha<\delta} p^1_\alpha \le^{\rm ap}_\kappa (\bigcup\limits_{\alpha<\delta} p^2_\alpha)$.

Proof. Straightforward. For example:

Clause (i):

So assume $x \in \{\mathrm{us}, \mathrm{pr}, \mathrm{ap}\}$ and $\kappa \in \Theta$ and $\{p_\varepsilon : \varepsilon < \zeta\} \subseteq Q$ and $q \in Q$ is an $\leq^{\mathrm{x}}_\kappa$-upper bound of $\{p_\varepsilon : \varepsilon < \zeta\}$. Let $p := \cup\{p_\varepsilon : \varepsilon < \zeta\}$; then we shall prove that $p \in Q$ and p is a $\leq^{\mathrm{x}}_\kappa$-upper bound of $\{p_\varepsilon : \varepsilon < \zeta\}$. This clearly suffices for proving sub-clauses $(\alpha), (\beta)$ of clause (i), and the $\leq^x_\kappa$-lub part, i.e., sub-clauses $(\gamma), (\delta)$ are left to the reader; for $(\gamma), (\delta)$, see 2.8(1B),(1A).

Now

$(*)_1$ p is a well-defined function with domain $\subseteq \mu$ and $p \subseteq q$.

[Why? As $\varepsilon < \zeta \Rightarrow p_\varepsilon \subseteq q$, i.e., as functions (by 2.5(1)(B)(a)) clearly $p \subseteq q$, as functions, so p is a well defined function with domain $\subseteq \mathrm{Dom}(q)$, but $\mathrm{Dom}(q) \subseteq \mu$ by 2.5(A)(a).]

$(*)_2$ if $i < \mu$ and $\theta \in \Theta$ then the cardinality of $(i/E_\theta) \cap \mathrm{Dom}(p)$ is $< \partial_\theta$.

[Why? Recall $p \subseteq q \in Q$, see above; so as $q \in Q$ by 2.5(1)(a) we have $|(i/E_\theta) \cap \mathrm{Dom}(p)| \leq |(i/E_\theta) \cap \mathrm{Dom}(q)| < \partial_\theta$.]

$(*)_3$ $p \in Q$.

[Why? By $(*)_1 + (*)_2$ recalling 2.5(1)(A).]

$(*)_4$ $p_\varepsilon \subseteq p$ for $\varepsilon < \zeta$.

[Why? By the choice of p.]

$(*)_5$ If $\varepsilon < \zeta$ and $\theta \in \Theta$, then $\{A \in \mu/E_\theta : A$ grows from p_ε to $p\}$ has cardinality $< \partial_\theta$.

[Why? Because, recalling $p \subseteq q$, this set is included in $\{A \in \mu/E_\theta : A$ grows from p_ε to $q\}$ which has cardinality $< \partial_\theta$ because $p_\varepsilon \leq q$, which holds as $p_\varepsilon \leq^x_\kappa q$.]

$(*)_6$ $p_\varepsilon \leq p$ for $\varepsilon < \zeta$.

[Why? By $(*)_4 + (*)_5$ recalling 2.5(1)(B).]

$(*)_7$ If $x =$ us then p is a $\leq$-upper bound of $\{p_\varepsilon : \varepsilon < \zeta\}$.

[Why? By $(*)_3 + (*)_6$.]

$(*)_8$ If $x =$ pr and $\varepsilon < \zeta$ then $p_\varepsilon \leq^{\mathrm{pr}}_\kappa p$.

[Why? If $\kappa = \mu$ then $\leq^{\mathrm{pr}}_\kappa$ is equality and $p_\varepsilon \leq^{\mathrm{pr}}_\kappa q$, hence $p_\varepsilon = q$; but $p_\varepsilon \subseteq p \subseteq q$, hence $p_\varepsilon = p$ so this is trivial, hence assume $\kappa < \mu$. We have to check 2.5(2)(A); now clause (a) there holds by $(*)_6$ and clause (b) there holds as no E_κ-equivalence class grows from p_ε to q (as $p_\varepsilon \leq^{\mathrm{pr}}_\kappa q$) and $p \subseteq q$.]

$(*)_9$ If $x =$ pr then p is a $\leq^{\mathrm{x}}_\kappa$-upper bound of $\{p_\varepsilon : \varepsilon < \zeta\}$.

[Why? By $(*)_8$.]

$(*)_{10}$ If $x =$ ap and $\varepsilon < \zeta$ then $p_\varepsilon \leq^{\rm ap}_\kappa p$.

[Why? If $\kappa = \mu$ then $\leq^{\rm ap}_\kappa = \leq^{\rm us}_\kappa$ and we are done by $(*)_7$. Assume $\kappa < \mu$. We have to check 2.5(2)(B). First, clause (a) there holds by $(*)_6$. Second, clause (b) there holds because if $A \in {\rm Dom}(p)/E_\kappa$ then $A \cap {\rm Dom}(p) \neq \emptyset$ by the definition, hence $A \cap {\rm Dom}(q) \neq \emptyset$ as $p \subseteq q$ by $(*)_1$; but this implies $A \cap {\rm Dom}(p_\varepsilon) \neq \emptyset$ because $p_\varepsilon \leq^{\rm ap}_\kappa q$, as required.]

$(*)_{11}$ If $x =$ ap then p is a $\leq^{\rm x}_\kappa$-upper bound of $\{p_\varepsilon : \varepsilon < \zeta\}$.

[Why? By $(*)_{10}$.]

The $\leq^{\rm x}_\kappa$-lub parts are easy too; for a limit ordinal δ see 2.8(1A).

Clause (j):

Let $\mathcal{U} = \{A : A \in \mu/E_\kappa$ and A grows from p to $q\}$. Recalling Definition 2.5(1)(B)(b), clearly, as $p \leq q$, we have $|\mathcal{U}| < \partial_\kappa$. But as $p \leq^{\rm ap}_\kappa q$ necessarily ${\rm Dom}(q) \setminus {\rm Dom}(p)$ is included in $\bigcup\{A : A \in \mathcal{U}\}$. Also, as $q \in Q$, by Definition 2.5(1)(A)(b) we have $A \in \mathcal{U} \Rightarrow |A \cap {\rm Dom}(q)| < \partial_\kappa$.

So ${\rm Dom}(q) \setminus {\rm Dom}(p)$ is included in $\bigcup\{A \cap {\rm Dom}(q) : A \in \mathcal{U}\}$, a union of $< \partial_\kappa$ sets each of cardinality $< \partial_\kappa$. But ∂_κ is regular by 2.1(C)(β), so we are done.]

Clause (m):

As p_1, p_2 are compatible in $\mathbb{Q}$, there is $r \in \mathbb{Q}$ such that $p_1 \leq r, p_2 \leq r$. Choose $t = \bigcup\{r{\restriction}(i/E_\kappa) : i/E_\kappa$ grow from p_2 to $r\} \cup p_2$, so $t \in Q$ and $p_2 \leq^{\rm ap}_\kappa t \leq^{\rm pr}_\kappa r$. Choose $q = \bigcup\{r{\restriction}(i/E_\kappa) : i/E_\kappa$ does not grow from p_1 to $r\} \cup r$, so $q \in Q$ and $p_1 \leq^{\rm pr}_\kappa q \leq^{\rm apr}_\kappa r$.

Now check. $\blacksquare_{2.7}$

Claim 2.8: *Let $\kappa \in \Theta$.*

(1) *$(Q, \leq^{\rm pr}_\kappa)$ is $(< \partial^\kappa)$-complete and, in fact, if $\bar{p} = \langle p_\alpha : \alpha < \delta\rangle$ is $<^{\rm pr}_\kappa$-increasing, δ a limit ordinal $< \partial^\kappa$ then $p_\delta := \bigcup\{p_\alpha : \alpha < \delta\}$ is a $\leq^{\rm pr}_\kappa$-lub and a $\leq$-lub of $\bar{p}$; we use $\kappa < \theta \in \Theta \Rightarrow \partial^\kappa \leq \partial_\theta$; see 2.4(2a).*

(1A) *If $\gamma(*) < \partial^\kappa$ and $p_\alpha \in Q$ for $\alpha < \gamma(*)$ and p_α, p_β has a common $\leq^{\rm pr}_\kappa$-lub for any $\alpha, \beta < \gamma(*)$ then $p_* = \bigcup\{p_\alpha : \alpha < \gamma(*)\}$ is a $\leq^{\rm pr}_\kappa$-lub of $\{p_\alpha : \alpha < \gamma(*)\}$.*

(1B) *If $\gamma(*) < \lambda$ then (1A) holds for $\leq^{\rm ap}_\kappa$.*

(2) *If $p \in \mathbb{Q}$ then $\mathbb{Q}_p := \mathbb{Q}_{\mathbf{p},p} = (\{q : p \leq^{\rm ap}_\kappa q\}, <^{\rm ap}_\kappa)$ satisfies[4] the $(\partial_\kappa)^+$-c.c.*

(3) *Moreover, if $\langle p_\alpha : \alpha < \partial^+_\kappa\rangle$ is $\leq^{\rm pr}_\kappa$-increasing continuous and $p_\alpha \leq^{\rm ap}_\kappa q_\alpha$ for $\alpha < \partial^+_\kappa$, then for some $\alpha < \beta$ the conditions q_α, q_β are compatible*

[4] Compare with [ShSt:608, 1.8].

in $\mathbb{Q}$; moreover, there is r such that $q_\alpha \le r$ and $q_\beta \le^{\rm ap}_\kappa r$ and $p_\alpha = p_\beta \Rightarrow q_\alpha \le^{\rm ap}_\kappa r \wedge q_\beta \le^{\rm ap}_\kappa r$.

(4) *Assume $p \in \mathbb{Q}_{\mathbf{p}}, \chi = |A| < \partial^\kappa, \kappa \in \Theta$ and $p \Vdash$ "$\underset{\sim}{f}$ is a function from $A \in \mathbf{V}$ to $\mathbf{V}$". Then we can find q such that:*

- (α) *$p \le^{\rm pr}_\kappa q$,*
- (β) *if $a \in A$ then $\mathcal{I}_{q,\underset{\sim}{f},a} := \{r : q \le^{\rm ap}_\kappa r$ and r forces a value to $\underset{\sim}{f}(a)\}$ is predense over q in $\mathbb{Q}_{\mathbf{q}}$,*
- (γ) *moreover, some subset $\mathcal{I}'_{q,\underset{\sim}{f},a}$ of $\mathcal{I}_{q,\underset{\sim}{f},a}$ of cardinality $\le \partial_\kappa$ is predense over q in $\mathbb{Q}_{\mathbf{q}}$ (really follows).*

Proof. (1) By (1A).

(1A) Let $q_{\alpha,\beta}$ be a common $\le^{\rm pr}_\kappa$-upper bound of p_α, p_β for $\alpha, \beta < \gamma(*)$. Why is $p_* \in Q$? Let us check Definition 2.5(1)(A).

Clearly p_* is a partial function from μ to $\{0,1\}$, so clause (a) there holds. For checking clause (b) there, assume $\theta \in \Theta$ and $A \in \mu/E_\theta$.

First, assume $\theta \le \kappa$ and $A \cap \operatorname{Dom}(p_*) \ne \emptyset$. Then for some $\alpha < \gamma(*)$ we have $A \cap \operatorname{Dom}(p_\alpha) \ne \emptyset$, hence

$$A \cap \operatorname{Dom}(p_*) = \bigcup\{A \cap \operatorname{Dom}(p_\beta) : \beta < \gamma(*)\} \subseteq \bigcup\{A \cap \operatorname{Dom}(q_{\alpha,\beta}) : \beta < \gamma(*)\},$$

but $p_\alpha \le^{\rm pr}_\kappa q_{\alpha,\beta}$ and $A \cap \operatorname{Dom}(p_\alpha) \ne \emptyset$, hence $A \cap \operatorname{Dom}(q_{\alpha,\beta}) = A \cap \operatorname{Dom}(p_\alpha)$. Together $A \cap \operatorname{Dom}(p_*)$ is equal to $A \cap \operatorname{Dom}(p_\alpha)$ which, because $p_\alpha \in Q$, has cardinality $< \partial_\theta$ as required in clause (b) of Definition 2.5(1)(A).

Second of course, if $A \cap \operatorname{Dom}(p_*) = \emptyset$ this holds, too.

Third, assume $\theta > \kappa$. Then $\alpha < \gamma(*) \Rightarrow p_\alpha \in Q \Rightarrow |A \cap \operatorname{Dom}(p_\alpha)| < \partial_\theta$, hence $|A \cap \operatorname{Dom}(p_*)| = |A \cap \bigcup_{\alpha<\gamma(*)} \operatorname{Dom}(p_\alpha)| \le \sum_{\alpha<\gamma(*)} |A \cap \operatorname{Dom}(p_\alpha)|$ which is $< \partial_\theta$ as $\gamma(*) < \partial^\kappa \le \partial_\theta = \operatorname{cf}(\partial_\theta)$, so again the desired conclusion of clause (b) of Definition 2.5(1)(A) holds. Together indeed $p_* \in Q$.

