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THERE MAY BE SIMPLE $P_{\aleph_1}$- and $P_{\aleph_2}$-POINTS AND THE RUDIN–KEISLER ORDERING MAY BE DOWNWARD DIRECTED

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We prove the consistency, relative to ZFC, of each of the following two (mutually contradictory) statements. (A) Every two non-principal ultrafilters on ω have a common image via a finite-to-one function. (B) Simple $P_{\aleph_1}$-points and simple $P_{\aleph_2}$-points both exist. These results, proved by the second author, answer questions of the first author and P. Nyikos, who had obtained numerous consequences of (A) and (B), respectively. In the models we construct, the bounding number is $\aleph_1$, while the dominating number, the splitting number, and the cardinality of the continuum are $\aleph_2$.

1. Introduction

The purpose of this paper is to prove the consistency, relative to ZFC, of each of the following two statements.

(A) *If $\mathcal{U}_1$ and $\mathcal{U}_2$ are non-principal ultrafilters on ω, then $f(\mathcal{U}_1) = f(\mathcal{U}_2)$ for some finite-to-one $f: \omega \to \omega$.*

(B) *There exist both a simple $P_{\aleph_1}$-point and a simple $P_{\aleph_2}$-point.*

For any regular uncountable κ, a *simple P_κ-point* is an ultrafilter (by which we always mean a non-principal ultrafilter on ω) generated by an almost decreasing (i.e. decreasing modulo finite sets) κ-sequence of subsets of ω. Clearly, every simple P_κ-point $\mathcal{U}$ is a *P-point*, i.e., for any countably many sets $A_n \in \mathcal{U}$, there is a set in $\mathcal{U}$ almost included in every A_n.

It is easy to check that, if $\mathcal{U}$ is a simple P_κ-point, then so is $f(\mathcal{U})$ for any $f: \omega \to \omega$. It is also easy to check that no ultrafilter can be a simple P_κ-point for two different values of κ. Thus, the models for (A) and for (B) cannot be the same. Nevertheless, the ideas involved in the two consistency proofs are very similar.

To discuss the origin of the statements (A) and (B) as well as the nature of our models for them, we must recall the definitions of three of the cardinal invariants

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of the continuum discussed in [11]: the dominating number, the bounding number, and the splitting number. A family $\mathcal{F}$ of functions $\omega \to \omega$ is *dominating* (resp. *unbounded*) if, for each $g: \omega \to \omega$ there exists an $f \in \mathcal{F}$ such that $g(n) < f(n)$ for all but finitely many (resp. for infinitely many) n. A family $\mathcal{S}$ of subsets of ω is *splitting* if, for each infinite $A \subseteq \omega$, there exists an $S \in \mathcal{S}$ such that both $A \cap S$ and $A - S$ are infinite. The dominating, bounding, and splitting numbers are the smallest cardinalities, d, b, and s, of dominating, unbounded, and splitting families. Both b and s are uncountable and $\leq d \leq 2^{\aleph_0}$ (see [11]), and it was shown in [8] that the relative order of b and s is independent of ZFC. The models constructed in this paper satisfy $b < s$ and thus provide an alternate proof for one of the main results of [8]. Nyikos has shown that $b < s$ is implied by (B); that the same inequality also holds in our model of (A) seems to be due merely to the similarity of the constructions of our two models.

The statement (A), or rather its negation, arose in the work of Blass and Weiss [2] on decompositions of ideals of Hilbert space operators. It was shown there that, if one could refute (A) in ZFC, then one could also eliminate the continuum hypothesis from the main theorem of that paper. Interest in (A) increased when van Douwen [10] pointed out that the negation of (A) was sufficient to prove, as Rudin [6] had done using the continuum hypothesis, that the indecomposable continuum $\beta([0, 1)) - [0, 1)$ has more than one composant. Since then, Blass [1] has proved the converses of these results (the latter was also obtained by Mioduszewski [3]) and several other equivalences involving (A). One of the equivalent formulations of (A) is that every non-principal ultrafilter on ω has an image, via a finite-to-one function, generated by fewer than d sets. The consistency of $u < d$, where u is the minimal number of sets that can generate an ultrafilter, was proved, using a different model, by the second author shortly before finding the proof presented here. That model is, for the study of u and d, preferable to the present one in that it allows u and d to be prescribed cardinals. On the other hand, it will not do for our present purpose, as M. Canjar has pointed out that it does not satisfy (A). The consistency of $u < d$ contrasts with Solomon's theorem [9] that no non-principal ultrafilter on ω is generated by fewer than b sets. In particular, (A) implies $b < d$.

Statement (B), or rather the statement that simple P_κ-points exist for two distinct cardinals κ, is the strongest of several statements whose consistency was asked about by Nyikos [4]. The question arises naturally from his theorem that, if a simple P_κ-point exists, then $\kappa = b$ or $\kappa = d$. Nyikos [5] has deduced from (B) or from weaker statements numerous consequences concerning the eventual-domination ordering on the set of functions $\omega \to \omega$.

The plan of this paper is as follows. In Section 2, we introduce a notion of forcing Q and establish some of its combinatorial properties, including properness and a partition theorem. In Section 3, we study the generic extension produced by forcing with Q. In particular, we show that, if there is a P-point in the ground model, then it generates a P-point in the extension and every two ultrafilters in the ground model have a common image via a finite-to-one function in the extension. In Section 4, we establish that countable-support interation of proper forcing preserves the property, "P-points in the ground model generate P-points in the extension", that was previously established for Q. Using this, we show in Section 5 that an $\aleph_2$-length countable-support iteration of Q forcing, over a ground model satisfying CH, yields a model of (A). Finally, in Section 6, we show how to modify the construction so as to produce a model of (B) instead.

Our notation is standard, except for the following points. Names in forcing languages are often denoted by boldface symbols, like $\mathbf{x}$. The corresponding lightface symbol then denotes the value of this name with respect to some generic subset of the forcing notion. When necessary, we shall be more explicit, writing $\mathbf{x}^{(G)}$ for the value of $\mathbf{x}$ with respect to the generic set G. This notation will also be used for partial evaluation of names in the context of iterated forcing. Thus, if $\mathbf{x}$ is a $P * \mathbf{Q}$-name and G is a V-generic subset of P, then $\mathbf{x}^{(G)}$ is the Q-name such that, for all $V[G]$-generic $H \subseteq Q$, $\mathbf{x}^{(G)(H)} = \mathbf{x}^{(G*H)}$.

We often let χ be an unspecified regular cardinal so large that $H(\chi)$, the collection of sets of hereditary cardinality $<\chi$, contains all sets of interest to us. In this situation, we often consider (countable) elementary submodels N of $H(\chi)$, by which we mean that $(N, \in)$ is an elementary submodel of $(H(\chi), \in)$. For all practical purposes, one can think of $H(\chi)$ as being the whole universe V; the reason for introducing $H(\chi)$ is that the concept of elementary submodel of V cannot be formalized in the usual language of set theory.

The theorems proved in this paper are due to the second author. The first author's contribution was to fill in some details, to ask the second author to fill in other details, and to write the paper.

2. The basic forcing

We shall construct a model for (A) by iterating, $\aleph_2$ times, with countable support, a certain forcing Q, which we introduce in this section. The essential properties of Q are (a) that it is proper, (b) that P-points in the ground model generate P-points in the generic extension, and (c) that it adjoins a finite-to-one $f:\omega \to \omega$ such that all ultrafilters in the ground model have the same f-image, provided a P-point exists in the ground model. In addition, Q has the property, inessential for (A) but essential for (B), (d) that it adjoins an infinite subset W of ω that is not split into two infinite pieces by any set in the ground model; this property will ensure that $s = \aleph_2$ in the iterated forcing model. The sets on which the function f of (c) is constant will be the intervals determined by successive elements of the set W of (d), so we think of forcing with Q as simply adjoining W. In this section, we define Q, verify (a), and establish a combinatorial lemma about Q. The proofs of (b), (c), and (d) will occupy Section 3.

For natural numbers $n < m$, let $K_{n,m}$ be the set of all binary relations

$t \subseteq \mathscr{P}(n) \times \mathscr{P}(m)$ such that, for each $a \subseteq n$,

$$(a, a) \in t, \quad \text{and} \quad \text{if } (a, b) \in t, \text{ then } b \cap n = a.$$

We think of each $t \in K_{n,m}$ as specifying, for each $a \subseteq n$, some permisible extensions of a to subsets of m, each extension being obtained by adjoining to a some members of $[n, m)$; the first requirement on t above says that adjoining nothing is always permissible. If $t \in K_{n,m}$ and $s \in K_{m,l}$, then we write ts for the ordinary composition of these binary relations, so $ts \in K_{n,l}$.

For $t \in K_{n,m}$ and $Y \subseteq [n, m)$, we define $t_Y \in K_{n,m}$ by

$$t_Y = \{(a, b) \in t \mid b - a \subseteq Y\}.$$

We define the *depth* $\mathrm{Dp}(t)$ of each $t \in K_{n,m}$ by the following induction

$\mathrm{Dp}(t) \geqslant 0$ always.

$\mathrm{Dp}(t) \geqslant 1$ if, for every $a \subseteq n$, there is $b \subseteq m$ such that $(a, b) \in t$ and $b \neq a$.

$\mathrm{Dp}(t) \geqslant d+1$, for $d \geqslant 1$, if, for every partition $\{Y, Z\}$ of $[n, m)$, at least one of $\mathrm{Dp}(t_Y)$ and $\mathrm{Dp}(t_Z)$ is $\geqslant d$.

Thus, t has depth $\geqslant d+1$ if and only if we have a winning strategy in the game played as follows. There are d moves. At each move our opponent partitions $[n, m)$ into two pieces and we choose one of the pieces. After d moves, let Y be the intersection of all the sets we chose. We win if and only if, for every $a \subseteq n$, t_Y contains (a, b) for some $b \neq a$. (Note that the game would be unchanged if, from the second move on, our opponent partitioned the set we had just chosen rather than $[n, m)$.) This game interpretation of depth makes the following lemmas quite easy.

Lemma 2.1. *For any $t \in K_{n,m}$ and $s \in K_{m,l}$,*

$$\max\{\mathrm{Dp}(t), \mathrm{Dp}(s)\} \leqslant \mathrm{Dp}(ts) \leqslant 1 + \max\{\mathrm{Dp}(t), \mathrm{Dp}(s)\}.$$

Proof. For the first inequality, we adopt a strategy of playing the game for ts by using our winning strategy in the game for the deeper of t and s, while ignoring the other 'half' of $[n, l)$. To prove the second inequality, consider what happens if our opponent's first move is to partition $[n, l)$ into $[n, m)$ and $[m, l)$. □

Lemma 2.2. *Let $t \in K_{n,m}$ have depth $>2d+1$, let $w \subseteq n$, and let $\mathscr{P}(m)$ be the union of two pieces. Then for at least one of the pieces, say X, the relation*

$$t' = \{(a, b) \in t \mid \textit{if } a = w \textit{ and } b \neq a, \textit{ then } b \in X\}$$

has depth $\geqslant d+1$.

Proof. Let the two pieces be X_0 and X_1, and suppose the conclusion fails for both

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of the corresponding relations t'_0 and t'_1. So we do not have a winning strategy in either of the associated d-move games described above. Since the games are finite, our opponent has winning strategies, say σ_0 and σ_1. Suppose our opponent plays the $2d$-move game for t by using σ_0 for the first d moves and then using σ_1 for the last d moves (as if each half were a separate game). No matter how we respond, the intersection Y of our moves will be $Y_0 \cap Y_1$ where Y_0, the intersection of our first d moves, is a possible outcome of the d-move game with our opponent using σ_0, and Y_1, the intersection of our last d moves is a possible outcome of the d-move game with our opponent using σ_1. As σ_0 and σ_1 are winning strategies for our opponent, there exist sets $a_0, a_1 \subseteq n$ such that, for $i = 0$ and 1,

$$t'_i \text{ contains no pair } (a_i, b) \text{ with } b \neq a_i \text{ and } b - a \subseteq Y_i.$$

If, for at least one i, $a_i \neq w$, then t contains no pair (a_i, b) with $b \neq a_i$ and $b - a \subseteq Y$ (as $Y \subseteq Y_i$ and t'_i agrees with t on pairs whose first component is not w), so we have lost the play of the $2d$-move game for t. If, on the other hand, $a_i = w$ for both values of i, then t contains no pair (w, b) with $b \neq w$ and $b - w \subseteq Y$, for such a pair would be in t'_0 or t'_1, according to whether $b \in X_0$ or X_1. So again we have lost the $2d$-move game for t. This shows that our opponent's strategy "first σ_0 then σ_1" is a winning one for him. This contradicts the hypothesis $\mathrm{Dp}(t) \geqslant 2d + 1$. □

We are now ready to define the notion of forcing Q. A condition in Q is a pair (w, T) consisting of a finite subset w of ω and a sequence $T = \langle t_l : l \in \omega \rangle$ such that, for some increasing function $n : \omega \to \omega$,

(a) $w \subseteq n(0)$,
(b) $t_l \in K_{n(l),n(l+1)}$ for each l, and
(c) $\mathrm{Dp}(t_l) \to \infty$ as $l \to \infty$.

(Notice that (w, T) determines the function n uniquely, since $\mathcal{P}(n(l))$ is the domain of t_l.) Another such condition (w', T') is an extension of (w, T) if and only if there is an increasing function $k : \omega \to \omega$ such that, writing t^*_l for $t_{k(l)} t_{k(l)+1} \cdots t_{k(l+1)-1}$, we have

(a) $(w, w') \in t_0, t_1 \cdots t_{k(0)-1}$, by which we mean $w = w'$ if $k(0) = 0$,
(b) $t'_l \in K_{n(k(l)),n(k(l+1))}$ for all $l \in \omega$, and
(c) $t'_l \subseteq t^*_l$ for all $l \in \omega$.

Thus, any extension of (w, T) is obtained by a succession of operations of the following three sorts.

Compose relations. Partition the sequence T into finite blocks of consecutive t_i's, and compose the t_i's within each block. Leave w unchanged. (In the description of extensions above, this is the special case where $k(0) = 0$ and $t'_l = t^*_l$.)

Shrink relations. Replace each t_l by a subset t'_l in $K_{n(l),n(l+1)}$, and leave w unchanged. Of course the t'_l must be big enough so that their depths tend to ∞ with l. (This is the special case where $k(l) = l$ for all l.)

Fix values. Replace w with some w' such that $(w, w') \in t_0 \cdots t_{m-1}$ for some m, and delete the initial segment $t_0, \ldots, t_{m-1}$ from T, so $t_l' = t_{k+l}$. (This is the special case where $k(l) = m + l$ and $t_l' = t_l^*$; it could be replaced by the even more special case where $m = 1$.)

By Lemma 2.1, the relations t_l^* obtained by composing have depth at least equal to the maximum depth of the relations $t_{k(l)}, \ldots, t_{k(l+1)-1}$ being composed. Because of this and the fact that $\mathrm{Dp}(t_l) \to \infty$ in any condition, we can always compose relations so as to make $\mathrm{Dp}(t_l^*)$ grow as rapidly as we want. Thus, for example, if $f : \omega \to \omega$, then the conditions with $\mathrm{Dp}(t_l) \geqslant f(l)$ for all l are dense in Q; so are the conditions with $\mathrm{Dp}(t_{l+1}) \geqslant f(\mathrm{Dp}(t_l))$ for all l. Furthermore, the extensions witnessing this density can all be taken to be of the 'compose relations' type.

We think of a condition (w, T) as providing the following information about the generic $W \subseteq \omega$ being produced.

$$W \cap n(0) = w.$$

$$\text{For each } l, \quad (W \cap n(l), W \cap n(l+1)) \in t_l.$$

(The first of these explains the terminology 'fixing values'.) Clearly, extensions in Q give more information about W. We call a natural number x *possible* for (w, T) if there exist $l \in \omega$ and $v \subseteq \omega$ such that $x \in v$ and $(w, v) \in t_0 \cdots t_l$. Thus, x is possible for (w, T) if and only if the information that (w, T) gives about W does not preclude the possibility that $x \in W$. An equivalent formulation is that (w, T) has an extension (w', T') (which can be taken to be a 'fixing values' extension) with $x \in w'$. We write $\mathrm{ps}(w, T)$ for $\{x \in \omega \mid x \text{ is possible for } (w, T)\}$.

It will be helpful to view a condition (w, T) as a labeled tree in which the root (at level 0) is labeled w and, if a node at level l is labeled with a set $a \subseteq n(l)$, then its immediate successors are labeled with the sets $b \subseteq n(l+1)$ such that $(a, b) \in t_l$. Thus, the set of labels at level m is

$$\mathrm{Lev}_{(w,T)}(m) = \{a \mid (w, a) \in t_0 \cdots t_{m-1}\}.$$

We omit the subscript when (w, T) is clear from the context. Note that a set that labels a node at some level also labels successor nodes at all higher levels. (This is why we say that sets label nodes, not that sets are nodes.) We write $\mathrm{Lev}'_{(w,T)}(m)$ for the set $\mathrm{Lev}_{(w,T)}(m) - \mathrm{Lev}_{(w,T)}(m-1)$ of new labels at level m. We also write $\mathrm{Tree}(w, T)$ for the set $\bigcup_m \mathrm{Lev}(m)$ of all the labels occurring in the tree. The information in (w, T) about W is that $\{W \cap n(l) \mid l \in \omega\}$ is the set of labels of a path through this tree. Although (w, T) contains information not captured in the tree, e.g., whether $(a, b) \in t_l$ when $a \notin \mathrm{Lev}(l)$, this additional information will be irrelevant for us, so it would do no harm to identify conditions with trees.

Our next goal is to show that fusion arguments can be carried out in Q, from which it will follow that Q is proper.

By an *m-extension* of a condition (w, T) we mean an extension (w', T') such that $w' = w$ and $t_l' = t_l$ for all $l < m$. (So the trees agree up to level m.) In particular, a 0-extension is an extension with the same first component. Call a subset D of Q *m-dense* if every condition in Q has an m-extension in D.

Proposition 2.3. *For each $m \in \omega$, let D_m be m-dense and closed under m-extensions. Then $\bigcap_{m\in\omega} D_m$ is dense (in fact 0-dense).*

Proof. Let any condition (w, T^0) be given. We inductively define extensions (w, T^m) such that, for each m,

(i) $(w, T^{m+1}) \in D_m$,

(ii) (w, T^{m+1}) is an m-extension of (w, T^m), and

(iii) $\mathrm{Dp}(t_m^{m+1}) \geq m$.

The induction is quite easy. Given (w, T^m), we use the m-density of D_m to get an m-extension $(w, T') \in D_m$. Then we compose relations to obtain (w, T^{m+1}) satisfying (iii) as well. Specifically, we compose $t'_m t'_{m+1} \cdots t'_r$ for some r such that $\mathrm{Dp}(t'_r) \geq m$. Then (iii) holds by Lemma 2.1, while (i) and (ii) hold because (w, T^{m+1}) is an m-extension of (w, T').

Having defined the sequence $\langle (w, T^m) : m \in w \rangle$, we obtain a condition (w, S) by setting

$$s_l = t_l^{l+1}.$$

This is easily seen to be a condition, since, by (iii), $\mathrm{Dp}(s_l) = l \to \infty$. This (w, S) is an m-extension of (w, T^m) for every m, so it is in $\bigcap_{m\in\omega} D_m$ and it extends (in fact 0-extends) (w, T^0). □

In Proposition 2.3, we can weaken the hypothesis from 'closed under m-extensions' to 'closed under $(m+1)$-extensions' provided we strengthen the density hypothesis to assert that every condition has m-extensions $(w, T) \in D_m$ with $\mathrm{Dp}(t_m) \geq m$. This is because the extra hypothesis lets us dispense with composing relations; if we choose T' in the proof in accordance with the stronger hypothesis, then we can set $T^{m+1} = T'$. Then, as (w, S) is an $(m+1)$-extension of (w, T^{m+1}), we will have $(w, S) \in D_m$.

In the next proposition, it will be convenient to have a short notation for conditions obtained by fixing values. Suppose (w', T') is obtained from (w, T) by fixing values, so for some (unique) r we have $(w, w') \in t_0 \cdots t_{r-1}$ and $t'_l = t_{l+r}$. In this situation, we write $T - r$ for T'.

Proposition 2.4. *Let $\boldsymbol{\tau}_i$ $(i < \omega)$ be Q-names of ordinals. Then every condition has a 0-extension (w, S) with the following property. If $l \in \omega$, if $n = n(l)$ is the number such that s_l has domain $\mathcal{P}(n)$, if $(w, w^*) \in s_0 \cdots s_{l-1}$, if $i < n$, and if $(w^*, S - l)$ has a 0-extension forcing a particular value for $\boldsymbol{\tau}_i$, then $(w^*, S - l)$ forces a particular value for $\boldsymbol{\tau}_i$.*

Proof. By Proposition 2.3, it suffices to show that, for each l, every condition has an l-extension with the stated property for that particular l. So let l be fixed and let (w, T) be given; we must construct an l-extension (w, S) of (w, T) satisfying the conclusion of the proposition for the fixed l. By definition of l-extension, $n(l)$ is the same for (w, S) as for (w, T), so n is also fixed. There are only finitely many $i < n$ and finitely many $w^* \subseteq n$ to consider. List the w^*'s in some arbitrary order. For the first w^* in the list, inductively define conditions (w, T^i) $(i \leq n)$ by

> $T^0 = T - l$.
> If (w^*, T^i) has a 0-extension forcing a particular value for τ_i, then let (w^*, T^{i+1}) be such a 0-extension; otherwise let $T^{i+1} = T^i$.

Let $T' = T^n$. Now repeat the process for the next w^* on the list, starting with T' in place of $T - l$. Continue in the same way with all the other w^*'s. If we let T'_{w^*} be the T' obtained at the end of the stage where w^* was used and we let T'' be the final T' (T'_{w^*} for the last w^* on the list), then, for each w^*:

> if $i < n$ and some 0-extension of (w^*, T'_{w^*}) forces a particular value for τ_i, then so does (w^*, T'_{w^*}), and (w^*, T'') is a 0-extension of (w^*, T'_{w^*}), so if $i < n$ and some 0-extension of (w^*, T'') forces a particular value for τ_i, then so does (w^*, T'').

Finally, we set $s_m = t_m$ for $m < l$ and $s_m = t''_{m-l}$ for $m \geq l$. Then (w, s) is an l-extension of (w, T) and $S - l = T''$ has the desired properties. □

Proposition 2.5. *Q is proper.*

Proof. Let χ be a regular cardinal so large that $H(\chi)$, the set of sets of hereditary cardinality $<\chi$, contains all the sets we are interested in. Let N be a countable elementary submodel of $H(\chi)$ containing Q. We must show that every condition $(w, T) \in N \cap Q$ has an (N, Q)-generic extension.

We shall use the proof of Proposition 2.4 with a little extra caution. That proof involved some arbitrary choices, namely the T^i's. Let us fix, once and for all, a choice function (on the nonempty subsets of Q, say) in N, to be used whenever such choices need to be made. The existence of a choice function in N follows, of course, from the elementarity of the submodel N of $H(\chi)$.

Let $\langle \tau_i : i < \omega \rangle$ be an enumeration of all the elements of N that are Q-names of ordinals. Proposition 2.4 gives us a 0-extension (w, S) of (w, T) having the 'value-deciding' property (as in the proposition) for the sequence $\langle \tau_i : i < \omega \rangle$. This sequence is, of course, not in N, so the construction of S cannot be carried out in N, but any finite initial segment of it can be. Indeed, for each fixed l, the proof of Proposition 2.4 for that l involves τ_i only for $i < n$, so it can be done within N. It is only in applying Proposition 2.3, i.e., in doing this proof repeatedly for ever larger values of l (hence of n), that we need the whole sequence of τ_i's and must therefore step outside N.

For each i, the construction (outside N) must eventually lead to a condition

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forcing a particular value for τ_i. Indeed, since τ_i is a name of an ordinal, some condition extending (w, S) forces a particular value of τ_i. Extending it further, we may assume that this condition is a 0-extension of $(w^*, S-l)$ for some $l>i$ and some w^* with $(w, w^*) \in s_0 \cdots s_{l-1}$. Then $(w^*, S-l)$ forces a particular value for τ_i. But this occurs at a finite stage of the construction and the construction up to that point could have been carried out in N. Thus, the value forced for τ_i is in N.

This shows that every particular value for τ_i that is forced by an extension of (w, S) is in N. Since τ_i ranges over all names in N for ordinals, we have shown that (w, S) is (N, Q)-generic. □

The following partition theorem is the combinatorial property of Q on which the rest of the proof hinges.

Theorem 2.6. *Let $(w, T) \in Q$ and let C map the finite subsets of ω into $\{0, 1\}$. Then either there is an extension (w', T') of (w, T) such that C maps* $\mathrm{Tree}(w', T')$ *to* 0 *or there is a 0-extension (w, T') of T such that C maps* $\mathrm{Tree}(w, T') - \{w\}$ *to* 1.