Why does the following hold: $\alpha < \gamma(*) \Rightarrow p_\alpha \le p_*$? We have to check 2.5(1)(B); obviously clause (a) there holds. Clause (b) there is proved as above.

Why does the following hold: $\alpha < \gamma(*) \Rightarrow p_\alpha \le^{\rm pr}_\kappa p_*$? We have to check Definition 2.5(2)(A). Now clause (a) there was just proved, and clause (b) there holds as in the proof of "$p_* \in Q$".

Next we show that p_* is a $\le^{\rm pr}_\kappa$-lub of $\bar{p}$, so assume $q \in Q$ and $\alpha < \delta \Rightarrow p_\alpha \le^{\rm pr}_\kappa q$. To show $p_* \le^{\rm pr}_\kappa q$ we have to check clauses (B)(a),(b) of 2.5(1) and (A)(b) of 2.5(2). As $p_* = \cup\{p_\alpha : \alpha < \gamma(*)\}$, clearly $p_* \subseteq q$ as a function, so 2.5(1)(B)(a) above holds. Also, if $A \in \mu/E_\kappa$ and A is represented in p_*, then it is represented in p_α for some $\alpha < \gamma(*)$; but $p_\alpha \leq^{\rm pr}_\kappa q$, so $q \restriction A = p_\alpha \restriction A$, but $(p_\alpha {\restriction} A) \subseteq (p_* {\restriction} A) \subseteq (q {\restriction} A)$, hence $q \restriction A = p_* \restriction A$ as required in 2.5(2)(A)(b).

Lastly, when $\theta \in \Theta$, 2.5(1)(B)(b) holds: if $\theta \leq \kappa$ because more was just proved and if $\theta > \kappa$ it is proved as in the proof of $p_* \in Q$.

(2) This is a special case of (3) when $\langle p_\alpha : \alpha < \partial^+_\kappa \rangle$ is constant (recalling 2.7(h)).

(3) So in particular $p_i \leq^{\rm ap}_\kappa q_i$ for $i < \partial^+_\kappa$. Hence by clause (j) of Claim 2.7 the set $u_i := \mathrm{Dom}(q_i) \backslash \mathrm{Dom}(p_i)$ has cardinality $< \partial_\kappa$. Hence by the Δ-system lemma (recalling that $(\partial_\kappa)^{<\partial_\kappa} = \partial_\kappa$ by 2.1(c)(β)) for some unbounded $\mathcal{U} \subseteq \partial^+_\kappa$ the sequence $\langle u_i : i \in \mathcal{U} \rangle$ is a Δ-system, with heart u_*. Moreover, since $2^{|u_*|} \leq \partial^{<\partial_\kappa}_\kappa = \partial_\kappa < \partial^+_\kappa$, we can assume that $q_i {\restriction} u_* = q_*$ for every $i \in \mathcal{U}$.

As each $E_{<\kappa}$-class has cardinality $\leq \partial_\kappa$ (see 2.4(2)(c),(e)), without loss of generality for every $i \neq j$ from $\mathcal{U}$, if $\alpha \in u_i \backslash u_*$ then $\alpha / E_{<\kappa}$ is disjoint to u_j. Now by 2.7(h) for every $i, j \in \mathcal{U}$, the function $q = q_i \cup q_j$ is a $\leq^{\rm ap}_\kappa$-lub of q_i, q_j for part (2), i.e., when $p_i = p_j$. Also, it is easy to check that for $i < j, q$ is a $\leq$-lub of q_i, q_j, which is $\leq^{\rm ap}_\kappa$-above q_j for part (3).

(4) If $\kappa = \mu$ then $\leq^{\rm ap}_\kappa = \leq$ by clause 2.7(a)(γ), recall $\mathbb{Q}_p = (\{q \in Q : p \leq q\}, \leq_{\mathbb{Q}_{\mathbf{p}}})$, so $q = p$ can serve, as $\mathbb{Q}_p$ satisfies the ∂^+_κ-c.c. by part (2); so we shall assume $\kappa < \mu$. Recall that $\partial_\kappa < \partial^\kappa$ by 2.4(2)(b). As $|A| < \partial^\kappa = \mathrm{cf}(\partial^\kappa)$, by part (1) of the claim and clause (f) of Claim 2.7 it is enough to consider the case $A = \{a\}$. Now we try to choose p_i, r_i, b_i by induction on $i < \partial^+_\kappa$; but r_i, b_i are chosen in stage $i+1$ together with p_{i+1}, such that:

$\circledast$ (a) $p_0 = p$,
(b) $\langle p_j : j \leq i \rangle$ is $\leq^{\rm pr}_\kappa$-increasing,
(c) $p_{i+1} \leq^{\rm ap}_\kappa r_i$,
(d) $p_{i+1} \Vdash$ "if $r_i \in \underset{\sim}{\mathbf{G}}_{\mathbf{Q}}$ then $\underset{\sim}{f}(a) = b_i$",
(e) $p_{i+1} \Vdash$ "if $r_i \in \underset{\sim}{\mathbf{G}}_{\mathbf{Q}}$ then for no $j < i$ do we have $r_j \in \underset{\sim}{\mathbf{G}}_{\mathbf{Q}}$",
(f) if i is a limit, then p_i is the union so a $\leq^{\rm pr}_\kappa$-lub of $\langle p_j : j < i \rangle$.

For $i = 0$ just use clause (a) of $\circledast$.

For i limit use clause (f) of $\circledast$, recalling part (1) of the claim and the fact that $\partial^+_\kappa \leq \partial^k$.

For $i = j+1$, try to choose q_i such that:

$$p_j \leq q_i$$

and

$$q_i \Vdash \text{“}r_{i_1} \notin \underset{\sim}{\mathbf{G}}_{\mathbf{Q}} \text{ for } i_1 < j\text{”}.$$

If we cannot, we have succeeded, i.e., p_i is as required from q with $\mathcal{I}_{p_i,\underset{\sim}{f},a} = \{p_i \cup r_j : j < i\}$. If we can, let (b_j, r_j) be such that $q_i \leq r_j$ and r_j forces $\underset{\sim}{f}(a) = b_j$; this is clearly possible. By clause (c) of Claim 2.7 applied to the pair (p_j, r_j) we choose[5] p_i such that $p_j \leq^{\rm pr}_\kappa p_i \leq^{\rm ap}_\kappa r_j$, and clearly we have carried out the induction. But if we carry out the induction, then we get a contradiction by part (3). So we have to be stuck for some $i < \partial^+_\kappa$, and, as said above, we then get the desired conclusion. $\blacksquare_{2.8}$

Conclusion 2.9: Forcing with $\mathbb{Q}_{\mathbf{p}}$

(a) does not collapse cardinals except possibly cardinals from the set $\Omega_{\mathbf{p}} = \{\theta : \lambda < \theta \leq \mu$ and for no $\kappa \in \Theta$ do we have $\partial_\kappa < \theta \leq \partial^\kappa\}$, so $\mu \notin \Omega_{\mathbf{p}}$,

(b) does not change cofinalities $\notin \Omega_{\mathbf{p}}$; moreover, if it changes the cofinality of $\theta \in$ Reg to $\chi < \theta$ then there is $\theta_1 \in \Omega_{\mathbf{p}}$ such that $\chi < \theta_1 \leq \theta$,

(c) does not add new sequences of length $< \lambda$,

(d) does not change 2^θ for $\theta \notin [\lambda, \mu)$,

(e) makes $2^\lambda = \mu$,

(f) also the set $\Omega'_{\mathbf{p}} := \bigcup\{(\kappa_1, 2^{\sup(\Theta \cap \kappa)}]$: for some $\kappa \in \Theta$, $\Theta \cap \kappa$ has no last member, so $\sup(\Theta \cap \kappa)$ is strong limit and $\kappa_1 = \min(\mathrm{Reg} \setminus \sup(\Theta \cap \kappa))\}$ is O.K. in clauses (a),(b),

(g) $\mathbb{Q}_{\mathbf{p}}$ has cardinality μ and satisfies the ∂^+_μ-c.c., recalling $\partial_\mu \leq \mu$.

Proof. First, $\mathbb{Q}_{\mathbf{p}}$ is $(< \lambda)$-complete, hence it adds no new sequences to ${}^{\lambda>}\mathbf{V}$, i.e., clause (c) holds so cardinals $\leq \lambda$ are preserved as well as cofinalities $\leq \lambda$ as well as 2^θ for $\theta < \lambda$.

Second, $|\mathbb{Q}_{\mathbf{p}}| = \mu$ as $p \in \mathbb{Q}_{\mathbf{p}} \Rightarrow p$ is a function from $\mathrm{Dom}(p) \subseteq \mu$ to $\{0, 1\}$ (see 2.5(1)(A)(a)) and $|\mathrm{Dom}(p)| < \partial_\mu = \mu$ by 2.5(1)(A)(b) and $\mu^{<\mu} = \mu$ by 2.1(a).

Third, by 2.8(2) the forcing notion $\mathbb{Q}_p$ satisfies the ∂^+_μ-c.c. but $\mathbb{Q} = \mathbb{Q}_p$ when $p = \emptyset$, so $\mathbb{Q}$ satisfies the ∂^+_μ-c.c. and, of course, $\partial_\mu \leq \mu$. This gives clauses (g) and (d) (recalling (c)).

Fourth, for clause (e), for any $\alpha < \mu$ let $\underset{\sim}{\eta}_\alpha \in {}^\lambda 2$ be defined by $p \Vdash$ "$\eta_\alpha(i) = \ell$" iff $i < \lambda \wedge \alpha + i \in \mathrm{Dom}(p) \wedge \ell = p(\alpha + i)$. By density, indeed $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta}_\alpha \in {}^\lambda 2$" and $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta}_\alpha \neq \underset{\sim}{\eta}_\beta$" for $\alpha \neq \beta < \mu$, so clearly clause (e) holds.

Fifth, use 2.8(2),(4) to prove clauses (a) and (b), toward a contradiction assume θ is regular in $\mathbf{V}$ and θ_1 is not in $\Omega_{\mathbf{p}}$ but $p \Vdash_{\mathbb{Q}}$ "$\chi = \mathrm{cf}(\theta) < \theta_1 \leq \theta$".

---

[5] We can use r'_j such that $p_j \leq^{\rm ap}_\kappa r'_j \leq^{\rm pr}_\kappa r_j$ where r_j is the $\leq$-lub of r'_j, p_{i+1}; this may be helpful but is not needed now.

If $\theta \leq \lambda$ or just $\chi < \lambda$ use clause (c), if $\theta > \mu$ use clause (g), so necessarily $\lambda \leq \chi < \theta_1 \leq \theta \leq \mu$. By the choice of $\Omega_{\mathbf{p}}$ there is $\kappa \in \Theta$ such that $\partial_\kappa < \theta_1 \leq \partial^\kappa$ and $\chi + \partial_\kappa < \theta_1 \leq \theta$; now, without loss of generality, $p \Vdash$ "$\underset{\sim}{f} : \chi \to \theta$ has range unbounded in θ". Apply 2.8(4) with $(p, \chi, \underset{\sim}{f}, \kappa)$ here standing for $(p, A, \underset{\sim}{f}, \kappa)$ there, and get $q, \langle \mathcal{I}_{q,\underset{\sim}{f},\alpha} : \alpha < \chi\rangle$ as there. By 2.8(3) we have $|\mathcal{I}_{q,\underset{\sim}{f},\alpha}| \leq \partial_\kappa$, and $\bigcup\{\mathcal{I}_{q,\underset{\sim}{f},\alpha} : \alpha < \chi\}$ has cardinality $\leq \chi + \partial_\kappa \leq \theta_1$. In any case, in $\mathbf{V}$ the set $\{\beta$: for some $\alpha < \chi$ and $q \nVdash$ "$\underset{\sim}{f}(\alpha) \neq \beta$"$\}$ has cardinality $< \theta_1 \leq \theta$, contradiction. So clauses (a) and (b) hold.

We are left with clause (f); it is not really needed, but still nice to have. Now if $\theta \in \mathrm{Reg} \cap (\lambda, \mu]$ is in $\Omega'_{\mathbf{p}}$ and κ witness it, then necessarily $\Theta \cap \kappa$, which is not empty, has no last element, so if $\theta_1 < \theta_2$ are from $\Theta \cap \theta$ then $\theta_1 \leq \partial_{\theta_2} = (\partial_{\theta_2})^{<\partial_{\theta_2}} \leq \theta_2$, hence $\sup(\Theta \cap \theta)$ is strong limit.

If $\theta = \kappa$ use clause (b). If $\theta \geq 2^{<\kappa}$ we repeat the proofs above for $\leq^{\mathrm{pr}}_{<\kappa}$, where

$$\leq^{\mathrm{pr}}_{<\kappa} = \bigcap\{\leq^{\mathrm{pr}}_\theta : \theta \in \Theta \cap \theta\}, \leq^{\mathrm{ap}}_{<\kappa} = \{(p,q) : p \leq q \quad \text{and}$$
$$\alpha \in \mathrm{Dom}(p) \backslash \mathrm{Dom}(p) \Rightarrow (\exists \theta \in \Theta \cap \theta)((\alpha/E_\theta \cap \mathrm{Dom}(p)) \neq \emptyset\}. \qquad \blacksquare_{2.9}$$

Definition 2.10: (1) If $p \leq q$ and $\kappa \in \Theta$, let $\mathrm{supp}_\kappa(p,q) := \bigcup\{i/E_\kappa : i \in \mathrm{Dom}(q)\backslash\mathrm{Dom}(p)\}$, so of cardinality $< \partial_\kappa$.

(2) We say $\mathbf{y} = \langle \kappa, \bar{p}, \bar{u}\rangle = \langle \kappa_{\mathbf{y}}, \bar{p}_{\mathbf{y}}, \bar{u}_{\mathbf{y}}\rangle$ is a **reasonable p-parameter** when:

$\circledast_1$ (a) $\kappa \in \Theta$ but $\kappa < \mu$,
- (b) $\bar{p} = \langle p_\alpha : \alpha < \gamma\rangle$ is a non-empty $\leq^{\mathrm{pr}}_\theta$-increasing continuous sequence, so we write $\gamma = \gamma_{\mathbf{y}}, \bar{p} = \bar{p}^{\mathbf{y}}$ and $p_\alpha = p^{\mathbf{y}}_\alpha$,
- (c) $\bar{u} = \langle u_\alpha : \alpha < \gamma\rangle$ is $\subseteq$-increasing continuous, so $u_\alpha = u^{\mathbf{y}}_\alpha, \bar{u} = \bar{u}_{\mathbf{y}}$,
- (d) $u_\alpha \subseteq \bigcup\{i/E_\kappa : i \in \mathrm{Dom}(p_\alpha)\}$ for $\alpha < \gamma$,
- (e) $|u_\alpha| \leq \partial^\kappa$ for $\alpha < \gamma$.

(3) For $\mathbf{y}$ as above we define $\mathbf{Q}_{\mathbf{y}}$ as $(Q_{\mathbf{y}}, \leq_{\mathbf{y}}, \leq^{\mathrm{pr}}_{\mathbf{y}}, \mathrm{ap}_{\mathbf{y}})$ (so $\mathbb{Q}_{\mathbf{y}} = (Q_{\mathbf{y}}, \leq_{\mathbf{y}})$ is $\mathbf{Q}_{\mathbf{y}}$ as a forcing notion), where:

$\circledast_2$ (a) $\theta = \theta_{\mathbf{y}} = \min(\Theta\backslash\kappa^+_{\mathbf{y}})$; notice that θ is well defined, as $\kappa_{\mathbf{y}} < \mu$ and $\mu \in \Theta$,
- (b) $Q_{\mathbf{y}} := \{q$: for some $\alpha < \gamma_{\mathbf{y}}$ we have $p_\alpha \leq^{\mathrm{ap}}_\theta q$ and $\mathrm{supp}_\theta(p_\alpha, q) \subseteq u_\alpha\}$,
- (c) $\leq_{\mathbf{y}} = \leq_{\mathbf{p}} \restriction Q_{\mathbf{y}}$,
- (d) for $q \in Q_{\mathbf{y}}$, let $\alpha_{\mathbf{y}}(q) = \min\{\alpha < \gamma_{\mathbf{y}} : p_\alpha \leq^{\mathrm{ap}}_\theta q$ and $\mathrm{supp}_\theta(p_\alpha, q) \subseteq u_\alpha\}$,
- (e) the two-place relation $\leq^{\mathrm{pr}}_{\mathbf{y}}$ is defined by $p \leq^{\mathrm{pr}}_{\mathbf{y}} q$ iff
  - (α) $p, q \in Q_{\mathbf{y}}$,
  - (β) $p \leq^{\mathrm{pr}}_{\mathbf{p},\kappa} q$,

(f) for $q \in Q_{\mathbf{y}}$ let $\mathrm{ap}_{\mathbf{y}}(q) = \mathrm{ap}_{\mathbf{Q_y}}(q) = \{r \in Q_{\mathbf{y}} : q \le^{\mathrm{ap}}_{\kappa} r$ and $\mathrm{supp}_\kappa(q,r) \subseteq \mathrm{supp}_\theta(p_{\alpha_{\mathbf{y}}(q)}, q)\}$.

Observation 2.11: *Let $\mathbf{y}$ be a reasonable $\mathbf{p}$-parameter.*

(0) *If $p_1 \le p_2 \le q_2 \le q_1$ and $\kappa_1 \ge \kappa_2$ are from Θ, then $\mathrm{supp}_{\kappa_2}(p_2,q_2) \subseteq \mathrm{supp}_{\kappa_1}(p_1,q_1)$.*

(0A) *If $p_1 \le p_2 \le p_3$, then $\mathrm{supp}_{\kappa_1}(p_1,p_3) = \mathrm{supp}_{\kappa_1}(p_1,p_2) \cup \mathrm{supp}_{\kappa_1}(p_2,p_3)$.*

(1) *For $q \in Q_{\mathbf{y}}$, the ordinal $\alpha_{\mathbf{y}}(q)$ is well defined $< \gamma_{\mathbf{y}}$.*

(2) *If $q_1 \le_{\mathbf{y}} q_2$ are from $Q_{\mathbf{y}}$, then $\alpha_{\mathbf{y}}(q_1) \le \alpha_{\mathbf{y}}(q_2)$.*

(2A) *If $q_1 \in Q_{\mathbf{y}}$ and $q_1 \le^{\mathrm{ap}}_{\mathbf{p},\kappa}$, then $q_2 \in Q_{\mathbf{y}}, q_1 \le_{\mathbf{y}} q_2$ and $\alpha_{\mathbf{y}}(q_1) = \alpha_{\mathbf{y}}(q_2)$.*

(3) *If $p \le^{\mathrm{pr}}_{\mathbf{y}} r$ and $q \in \mathrm{ap}_{\mathbf{y}}(p)$, then $s := q \cup r$ belongs to $Q_{\mathbf{y}}$, $s \in \mathrm{ap}_{\mathbf{y}}(r)$ and $q \le^{\mathrm{pr}}_{\mathbf{y}} s$.*

Proof. (0), (0A) Should be easy.

(1) By the definitions of $q \in Q_{\mathbf{y}}$ and of $\alpha_{\mathbf{y}}(q)$.

(2) For $\ell = 1,2$, letting $\alpha_\ell = \alpha_{\mathbf{y}}(q_\ell)$ we have $p_{\alpha_\ell} \le^{\mathrm{ap}}_\theta q_\ell \wedge \mathrm{supp}_\theta(p_{\alpha_\ell}, q_\ell) \subseteq u_{\alpha_\ell}$. If $\alpha_2 < \alpha_1$, then $p_{\alpha_2} \le p_{\alpha_1} \le^{\mathrm{ap}}_\theta q_1 \le q_2 \wedge p_{\alpha_2} \le^{\mathrm{ap}}_\theta q_2$ hence $p_{\alpha_2} \le^{\mathrm{ap}}_\theta q_1$ (by 2.7(k)) and $\mathrm{supp}_\theta(p_{\alpha_2}, q_1) \subseteq \mathrm{supp}_\theta(p_{\alpha_2}, q_2) \subseteq u_{\alpha_2}$ by the definition of 2.10(1) of supp, contradicting the choice of α_1.

(2A) We know $p_{\alpha_{\mathbf{y}}}(q_1) \le^{\mathrm{ap}}_{\mathbf{p},\kappa} q_1$ by the definition $\alpha_{\mathbf{y}}(q_1)$ but we assume $q_1 \le^{\mathrm{ap}}_{\mathbf{p},\kappa} q_2$ and $\le^{\mathrm{pr}}_{\mathbf{p},\kappa}$ is a quasi order hence $p_{\alpha_{\mathbf{y}}}(q_1) \le^{\mathrm{ap}}_{\mathbf{p},\kappa} q_1$. So by the definition, $q_2 \in Q_{\mathbf{y}} \wedge \alpha_{\mathbf{y}}(q_1) \ge \alpha_{\mathbf{y}}(q_2)$. Also clearly $q_1 \le_{\mathbf{p}} q_2$ hence $q_1 \le_{\mathbf{y}} q_2$ hence by part (2), $\alpha_{\mathbf{y}}(q_1)^{\mathbf{P}} \le \alpha_{\mathbf{y}}(q_2)$, together we are done.

(3) Let $\kappa = \kappa_{\mathbf{y}}$ and $\theta = \theta_{\mathbf{y}}, p_\alpha = p^{\mathbf{y}}_\alpha$. By Definition 2.10(3)(e),(f) we know that $p \le^{\mathrm{pr}}_{\mathbf{p},\kappa} r$ and $p \le^{\mathrm{ap}}_{\mathbf{p},\kappa} q$. By Claim 2.7(f) we know that $s \in Q_{\mathbf{p}}$ and $p \le^{\mathrm{ap}}_{\mathbf{p},\kappa} q \le^{\mathrm{pr}}_{\mathbf{p},\kappa} s$ and $p \le^{\mathrm{pr}}_{\mathbf{p},\kappa} r \le^{\mathrm{ap}}_{\mathbf{p},\kappa} s$ recalling $s = q \cup r$, note

$(*)_1$ [why? As $r \in Q_{\mathbf{y}}$]

$(*)_2$ $\alpha_{\mathbf{y}}(s) = \alpha_{\mathbf{y}}(r) = \beta$.

[Why? As $p \in Q_{\mathbf{y}}$ the ordinal $\beta := \alpha_{\mathbf{y}}(r) < \gamma_{\mathbf{y}}$ is well defined and the ordinal $\alpha := \alpha_{\mathbf{y}}(p) < \gamma_{\mathbf{y}}$ is well defined and, by part (2), we have $\alpha \le \beta$. So clearly $p_\beta \le^{\mathrm{ap}}_{\mathbf{p},\theta} r$ by the choice of β and $r \le^{\mathrm{ap}}_{\mathbf{p},\kappa} s$ as said above, hence by 2.7(e) recalling $\kappa < \theta$, we have $\le^{\mathrm{ap}}_{\mathbf{p},\kappa} \subseteq \le^{\mathrm{ap}}_\theta$ hence $r \le^{\mathrm{ap}}_{\mathbf{p},\theta} s$ so together $p_\beta \le^{\mathrm{ap}}_{\mathbf{p},\theta} s$. Also, $s = q \cup r$ hence $\mathrm{supp}_\theta(r,s) \subseteq \mathrm{supp}_\theta(p,q)$ and, as $q \in \mathrm{ap}_{\mathbf{y}}(p)$, necessarily $p \le^{\mathrm{ap}}_{\mathbf{p},\kappa} q$, hence $p \le^{\mathrm{ap}}_{\mathbf{p},\theta} q$, hence by part (2A) $\mathrm{supp}_\theta(p,q) \subseteq \mathrm{supp}_\theta(p_\alpha, q) \subseteq u^{\mathbf{y}}_{\alpha_{\mathbf{y}}(q)} = u^{\mathbf{y}}_{\alpha_{\mathbf{y}}(p)} = u^{\mathbf{y}}_\alpha$, but $u^{\mathbf{y}}_\alpha \subseteq u^{\mathbf{y}}_\beta$ as $\alpha \le \beta$. Together $\mathrm{supp}_\theta(r,s) \subseteq u_\beta$, and by the choice of β

clearly $\mathrm{supp}_\theta(p_\beta, r) \subseteq u_\beta$, hence $\mathrm{supp}_\theta(p_\beta, s) \subseteq \mathrm{supp}_\theta(p_\beta, r) \cup \mathrm{supp}_\theta(r, s) \subseteq u_\beta \cup u_\beta = u_\beta$. As we have shown earlier that $p_\beta \leq^{\mathrm{ap}}_{\mathbf{p},\theta} s$ it follows that $s \in Q_{\mathbf{y}}$ and $\alpha_{\mathbf{y}}(s) \leq \beta$. But $r \leq_{\mathbf{p}} s$, hence by part (2) we know that $\beta = \alpha_{\mathbf{y}}(r) \leq \alpha_{\mathbf{y}}(s)$, so necessarily $\alpha_{\mathbf{y}}(s) = \alpha_{\mathbf{y}}(r) = \beta$, i.e., $(*)$ holds.]