Proof. Let C be given. The theorem asserts that the set of conditions (w, T) such that either C is constantly 1 on $\mathrm{Tree}(w, T) - \{w\}$ or there is an extension on whose tree C is constantly 0 is 0-dense. By the remark following Proposition 2.3, it suffices to show that, for each m, the set

$$D_m = \{(w, T) \mid \text{either } (w, T) \text{ has an extension on whose tree } C \text{ is identically 0 or } C \text{ is identically 1 on } \mathrm{Lev}'_{(w,T)}(m+1)\}$$

is m-dense in the strong sense that every condition (w, T) has an m-extension (w, T') in D_m with $\mathrm{Dp}(t'_m) \geq m$. (Note that D_m is trivially closed under $(m+1)$-extensions.) Let m and (w, T) be given. We construct a suitable (w, T'). We may assume, by composing relations if necessary, that $\mathrm{Dp}(t_l) \geq l$ for all $l \geq m$.

The core of the construction is the following definition.

Let q be a positive integer, $\zeta = \langle \zeta(n) : n \in \omega \rangle$ an increasing sequence of natural numbers, and $\eta = \langle \eta(n) : n \in \omega \rangle$ a subsequence of ζ with $\eta(0) = \zeta(0)$. We say that η is *q-thin* in ξ if the hypotheses

(i) $\zeta(k) < \zeta(l)$ are consecutive terms in the subsequence η,

(ii) $s_m \subseteq t_m$ for every $m \in [\zeta(k), \zeta(l))$,

(iii) for each $p \in [k, l)$ at least one $m \in [\zeta(p), \zeta(p+1))$ has $\mathrm{Dp}(s_m) \geq q + m - \zeta(k)$, and

(iv) $t^* := \{(a, b) \in s_{\zeta(k)} s_{\zeta(k)+1} \cdots s_{\zeta(l)-1} \mid b = a \text{ or if } a \in \mathrm{Tree}(w, T) \text{ then } C(b) = 1\}$

imply that $\mathrm{Dp}(t^*) \geq q$.

We consider two cases according to whether or not every increasing sequence ζ of non-negative integers has a 1-thin subsequence.

Case 1: *Some ζ has no* 1-*thin subsequence.*

Fix such a ζ. We attempt to define inductively a 1-thin subsequence by setting $\eta(0)=\zeta(0)$ and, after $\eta(n)$ has been chosen as $\zeta(k)$, say, setting $\eta(n+1)=\zeta(l)$ for some l such that hypotheses (ii) to (iv) in the definition of 1-thin imply $\mathrm{Dp}(t^*)\geqslant q$. By the case hypothesis, this attempt fails; at some stage, no suitable l exists. Consider such a stage and fix k as above, so $\zeta(k)$ is the last $\eta(n)$ defined. For every $l>m$, there exist

$$s_m^{(l)}\subseteq t_m \quad \text{for every } m\in[\zeta(k),\,\zeta(l))$$

such that, for each $p\in[k,l)$, at least one $m\in[\zeta(p),\,\zeta(p+1))$ has $\mathrm{Dp}(s_m^{(l)})\geqslant 1+m-\zeta(k)$, but t^*, defined by (iv), has depth 0. Recall that depth 0 means that, for some $a^{(l)}\subseteq n(\zeta(k))$ [= the n such that $t_{\zeta(k)}$ has domain $\mathscr{P}(n)$], the only b such that $(a^{(l)},b)\in t^*$ is $b=a_l$. Clearly $a^{(l)}$ must be in Tree(w,T) for otherwise t^* would contain all pairs $(a^{(l)},b)\in s^{(l)}_{\zeta(k)}\cdots s^{(l)}_{\zeta(l)-1}$ which has depth $\geqslant 1$ by hypothesis (iii) and Lemma 2.1. We thus have that, if $(a^{(l)},b)\in s^{(l)}_{\zeta(k)}\cdots s^{(l)}_{\zeta(l)-1}$ and $b\neq a^{(l)}$, then $C(b)=0$.

As l varies whle m is fixed, there are only finitely many possibilities for $s_m^{(l)}$, namely the subsets of t_m; similarly, there are only finitely many possibilities for $a^{(l)}$, namely the members of $\mathrm{Lev}_{(w,T)}(\zeta(k))$. By König's infinity lemma, there is a single infinite sequence $\langle s_m : m\geqslant\zeta(k)\rangle$ and there is a single $a\in\mathrm{Lev}(\zeta(k))$ such that

(ii′) $s_m\subseteq t_m$ for all m,

(iii′) for each $p\geqslant k$ there is $m\in[\zeta(p),\,\zeta(p+1))$ with $\mathrm{Dp}(s_m)\geqslant 1+m-\zeta(k)$,

(iv′) for each $l\geqslant k$, if $(a,b)\in s_{\zeta(k)}s_{\zeta(k)+1}\cdots s_{\zeta(l)-1}$ and $b\neq a$, then $C(b)=0$.

For $p\geqslant k$, let $s_p^*=s_{\zeta(p)}\cdots s_{\zeta(p+1)-1}$, and let S^* be the sequence $\langle s^*_{k+l} : l\in\omega\rangle$. Then (a,S^*) is an extension of (w,T) (obtained by fixing values to a, shrinking relations to s, and composing relations to s^*). Note that (iii′) and Lemma 2.1 ensure that $\mathrm{Dp}(s^*_{k+l})\geqslant 1+\zeta(k+l)-\zeta(k)\to\infty$, as $l\to\infty$, so (a,S^*) is a condition. Furthermore, (iv′) tells us that all the sets $b\in\mathrm{Tree}(a,S^*)$ except a itself have $C(b)=0$. The exception can be eliminated by extending (a,S^*) by fixing values. So we have an extension of (w,T) on whose tree C is identically 0. So $(w,T)\in D_m$.

Case 2: *Every ζ has a* 1-*thin subsequence.*

Lemma 2.7. *If θ is a q-thin subsequence of η and η is a q-thin subsequence of ζ, then θ is a $q+1$-thin subsequence of ζ.*

Proof. Let $\zeta(k)$, $\zeta(l)$, s_m, and t^* be as in hypotheses (i) to (iv) with θ in place of η and $q+1$ in place of q. We must show that $\mathrm{Dp}(t^*)\geqslant q+1$. Let $\{Y,Z\}$ be a partition of $[n(\zeta(k)),\,n(\zeta(l)))$ into two pieces; we must show that either t_Y^* or t_Z^*

has depth $\geq q$. Observe that

$$t_Y^* = \{(a, b) \in (s_{\zeta(k)})_Y \cdots (s_{\zeta(l)-1})_Y \mid b = a \text{ or if } a \in \text{Tree}(w, T), \text{ then } C(b) = 1\},$$

and similarly for t_Z^*.

By hypothesis (iii) and definition of depth, we have, for each $p \in [k, l)$, at least one $m = m^{(p)} \in [\zeta(p), \zeta(p+1))$ and at least one $X = X^{(p)} \in \{Y, Z\}$ such that $\text{Dp}((s_m)_X) \geq q + m - \zeta(k)$.

If there are $k' < l' \in [k, l)$ such that $\zeta(k')$ and $\zeta(l')$ are consecutive terms of η and $X^{(p)} = Y$ for all $p \in [k', l')$, then k', l', $(s_m)_Y$, and t_Y^* satisfy hypotheses (i) to (iv) in the definition of "η is q-thin in ζ", so we have $\text{Dp}(t_Y^*) \geq q$, as desired.

Otherwise, let k'' and l'' be such that $\eta(k'') = \zeta(k)$ and $\eta(l'') = \zeta(l)$. Clearly, k'', l'', $(s_m)_Z$, and $(t^*)_Z$ satisfy hypotheses (i), (ii), and (iv) in the definition of "θ is q-thin in η". If we verify that (iii) also holds, then that definition allows us to conclude $\text{Dp}(t_Z^*) \geq q$, as desired. To verify (iii), let $p \in [k'', l'')$ be given, and let k' and l' be such that $\zeta(k') = \eta(p)$ and $\zeta(l') = \eta(p+1)$. Since the hypothesis of the preceding paragraph fails, there must be $p' \in [k', l')$ with $X^{(p')} = Z$. So there is, by definition of $X^{(p')}$, at least one $m \in [\zeta(p'), \zeta(p'+1)) \subseteq [\zeta(k'), \zeta(l')) = [\eta(p), \eta(p+1))$ with $\text{Dp}((s_m)_Z) \geq q + m - \zeta(k)$, so (iii) is verified and the lemma is proved. □

Corollary 2.8. *For every q, every increasing sequence ζ has a q-thin subsequence.*

Proof. Immediate by the lemma and the case hypothesis. □

Let ζ be the sequence $\langle m, m+1, m+2, \ldots \rangle$ so $\zeta(k) = m + k$, and let η be an m-thin subsequence. Apply the definition of m-thin with $k = 0$ (so $\zeta(k) = m = \eta(0)$) and l such that $\zeta(l) = m + l = \eta(1)$, and with $s_m = t_m$. Hypotheses (i) and (ii) are clear, and (iii) asserts that for each $p \in [0, l)$

$$\text{Dp}(t_{p+m}) \geq m + (p + m) - m = p + m,$$

which is true by our initial normalization of T. So, if we define t^* by (iv), we have $\text{Dp}(t^*) \geq m$. Define T' by

$$t'_r = \begin{cases} t_r & \text{if } r < m, \\ t^* & \text{if } r = m, \\ t_{r-l} & \text{if } r > m. \end{cases}$$

Then (w, T') is an m-extension of (w, T), has $\text{Dp}(t'_m) \geq m$, and lies in D_m because C is identically 1 on $\text{Lev}_{(w,T')}(m+1)$ by definition of t^*. This completes the proof of the theorem. □

We apply Theorem 2.6 to obtain an improvement of Proposition 2.4 in the case that the names $\boldsymbol{\tau}_i$ have only finitely many values.

Proposition 2.9. *Let* $\boldsymbol{A}$ *be a* Q*-name for a subset of* ω*. Then every condition has a* 0*-extension* (w, S) *with the following property. If* $l \in \omega$*, if* $n = n(l)$ *is the number such that* s_l *has domain* $\mathcal{P}(n)$*, if* $(w, w^*) \in s_0 \cdots s_{l-1}$*, and if* $i < n$*, then* $(w^*, S - l)$ *decides whether* $i \in \boldsymbol{A}$*.*

Proof. Let $\boldsymbol{\tau}_i$ be the Q-name forced (by all conditions) to be 1 if $i \in \boldsymbol{A}$ and 0 if $i \notin \boldsymbol{A}$. So to decide whether $i \in \boldsymbol{A}$ is the same as to force a particular value for $\boldsymbol{\tau}_i$. Let (w, S) be as in Proposition 2.4 for this sequence $\langle \boldsymbol{\tau}_i : i \in \omega \rangle$. Let l, n, w^*, and i be as in the present proposition. The desired conclusion would follow, by 2.4, if we knew that some 0-extension of $(w^*, S - l)$ decides whether $i \in \boldsymbol{A}$.

We apply Theorem 2.6 with $(w^*, S - l)$ in the role of (w, T) and with the function C defined by

$$C(a) = 1 \quad \text{iff} \quad \text{for some } r,\ (a, S - r) \text{ decides whether } i \in A.$$

The first alternative in the theorem is that $(w^*, S - l)$ has an extension (w', T') on whose tree C is identically zero. Let (a, T'') be an extension of (w', T') deciding whether $i \in \boldsymbol{A}$. Then a is in the tree of (w', T'), so $C(a) = 0$. But (a, T'') is an extension of $(w^*, S - l)$, hence is a 0-extension of $(a, S - r)$ for some r (the lowest level of a in the tree of (w^*, S)). So $C(a) = 1$. This contradiction shows that the first alternative does not occur.

So we have the second alternative in the theorem. That is, we have a 0-extension (w^*, T') of $(w^*, S - l)$ such that C is identically 1 on the set $L = \text{Tree}(w^*, T') - \{w^*\}$. Thus, for each $a \in L$, $(a, S - r)$ decides a value $v(a) \in \{0, 1\}$ for $\boldsymbol{\tau}_i$, provided r is large enough. Our choice of S ensures that we can take the smallest posssible r, namely the one with $a \in \text{Lev}'_{(w,S)}(r)$, and still have $(a, S - r)$ forcing $\boldsymbol{\tau}_i = v(a)$.

Apply Lemma 2.2 to each of the terms in the sequence $S - l$, using w^* in the role of w and using the partition given by the function v. the result is a sequence of sub-relations $s'_k \subseteq s_k$ for $k \geq l$, with $\text{Dp}(s'_k) \geq \frac{1}{2}\text{Dp}(s_k)$, such that if $(w^*, b) \in s'_k$ and $w^* \neq b$, then $v(b) = \bar{v}$ where $\bar{v}$ depends only on k. Since $\text{Dp}(s'_k) \to \infty$, $(w^*, \langle s'_{l+r} : r \in \omega \rangle) = (w, S')$ is a condition, a 0-extension of (w^*, S). Composing relations, we can arrange that $\bar{v}$ is independent of k.