So $p_{\alpha_{\mathbf{y}}(s)} \leq^{\mathrm{ap}}_{\mathbf{p},\theta} s$ and $\mathrm{supp}_\theta(p_{\alpha_{\mathbf{y}}(s)}, s) = \mathrm{supp}_\theta(p_\beta, s) \subseteq u_\beta = u_{\alpha_{\mathbf{y}}(s)}$, hence together $s \in Q_{\mathbf{y}}$, the first statement in the conclusion.

Also $q \leq^{\mathrm{pr}}_{\mathbf{y}} s$; for this check (e)$(\alpha)+(\beta)$ of Definition 2.10(3); for clause (α): $q \in Q_{\mathbf{y}}$ is assumed, $s \in Q_{\mathbf{y}}$ was just proved; for clause (β): "$q \leq^{\mathrm{pr}}_{\mathbf{p},\kappa} s$" was proved in the beginning of the proof; so the third statement in the conclusion holds.

Lastly, we check that $s \in \mathrm{ap}_{\mathbf{y}}(r)$. For this we have to check the two demands in 2.10(3)(f). Now "$s \in Q_{\mathbf{y}}$" was proved above, "$r \leq^{\mathrm{ap}}_{\mathbf{p},\kappa} s$" was proved in the beginning of the proof and "$\mathrm{supp}_\kappa(r, s) \subseteq \mathrm{supp}_\theta(p_{\alpha_{\mathbf{y}}(r)}, s)$" holds as $\mathrm{supp}_\kappa(r, s) \subseteq \mathrm{supp}_\theta(r, s) \subseteq \mathrm{supp}_\theta(p_{\alpha_{\mathbf{y}}(r)}, s) = \mathrm{supp}_\theta(p_\beta, s) = \mathrm{supp}_\theta(p_{\alpha_{\mathbf{y}}(s)}, s)$ is as required. $\blacksquare_{2.11}$

Claim 2.12: (1) *Assume $\kappa < \theta$ are successive members of $\Theta_{\mathbf{p}}$ and $(\forall \alpha < \partial_\theta)(|\alpha|^{<\partial_\kappa} < \partial_\theta)$, and $\mathbf{y}$ is a reasonable $\mathbf{p}$-parameter, $\kappa = \kappa_{\mathbf{y}}$, hence $\theta_{\mathbf{y}} = \theta$ and $\bar{p}_{\mathbf{y}}$ is $\leq^{\mathrm{pr}}_\theta$-increasing (hence also $\leq^{\mathrm{pr}}_\kappa$-increasing) and $\gamma_{\mathbf{y}}$ is a successor or a limit ordinal of cofinality $\geq \partial_\theta$. Then $\mathbf{Q}_{\mathbf{y}}$ is a $(\partial^+_\theta, \partial_\theta, < \partial_\theta)$-forcing.*

(2) *If, in addition, $\gamma_{\mathbf{y}} = \alpha_* + 1$ then*

$$p_{\alpha_*} \Vdash \text{“}\underset{\sim}{\mathbf{G}}_{\mathbf{Q}} \cap Q_{\mathbf{y}} \text{ is a subset of } \mathbf{Q}_{\mathbf{y}} \text{ generic over } \mathbf{V}\text{”}.$$

Proof. (1) We should check for $\mathbf{Q} = \mathbf{Q}_{\mathbf{y}}$ (defined in 2.10) each of the clauses of Definition 1.2. Let $p_\alpha = p^{\mathbf{y}}_\alpha, u_\alpha = u^{\mathbf{p}}_\alpha$.

Clause (a): Trivial, just $\mathbf{Q}_{\mathbf{y}}$ has the right form, a quadruple.

Clause (b): $(Q_{\mathbf{y}}, \leq_{\mathbf{y}})$ is a forcing notion.

Why? By $\circledast_2$(b)+(c) from 2.10(3), i.e., $Q_{\mathbf{y}}$ is a non-empty subset of $Q_{\mathbf{p}}$ because $\gamma_{\mathbf{y}} > 0$, so $p^{\mathbf{y}}_0 = p \in Q_{\mathbf{y}}$ and $\leq_{\mathbf{y}}$ being $\leq_{Q_{\mathbf{p}}} \restriction Q_{\mathbf{y}}$ is a quasi order.

Clause (c): $\leq^{\mathrm{pr}}_{\mathbf{y}}$ is a quasi order on $Q_{\mathbf{y}}$ and $p \leq^{\mathrm{pr}}_{\mathbf{y}} q \Rightarrow p \leq_{\mathbf{y}} q \Rightarrow p \leq_{\mathbf{p}} q$.

Why? The first half holds because if $p_1 \leq^{\mathrm{pr}}_{\mathbf{y}} p_2 \leq^{\mathrm{pr}}_{\mathbf{y}} p_3$ then: we should check that $p_1 \leq^{\mathrm{pr}}_{\mathbf{y}} p_3$, i.e., clauses $(\alpha), (\beta)$ of $\circledast_2$(e) of 2.10(3) hold. Now clause (α) is obvious. For clause (β) note $p_1 \leq^{\mathrm{pr}}_{\mathbf{p},\kappa} p_2 \leq^{\mathrm{pr}}_{\mathbf{p},\kappa} p_3$ and $\leq^{\mathrm{pr}}_{\mathbf{p},\kappa}$ is a partial order of $Q_{\mathbf{p}}$, so $p_1 \leq^{\mathrm{pr}}_{\mathbf{p},\kappa} p_3$, and hence (β) holds.

The second part of clause (c), which says $p \leq^{\mathrm{pr}}_{\mathbf{y}} q \Rightarrow p \leq_{\mathbf{y}} q$ (recalling Claim 2.7(a)(β)), holds by the definition of $\leq_{\mathbf{y}}, \leq^{\mathrm{pr}}_{\mathbf{y}}$ in $\circledast_2$(c),(e) of 2.10(3).

Clause (d)(α): $\mathrm{ap}_{\mathbf{y}}$ is a function with domain $Q_{\mathbf{y}}$.

Why? By $\circledast_2(f)$ of 2.10(3).

Clause (d)(β): If $q \in Q_{\mathbf{y}}$ then $q \in \mathrm{ap}_{\mathbf{y}}(q) \subseteq Q_{\mathbf{y}}$.

Why? By $\circledast_2$(f) of 2.10(3) trivially $\mathrm{ap}_{\mathbf{y}}(q) \subseteq Q_{\mathbf{y}}$. Also, we can check that $q \in \mathrm{ap}_{\mathbf{y}}(q)$: $q \in Q_{\mathbf{y}}$ by an assumption and $q \le^{\mathrm{ap}}_{\kappa} q$ as $\le^{\mathrm{ap}}_{\kappa}$ is a quasi order on $Q_{\mathbf{p}}$ and "$\mathrm{supp}_\kappa(q,q) \subseteq \mathrm{supp}_\theta(p_{\alpha_{\mathbf{y}}(q)}, q)$" trivially, because $\mathrm{supp}_\kappa(q,q) = \emptyset$.

Clause (d)(γ): If $r \in \mathrm{ap}_{\mathbf{y}}(q)$ and $q \in Q_{\mathbf{y}}$, then r, q are compatible in $\mathbb{Q}_{\mathbf{y}}$.

Why? As $r \in \mathrm{ap}_{\mathbf{y}}(q) \Rightarrow (q \le^{\mathrm{ap}}_{\kappa} r \wedge \{r,q\} \subseteq Q_{\mathbf{y}}) \Rightarrow q \le_{\mathbf{y}} r$.

Clause (d)$(\gamma)^+$: If $r \in \mathrm{ap}_{\mathbf{y}}(q)$ and $q \le^{\mathrm{pr}}_{\mathbf{y}} q^+$, then q^+, r are compatible in $(Q_{\mathbf{y}}, \le_{\mathbf{y}})$; moreover, there is $r^+ \in \mathrm{ap}_{\mathbf{Q}_{\mathbf{y}}}(q^+)$ such that

$$q^+ \Vdash_{\mathbb{Q}_{\mathbf{y}}} \text{"}r^+ \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}_{\mathbf{y}}} \Rightarrow r \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}_{\mathbf{y}}}\text{"}.$$

This follows from 2.11(3), by defining $s = r^+ = r \cup q^+$, which gives more.

Clause (e): $(Q_{\mathbf{y}}, \le^{\mathrm{pr}}_{\mathbf{y}})$ is $(< \partial_\theta)$-complete, recalling $\partial_\theta = \partial^\kappa$.

So assume $\langle q_\varepsilon : \varepsilon < \delta \rangle$ is $\le^{\mathrm{pr}}_{\mathbf{y}}$-increasing and δ is a limit ordinal $< \partial_\theta$; now $(Q_{\mathbf{p}}, \le^{\mathrm{pr}}_{\kappa})$ is $(< \partial^\kappa)$-complete by Claim 2.8(1) and $\langle q_\varepsilon : \varepsilon < \delta \rangle$ is also $\le^{\mathrm{pr}}_{\mathbf{p},\kappa}$-increasing by clause $\circledast_2$(e)(β) of Definition 2.10(3), hence $q_\delta := \bigcup\{q_\varepsilon : \varepsilon < \delta\}$ is a $\le^{\mathrm{pr}}_{\mathbf{p},\kappa}$-lub of the sequence by 2.8(1). Now $\langle \alpha_\varepsilon := \alpha_{\mathbf{y}}(q_\varepsilon) : \varepsilon < \delta \rangle$ is an $\le$-increasing sequence of ordinals $< \gamma_{\mathbf{y}}$ by Observation 2.11(2).

Also, by an assumption of 2.12(1), the ordinal $\gamma_{\mathbf{y}}$ is a successor ordinal or limit of cofinality $\ge \partial_\theta$, but then $\delta < \mathrm{cf}(\gamma_{\mathbf{y}})$. So in both cases $\alpha_* = \sup\{\alpha_\varepsilon : \varepsilon < \delta\}$ is an ordinal $< \gamma_{\mathbf{y}}$. But $\bar{p}^{\mathbf{y}}$ is $\le^{\mathrm{pr}}_{\mathbf{p},\kappa}$-increasing continuous, hence $p_{\alpha_*} = \bigcup\{p_{\alpha_\varepsilon} : \varepsilon < \delta\}$ and similarly $u_{\alpha_*} = \bigcup\{u_{\alpha_\varepsilon} : \varepsilon < \delta\}$. Now easily q_δ is a $\le^{\mathrm{ap}}_\theta$-extension of $p^{\mathbf{y}}_{\alpha_*}$, and $\mathrm{supp}_\theta(p^{\mathbf{y}}_{\alpha_*}, q_\delta) \subseteq \bigcup\{\mathrm{supp}_\theta(p_{\alpha_{\mathbf{y}}(q_\varepsilon)}, q_\varepsilon) : \varepsilon < \delta\} \subseteq \bigcup\{u_{\alpha_\varepsilon} : \varepsilon < \delta\} = u_{\alpha_*}$ which has card $< \partial_\theta$, hence $q_\delta \in Q_{\mathbf{y}}$. Easily q_δ is as required.

Clause (f): $(Q_{\mathbf{y}}, \le^{\mathrm{pr}}_{\mathbf{y}})$ satisfies the ∂^+_θ-c.c.

Why? Let $q_\varepsilon \in Q_{\mathbf{y}}$ for $\varepsilon < \partial^+_\theta$, so $\alpha_\varepsilon := \alpha_{\mathbf{y}}(q_\varepsilon)$ is well defined and, without loss of generality, $\langle \alpha_\varepsilon : \varepsilon < \partial^+_\theta \rangle$ is constant or increasing; also $p_{\alpha_\varepsilon} \le^{\mathrm{ap}}_\theta q_\varepsilon$, so by Definition 2.5 the set $\mathrm{supp}_\theta(p_{\alpha_{\mathbf{y}}(q_\varepsilon)}, q_\varepsilon)$ has cardinality $< \partial_\theta$, so by the Δ-system lemma, as in the proof of 2.8(3), there are $\varepsilon(1) < \varepsilon(2) < \partial^+_\theta$ such that:

- $(*)$ if $i_1 \in \mathrm{supp}_\theta(p_{\alpha_{\varepsilon(1)}}, q_{\varepsilon(1)})$ and $i_2 \in \mathrm{supp}_\theta(p_{\alpha_{\varepsilon(2)}}, q_{\varepsilon(2)})$, then:
  - (α) if $i_1 = i_2$ then $q_{\varepsilon(1)}(i) = q_{\varepsilon(2)}(i)$,
  - (β) if $i_1 E_\kappa i_2$ then $i_1, i_2 \in \mathrm{supp}_\theta(p_{\alpha_{\varepsilon(1)}}, q_{\varepsilon(1)}) \cap \mathrm{supp}_\theta(p_{\alpha_{\varepsilon(2)}}, q_{\varepsilon(2)})$.

So $\varepsilon(1) < \varepsilon(2), \alpha_{\varepsilon(1)} \le \alpha_{\varepsilon(2)}, p_{\alpha_{\varepsilon(1)}} \le^{\rm ap}_\theta q_{\varepsilon(1)}, p_{\alpha_{\varepsilon(2)}} \le^{\rm ap}_\theta q_{\varepsilon(2)}$.

Hence $q := q_{\varepsilon(1)} \cup q_{\varepsilon(2)}$ belongs to $Q_{\mathbf{p}}$, is a $\le^{\rm ap}_\theta$-lub of $\{q_{\varepsilon(1)}, q_{\varepsilon(2)}\}$ and $q_{\alpha_{\varepsilon(2)}} \le^{\rm ap}_\theta q$, hence $q \in Q_{\mathbf{y}}$. Also, if $i \in \mathrm{Dom}(q) \setminus \mathrm{Dom}(p_{\varepsilon(\ell)})$ then i/E_κ is disjoint to $\mathrm{Dom}(p_{\varepsilon(\ell)})$ by $(*)(\beta)$; this implies $p_{\varepsilon(\ell)} \le^{\rm pr}_\kappa q$, which means $p_{\varepsilon(\ell)} \le^{\rm pr}_{\mathbf{y}} q$ by 2.10(3)(e), for $\ell = 1, 2$, so $q_{\varepsilon(1)}, q_{\varepsilon(2)}$ are indeed compatible in $(Q_{\mathbf{y}}, \le^{\rm pr}_{\mathbf{y}})$.

Clause (g): If $\bar q = \langle q_\varepsilon : \varepsilon < \partial_\theta\rangle$ is $\le^{\rm pr}_{\mathbf{y}}$-increasing, then for stationarily many limit $\zeta < \partial_\theta$ the sequence $\bar q \restriction \zeta$ has an exact $\le^{\rm pr}_{\mathbf{y}}$-upper bound (recalling that ∂_θ here stands for θ in Definition 1.2).

Why? We prove more, that if $\mathrm{cf}(\zeta) = \partial_\kappa$ and $\langle q_\varepsilon : \varepsilon < \zeta\rangle$ is $\le^{\rm pr}_{\mathbf{y}}$-increasing then the union $q = \bigcup\{q_\varepsilon : \varepsilon < \zeta\}$ is an exact $\le^{\rm pr}_{\mathbf{y}}$-upper bound. This suffices as $\partial_\kappa < \partial_\theta$ and both are regular. Now by 2.11(2) the sequence $\langle \alpha_{\mathbf{y}}(q_\varepsilon) : \varepsilon < \zeta\rangle$ is $\le$-increasing, hence $\langle u_{\alpha_{\mathbf{y}}(q_\varepsilon)} : \varepsilon < \zeta\rangle$ is $\subseteq$-increasing and, letting $\alpha_* = \bigcup\{\alpha_{\mathbf{y}}(q_\varepsilon) : \varepsilon < \zeta\}$, we have $\alpha_* < \gamma_{\mathbf{y}}$ as $\gamma_{\mathbf{y}}$ is a successor ordinal or limit of cofinality $\ge \partial_\theta$; hence $u_{\alpha_*} = \bigcup\{u_{\alpha_{\mathbf{y}}(q_\varepsilon)} : \varepsilon < \zeta\}$; see 2.10(2)(c).

By the proof of clause (e) which we have proved above, clearly $q \in Q_{\mathbf{y}}$ and is a $\le^{\rm pr}_{\mathbf{y}}$-upper bound of $\langle q_\varepsilon : \varepsilon < \zeta\rangle$. But what about "exact"? We should check Definition 1.2(2). So assume $p \in \mathrm{ap}_{\mathbf{y}}(q)$, and we should prove that for some $\varepsilon < \zeta$ and $p' \in \mathrm{ap}_{\mathbf{y}}(q_\varepsilon)$ we have $\Vdash_{\mathbb{Q}_{\mathbf{y}}}$ "if $q, p' \in \utilde{\mathbf{G}}_{\mathbb{Q}_{\mathbf{y}}}$ then $p \in \utilde{\mathbf{G}}_{\mathbb{Q}_{\mathbf{y}}}$".

Note that $q \le^{\rm ap}_{\mathbf{p},\kappa} p$ and $\mathrm{supp}_\theta(q,p) \subseteq u_{\alpha_*}$ by the definition of $\mathrm{ap}_{\mathbf{y}}(q)$; hence $u := \mathrm{supp}_\kappa(q,p)$ is a subset of $\mathrm{supp}_\theta(q,p) \subseteq u^{\mathbf{y}}_{\alpha_*}$ of cardinality $< \partial_\kappa$. As $\langle u^{\mathbf{y}}_{\alpha_\varepsilon} : \varepsilon < \zeta\rangle$ is $\subseteq$-increasing with union $u^{\mathbf{y}}_{\alpha_*}$, necessarily for some $\varepsilon < \zeta$ we have $u \subseteq u_{\alpha_\varepsilon}$. Let $p' = p \restriction \mathrm{Dom}(p_\varepsilon)$, and check (as in earlier cases).

Clause (h): If $\langle q_\varepsilon : \varepsilon < \partial_\theta\rangle$ is $\le^{\rm pr}_{\mathbf{y}}$-increasing and $r_\varepsilon \in \mathrm{ap}_{\mathbf{y}}(q_\varepsilon)$ for $\varepsilon < \partial_\theta$ and $\xi < \partial_\theta$ then for some $\zeta < \partial_\theta$ we have $q_\zeta \Vdash_{\mathbb{Q}_{\mathbf{y}}}$ "if $r_\zeta \in \utilde{\mathbf{G}}_{\mathbb{Q}_{\mathbf{y}}}$ then $\xi \le \mathrm{otp}\{\varepsilon < \zeta : p_\varepsilon \in \utilde{\mathbf{G}}_{\mathbb{Q}_{\mathbf{y}}}\}$".

This follows from 2.8(3).

Clause (i): $\mathrm{ap}_{\mathbf{y}}(q)$ has cardinality $< \partial_\theta$.

Should be clear as $\alpha < \partial_\theta \Rightarrow |\alpha|^{<\partial_\kappa} < \partial_\theta$ by an assumption of the claim and $\alpha < \partial_\theta \Rightarrow |u_\alpha| < \partial_\theta$ (see 2.10(3)(f)) and the definition of $\mathrm{ap}_{\mathbf{y}}(q)$ in $\circledast_2$(e) of 2.10(3).

Let $\alpha = \alpha_{\mathbf{y}}(q)$, so $\alpha < \gamma_{\mathbf{y}}$ and

$$|\mathrm{ap}_{\mathbf{y}}(q)| = |\{s : q \le^{\rm ap}_\kappa s \text{ and } \mathrm{supp}_\kappa(q,s) \subseteq \mathrm{sup}_\theta(p_{\alpha_{\mathbf{y}}(q)}, q)\}| \le |\mathrm{supp}_\theta(p_{\alpha_{\mathbf{y}}(q)}, q)|^{<\kappa},$$

but $|\mathrm{supp}_\theta(p_{\alpha_{\mathbf{y}}(q)}, q)| < \partial_\theta$ and so, by an assumption of the claim, $|\mathrm{supp}_\theta(p_{\alpha_{\mathbf{y}}(q)}, q)|^{<\kappa} < \partial_\theta$, hence we are done.

Clause (j): Let $q_* \le_{\mathbf{y}} r$, so $\alpha \le \beta$ where $\alpha := \alpha_{\mathbf{y}}(q_*), \beta := \alpha_{\mathbf{y}}(r)$.

By 2.7(c) we can find a pair (q,p) such that $q_* \le^{\rm pr}_{\mathbf{p},\kappa} q \le^{\rm ap}_{\mathbf{p},\kappa} r, q_* \le^{\rm ap}_{\mathbf{p},\kappa} p \le^{\rm pr}_{\mathbf{p},\kappa} r$, $r = p \cup q$. Now check.

(2) Let $Q'' = \{p : p_{\alpha_*} \le^{\rm ap}_\theta p\}$. So clearly $Q'' \subseteq Q_{\mathbf{y}}$ and then $(\forall p \in Q_{\mathbf{y}})(\exists q \in Q'')[p \le_{\mathbf{y}} q]$, by clause (f) of Claim 2.7, i.e., Q'' is a dense subset of $Q_{\mathbf{y}}$ (by $\le_{\mathbb{Q}_{\mathbf{y}}} = \le_{\mathbb{Q}_{\mathbf{p}}} \restriction Q_{\mathbf{y}}$). Really $q_1 \in Q'' \wedge q_1 \le q_2 \in Q_{\mathbf{y}} \Rightarrow q_2 \in Q''$ by 2.11(2).

Suppose $\mathcal{I}$ is a dense open subset of $\mathbb{Q}_{\mathbf{y}}$, so $\mathcal{I}_1 := \mathcal{I} \cap Q''$ is dense in $\mathbb{Q}_{\mathbf{y}}$.

Let $\mathbf{G}$ be a subset of $\mathbb{Q}$ generic over $\mathbf{V}$ such that p_{α_*} belongs to it. If $\mathcal{I} \cap \mathbf{G} \ne \emptyset$ we are done, otherwise some $q_1 \in \mathbf{G}$ is incompatible (in $\mathbb{Q}$) with every $q \in \mathcal{I}$. As $\mathbf{G}$ is directed, there is $q_2 \in \mathbf{G}$ such that $p_{\alpha_*} \le q_2 \wedge q_1 \le q_2$. As $p_{\alpha_*} \le q_2$, by clause (c) of Claim 2.7 there is a $r_2 \in \mathbb{Q}$ such that $p_{\alpha_*} \le^{\rm ap}_\theta r_2 \le^{\rm pr}_\theta q_2$. So $r_2 \in Q''$, hence by the assumption on $\mathcal{I}$ there is $r_3 \in \mathcal{I}$ such that $r_2 \le r_3$. Now as $r_3 \in \mathcal{I}$, necessarily $p_{\alpha_*} \le^{\rm ap}_\theta r_3$ and, of course, $p_{\alpha_*} \le r_2 \le r_3$, hence by clause (k) of Claim 2.7 we have $r_2 \le^{\rm ap}_\theta r_3$. Recalling $r_2 \le^{\rm pr}_\kappa q_2$ it follows by clause (f) of 2.7 that there is $q_3 \in \mathbf{Q}$ such that $q_2 \le q_3 \wedge r_3 \le q_3$, hence $q_3 \Vdash$ "$\underset{\sim}{\mathbf{G}} \cap \mathcal{I} \ne \emptyset$" and $q_1 \le q_3$, contradicting the choice of q_1. $\blacksquare_{2.12}$

Claim 2.13: *If $\kappa \in \Theta \backslash \{\mu\}, \theta = \min(\Theta \backslash \kappa^+)$ and $\theta = \mu \Rightarrow \partial_\theta < \mu$ and $(\forall \alpha < \partial_\theta)[|\alpha|^{<\partial_\kappa} < \partial_\theta]$ and $\xi < \partial_\theta, \sigma < \partial_\theta$ then $\Vdash_{\mathbb{Q}_{\mathbf{p}}}$ "$\partial^+_\theta \to (\xi, (\xi;\xi)_\sigma)^2$".*

Proof. Let $\sigma < \partial_\theta$ and $\xi < \partial_\theta$ and we shall prove $\Vdash_{\mathbb{Q}_{\mathbf{p}}}$ "$\partial^+_\theta \to (\xi, (\xi;\xi)_\sigma)^2$". Toward this assume $\underset{\sim}{\mathbf{c}}$ is a $\mathbb{Q}_{\mathbf{p}}$-name and $q^* \in \mathbb{Q}_{\mathbf{p}}$ forces that $\underset{\sim}{\mathbf{c}}$ is a function from $[\partial^+_\theta]^2$ to $1+\sigma$. Now we shall apply Claim 2.8(4) with θ here standing for κ there. We choose (p_i, u_i) by induction on $i < \partial^+_\theta$ such that:

$\circledast_1$ (a) $p_i \in \mathbb{Q}_{\mathbf{p}}$ is $\le^{\rm pr}_\theta$-increasing continuous with i and $p_0 = q^*$,

(b) for every $i < j < \partial^+_\theta$ the set $\mathcal{I}_{i,j}$ is predense above p_{j+1} where
$$\mathcal{I}_{i,j} = \{r : p_{j+1} \le^{\rm ap}_\theta r \text{ and } r \text{ forces a value to } \underset{\sim}{\mathbf{c}}\{i,j\}\},$$

(c) moreover, $\mathcal{I}_{i,j}$ has a subset $\mathcal{I}'_{i,j}$ of cardinality $\le \partial_\theta$ which is predense over p_{j+1},

(d) u_i is $\subseteq$-increasing continuous and $u_i \subseteq \cup\{\alpha/E_\kappa : \alpha \in \mathrm{Dom}(p_i)\}$ and $|u_i| \le \partial_\theta$ for $i < \partial^+_\theta$,

(e) $\alpha \in u_i \Rightarrow (\alpha/E_\kappa) \subseteq u_i$,

(f) $q \in \mathcal{I}'_{i,j} \Rightarrow \mathrm{supp}_\kappa(p_{j+1}, q) \subseteq u_{j+1}$.