We complete the proof by checking that (w^*, S') forces "$\boldsymbol{\tau}_i = \bar{v}$". If not, then some extension (b, S'') forces "$\boldsymbol{\tau}_i = 1 - \bar{v}$", and, by extending further (fixing values) we may assume $b \neq w^*$. But then $v(b) = \bar{v}$, so $(b, S - r)$ forces "$\boldsymbol{\tau}_i = \bar{v}$". This is absurd, as (b, S'') is an extension of $(b, S - r)$. This contradiction completes the proof of the proposition. □

3. The extension by Q

This section is devoted to the study of the forcing extension $V[G]$ produced by adjoining to the universe V a V-generic subset G of Q. In this extension, we define

$$W = \bigcup \{w \mid (w, T) \in G\}.$$

A trivial genericity argument shows that W is an infinite subset of ω.

Proposition 3.1. *For every set $X \subseteq \omega$ in the ground model V, either $W \cap X$ or $W - X$ is finite.*

Proof. Let X be given; we show that the conditions forcing the desired conclusion form a dense set. Let (w, T) be any condition. For each k, let $t'_k = (t_k)_X$ or $(t_k)_{\omega - X}$, whichever has the greater depth (either one in case of equal depth). By definition of depth, $\mathrm{Dp}(t'_k) \geqslant \mathrm{Dp}(t_k) - 1$, so (w, T') is a condition extending (w, T). Suppose $t'_k = (t_k)_X$ for infinitely many X. (The $\omega - X$ case is analogous.) Let T'' be the subsequence of T' consisting of only those t'_k that equal $(t_k)_X$. (The codomain of such a t'_k may need to be defined differently in T'' than in T', to match the domain of the next term in T''.) Then (w, T'') is an extension of (w, T') (obtainable by composing and then shrinking relations), and it clearly forces $W - X \subseteq w$. □

The main theorem of this section will assert that any P-point ultrafilter in V generates a P-point in $V[G]$. We first check that it will suffice to prove that it generates an ultrafilter.

Lemma 3.2. *Let $\mathcal{U}$ be a P-point, and let H be a V-generic subset of some proper notion of forcing. If $\mathcal{U}$ generates an ultrafilter $\bar{\mathcal{U}}$ in $V[H]$, then this $\bar{\mathcal{U}}$ is a P-point in $V[H]$.*

Proof. Since every set in $\bar{\mathcal{U}}$ has a subset in $\mathcal{U}$, it suffices to show that, if $\langle X_n : n \in \omega \rangle$ is a sequence in $V[H]$ of sets in $\mathcal{U}$, then some set $Y \in \mathcal{U}$ is almost included (i.e., included modulo a finite set) in every X_n. By [7, III.1.16], there is a countable set $S \in V$ such that each $X_n \in S$. As $\mathcal{U}$ is a P-point in V, there is $Y \in \mathcal{U}$ almost included in every set in $S \cap \mathcal{U}$, hence almost included in every X_n. □

As Q is proper, Lemma 3.2 can be applied to it. We have stated the lemma for arbitrary proper forcing notions in order to apply it to iterations of Q in the next section.

Theorem 3.3. *Every P-point in V generates a P-point in $V[G]$.*

Proof. Let $\mathcal{U}$ be a P-point in V. By the preceding lemma, we need only show that the filter $\bar{\mathcal{U}}$ generated by $\mathcal{U}$ in $V[G]$ is an ultrafilter. By genericity, it suffices to show that, if (w, T) forces "$\mathbf{A} \subseteq \omega$", then some extension forces either

"$B \subseteq A$" or "$B \cap A = \emptyset$" for some $B \in \mathcal{U}$. According to Proposition 2.9, we may assume that, if $l \in \omega$, if $n = n(l)$ is the number such that t_l has domain $\mathcal{P}(n)$, if $(w, w^*) \in t_0 \cdots t_{l-1}$, and if $i < n$, then $(w^*, T - l)$ decides whether $i \in A$.

Consider any $w^* \in \text{Tree}(w, T)$. Then $w^* \in \text{Lev}_{(w,T)}(l)$ for all sufficiently large l. Thus, for any fixed $i \in \omega$, $(w^*, T - l)$ will decide whether $i \in A$ once l is large enough; of course the decisions agree as l varies, since $(w^*, T - l')$ extends $(w^*, T - l)$ (by fixing values) if $l' \geq l$. Let $A(w^*)$ be the set of those $i \in \omega$ for which the decision is positive, i.e., $(w^*, T - l) \Vdash$ "$i \in A$" for all sufficiently large l. Partition Tree(w, T) by putting into one class all those w^* for which $A(w^*) \in \mathcal{U}$. By Theorem 2.6, we can extend (w, T) to arrange that all of Tree(w, T) is in a single class. Note that, when we form this extension, we do not destroy the fact that, for $i \in A(w^*)$ (resp. $i \notin A(w^*)$), $(w^*, T - l) \Vdash$ "$i \in A$" ("$i \notin A$") for all sufficiently large l. We assume henceforth that $A(w^*) \in \mathcal{U}$ for all $w^* \in$ Tree(w, T); the case that $\omega - A(w^*) \in \mathcal{U}$ for all $w^* \in \text{Tree}(w, T)$ is handled analogously, with A replaced by its complement. As $\mathcal{U}$ is a P-point, let $B \in \mathcal{U}$ be almost included in each $A(w^*)$.

Inductively define a sequence $\langle \zeta(n) : n \in \omega \rangle$ of natural numbers, starting with $\zeta(0) = 0$, and increasing so rapidly that, if $w^* \in \text{Lev}_{(w,T)}(\zeta(n))$, then

(i) $B - A(w^*) \subseteq \zeta(n+1)$, and

(ii) if $i \in A(w^*)$ and $i < \zeta(n)$, then $(w^*, T - \zeta(n+1)) \Vdash$ "$i \in A$".

Think of ζ as partitioning ω into blocks $[\zeta(n), \zeta(n+1))$ and consider the four sets obtainable by taking the union of every fourth block:

$$X_j = \bigcup \{[\zeta(n), \zeta(n+1)) \mid n \equiv j \pmod 4\}.$$

As $\mathcal{U}$ is an ultrafilter, it contains exactly one of these sets. By omitting a few terms (at most 3) from the ζ sequence, we may assume $X_2 \in \mathcal{U}$. Replacing B with $X_2 \cap B$, which is also in $\mathcal{U}$, we may assume $B \subseteq X_2$.

We define an extension (w, T') of (w, T) as follows. Let n be the function such that $t_l \in K_{n(l),n(l+1)}$ for all l. Then t'_k is to be the element of $K_{n(\zeta(4k)),n(\zeta(4k+4))}$ given by $t'_k = t_{\zeta(4k)}$ (as relations). This defines a condition because $\text{Dp}(t'_k) = \text{Dp}(t_{\zeta(4k)}) \to \infty$. Notice that $\text{Lev}_{(w,T')}(k+1) \subseteq \text{Lev}_{(w,T)}(\zeta(4k)+1) \subseteq \text{Lev}_{(w,T)}(\zeta(4k+1))$.

To complete the proof of the theorem, we show that (w, T') forces "$B \subseteq A$". Suppose it did not, and fix an element $i \in B$ and an extension (v, S) of (w, T') forcing "$i \notin A$". Since $B \subseteq X_2$, let k be such that $i \in [\zeta(4k+2), \zeta(4k+3))$. Let $w^* = v \cap n(\zeta(4k+4))$. So $w^* \in \text{Lev}_{(w,T')}(k+1) \subseteq \text{Lev}_{(w,T)}(\zeta(4k+1))$ and $(w^*, T' - (k+1))$, being compatible with (v, S), cannot force "$i \in A$". We will obtain a contradiction by showing that $(w^*, T - \zeta(4k+4))$, of which $(w^*, T' - (k+1))$ is an extension, does force "$i \in A$". Since $w^* \in \text{Lev}_{(w,T)}(\zeta(4k+1))$ and $i \in B$ and $i \geq \zeta(4k+2)$, clause (i) in the definition of ζ implies that $i \in A(w^*)$. Since also $w^* \in \text{Lev}_{(w,T)}(\zeta(4k+3))$ and $i \leq \zeta(4k+3)$, clause (ii) in the same definition implies that $(w^*, T - \zeta(4k+4)) \Vdash$ "$i \in A$", the desired contradiction. □

Our next goal is to show, under suitable hypotheses, that every two ultrafilters in V have a common finite-to-one image in $V[G]$. In fact, for this purpose, a single finite-to-one function works for all ultrafilters simultaneously. Specifically, let $f \in V[G]$ be a function from ω to ω that is constant precisely on intervals $[a, b)$ where a and b are consecutive elements of W (or $a = 0$ and b is the first element of W); for example

$$f(x) = \text{the number of elements of } W \text{ that are } \leqslant x.$$

Proposition 3.4. *Let X be an infinite subset of ω in V, and let $\mathcal{U}$ be an ultrafilter in V. There exists $Y \in \mathcal{U}$ such that $f(X) \supseteq f(Y)$.*

Proof. Let X, $\mathcal{U}$, and a condition $(w, T) \in Q$ be given. We shall find $Y \in \mathcal{U}$ and an extension of (w, T) forcing "$\boldsymbol{f}(X) \supseteq \boldsymbol{f}(Y)$"; by genericity of G, this will suffice to prove the proposition. By a preliminary extension of (w, T) (fixing values), we may assume that the first element of W is forced to be a particular number p, and we decide to put no smaller numbers into Y. Then the statement to be forced, "$\boldsymbol{f}(X) \supseteq \boldsymbol{f}(Y)$", is equivalent to "If a and b are consecutive elements of $\boldsymbol{W}$ and if Y has elements in $[a, b)$, then so does X".

Extending (w, T) by composing relations and fixing values, we may assume that the function $n : \omega \to \omega$, such that t_l has domain $\mathcal{P}(n(l))$, grows so rapidly that each interval $[n(l), n(l+1))$ meets X and that $n(0) > p$. Let Y_0 and Y_1 be defined by

$$Y_i = \bigcup \{[n(l), n(l+1)) \mid l \equiv i \pmod 2\}.$$

These two sets constitute a partition of $[p, \omega)$, so one of them in $\mathcal{U}$; assume for notational simplicity that it is Y_1. Let $t'_l = t_{2l}$. Then (w, T') is an extension of (w, T). Notice that the set $\mathrm{ps}(w, T')$ of possible elements of W (given (w, T')) is included in $n(0) \cup Y_0$. Thus, (w, T') forces "$\boldsymbol{W} \subseteq n(0) \cup Y_0$", which implies "If $y \in [n(k), n(k+1))$ with k odd and if $a \leqslant y \leqslant b$ with $a, b \in \boldsymbol{W}$, then $a < n(k)$ and $b \geqslant n(k+1)$", which in turn implies "If $y \in Y_1 \cap [a, b)$ with $a, b \in \boldsymbol{W}$, then $[a, b)$ includes an interval $[n(k), n(k+1))$ for some k". Finally, since every interval $[n(k), n(k+1))$ meets X, we see that (w, T') forces "If Y_1 meets $[a, b)$ with $a, b \in \boldsymbol{W}$, then X also meets $[a, b)$", as desired. □

If $\mathcal{U}$ is an ultrafilter in V, let $\bar{\mathcal{U}}$ be the filter it generates in $V[G]$. Observe that, for any function $g : \omega \to \omega$, $g(\bar{\mathcal{U}})$ is generated by the sets $g(X)$ with $X \in \mathcal{U}$.

Corollary 3.5. *Assume that there is a P-point in V. Then, for any two ultrafilters $\mathcal{U}_1$ and $\mathcal{U}_2$ in V, $f(\bar{\mathcal{U}}_1) = f(\bar{\mathcal{U}}_2)$.*

Proof. Since "$f(\mathcal{U}_1) = f(\mathcal{U}_2)$" is an equivalence relation on ultrafilters, it suffices to prove the corollary under the additional hypothesis that $\mathcal{U}_2$ is itself a P-point. By Theorem 3.3, the filter $\bar{\mathcal{U}}_2$ is an ultrafilter (in fact a P-point); hence so is

$f(\bar{\mathcal{U}}_2)$. So it suffices to prove that $f(\bar{\mathcal{U}}_2) \subseteq f(\bar{\mathcal{U}}_1)$, and by the observation immediately preceding the corollary, it suffices to show that $f(X) \in f(\bar{\mathcal{U}}_1)$ for every $X \in \mathcal{U}_2$. But this is immediate by Proposition 3.4. □

Proposition 3.6. *There is no function $g : \omega \to \omega$ in V such that, for all $n \in \omega$, the* nth *element of W is $\leqslant g(n)$.*

Proof. Suppose (w, T) forced "for all $n \in \omega$, the nth element of $\boldsymbol{W}$ is $\leqslant g(n)$", for a certain g. Let n_0 be larger than the number of elements of w. For sufficiently large k (e.g., any $k > g(n_0)$), the extension $(w, T-k)$ of (w, T) has $\mathrm{ps}(w, T-k)$ consisting of elements of w and numbers larger than $g(n_0)$. Thus, $(w, T-k)$ forces "the n_0th element of $\boldsymbol{W}$ is $> g(n_0)$", a contradiction. □

4. Preservation of *P*-points in iterations

In this section, which is nearly independent of the preceding ones, we show that, for proper notions of forcing, countable-support iteration preserves the property that a P-point in the ground model generates a P-point in the extension.