[Why is this possible? For $i = 0$ let $p_0 = q^*$, for i limit let $u_i = \bigcup\{u_j : j < i\}$ and $i < \partial^+_\theta$, and we like to apply 2.8(1) with κ there standing for θ here; so if $\partial_\theta^+ \le \partial^\theta$ this is fine, otherwise by 2.4(2)(h) necessarily $\theta = \mu \wedge \partial_\theta = \mu = 2^\theta$, contradicting an assumption. Lastly, if $i = \iota + 1$ then we have to deal with $\underset{\sim}{\mathbf{c}}\{\zeta, \iota\}$ for $\zeta < \iota$, i.e., with $\le \partial_\theta$ names of ordinals $< \sigma$. So we apply 2.8(4) with $(p_\iota, \iota, \langle \underset{\sim}{\mathbf{c}}(j, \iota) : j < \iota\rangle, \theta)$ here standing for $(p, A, \underset{\sim}{f}, \kappa)$ there and get $p_i, \langle \mathcal{I}_{j,\iota}, \mathcal{I}'_{j,\iota} : j < \iota\rangle$ here standing for $q, \langle \mathcal{I}_{q,\underset{\sim}{f},a}, \mathcal{I}'_{q,\underset{\sim}{f},a} : a \in A\rangle$ there. So the relevant parts of clauses (a),(b),(c) hold. Define u_i as in clauses (d),(e),(f), possible as $|\mathcal{I}'_{j,\iota}| \le \partial_\theta, r \in \mathcal{I}'_{j,\iota} \Rightarrow |\mathrm{supp}_\kappa(p_i, q)| \le \partial_\kappa < \partial_\theta$. So we are done carrying the induction.]

Let $\bar{p} = \langle p_i : i < \partial_\theta^+\rangle$ and $\bar{u} = \langle u_i : i < \partial_\theta^+\rangle$.

So this will help to translate the problem from the forcing $\mathbb{Q}$ to the forcing $\mathbb{Q}_{\mathbf{y}}$.

We define $\mathbf{y} = (\kappa, \langle p_\alpha : \alpha < \partial_\theta^+\rangle, \langle u_\alpha : \alpha < \partial_\theta^+\rangle)$, so:

$\circledast_2$ $\mathbf{y}$ is a reasonable $\mathbf{p}$-parameter.

[Why? Check, see Definition 2.10(2).]

$\circledast_3$ $\mathbf{Q}_{\mathbf{y}}$ is a $(\partial_\theta^+, \partial_\theta, < \partial_\theta)$-forcing.

[Why? By Claim 2.12(1).]

Now for $i < j < \partial_\theta^+$:

$(*)$ (a) $\mathcal{I}_{i,j}$ is predense in $\mathbb{Q}_{\mathbf{y}}$,
(b) if $q_1, q_2 \in \mathcal{I}_{i,j}$ or just $\in \mathbb{Q}_{\mathbf{y}}$, then q_1, q_2 are compatible in $\mathbb{Q}_{\mathbf{p}}$ iff they are compatible in $\mathbb{Q}_{\mathbf{y}}$.

[Why? The first clause (a) holds by our definitions. For the second clause (b), assume $q_1, q_2 \in \mathbb{Q}_{\mathbf{y}}$. If they are compatible in $\mathbb{Q}_{\mathbf{y}}$, then clearly they are compatible in $\mathbb{Q}_{\mathbf{p}}$. To show the other direction, let q be $q_1 \cup q_2$. If $q \in \mathbb{Q}_{\mathbf{y}}$ we are done, since $q_1, q_2 \le_{\mathbf{y}} q$. So let us prove that $q \in \mathbb{Q}_{\mathbf{y}}$. Denote $\alpha_1 = \alpha_{\mathbf{y}}(q_1), \alpha_2 = \alpha_{\mathbf{y}}(q_2)$ and, without loss of generality, $\alpha_1 \le \alpha_2$. So $p_{\alpha_1} \le_\theta^{\mathrm{ap}} q_1, p_{\alpha_2} \le_\theta^{\mathrm{ap}} q_2$ and also $p_{\alpha_1} \le_\theta^{\mathrm{pr}} p_{\alpha_2}$, and it follows from 2.7(f)(δ) that $p_{\alpha_2} \le_\theta^{\mathrm{ap}} q$. Moreover, $\mathrm{supp}_\theta(p_{\alpha_2}, q) \subseteq \mathrm{supp}_\theta(p_{\alpha_1}, q_1) \cup \mathrm{supp}_\theta(p_{\alpha_2}, q_2) \subseteq u_{\alpha_1} \cup u_{\alpha_2} = u_{\alpha_2}$. Together, $q \in \mathbb{Q}_{\mathbf{y}}$ and we are done.]

So we can define a $\mathbb{Q}_{\mathbf{y}}$-name $\underset{\sim}{\mathbf{c}}'$ as follows: for $q \in \mathbb{Q}_{\mathbf{y}}$

$$q \Vdash_{\mathbb{Q}_{\mathbf{y}}} \text{“}\underset{\sim}{\mathbf{c}}'\{i, j\} = t\text{” iff } q \Vdash_{\mathbb{Q}_{\mathbf{p}}} \text{“}\underset{\sim}{\mathbf{c}}\{i, j\} = t\text{”}.$$

So by $(*)$

$$\Vdash_{\mathbb{Q}_{\mathbf{y}}} \text{“}\underset{\sim}{\mathbf{c}}' : [\partial_\theta^+]^2 \to \sigma\text{”}.$$

Now by Claim 1.4, for some $\mathbb{Q}_{\mathbf{y}}$-name and a sequence $\langle \underset{\sim}{\alpha}_\varepsilon, \underset{\sim}{\beta}_\varepsilon : \varepsilon < \xi\rangle$ we have

$$\Vdash_{\mathbb{Q}_{\mathbf{y}}} \text{“the sequence } \langle \underset{\sim}{\alpha}_\varepsilon, \underset{\sim}{\beta}_\varepsilon : \varepsilon < \xi\rangle \text{ is as required in Definition 0.3} \\ \text{(for } \partial_\theta^+ \to (\xi, (\xi;\xi)_\sigma)^2\text{)”.}$$

So for each $\varepsilon < \xi$ there is a maximal antichain $\mathcal{J}_\varepsilon$ of $\mathbb{Q}_{\mathbf{y}}$ of elements forcing a value to $(\underset{\sim}{\alpha}_\varepsilon, \underset{\sim}{\beta}_\varepsilon)$ by $\mathbb{Q}_{\mathbf{y}}$.

But $\mathbb{Q}_{\mathbf{y}}$ satisfies the ∂_θ^+-c.c., so $|\mathcal{J}_\varepsilon| \le \partial_\theta$ hence, for some $\alpha_* < \partial_\theta^+$ we have:

(∗) $\mathcal{J}_\varepsilon \subseteq \{q : (\exists \alpha \le \alpha_*)(p_\alpha \le^{\rm ap}_{\mathbf{Q}} q)\}$ for any $\varepsilon < \xi$.

Recall that (by 2.12)

(∗) $p_{\alpha_*} \Vdash$ “$\underset{\sim}{\mathbf{G}}_{\mathbf{Q}} \cap Q_{\mathbf{y}\restriction(\alpha_*+1)}$ is a subset of $Q_{\mathbf{y}\restriction(\alpha_*+1)}$ generic over $\mathbf{V}$”,

so we are done. $\blacksquare_{2.13}$

Remark 2.14: (1) We can replace the exponent 2 by $n \ge 2$, so getting suitable polarized partition relations; we intend to continue elsewhere.

(2) For exact such results provable in ZFC, see [EHMR84] and [Sh:95].

3. Simultaneous partition relations and general topology

Recall (to simplify results we define $\mathrm{hL}^+(X) > \lambda >$ cf(λ) using an elaborate definition for regulars)

Definition 3.1: Let X be a topological space:

(a) the density of X is:
$d(X) = \min\{|S| : S \subseteq X \text{ and } S \text{ is dense in } X\}$,

(b) the hereditary density of X is:
$\mathrm{hd}(X) = \sup\{\lambda : X \text{ has a subspace of density} \ge \lambda\}$,

(c) $\mathrm{hd}^+(X) = \widehat{\mathrm{hd}}(X) = \sup\{\lambda^+ : X \text{ has a subspace of density} \ge \lambda\}$,

(d) X is not λ-Lindelof if there is a family $\{\mathcal{U}_\alpha : \alpha < \lambda\}$ of open susets of X whose union is X but $w \subseteq \lambda \wedge |w| < \lambda \Rightarrow \bigcup\{\mathcal{U}_\alpha : \alpha \in w\} \ne X$,

(e) the hereditarily Lindelof number of X is:
$\mathrm{hL}(X) = \widehat{\mathrm{hL}}(X) = \sup\{\lambda$: there are $x_\alpha \in X$ and $\mathcal{U}_\alpha \in$ open(X) for $\alpha < \lambda$, such that $x_\alpha \in \mathcal{U}_\alpha$ and $\alpha < \beta \Rightarrow x_\beta \notin \mathcal{U}_\alpha\}$,

(f) $\mathrm{hL}^+(X) = \sup\{\lambda^+$: there are $x_\alpha \in X, \mathcal{U}_\alpha$ for $\alpha < \lambda$ as above$\}$,

(g) the spread of X is $s(X) = \sup\{\lambda : X$ has a discrete subset with λ points$\}$; $s^+(X) = \hat{s}(X) = \sup\{\lambda^+ : X$ has a discrete subspace with λ points$\}$.

Our starting point was the following problem (0.1) of Juhasz–Shelah [JuSh:899].

Problem 3.2: Assume $\aleph_1 < \lambda < 2^{\aleph_0}$. Does there exist a hereditarily Lindelof Hausdorff regular space of density λ?

We answer negatively by a consistency result, but then look again at related problems on hereditary density, Lindelofness and spread; our main theorem is 3.10 getting consistency for all cardinals.

We also try to clarify the relationships of this and related partition relations to $\chi \to [\theta]^2_{2\kappa,2}$, recalling that by [Sh:276], consistently, e.g., $2^{\aleph_0} \to [\aleph_1]^2_{n,2}$ for $n < \omega$. Now, see 3.13 below, $2^{\aleph_0} \to [\aleph_1]_{2n,2}$ implies $2^{\aleph_0} \to (\aleph_1, (\aleph_1;\aleph_1)_n)^2$ and by 3.14 it implies $\gamma < \aleph_1 \Rightarrow 2^{\aleph_0} \to (\gamma)^2_n$; see on the consistency of this Baumgartner-Hajnal in [BH73], and Galvin in [Gal75].

On cardinal invariants in general topology, in particular, $s(X)$,$\mathrm{hd}(X)$,$\mathrm{hL}(X)$; see Juhasz [Juh80]. In particular, recall the obvious

Observation 3.3: *For a Hausdorff topological space X:*

(a) *$hL(X) \geq s(X)$,*
(b) *$hd(X) \geq s(X)$,*
(c) *for λ regular, X is hereditarily λ-Lindelof (i.e., every subspace is λ-Lindelof) iff there is $x_\alpha \in X, \mathcal{U}_\alpha$ for $\alpha < \lambda$ as in (e) of Definition 3.1,*
(d) *we choose the second statement in (c) as the definition of "X is hereditarily λ-Lindelof", in which case 3.7 and 3.9 hold also for λ singular.*

Conclusion 3.4: Assume $\lambda = \lambda^{<\lambda} < \mu = \mu^{<\mu}$ and G.C.H. holds in $[\lambda, \mu]$, so $\lambda \leq \theta = \mathrm{cf}(\theta) \leq \mu \Rightarrow \theta = \theta^{<\theta}$ and $\{\lambda, \mu\} \subseteq \Theta \subseteq \mathrm{Reg} \cap [\lambda, \mu]$ and, for $\theta \in \Theta$, we let $\partial_\theta = \theta$ and let $\mathbf{p} = (\lambda, \mu, \Theta, \langle \partial_\theta : \theta \in \Theta \rangle)$. Then

(a) $\mathbf{p}$ is as required in Hypothesis 2.1,
(b) the forcing notion $\mathbb{Q}_{\mathbf{p}}$ satisfies:
 (α) $\mathbb{Q}_{\mathbf{p}}$ is of cardinality μ,
 (β) $\mathbb{Q}_{\mathbf{p}}$ is $(< \lambda)$-complete (hence no new sequence of length $< \lambda$ is added),
 (γ) no cardinal is collapsed, no cofinality is changed,
 (δ) in $\mathbf{V}^{\mathbb{Q}_{\mathbf{p}}}$ we have $\lambda = \lambda^{<\lambda}, 2^\lambda = \mu$ and $\chi \notin [\lambda, \mu) \Rightarrow 2^\chi = (2^\chi)^{\mathbf{V}}$,
 (ε) if $\kappa < \theta$ are successive members of Θ and θ is not a successor of singular (or just $\theta = \chi^+ \Rightarrow \chi^{<\kappa} = \chi$) then $\lambda \to (\xi, (\xi;\xi)_\sigma)^2$ for any $\xi, \sigma < \theta$.

Proof. By 2.9 and 2.13. $\blacksquare_{3.4}$

The topological consequences from 3.4 in 3.5 hold by 3.7 and 3.9 below, that is

Conclusion 3.5: We can add in 3.4 that

(b)(ζ) if $\theta \in [\lambda, \mu) \cap \Theta$ is the successor of the regular κ, then for any Hausdorff regular topological space X we have $\mathrm{hd}(X) \geq \theta^+ \Rightarrow s^+(X) \geq \theta$ and also $\mathrm{hL}(X) \geq \theta^+ \Rightarrow s^+(X) \geq \theta$, so recalling $\theta = \kappa^+$ we have $\mathrm{hd}(X) \geq \theta^+ \Rightarrow \mathrm{hL}(X) \geq s(X) \geq \kappa$, $\mathrm{hL}(X) \geq \theta^+ \Rightarrow \mathrm{hd}(X) \geq s(X) \geq \kappa$,

(η) if $\theta \in (\lambda, \mu]$ is a limit cardinal, then $\mathrm{hd}(X) \geq \theta \vee \mathrm{hL}(X) \geq \theta \Rightarrow s(X) \geq \theta$.

Observation 3.6: (1) *If* $\lambda_1 \to (\xi_1; \xi_1)^2_{\kappa_1}$ *and* $\lambda_2 \geq \lambda_1, \xi_2 \leq \xi_1, \kappa_2 \leq \kappa_1$ *then* $\lambda_2 \to (\xi_2; \xi_2)^2_{\kappa_2}$.

(1A) *Similarly for* $\lambda \to (\xi, (\xi; \xi)_\kappa)^2$.

(2) *If* $\lambda \to (\xi, (\xi; \xi)_\kappa)^2$ *then* $\lambda \to (\xi; \xi)^2_{1+\kappa}$.

(3) $\lambda \to (\xi + \xi; \xi + \xi)^2_\kappa$ *implies* $(\lambda, \lambda) \to (\xi, \xi)^{1,1}_\kappa$, *the polarized partition.*

Claim 3.7: *X has a discrete subspace of size μ, i.e., $s^+(X) > \mu$ (hence is not hereditarily μ-Lindelof) when:*

(a) $\lambda \to (\mu, (\mu; \mu))^2$,

(b) *X is a Hausdorff; moreover, a regular ($= T_3$) topological space,*

(c) *X has a subspace of density $\geq \lambda$.*

Remark 3.8: The proofs of 3.7 and 3.9 are similar to older proofs.

Proof. X has a subspace Y with density $\geq \lambda$, by clause (c) of the assumption. We choose x_α, C_α by induction on $\alpha < \lambda$ such that

$\circledast$ (α) $x_\alpha \in Y$,

(β) $C_\alpha =$ the closure of $\{x_\beta : \beta < \alpha\}$,

(γ) $x_\alpha \notin C_\alpha$.

This is possible as Y has density $\geq \lambda$.

Let u^1_α be an open neighborhood of x_α disjoint to C_α.

Let u^2_α be an open neighborhood of x_α whose closure, $c\ell(u^2_\alpha)$, is $\subseteq u^1_\alpha$. Why does it exist? As X is a regular ($= T_3$) space.

We define $\mathbf{c} : [\lambda]^2 \to \{0, 1\}$ as follows:

($*$) if $\alpha < \beta$ then $\mathbf{c}\{\alpha, \beta\} = 1$ iff $x_\beta \in u^2_\alpha$.

By the assumption $\lambda \to (\mu, (\mu;\mu))^2$ at least one of the following cases occurs.

CASE 1: There is an increasing sequence $\langle \alpha_\varepsilon : \varepsilon < \mu \rangle$ of ordinals $< \lambda$ such that $\varepsilon < \zeta < \mu \Rightarrow \mathbf{c}\{\alpha_\varepsilon, \alpha_\zeta\} = 0$.

This means that $\varepsilon < \zeta < \mu \Rightarrow x_{\alpha_\zeta} \notin u^2_{\alpha_\varepsilon}$. But if $\varepsilon < \zeta < \mu$, then $u^2_{\alpha_\zeta}$ is an open neighborhood of x_{α_ζ} included in $u^1_{\alpha_\zeta}$ which is disjoint to C_{α_ζ} and $x_{\alpha_\varepsilon} \in C_{\alpha_\zeta}$, so $x_{\alpha_\varepsilon} \notin u^2_{\alpha_\zeta}$.

Lastly, $x_{\alpha_\varepsilon} \in u^2_{\alpha_\varepsilon}$ by the choice of $u^2_{\alpha_\varepsilon}$. Together we are done, i.e., $\langle (x_{\alpha_\varepsilon}, u^2_{\alpha_\varepsilon}) : \varepsilon < \mu \rangle$ is as required.

CASE 2: There is a sequence $\langle (\alpha_\varepsilon, \beta_\varepsilon) : \varepsilon < \mu \rangle$ such that:

$(*)_1$ $\varepsilon < \zeta < \mu \Rightarrow \alpha_\varepsilon < \beta_\varepsilon < \alpha_\zeta < \lambda$,

$(*)_2$ $\varepsilon < \zeta \Rightarrow \mathbf{c}\{\alpha_\varepsilon, \beta_\zeta\} = 1$; really $\varepsilon \leq \zeta$ suffices.

So

$(*)_3$ $\varepsilon < \zeta \Rightarrow x_{\beta_\zeta} \in u^2_{\alpha_\varepsilon}$,

but now for every $\varepsilon < \mu$ let

$(*)_4$ $y_\varepsilon := x_{\beta_{2\varepsilon}}$ and $u^3_\varepsilon := u^2_{\beta_{2\varepsilon}} \backslash c\ell(u^2_{\alpha_{2\varepsilon+1}})$.

So

(a) $u^3_\varepsilon = u^2_{\beta_{2\varepsilon}} \backslash c\ell(u^2_{\alpha_{2\varepsilon+1}})$ is open (as open minus closed),

(b) $y_\varepsilon \in u^3_\varepsilon$.

[Why? Recall $y_\varepsilon = x_{\beta_{2\varepsilon}}$ belongs to $u^2_{\beta_{2\varepsilon}}$ (by the choice of $u^2_{\beta_{2\varepsilon}}$) and not to $u^1_{\alpha_{2\varepsilon+1}}$ (as $u^1_{\alpha_{2\varepsilon+1}}$ is disjoint to $C_{\alpha_{2\varepsilon+1}}$ while $x_{\beta_{2\varepsilon}} \in C_{\alpha_{2\varepsilon+1}}$), hence not to $c\ell(u^2_{\alpha_{2\varepsilon+1}})$, being a subset of $u^1_{\alpha_{2\varepsilon+1}}$. Together y_ε belongs to $u^2_{\beta_{2\varepsilon}} \backslash c\ell(u^2_{\alpha_{2\varepsilon+1}}) = u^3_\varepsilon$.]

(c) If $\varepsilon < \zeta < \mu$ then $y_\zeta \notin u^3_\varepsilon$.

[Why? Now $y_\zeta = x_{\beta_{2\zeta}}$ belongs to $u^2_{\alpha_{2\varepsilon+1}}$ by $(*)_3$ as $2\varepsilon + 1 < 2\zeta$, which follows from $\varepsilon < \zeta$, hence y_ζ belongs to $c\ell(u^2_{\alpha_{2\varepsilon+1}})$, hence $y_\zeta \notin u^3_\varepsilon$ by the definition of u^3_ε.]

(d) If $\zeta < \varepsilon < \mu$ then $y_\zeta \notin u^3_\varepsilon$.

[Why? As $u^3_\varepsilon \subseteq u^2_{\beta_{2\varepsilon}}$ and the latter is disjoint to $C_{\beta_{2\varepsilon}}$, to which $x_{\beta_{2\zeta}} = y_\zeta$ belongs.]

Together $\langle (y_\varepsilon, u^3_\varepsilon) : \varepsilon < \mu \rangle$ exemplifies that we are done. $\blacksquare_{3.7}$

CLAIM 3.9: *X has a discrete subspace of size μ when:*

(a) $\lambda \to (\mu, (\mu;\mu))^2$,

(b) *X is a Hausdorff; moreover, a regular ($= T_3$) topological space,*

(c) $\mathrm{hL}^+(X) > \lambda$, *i.e., if* λ *is a regular cardinal this means that* X *is not hereditarily* λ*-Lindelof.*

Proof. Similar to 3.7. We choose $\langle (x_\alpha, u^1_\alpha) : \alpha < \lambda\rangle$ such that u^1_α is an open subset of $X, x_\alpha \in u^1_\alpha$ and $u^1_\alpha \cap \{x_\beta : \beta \in (\alpha,\lambda)\} = \emptyset$. We can choose them as $\mathrm{hL}^+(X) > \lambda$. We then choose an open neighborhood u^2_α of x_α such that $c\ell(u^2_\alpha) \subseteq u^1_\alpha$. We then define $\mathbf{c} : [\lambda]^2 \to \{0,1\}$ as follows:

(∗) if $\alpha < \beta$ then $\mathbf{c}\{\alpha,\beta\} = 1$ iff $x_\alpha \in u^2_\beta$.

We continue as in the proof of 3.7, but now, in Case 2,

$(*)'_3$ $\varepsilon < \zeta \Rightarrow x_{\alpha_\varepsilon} \in u^2_{\beta_\zeta}$,

and let

$(*)'_4$ $y_\varepsilon := x_{\alpha_{2\varepsilon}}, u^3_\varepsilon := u^2_{\alpha_{2\varepsilon}} \backslash c\ell(u^2_{\beta_{2\varepsilon+1}})$. $\blacksquare_{3.9}$

Now we come to our main result.

Theorem 3.10 (The Main Theorem): *It is consistent (using no large cardinals) that:*

(∗) (α) 2^μ *is* μ^+ *if* μ *is strong limit singular and always* 2^μ *is the successor of a singular cardinal,*

(β) *for every* μ *we have* $\mu \le \chi < 2^\mu \Rightarrow 2^\chi = 2^\mu$,

(γ) $\mathrm{hd}(X) \ge \theta \Leftrightarrow \mathrm{hL}(X) \ge \theta \Leftrightarrow \mathrm{s}(X) \ge \theta$ *for any limit cardinal* θ *and Hausdorff regular* $(= T_3)$ *topological space* X,

(δ) $\mathrm{hd}(X) \le \mathrm{s}(X)^{+3}$ *and* $\mathrm{hL}(X) \le \mathrm{s}(X)^{+3}$ *for any Hausdorff regular* $(= T_3)$ *topological space,*

(ε) *in* (δ) *we can replace* $\mathrm{s}(X)^{+3}$ *by* $\mathrm{s}(X)^{+2}$, *except when* $2^{s(X)}$ *is regular,*

(ζ) *in particular, if* X *is a (Hausdorff regular topological space which is) Lindelof or of countable density or just* $s(X) = \aleph_0$ *then*

$$hd(X) + \mathrm{hL}(X) \le \aleph_2,$$

(η) *if* X *is a Hausdorff space*[6] *then* $|X| < 2^{(\mathrm{hd}(X)^+)}$,

(θ) *if* X *is a Hausdorff space then* $w(X) \le 2^{(\mathrm{hL}(X)^+)}$,

(ι) *if* $2^\mu > \mu^+$ *then* $\mu^{++} \to (\xi, (\xi;\xi)_\mu)^2$ *for* $\xi < \mu^+$.

Remark 3.11: In Theorem 3.10 above:

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[6] This is interesting because usually $2^\chi = 2^{(\chi^+)}$; see clause (α).

(1) If we use less sharp results in §§1–3, above we should just use $(\mathrm{hd}(X))^{+n(*)}$ for large enough $n(*)$.