Throughout this section, $\langle P_\alpha, \boldsymbol{Q}_\alpha : \alpha < \lambda \rangle$ is a countable-support proper forcing iteration of limit length λ. That is,

P_0 is the trivial notion of forcing (with just one element),
$P_{\alpha+1} = P_\alpha * \boldsymbol{Q}_\alpha$ (the two-step iteration),
$P_\beta =$ direct limit of $(P_\alpha)_{\alpha<\beta}$ if β has uncountable cofinality,
$P_\beta =$ inverse limit of $(P_\alpha)_{\alpha<\beta}$ if β has cofinality ω, and
$P_\alpha \Vdash$ "$\boldsymbol{Q}_\alpha$ is a proper notation of forcing".

We write P_λ for the inverse or direct limit of $(P_\alpha)_{\alpha<\lambda}$ according as the cofinality of λ is ω or larger. By [7, III.3.2], each P_α $(\alpha \leqslant \lambda)$ is proper.

Theorem 4.1. *For any countable-support proper forcing iteration $\langle P_\alpha, \boldsymbol{Q}_\alpha : < \lambda \rangle$ and any P-point $\mathcal{U}$, if, for each $\alpha < \lambda$,*

$$P_\alpha \Vdash \text{“}\mathcal{U} \textit{ generates a P-point''},$$

then also

$$P_\lambda \Vdash \text{“}\mathcal{U} \textit{ generates a P-point''}.$$

Proof. By Lemma 3.2, it suffices to prove that

$$P_\lambda \Vdash \text{“}\mathcal{U} \text{ generates an ultrafilter''}.$$

The case $\mathrm{cf}(\lambda) > \omega$ is an easy consequence of the fact that, in this case, every real added by P_λ is already added by P_α for some $\alpha < \lambda$ [7, V.4.4], hence either

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includes or is disjoint from a member of $\mathcal{U}$. We may therefore assume that $\mathrm{cf}(\lambda) = \omega$. Indeed we may assume that $\lambda = \omega$ by passing to a cofinal ω-subsequence of the iteration. More precisely, let $\langle \alpha_n : n \in \omega \rangle$ be an increasing ω-sequence cofinal in λ, let $P'_n = P_{\alpha_n}$ and let $Q'_n = P_{\alpha_n \alpha_{n+1}}$. Then $\langle P'_n, Q'_n : n < \omega \rangle$ is a proper forcing iteration of length ω with (inverse) limit $P'_\omega = P_\lambda$, and each P'_n forces "$\mathcal{U}$ generates a P-point". So if the theorem is true for ω-length iterations, we obtain that P'_ω $(= P_\lambda)$ forces "$\mathcal{U}$ generates a P-point", as desired.

Henceforth, we assume $\lambda = \omega$. We are given that

$$P_n \Vdash \text{"}\mathcal{U} \text{ generates a } P\text{-point"}$$

for each $n \in \omega$, and we wish to prove, for the limit forcing,

$$P_\omega \Vdash \text{"}\mathcal{U} \text{ generates an ultrafilter"}.$$

For this we consider an aribtrary P_ω-name $\mathbf{A}$ and an arbitrary condition $p \in P_\omega$ forcing "$\mathbf{A} \subseteq \omega$", and we find a set $B \in \mathcal{U}$ and an extension q of p forcing that "$B \subseteq \mathbf{A}$ or $B \cap \mathbf{A} = \emptyset$".

We claim that it will suffice to carry out the proof under the following additional assumption.

Hypothesis 4.3. *For every $k < \omega$, every generic (over V) $G_k \subseteq P_k$ containing $p \restriction k$, and every $r \in P_\omega / G_k$ extending $p \restriction [k, \omega)$, there exists $B \in \mathcal{U}$ such that, for each $n \in \omega$, some extension r' of r in P_ω / G_k* forces "$B \cap n \subseteq \mathbf{A}^{(G_k)}$".

To see that this hypothesis entails no loss of generality, suppose that the desired result had been established under the hypothesis but that, for the particular p and $\mathbf{A}$ under consideration, the hypothesis fails. Let k, G_k, and r constitute a counterexample to the hypothesis. We work temporarily in $V' = V[G_k]$. In this universe, we have a P-point $\mathcal{U}'$ generated by $\mathcal{U}$, an ω-length iteration of proper forcing

$$\langle P'_n, Q'_n : n \in \omega \rangle = \langle P_{k+n} / G_k, Q^{(G)_k}_{k+n} : n \in \omega \rangle$$

with inverse limit $P'_\omega = P_\omega / G_k$, a condition $p' = r \in P'_\omega$ (the r from the failure of Hypothesis 4.3), and a P'_ω-name $\mathbf{A}'$ for the complement of $\mathbf{A}^{(G_k)}$ (so all conditions force "$\mathbf{A}' = \omega - \mathbf{A}^{(G_k)}$"). Consider Hypothesis 4.3', obtained by putting these primed objects in place of the corresponding unprimed objects in Hypothesis 4.3.

If 4.3' holds, then, by our supposition, so does the primed version of the desired conclusion. That is, we have $B' \in \mathcal{U}'$ and an extension q' of p' in P'_ω forcing "$B' \subseteq \mathbf{A}'$ or $B' \cap \mathbf{A}' = \emptyset$". As $\mathcal{U}$ generates $\mathcal{U}'$, let B be a subset of B'. Then we have (still in $V' = V[G_k]$)

$$q' \text{ extends } p' \text{ in } P'_\omega, \text{ and}$$

$$q' \Vdash \text{"}B \subseteq \mathbf{A}' \text{ or } B \cap \mathbf{A}' = \emptyset\text{"}.$$

Returning to V, find $s \in G_k$ forcing the facts just displayed. As $p \restriction k \in G_k$, we may assume that s extends $p \restriction k$. Then $q=(s, q')$ is a condition in P_ω extending p (as $q \restriction k = s$ extends $p \restriction k$ and s forces $q \restriction [k, \omega) = q'$ to extend $p' = r$ which extends $p \restriction [k, \omega)$) and forcing "$B \cap \boldsymbol{A} = \emptyset$ or $B \subseteq \boldsymbol{A}$". Thus, B and q are as desired.

There remains the case that Hypothesis 4.3′ fails. Let k', $G_{k'}$, and r' be a counterexample in $V' = V[G_k]$. Thus, $G_{k'}$ is a V'-generic subset of $P'_{k'} = P_{k+k'}/G_k$ containing $p' \restriction k' = r \restriction [k, k+k')$, and r' is an extension of $p' \restriction [k', \omega) = r \restriction [k', \omega)$ in $P'_\omega/G_{k'} = (P_\omega/G_k)/G_{k'}$. Let H be the V-generic subset $G_k * G_{k'}$ of $P_{k+k'}$. Then $r' \in P_\omega/H$. Working in $V[H] = V[G_k, G_{k'}]$, construct a sequence $\langle q_n : n \in \omega \rangle$ of conditions in P_ω/H such that:

$q_0 = r'$.

q_{n+1} extends q_n.

For each $m < n$, $q_n \Vdash m \in \boldsymbol{A}^{(H)}$ or $q_n \Vdash m \notin \boldsymbol{A}^{(H)}$.

Let $\hat{A} = \{m \in \omega \mid$ for some, hence for all, $n > m$, $q_n \Vdash m \in \boldsymbol{A}^{(H)}\}$. Thus, $q_n \Vdash \boldsymbol{A}^{(H)} \cap n = \hat{A} \cap n$. Since $\hat{A}$ is in $V[H]$ and $\mathcal{U}$ generates an ultrafilter there, we have a $B \in \mathcal{U}$ included in or disjoint from $\hat{A}$.

Suppose first that $B \subseteq \hat{A}$. For every $n \in \omega$, q_n forces $\boldsymbol{A}^{(H)} \cap n = \hat{A} \cap n \supseteq B \cap n$. That q_n extends r' and forces $B \cap n \subseteq \boldsymbol{A}^{(H)}$, being true in $V[H] = V[G_k, G_{k'}]$, must be forced over $V[G_k]$ by some $s \in G_{k'}$. This implies that the condition $(s, q_n) \in P_\omega/G_k$ forces (over $V[G_k]$) $B \cap n \subseteq \boldsymbol{A}^{(G_k)}$. We can arrange for s to be an extension of $p' \restriction k'$ as $p' \restriction k' \in G_{k'}$. But then (s, q_n) is an extension of r (as s extends $p' \restriction k' = r \restriction [k, k+k')$ and s forces q_n to extend r' which extends $r \restriction [k+k', \omega)$) forcing $B \cap n \subseteq \boldsymbol{A}^{(G_k)}$. That this can be done for every $n \in \omega$ contradicts our supposition that k, G_k, r constitute a counterexample to 4.3.

There remains the case that B is disjoint from $\hat{A}$. Now, for every $n \in \omega$, q_n extends r' and forces $B \cap n \subseteq \omega - \boldsymbol{A}^{(H)} = \boldsymbol{A}'^{(G_{k'})}$. that this can be done for every $n \in \omega$ contradicts our supposition that k', $G_{k'}$, r' constitute a counterexample to 4.3′.

These contradictions show that, if 4.3 fails then 4.3′ must hold, and so we always get the desired extension of p forcing "$B \subseteq \boldsymbol{A}$ or $B \cap \boldsymbol{A} = \emptyset$".

Thus, we may, and henceforth do, assume Hypothesis 4.3. Before constructing, under this hypothesis, the desired $B \in \mathcal{U}$ and extension q of p forcing "$B \subseteq \boldsymbol{A}$" (the alternative $B \cap \boldsymbol{A} = \emptyset$ was needed only to make 4.3 and 4.3′ symmetric), we need some more preliminary information.

Lemma 4.4. *If $\mathcal{U}$ is a P-point and $X_n \in \mathcal{U}$ for each $n \in \omega$, then there exists $Y \in \mathcal{U}$ such that, for infinitely many $n \in \omega$, $Y - n \subseteq X_n$.*

Proof. As $\mathcal{U}$ is a p-point, there exists $Z \in \mathcal{U}$ almost included in each X_n. Inductively define an increasing sequence of natural numbers $n(l)$ by setting $n(0) = 0$ and choosing $n(l+1)$ so large that $Z - n(l+1) \subseteq X_{n(l)}$. Being an

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ultrafilter, $\mathcal{U}$ must contain either $\bigcup_l [n(2l), n(2l+1))$ or its complement $\bigcup_l [n(2l+1), n(2l+2))$. Assume the latter; the other case is analogous. Then $\mathcal{U}$ contains the set $Y = Z \cap \bigcup_l [n(2l+1), n(2l+2))$. Since Y is disjoint from $[n(2l), n(2l+1))$ for every l, we have

$$Y - n(2l) = Y - n(2l+1) \subseteq Z - n(2l+1) \subseteq X_{n(2l)}.$$

So Y is as required. $\square$

Fix a regular cardinal χ big enough so that all the sets we shall need to consider belong to $H(\chi)$. Fix a countable elementary submodel N of $H(\chi)$ that contains the ultrafilter $\mathcal{U}$, the forcing sequence $\langle P_n, Q_n : n \in \omega \rangle$, the name A, and the condition $p \in P_\omega$ fixed earlier. Since N is countable and $\mathcal{U}$ is a P-point, choose $B^* \in \mathcal{U}$ such that, for all $X \in \mathcal{U} \cap N$, B^* is almost included in X. Let B be as in Hypothesis 4.3 with $k = 0$ (hence G_k trivial) and $r = p$; replacing B^* with $B^* \cap B$, we can arrange that

4.5. *For each $n \in \omega$, some extension r of p in P_ω forces "$B^* \cap n \subseteq A$".*

Lemma 4.6. *Let $k \in \omega$, let $q \in P_k$ be (N, P_k)-generic, let G_k be a V-generic subset of P_k containing q, and let $\langle X_n : n \in \omega \rangle$ be a sequence in $N[G_k]$ of sets $X_n \in \mathcal{U}$. Then, for infinitely many $n \in \omega$, $B^* - n \subseteq X_n$.*

Proof. Since the forcing P_k is proper, [7, III.2.11] tells us that $N[G_k]$ is an elementary submodel of $H(\chi)^{V[G_k]}$. Therefore, it is true in $N[G_k]$ that $\mathcal{U}$ generates a P-point. Applying Lemma 4.4 in $N[G_k]$, we obtain Y in $\mathcal{U} \cap N[G_k]$ such that $Y - n \subseteq X_n$ for infinitely many n. Note that we can take Y to be in $\mathcal{U}$, not merely in the ultrafilter in $N[G_k]$ generated by $\mathcal{U}$, by shrinking Y if necessary. Since $Y \in V \cap N[G_k]$, we have, by [7, III.2.12(c)], $Y \in N$. Therefore B^* is almost included in Y. For the infinitely many n that satisfy $Y - n \subseteq X_n$ and are larger than all the (finitely many) elements of $B^* - Y$, we have $B^* - n \subseteq X_n$. $\square$.