(2) We may like to improve clause (η) to $\leq 2^{\mathrm{hd}(X)}$. If below we choose $\mu_{\varepsilon+1}$ strongly inaccessible (so we need to assume $\mathbf{V} \models$ "there are unboundedly many strong inaccessible cardinals and clause (α) is changed"), nothing is lost; we have $\lambda_{\varepsilon+1} = \mu_{\varepsilon+1}$, then we can add

$(\eta)^+$ for any Hausdorff space $X, |X| < 2^{\mathrm{hd}(X)}$ except (possibly) when $\mathrm{hd}(X)$ is strong limit singular.

(3) Similarly of clause (θ) about $w(X) \leq 2^{\mathrm{hL(X)}}$.

(4) Probably using large cardinals we can eliminate also the exceptional case in $(\eta)^+$; it seemed that a similar situation is that in Cummings–Shelah [CuSh:541], but we have not looked into this.

(5) We may wonder whether in clause (ζ) we can replace $\aleph_2$ by $\aleph_1$ and similarly for other cardinals; hopefully see [Sh:F884].

Proof. We can assume $\mathbf{V}$ satisfies G.C.H. We choose $\langle(\lambda_\varepsilon, \mu_\varepsilon) : \varepsilon \text{ an ordinal}\rangle$ such that:

⊛ (a) $\lambda_0 = \mu_0 = \aleph_0$,
 (b) $\lambda_\varepsilon < \mathrm{cf}(\mu_{\varepsilon+1}) < \mu_{\varepsilon+1}$,
 (c) $\lambda_{\varepsilon+1}$ is the first regular $\geq \mu_{\varepsilon+1}$,
 (d) for limit ε we have λ_ε is the first regular cardinal $\geq \mu_\varepsilon := \bigcup\{\lambda_\zeta : \zeta < \varepsilon\}$.

Now let $\mathbf{p}_\varepsilon = (\lambda_\varepsilon, \lambda_{\varepsilon+1}, \Theta_\varepsilon, \bar{\partial}_\varepsilon)$, where $\Theta_\varepsilon, \bar{\partial}_\varepsilon$ are defined by $\Theta_\varepsilon = \mathrm{Reg} \cap [\lambda_\varepsilon, \lambda_{\varepsilon+1}]$, $\bar{\partial}_\varepsilon = \langle \partial^\varepsilon_\theta : \theta \in \Theta_\varepsilon\rangle, \partial^\varepsilon_\theta = \theta$, so are chosen as in 3.4.

Hence $\langle \mathbf{p}_\varepsilon : \varepsilon \text{ an ordinal}\rangle$ is a class. We define an Easton support iteration $\langle \mathbb{P}_\varepsilon, \underset{\sim}{\mathbb{Q}}_\varepsilon : \varepsilon \in \mathrm{Ord}\rangle$, so $\bigcup\{\mathbb{P}_\varepsilon : \varepsilon \in \mathrm{Ord}\}$ is a class forcing, choosing the $\mathbb{P}_\varepsilon$-name $\underset{\sim}{\mathbb{Q}}_\varepsilon$ such that $\Vdash_{\mathbb{P}_\varepsilon}$ "$\underset{\sim}{\mathbb{Q}}_\varepsilon = \mathbb{Q}_{\mathbf{p}_\varepsilon}$, i.e., $\underset{\sim}{\mathbb{Q}}_\varepsilon$ is defined as in Definition 2.5 for the parameter $\mathbf{p}_\varepsilon$ (in the universe $\mathbf{V}^{\mathbb{P}_\varepsilon}$ of course)".

As in $\mathbf{V}^{\mathbb{P}_\varepsilon}$, section two is applicable for $\mathbf{p}_\varepsilon$, so in $\mathbf{V}^{\mathbb{P}_{\varepsilon+1}}$, the conclusions of 3.4 and 3.5 hold and $2^{\lambda_\varepsilon} = \lambda_{\varepsilon+1}$, so cardinal arithmetic should be clear, in particular, clause (α) holds. Of course, forcing with $\mathbb{P}_\infty/\mathbb{P}_{\varepsilon+1}$ does not change those conclusions as it is $\lambda_{\varepsilon+1}$-complete.

In $\mathbf{V}^{\mathbb{P}_\infty}$ we have enough cases of $\theta^+ \to (\xi, (\xi;\xi))^2$, i.e., clause (γ) by 2.13. So, first, if $\chi \geq \mathrm{s}(X)$ belongs to $[\lambda_\varepsilon, \mu_{\varepsilon+1})$ and is regular we have $\chi^{+2} \to (\chi; (\chi;\chi))^2$

and $\mathrm{hd}(X)$, $\mathrm{hL}(X) \leq \chi^{+2}$. But if $s(X) \in [\lambda_\varepsilon, \mu_{\varepsilon+1})$ then $s(X)^+ < \mu_{\varepsilon+1}$ recalling μ_ε is singular hence $\mathrm{hd}(X), \mathrm{hL}(X) \leq s(X)^{+3} < \mu_{\varepsilon+2}$

Second, if $\chi = s(X)$ belongs to no such interval, then $\chi^+ = \lambda_\varepsilon, \chi = \mu_\varepsilon > \mathrm{cf}(\mu_\varepsilon)$ for some ε, hence recalling $\lambda_\varepsilon = \lambda_\varepsilon^{<\lambda_\varepsilon} = 2^\chi$ (in $\mathbf{V}^{\mathbb{P}_\infty}$) we have the conclusion. So clause (δ) follows, hence also clauses (γ) and (ε).

Let us deal with clause (η); set $\chi = \mathrm{hd}(X)$. First, if $\chi \in [\lambda_\varepsilon, \mu_{\varepsilon+1})$ we get $\mathrm{hL}(X) \leq \chi^{+3} < \mu_{\varepsilon+1}$, hence $|X| \leq 2^{\chi^{+3}} = 2^\chi$ by the classical inequality of de-Groot ($|X| \leq 2^{\mathrm{hL}(X)}$; see [Juh80]). Second, if χ belongs to no such interval, then $\chi = \mu_\varepsilon \wedge \chi^+ = \lambda_\varepsilon, 2^{\mu_\varepsilon} = 2^\chi$ for some ε. So $|X| \leq 2^{2^{\mathrm{hL}(X)}} \leq 2^{2^\chi} = 2^{\chi^+}$ as required.

Clause (θ) is proved similar. $\blacksquare_{3.10}$

Theorem 3.12: *If in* $\mathbf{V}$ *there is a class of (strongly) inaccessible cardinals, then in some forcing extension:*

- $(*)$ (α) 2^μ *is* μ^+ *when* μ *is a strong limit singular cardinal and is a weakly inaccessible cardinal otherwise,*
- $(*)$ $(\beta) - (\iota)$ *as in Theorem 3.10.*

Proof. As in the proof of Theorem 3.10. $\blacksquare$

Claim 3.13: *Assume* $\chi \to [\theta]^2_{2\kappa,2}$, *where* $\kappa \geq 2, \chi \leq 2^\lambda$ *and* $\lambda = \lambda^{<\lambda} < \theta = \mathrm{cf}(\theta)$. *Then* $\chi \to (\theta, (\theta;\theta)_\kappa)^2$.

Proof. Let $\mathbf{c} : [\chi]^2 \to \kappa$ be given.

Let $\eta_\alpha \in {}^\lambda 2$ for $\alpha < \chi$ be pairwise distinct. We define $\mathbf{d} : [\chi]^2 \to 2\kappa$ by: for $\alpha < \beta < \chi$ let $\mathbf{d}\{\alpha,\beta\}$ be $2\varepsilon + \ell$ when $\mathbf{c}\{\alpha,\beta\} = \varepsilon$ and $\ell = 1$ iff $\ell \neq 0$ iff $\eta_\alpha <_{\mathrm{lex}} \eta_\beta$ (i.e., $\eta_\alpha(\ell g(\eta_\alpha \cap \eta_\beta)) < \eta_\beta(\ell g(\eta_\alpha \cap \eta_\beta))$. As we are assuming $\chi \to [\theta]^2_{2\kappa,2}$ there is $\mathcal{U} \in [\chi]^\theta$ such that $\mathrm{Rang}(\mathbf{d} \restriction [\mathcal{U}]^2)$ has ≤ 2 members; without loss of generality $\mathrm{otp}(\mathcal{U}) = \theta$. If the number of members of $\mathrm{Rang}(\mathbf{d} \restriction [\mathcal{U}]^2)$ is one we are done, so assume it is $\{2\varepsilon_0 + \ell_0, 2\varepsilon_1 + \ell_1\}$ where $\varepsilon_0, \varepsilon_1 < \kappa$ and $\ell_0, \ell_1 < 2$. But we cannot have $\ell_1 = \ell_2$ by the Sierpinski coloring properties as $\theta > \lambda$, hence without loss of generality $\ell_0 = 0, \ell_1 = 1$. If $\varepsilon_0 = \varepsilon_1 = 0$ we are done, as then Case $(c)_0$ of Definition 0.2(2) holds, so assume $\ell \in \{0,1\} \Rightarrow \varepsilon_\ell \neq 0$. Let $\Lambda = \{\eta \in {}^{\lambda>}2 :$ for θ ordinals $\alpha \in \mathcal{U}$ we have $\eta \triangleleft \eta_\alpha\}$. Now Λ has two $\triangleleft$-incomparable members (otherwise we get a contradiction by $\mathrm{cf}(\theta) > \lambda$), say $\nu_0, \nu_1 \in \Lambda$ are $\triangleleft$-incomparable, and without loss of generality $\nu_0 <_{\mathrm{lex}} \nu_1$.

So

($*$) if $\nu_0 \trianglelefteq \eta_\alpha$ and $\nu_1 \triangleleft \eta_\beta$ and $\alpha < \beta$, then $\mathbf{c}\{\alpha, \beta\} = \varepsilon_0$,

($*$) if $\nu_1 \triangleleft \eta_\alpha, \nu_0 \triangleleft \eta_\alpha$ and $\alpha < \beta$, then $\mathbf{c}\{\alpha, \beta\} = \varepsilon_1$.

As θ is regular and $\mathrm{otp}(\mathcal{U}) = \theta$, we can choose $\alpha_\varepsilon, \beta_\varepsilon$ by induction on $\varepsilon < \theta$ such that:

$\odot$ (a) $\alpha_\varepsilon \in \mathcal{U}$ and $\alpha_\varepsilon > \sup\{\beta_\zeta : \zeta < \varepsilon\}$,
(b) $\nu_0 < \eta_{\alpha_\varepsilon}$,
(c) $\beta_\varepsilon \in \mathcal{U}$ is $> \alpha_\varepsilon$,
(d) $\nu_1 \triangleleft \eta_{\beta_\alpha}$.

So Case (c)$_1$ of Definition 0.2(2) holds. We are done. $\blacksquare_{3.13}$

We can note also

Claim 3.14: *Assume $\lambda = \lambda^{<\lambda} < \mathrm{cf}(\theta)$ and $\chi \leq 2^\lambda$ and $\chi \rightarrow [\theta]^2_{2\kappa,2}$. Then for every ordinal $\gamma < \lambda^+$ we have $\chi \rightarrow (\gamma)^2_\kappa$.*

Proof. Without loss of generality $\kappa \geq 2$.

So let $\mathbf{c} : [\chi]^2 \rightarrow \kappa$. Choose $\langle \eta_\alpha : \alpha < \chi \rangle$ and $\mathbf{d}$ as in the proof of 3.13, and let $\mathcal{U} \subseteq \chi$ of order type θ and $\{2\varepsilon_0, 2\varepsilon_1 + 1\}$ be as there, so $\varepsilon_0, \varepsilon_1 < \kappa$.

As $\{\eta_\alpha : \alpha \in \mathcal{U}\}$ is a subset of ${}^{\lambda >}2$ of cardinality $\theta > \lambda = \lambda^{<\lambda}$, clearly (e.g., prove by induction on $\gamma < \lambda^+$ that) for every such $\mathcal{U}$ there is $\mathcal{U}' \subseteq \mathcal{U}$ of order type γ such that $\langle \eta_\alpha : \alpha \in \mathcal{U}' \rangle$ is $<_{\mathrm{lex}}$-increasing. So $\mathcal{U}'$ is as required, i.e., $\mathbf{c} \restriction [\{\eta_\alpha : \alpha \in \mathcal{U}'\}]^2$ is constantly ε_0 (of course also ε_1 is O.K. if we use $<_{\mathrm{lex}}$-decreasing sequence). $\blacksquare_{3.14}$

Remark 3.15: If we use versions of $\chi \rightarrow [\theta]^2_{\kappa,2}$ with privilege positions for the value 0, we can get corresponding better results in 3.13 and 3.14.

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