We shall complete the proof of Theorem 4.1 by constructing an extension q of p in P_ω forcing "$B^* \subseteq A$". The construction is an inductive one, producing one component of q at a time. After k steps, we shall have an approximation p^k to q, correct in the first k components. In detail, we shall define a sequence of conditions $p^k \in P_\omega$, starting with $p^0 = p$, and satisfying, for all $k \in \omega$,

(1) p^{k+1} is an extension of p^k, and $p^{k+1} \restriction k = p^k \restriction k$,

(2) $p^k \Vdash$ "$B^* \cap k \subseteq A$",

(3) $p^k \restriction k$ is (N, P_k)-generic,

(4) $p^k \restriction k \Vdash$ "$p^k \restriction [k, \omega) \in N[G_k]$", and

(5) $p^k \restriction k \Vdash$ "for every $n \in \omega$, there is an extension t of $p^k \restriction [k, \omega)$ in P_ω / G_k forcing '$B^* \cap n \subseteq A^{(G_k)}$' ".

Before proceeding with the construction, we make a few explanatory remarks.

First, as incdicated by the context, $\Vdash$ refers to forcing with P_ω in (2) and to forcing with P_k in (4) and (5); the word 'forcing' in (5) refers to forcing with P_ω/G_k. Second, the choice of p^0 as p is consistent with requirements (1) to (5). Indeed, (1) to (3) are trivial for $k=0$ (as P_0 is trivial), (4) says $p \in N$ which is true by our choice of N, and (5) is exactly fact 4.5 above. Third, (1) implies that, for each k, $p^n \restriction k$ is independent of n once $n \geqslant k$. Since P_ω is an inverse limit, we can define $q \in P_\omega$ by

$$q \restriction k = p^k \restriction k;$$

then q extends every p^k, in particular p, and it forces "$\boldsymbol{B}^* \subseteq \boldsymbol{A}$" by (2). Thus, q will be as desired, so the proof will be complete once we construct the p^k's.

Suppose that, for a certain k, p^k has been constructed and satisfies (2) through (5). We wish to construct p^{k+1} so that the induction hypotheses continue to hold. By (1) we have no choice about the first k components; $p^{k+1} \restriction k$ must equal $p^k \restriction k$. The rest of p^{k+1}, which we must construct, is best viewed as consisting of two parts, the component $r = p^{k+1}(k)$ and the rest $s = p^{k+1} \restriction [k+1, \omega)$. Here

$$p^k \restriction k \Vdash \text{“}r \in \boldsymbol{Q}_k \text{ and } r \Vdash \text{‘}s \in (P_\omega/\boldsymbol{G}_k)/\boldsymbol{H}\text{’ ”}$$

where $\boldsymbol{G}_k$ and $\boldsymbol{H}$ are the names of the canonical generic subsets of P_k and Q_k respectively. In terms of r and s, the five requirements on p^{k+1} are as follows.

(1) $p^k \restriction k \Vdash$ "(r, s) extends $p^k \restriction [k, \omega)$",

(2) $p^k \restriction k \Vdash$ "$(r, s) \Vdash$ '$\boldsymbol{B}^* \cap (k+1) \subseteq \boldsymbol{A}^{(\boldsymbol{G}_k)}$' ",

(3) $(p^k \restriction k, r)$ is (N, P_{k+1})-generic,

(4) $p^k \restriction k \Vdash$ "$r \Vdash$ '$s \in N[\boldsymbol{G}_{k+1}]$' ", and

(5) $p^k \restriction k \Vdash$ "$r \Vdash$ 'for every $n \in \omega$, there is an extension t of s in $P_\omega/\boldsymbol{G}_{k+1}$ forcing $\langle\!\langle \boldsymbol{B}^* \cap n \subseteq \boldsymbol{A}^{(\boldsymbol{G}_{k+1})} \rangle\!\rangle$' ".

We can make several simplifications here. The only requirement not of the form "$p^k \restriction k \Vdash \cdots$" is (3), which follows from

$$p^k \restriction k \Vdash \text{“}r \text{ is } (N[\boldsymbol{G}_k], \boldsymbol{Q}_k)\text{-generic”}, \tag{3'}$$

by [7, p. 91], since $p^k \restriction k$ is (N, P_k)-generic. To satisfy all these requirements, it suffices to work in a generic extension $V[G_k]$ where $G_k \subseteq P_k$ is a V-generic set containing $p^k \restriction k$; if we can produce r and s in $V[G_k]$, having all the five properties that we want $p^k \restriction k$ to force, then the "forcing = truth" and maximum principles produce P_k-names forced by $p^k \restriction k$ to denote such r and s. Henceforth, we work in $V[G_k]$, with G_k as above, and we adopt the notational convention that the value, with respect to G_k, of a P_k-name such as $\boldsymbol{Q}_k$ will be denoted by the same symbol in lightface, e.g., Q_k. Names that were not boldface to begin with, like $p^k(k)$, will have their G_k-values denoted by the same symbol. (This ambiguity seems to cause less difficulty than any attempt to resolve it would.)

In $V[G_k]$, we seek r and s such that

(1) (r, s) extends $p^k \restriction [k, \omega)$ in P_ω/G_k,

(2) $(r, s) \Vdash$ "$\boldsymbol{B}^* \cap (k+1) \subseteq \boldsymbol{A}^{(G_k)}$",

(3′) r is $(N[G_k], Q_k)$-generic,

(4) $r \Vdash$ "$s \in N[G_k][H]$", and

(5) $r \Vdash$ "for every $n \in \omega$, there is an extension t of s in $(P_\omega/G_k)/H$ forcing '$B^* \cap n \subseteq \boldsymbol{A}^{(G_k)(H)}$' ".

(Here H is, in accordance with our convention, the G_k-value of $\boldsymbol{H}$. Though it is lightface, it is still a name, the Q_k-name of the canonical generic subset of Q_k.)

We simplify the problem further by replacing (3′) and (4) by

(6) $(r, s) \in N[G_k]$.

(6) implies (4) by definition of $N[G_k][H]$. It does *not* imply (3′), but if we have (6), or just $r \in N[G_k]$, then we can obtain (3′) by extending r, since Q_k is proper and $N[G_k]$ is an elementary submodel of $H^{V[G_k]}(\chi)$ by [7, III.2.11]. Of course extending r preserves the truth of (1), (2), (4), and (5).

Thus, our goal is to produce some r and s satisfying requirements (1), (2), (5), and (6) in $V[G_k]$. Because $p^k \restriction k \in G_k$, induction hypotheses (2), (4), and (5) imply the following facts, in which we have abbreviated $p^k \restriction [k, \omega)$ as $\bar{p}$ and $\boldsymbol{A}^{(G_k)}$ as $\bar{\boldsymbol{A}}$.

4.7. $\bar{p} \Vdash$ "$B^* \cap k \subseteq \bar{\boldsymbol{A}}$".

4.8. $\bar{p} \in N[G_k]$.

4.9. *For every $n \in \omega$, there is an extension t of $\bar{p}$ in P_ω/G_k forcing "$B^* \cap n \subseteq \bar{\boldsymbol{A}}$".*

We obtain r and s by first constructing a multitude of candidates and then selecting appropriate ones. For each finite sequence η of zeros and ones, choose, if possible, an extension p_η of $\bar{p}$ in P_ω/G_k that forces η to be an initial segment of (the characteristic function of) $\bar{\boldsymbol{A}}$, i.e., forcing

$$\text{"}\bar{\boldsymbol{A}} \cap \operatorname{length}(\eta) = \{i < \operatorname{length}(\eta) \mid \eta(i) = 1\}\text{"}.$$

Let T be the set of those η for which p_η exists. Clearly T is a tree (closed under initial segments). Being defined from P_ω/G_k, $\bar{\boldsymbol{A}}$, and $\bar{p}$, all of which are in $N[G_k]$ (as P_ω, $\boldsymbol{A} \in N$ and 4.8 holds), T belongs to $N[G_k]$. By 4.9, every initial segment of (the characteristic function of) B^* is majorized componentwise by at least one $\eta \in T$; in particular, T has infinite height.

For each $\eta \in T$, consider an arbitrary $V[G_k]$-generic $H \subseteq Q_k$ with $p_\eta(k) \in H$. We wish to apply hypothesis 4.3 with $k+1$ in place of k, $G_k * H$ in place of G_k, and $p_\eta \restriction [k+1, \omega)$ in place of r. To see that the hypothesis is applicable, we must check that $p \restriction (k+1) \in G_k * H$ and that $p_\eta \restriction [k+1, \omega)$ extends $p \restriction [k+1, \omega)$. Both verifications are easy because G_k contains both $p^k \restriction k$, an extension of $p \restriction k$, and conditions forcing p_η to extend $\bar{p}$ which in turn extends $p \restriction [k, \omega)$. Applying 4.3, we obtain a set $C_\eta \in \mathcal{U}$ such that

4.10. *For each $n \in \omega$, some extension t of $p_\eta \restriction [k+1, \omega)$ in $(P_\omega/G_k)/H$ forces "$C_\eta \cap n \subseteq \bar{\boldsymbol{A}}^{(H)}$".*

Since this statement is true in $V[G_k][H]$, it is forced over $V[G_k]$ by some $r_\eta \in H$; since $p_\eta(k) \in H$, we can arrange that r_η extends $p_\eta(k)$.

Since $N[G_k]$ is an elementary submodel of $H^{V[G_k]}(\chi)$, we can and do take $\langle p_\eta, C_\eta, r_\eta : \eta \in T\rangle$ to be in $N[G_k]$.

For each $m \in \omega$, let C_m be the intersection of the C_η's for $\eta \in T$ of length m. As an intersection of finitely many sets in $\mathcal{U}$, C_m is also in $\mathcal{U}$. Furthermore, the sequence $\langle C_m : m \in \omega\rangle$ is in $N[G_k]$. By Lemma 4.6, with $p^k \restriction k$ in place of q, there are infinitely many $m \in \omega$ with $B^* - m \subseteq C_m$. Fix such an $m > k$, and fix an $\eta \in T$ of length m majorizing the characteristic function of B^* (restricted to m) componentwise, i.e., $B^* \cap m \subseteq \{i < m \mid \eta(i) = 1\}$. Recall that such an η exists by (4.9). We set $r = r_\eta$ and $s = p_\eta \restriction [k+1, \omega)$. We arranged in the preceding paragraph that these be in $N[G_k]$, so (6) holds, Since r extends $p_\eta(k)$, (r, s) extends p_η which extends $\bar{p} = p^k \restriction [k, \omega)$, so (1) holds. Since $m > k$,

$$B^* \cap (k+1) \subseteq B^* \cap m \subseteq \{i < m \mid \eta(i) = 1\}$$

and this last set is forced to be a subset of $\bar{A}$ by p_η, hence also by the extension (r, s). Since $\bar{A} = A^{(G_k)}$, we have verified requirement (2). It remains to check (5):

$$r \Vdash \text{“For every } n \in \omega, \text{ there is an extension } t \text{ of } s \text{ in } (P_\omega/G_k)/H \text{ forcing ‘} B^* \cap n \subseteq \bar{A}^{(H)}\text{’ ”.}$$

If B^* were changed to C_η, the resulting statement would be true simply by definition of $r = r_\eta$. Now m was chosen so that

$$B^* - m \subseteq C_m \subseteq C_\eta.$$

Thus, to obtain (5), i.e., to change C_η back to B^*, it suffices to prove that

$$r \Vdash \text{“} s \Vdash \text{‘} B^* \cap m \subseteq \bar{A}^{(H)}\text{’ ”.}$$

But $B^* \cap m \subseteq \{i < m \mid \eta(i) = 1\}$, and r extends $p_\eta(k)$, so it more than suffices to prove

$$p_\eta(k) \Vdash \text{“} s \Vdash \text{‘}\{i < m \mid \eta(i) = 1\} \subseteq \bar{A}^{(H)}\text{’ ”.}$$

Since $s = p_\eta \restriction [k+1, \omega)$, this reduces to

$$p_\eta \Vdash \text{“}\{i < m \mid \eta(i) = 1\} \subseteq \bar{A}\text{”,}$$

which is true by definition of p_η.

This completes the proof that r and s have the required properties; it thus completes the proof of Theorem 4.1. □

5. The consistency of (A)

Assume the continuum hypothesis in the ground model V. Let $\langle P_\alpha, Q_\alpha : \alpha < \aleph_2\rangle$ be a countable support forcing iteration in which each Q_α denotes the basic forcing Q of Section 2 in the forcing extension V^{P_α}. Let $P = P_{\aleph_2}$ be the (direct) limit of this iteration, and let G be a V-generic subset of P. Our goal in this section is to prove

Theorem 5.1. *If $\mathcal{U}_1$ and $\mathcal{U}_2$ are non-principal ultrafilters in $V[G]$, then there exists a finite-to-one $f: \omega \to \omega$ in $V[G]$ such that $f(\mathcal{U}_1) = f(\mathcal{U}_2)$.*

Before embarking on the proof, we fix some notation and obtain some preliminary information about P and the forcing extension $V[G]$. For each $\alpha < \aleph_2$, let G_α be the restriction of G to P_α, a V-generic subset of P_α. Let $Q_\alpha \in V[G_\alpha]$ be the G_α-value of the name $\mathbf{Q}_\alpha$, and let H_α be the $V[G_\alpha]$-generic subset of Q_α given by G; thus $G_{\alpha+1} = G_\alpha * H_\alpha$. Since Q_α is the Q of Section 2, as calculated in $V[G_\alpha]$, all the results about forcing with Q in Sections 2 and 3 are applicable to the forcing by Q_α over $V[G_\alpha]$ that produces $V[G_{\alpha+1}]$.

By Proposition 2.5, $P_\alpha \Vdash$ "$\mathbf{Q}_\alpha$ is proper" for every $\alpha < \aleph_2$. By [7, III.3.2], each P_α, including $P_{\aleph_2} = P$, is proper.

By Theorem 3.3, every P-point in $V[G_\alpha]$ generates a P-point in $V[G_{\alpha+1}]$. Using this fact at sucessor stages and Theorem 4.1 at limit stages, one sees by induction on α that

5.2. *Every P-point in V generates P-points in all $V[G_\alpha]$ ($\alpha < \aleph_2$) and in $V[G]$.*

The definition of Q in Section 2 makes it obvious that Q has the cardinality of the continuum; thus $P_\alpha \Vdash$ "$\mathbf{Q}_\alpha$ has the cardinality of the continuum". Since we have assumed the continuum hypothesis in the ground model V, we find by [7, III.4.1.], that:

5.3. *For every $\alpha < \aleph_2$, P_α has a dense subset of cardinality $\leq \aleph_1$. So $P_\alpha \Vdash 2^{\aleph_0} = \aleph_1$.*

5.4. *P satisfies the $\aleph_2$ chain condition.*

(In [7, III.4.1], a cardinality bound on $\mathbf{Q}_i$ in V^{P_i} is assumed; in the present context, this seems to require that we know the continuum hypothesis in V^{P_i} *a priori*, so the proof seems circular. But in fact, in order to get the continuum hypothesis in V^{P_i} one needs the cardinality bounds only for $\mathbf{Q}_j$ in V^{P_j} for $j < i$, so the apparent circularity reduces to a legitimate induction.) The chain condition (5.4) implies that forcing with P preserves all cardinals $\geq \aleph_2$; properness implies that it also preserves $\aleph_1$. Thus,

5.5. *Cardinals are absolute between V and $V[G]$.*

We shall also need the following immediate consequence of [7, V.4.4].

5.6. *For any real $x \in V[G]$, there exists $\alpha < \aleph_2$ with $x \in V[G_\alpha]$, and the smallest such α has cofinality $\leq \omega$.*

The next lemma is implicit in the proofs of III.3.2 and III.4.1 in [7]. For all $\alpha \leq \aleph_2$, we consider P_α-names x of reals (viewed as functions $\omega \to 2$) as being specified by giving, for each $n \in \omega$, a maximal antichain $A(x, n)$ in P_α and, for each $p \in A(x, n)$, a value $v(x, n, p) \in \{0, 1\}$ such that $p \Vdash$ "$x(n) = v(x, n, p)$". It is well known that every name of a real is equivalent, in the sense of equality forced by all conditions, to one of this sort. When we are interested only in conditions extending a particular q, then the antichains $A(x, n)$ need to be maximal only in the weaker sense that no extension of q can be added to them; we then refer to x as a name for a real relative to q. We call such a name x *hereditarily countable* if, for each n, $A(x, n)$ is countable and all the P_β-names of reals, occurring in the conditions p_β constituting any $p = \langle p_\beta : \beta < \alpha \rangle \in A(x, n)$, are hereditarily countable.

As usual, let χ be a regular cardinal so large that $H(\chi)$ contains all sets of interest to us.

Lemma 5.7. *Let N be a countable elementary submodel of $H(\chi)$ that contains P, and let p be an (N, P)-generic condition. Then for every ordinal $\alpha \leq \aleph_2$ in N and every P_α-name $x \in N$ for a real, there is a hereditarily countable P_α-name y relative to p such that $p \Vdash$ "$x = y$".*

Proof. The (N, P)-genericity of p implies that $p \restriction \alpha$ is (N, P_α)-generic for all $\alpha \in N$. The lemma is proved by induction on α. To obtain y, first replace each of the antichains $A(x, n)$ by its intersection with N. Genericity ensures that these intersections are maximal relative to $p \restriction \alpha$, in the sense described above, and that the name x' obtained in this way is forced by $p \restriction \alpha$ to equal x. Since N is countable, the antichains $A(x', n) = A(x, n) \cap N$ are countable. If $q = \langle q_\beta : \beta < \alpha \rangle$ is in one of these antichains, hence in N, then N also contains an enumeration, in an ω-sequence, of all the (countably many, as the iteration has countable support) non-trivial components q_β. Thus, each of these components q_β is in N and can therefore, by induction hypothesis, be replaced by a hereditarily countable P_β-name. Doing this simultaneously for all such q and β, we obtain the desired name y. □

Corollary 5.8. *If x is a P-name for a real, then the set of conditions that force $x = y$ for some hereditarily countable y is dense.*

Proof. Given x and an arbitrary condition q, let N be a countable elementary submodel of $H(\chi)$ containing P, q, and x. As P is proper, q has an (N, P)-generic extension p. By the lemma, $p \Vdash$ "$x = y$" for some hereditarily countable y. □

Lemma 5.9. *For each $\alpha < \aleph_2$, there are only $\aleph_1$ hereditarily countable P_α-names, relative to any particular condition.*

Proof. We proceed by induction on α. A hereditarily countable name $\boldsymbol{x}$ is determined by countably many countable antichains $A(\boldsymbol{x}, n)$ and the function $v(\boldsymbol{x}, n, p)$. Since we are assuming the continuum hypothesis, and therefore $\aleph_1^{\aleph_0} = \aleph_1$, it suffices to check that there are at most $\aleph_1$ conditions $p = \langle p_\beta : \beta < \alpha \rangle$ that can occur in these antichains. Since our iteration uses countable supports, we need only check that there are, for each β, at most $\aleph_1$ possibilities for p_β. But p_β is required to be a hereditarily countable name, so the induction hypothesis gives us what we need. □

Lemma 5.10. *Let $\mathscr{F} \in V[G]$ be a set of reals. There is an $\aleph_1$-closed unbounded set of ordinals $\alpha < \aleph_2$ for which $\mathscr{F} \cap V[G_\alpha] \in V[G_\alpha]$.*

Proof. Fix a P-name $\boldsymbol{\mathscr{F}}$ for $\mathscr{F}$. For each $\gamma < \aleph_2$ and each hereditarily countable P_γ-name $\boldsymbol{y}$ for a real, consider a maximal antichain $B(\boldsymbol{y})$ of conditions that decide whether $\boldsymbol{y} \in \boldsymbol{\mathscr{F}}$. By 5.4 and 5.9, there are only $\aleph_1$ conditions altogether in these antichains. Since P is the direct limit of the P_β's, there is a single P_β $(\gamma < \beta < \aleph_2)$ containing all these conditions. We claim that $\mathscr{F} \cap V[G_\gamma] \in V[G_\beta]$. Indeed, we claim that

$$\mathscr{F} \cap V[G_\gamma] = \{\boldsymbol{y}^{(G_\gamma)} \mid \boldsymbol{y} \text{ is a hereditarily countable } P_\gamma\text{-name and some } p \in G_\beta \text{ forces } \boldsymbol{y} \in \boldsymbol{\mathscr{F}}\}.$$

To see this, note first that the '$\supseteq$' direction is obvious. For the converse, consider any element of $\mathscr{F} \cap V[G_\gamma]$. By Corollary 5.8 and genericity of G, it has a hereditarily countable name $\boldsymbol{y}$. By the choice of β and genericity of G_β, some condition p in G_β decides whether $\boldsymbol{y} \in \boldsymbol{\mathscr{F}}$. Since $\boldsymbol{y}^{(G)} = \boldsymbol{y}^{(G_\gamma)} \in \mathscr{F} = \boldsymbol{\mathscr{F}}^{(G)}$, no condition in $G_\beta \subseteq G$ can force $\boldsymbol{y} \notin \boldsymbol{\mathscr{F}}$, so p must force $\boldsymbol{y} \in \boldsymbol{\mathscr{F}}$. Therefore $\boldsymbol{y}^{(G_\gamma)} \in \mathscr{F}$, as desired.

For each $\gamma < \aleph_2$, let $h(\gamma)$ be a $\beta < \aleph_2$ as in the preceding paragraph. Let C be the set of ordinals $<\aleph_2$ that are closed under h. Then C is closed and unbounded, so the set of ordinals in C with cofinality $\aleph_1$ is $\aleph_1$-closed and unbounded. Consider any such ordinal α. By 5.6, each real in $V[G_\alpha]$ is already in $V[G_\gamma]$ for some $\gamma < \alpha$. Therefore, by choice of h,

$$\mathscr{F} \cap V[G_\alpha] = \bigcup_{\gamma<\alpha} \{\boldsymbol{y}^{(G_\gamma)} \mid \boldsymbol{y} \text{ is a hereditarily countable } P_\gamma\text{-name and some } p \in G_{h(\gamma)} \text{ forces } \boldsymbol{y} \in \boldsymbol{\mathscr{F}}\},$$

which is in $V[G_\alpha]$ since $h(\gamma) < \alpha$ for all $\gamma < \alpha$. □

Proof of Theorem 5.1. For each $\alpha < \aleph_2$, let

$$W_\alpha = \bigcup \{w \mid (w, T) \in H_\alpha\},$$

and, for $x \in \omega$,

$$f_\alpha(x) = \text{the number of elements of } W_\alpha \text{ that are } \leqslant x.$$

Thus, these are the W and f of Section 3, in the context of the Q_α-forcing that adjoins H_α to $V[G_\alpha]$. Each f_α is a finite-to-one function $\omega \to \omega$, and we shall find, for each pair of ultrafilters, $\mathcal{U}_1$ and $\mathcal{U}_2$, in $V[G]$, an α such that $f_\alpha(\mathcal{U}_1) = f_\alpha(\mathcal{U}_2)$.

Since we are assuming the continuum hypothesis in the ground model, there is a P-point in V, and it generates a P-point $\mathcal{U}_0$ in $V[G]$ by 5.2. To complete the proof, it suffices to find, for any ultrafilter $\mathcal{U}_1$ in $V[G]$, an $\aleph_1$-closed unbounded set of α's $<\aleph_2$ for which $f_\alpha(\mathcal{U}_1) = f_\alpha(\mathcal{U}_0)$. Indeed, if we do this for both $\mathcal{U}_1$ and $\mathcal{U}_2$, then, since the two $\aleph_1$-closed unbounded sets intersect (in an $\aleph_1$-closed unbounded set) there exists (an $\aleph_1$-closed unbounded set of) $\alpha < \aleph_2$ with $f_\alpha(\mathcal{U}_1) = f_\alpha(\mathcal{U}_0) = f_\alpha(\mathcal{U}_2)$.

So let an ultrafilter $\mathcal{U}_1 \in V[G]$ be given. By Lemma 5.10, there is an $\aleph_1$-closed unbounded set of $\alpha < \aleph_2$ such that $\mathcal{U}_1 \cap V[G_\alpha] \in V[G_\alpha]$. For such an α, $\mathcal{U}_1 \cap V[G_\alpha]$ is clearly an ultrafilter in the sense of $V[G_\alpha]$. So is $\mathcal{U}_0 \cap V[G_\alpha]$, since the P-point in V that generates $\mathcal{U}_0$ also generates P-points in every $V[G_\alpha]$, by 5.2. By Corollary 3.5, applied to Q_α-forcing over $V[G_\alpha]$,

$$f_\alpha(\mathcal{U}_0 \cap V[G_\alpha]) = f_\alpha(\mathcal{U}_1 \cap V[G_\alpha]) \subseteq f_\alpha(\mathcal{U}_1).$$

Since $\mathcal{U}_0 \cap V[G_\alpha]$ generates $\mathcal{U}_0$, $f_\alpha(\mathcal{U}_0 \cap V[G])$ generates $f_\alpha(\mathcal{U}_0)$. Therefore $f_\alpha(\mathcal{U}_0) \subseteq f_\alpha(\mathcal{U}_1)$. Finally, since $f_\alpha(\mathcal{U}_0)$ is an ultrafilter and $f_\alpha(\mathcal{U}_1)$ a filter, it follows that $f_\alpha(\mathcal{U}_0) = f_\alpha(\mathcal{U}_1)$. □

We conclude this section with some additional information about the model $V[G]$.

Theorem 5.2. *In $V[G]$, the following are true.*

(a) $2^{\aleph_0} = \aleph_2$.

(b) *There is a P-point generated by $\aleph_1$ sets.*

(c) $b = \aleph_1$.

(d) $d = \aleph_2$.

(e) $s = \aleph_2$.

Proof (a) By 5.6, all reals of $V[G]$ are in some $V[G_\alpha]$, and by 5.3 each of the $\aleph_2$ $V[G_\alpha]$'s contributes only $\aleph_1$ reals. So in $V[G]$, $2^{\aleph_0} \leqslant \aleph_2$. On the other hand, each step in the iteration adds a new real (cf. 3.1) and cardinals are preserved (5.5), so $2^{\aleph_0} = \aleph_2$ in $V[G]$.

(b) The $\mathcal{U}_0$ in the proof of Theorem 5.1 is generated by sets in the ground model, and there are only $\aleph_1$ such sets.

(c) b is always uncountable, and, by a theorem of Solomon [9], no ultrafilter is generated by fewer than b sets. So (c) follows from (b).

(d) By 5.6, any family of $\aleph_1$ reals is included in $V[G_\alpha]$ for some $\alpha < \aleph_2$. But then by 3.6 that family fails to dominate the function

$$n \mapsto \text{the } n\text{th element of } W_\alpha.$$

(e) Again, any family of $\aleph_1$ subsets of ω lies in some $V[G_\alpha]$ and therefore, by 3.1, fails to split W_α. □

6. The consistency of (B)

In this section, we show how to modify the previous construction so as to obtain a model containing both a simple $P_{\aleph_1}$-point and a simple $P_{\aleph_2}$-point.

As before, we assume the continuum hypothesis in the ground model V, and we let G be a V-generic subset of the direct limit $P = P_{\aleph_2}$ of a certain countable support iteration $\langle P_\alpha, \mathbf{Q}_\alpha : \alpha < \aleph_2 \rangle$. This iteration differs from the previous one in that, in V^{P_α}, $\mathbf{Q}_\alpha$ is not the Q described in Section 2 but rather the subset

$$\{(w, T) \mid \text{for each } \beta < \alpha, \text{ there exists an extension } (w', T') \text{ of } (w, T) \text{ such that } \operatorname{ps}(w', T') \subseteq^* \mathbf{W}_\beta\}.$$

Here, $\subseteq^*$ denotes inclusion modulo a finite set, and $\mathbf{W}_\beta$ is the $P_{\beta+1}$-name (hence also P_α-name) for the subset $\bigcup \{w \mid (w, T) \in \mathbf{G}_\beta\}$ adjoined to V^{P_β} by forcing with $\mathbf{Q}_\beta$. The ordering of $\mathbf{Q}_\alpha$ is the same as for Q.

Let G be a V-generic subset of P, and, as in the previous section, let G_α and H_α be the corresponding V-generic subset of P_α and $V[G_\alpha]$-generic subset of $Q_\alpha = \mathbf{Q}_\alpha^{(G_\alpha)}$, respectively.

Theorem 6.1. *In $V[G]$, there exist simple $P_{\aleph_1}$-points and simple $P_{\aleph_2}$-points.*

Before starting the proof of this theorem, we need some preliminary information about the forcing notions involved.

To say that a condition (w, T) in Q has an extension (w', T') with $\operatorname{ps}(w', T') \subseteq^* X$ implies that we can obtain, by composing relations in (w, T) a condition (w, T^*) such that $\operatorname{Dp}((t_n^*)_X)$ is an unbounded function of n. To see this, note that any extension (w', T') is obtainable from a composing-relations extension by shrinking relations and fixing values. These last two operations cannot convert a (w, T^*) with $\operatorname{Dp}((t_n^*)_X)$ bounded to a (w', T') with $\operatorname{Dp}((t_n')_X)$ unbounded, but this unboundedness is needed if we are to have $\operatorname{ps}(w', T') \subseteq^* X$.

On the other hand, if (w, T) has a composing-relations extension (w, T^*), with $\operatorname{Dp}((t_n^*)_X)$ unbounded, then we can find, for any prescribed m, an n-extension (w, T') of (w, T) with $\operatorname{ps}(w, T') \subseteq^* X$. To see this, first note that T^* can be taken to agree with T in the first m components, since only the behavior for large n of t_n^* matters. Then compose relations again, beyond t_m^*, to arrange that $\operatorname{Dp}((t_n^*)_X) \to \infty$. Finally replace t_n^* with $t_n' = (t_n^*)_X$ for all $n \geq m$. Observe also that, if $Y \subseteq^* X$, and $\operatorname{Dp}((t_n)_Y)$ is unbounded, then $\operatorname{Dp}((t_n')_Y)$ is also unbounded, because this unboundedness is preserved both by composition of relations and by shrinking t_n to $(t_n)_X$.

Lemma 6.2. *If $\beta < \gamma < \aleph_2$, then $W_\gamma \subseteq^* W_\beta$.*

Proof. We show by induction on α that $\beta<\gamma<\alpha$ implies $W_\gamma\subseteq^* W_\beta$. It suffices to deduce, from this induction hypothesis, that $W_\beta\subseteq^* W_\alpha$ for all $\beta<\alpha$. We work in $V[G_\alpha]$ and consider any $(w, T)\in Q_\alpha$. Given any $\beta<\alpha$, we have an extension $(w', T')\in Q$ with $\mathrm{ps}(w', T')\subseteq^* W_\beta$. By induction hypothesis, it follows that $\mathrm{ps}(w', T')\subseteq^* W_\delta$ for all $\delta\leq\beta$. The induction hypothesis and the remark preceding the lemma show that we can construct (w', T') so that it has extensions satisfying $\mathrm{ps}(w'', T'')\subseteq^* W_\gamma$ for any prescribed $\gamma\in[\beta, \alpha)$. Thus, we can construct (w', T') so that it is in Q_α. Thus, we have shown that, for each $\beta<\alpha$, the set

$$\{(w', T')\in Q_\alpha \mid \mathrm{ps}(w', T')\subseteq^* W_\beta\}$$

is dense in Q_α. Every condition in this dense set clearly forces "$\mathbf{W}_\alpha\subseteq^* W_\beta$", so $W_\alpha\subseteq^* W_\beta$. □

The core of the proof of Theorem 5.1 is the following proposition, which carries over to the present forcing construction some crucial properties of the construction in the preceding section.

Proposition 6.3. *For each $\alpha<\aleph_2$ and each non-limit $\beta<\alpha$:*
(a) *$P_{\alpha+1}/P_\beta$ is a proper notion of forcing in V^{P_β}.*
(b) *Every P-point in V^{P_β} generates a P-point in $V^{(P_{\alpha+1})}$.*
(c) *For every $A\subseteq\omega$ in $V[G_\alpha]$, either $W_\alpha\subseteq^* A$ or $W_\alpha\subseteq^*\omega-A$.*

Proof. We remark first that (a) is sufficient to ensure that P_λ/P_β is proper, for $\beta<\lambda$, when λ is a limit ordinal and β is not. This can be seen by looking carefully at the proof [7, III.3.2] of the preservation of properness under iteration; that proof essentially uses only (a), not properness of Q_α. See also [7, X.2.3 and X.2.6]. Similar remarks apply to the conjunction of (a) and (b); P-points in $V[G_\beta]$ for non-limit β generate P-points in $V[G_\gamma]$ for limit $\lambda>\beta$, by essentially the same proof as in Section 4.

To prove 6.3, we proceed by induction on α.

If α is a successor, say $\alpha=\gamma+1$, then the forcing notion Q_α in $V[G_\alpha]$ consists of all (w, T) having an extension satisfying $\mathrm{ps}(w', T')\subseteq^* W_\gamma$, because of Lemma 6.2. By the proof of that lemma, this (w', T') can be taken to be in Q_α also. Thus,

$$Q'_\alpha=\{(w, T)\mid \mathrm{ps}(w, T)\subseteq^* W_\gamma\}$$

is dense in Q_α. The extensions of any $(w, T)\in Q'_\alpha$ are the same in Q'_α as in Q. Thus, forcing with Q_α, with Q'_α, and with Q are all equivalent. Now (a) follows from the properness of $P_{\gamma+1}/P_\beta=P_\alpha/P_\beta$ and Proposition 2.5. (b) follows from P-point preservation from $V[G_\beta]$ to $V[G_\alpha]$ and Theorem 3.3. And (c) follows from Proposition 3.1.

If α is a limit ordinal of cofinality ω, let $\langle\gamma(n):n\in\omega\rangle$ be cofinal in α. Q_α consists, by Lemma 2.2, of all (w, T) having, for each n, an extension with

$\mathrm{ps}(w', T') \subseteq^* W_{\gamma(n)}$. As we saw before Lemma 2.2, each such (w, T) has, for each $n \in \omega$, an n-extension $(w', T') \in Q_\alpha$ with $\mathrm{ps}(w', T') \subseteq^* W_{\gamma(n)}$. This fact allows us to do a fusion argument (cf. Proposition 2.3) to obtain an extension (w'', T'') with $\mathrm{ps}(w'', T'') \subseteq^* W_{\gamma(n)}$ for all $n \in \omega$ simultaneously. Thus,

$$Q'_\alpha = \{(w, T) \mid \text{for all } n \in \omega,\ \mathrm{ps}(w, T) \subseteq^* W_{\gamma(n)}\}$$

is dense in Q_α. The extensions in Q of any $(w, T) \in Q'_\alpha$ are all in Q', so forcing with Q_α, with Q'_α, and with Q are all equivalent. As in the successor case, this completes the proof.

Finally, we consider the case that α has cofinality $\aleph_1$. Let K_β be any V-generic subset of P_β. To prove (a), it suffices, since K_β is arbitrary, to show that $P_{\alpha+1}/K_\beta$ is proper in $V[K_\beta]$. In $V[K_\beta]$, let N be a countable elementary submodel of $H(\chi)$ containing $P_{\alpha+1}/K_\beta$, where χ is, as usual, a sufficiently large regular cardinal. Let a condition $(q^0, \boldsymbol{r}^0) \in (P_{\alpha+1}/K_\beta) \cap N$ be given, where $q^0 \in P_\alpha/K_\beta$ and $q^0 \Vdash$ "$\boldsymbol{r}^0 \in \boldsymbol{Q}_\alpha$". Since q^0 has countable support and α has uncountable cofinality, $q^0 \in P_\gamma/K_\beta$ for some $\gamma < \alpha$. Since N is an elementary submodel of $H(\chi)$, we can take γ to be an element of N. Let μ be the supremum of $\alpha \cap N$. Since N is countable and $\mathrm{cf}(\alpha)$ is not, and since $\gamma \in N$, we have $\gamma < \mu < \alpha$. By the induction hypothesis and the first paragraph of this proof, P_μ/K_β is proper. Let q be an $(N, P_\mu/K_\beta)$-generic extension of q^0 in P_μ/K_β. We intend to define $\boldsymbol{p}$ and $\boldsymbol{r}$ such that $q \Vdash$ "$\boldsymbol{p} \in P_\alpha/\boldsymbol{G}_\mu$", $(q, \boldsymbol{p}) \Vdash$ "$\boldsymbol{r}$ is an extension of $\boldsymbol{r}^0$ in $\boldsymbol{Q}_\alpha$", and $(q, \boldsymbol{p}, \boldsymbol{r})$ is $(N, P_{\alpha+1}/K_\beta)$-generic. This will suffice to complete the proof of (a), since $(q, \boldsymbol{p}, \boldsymbol{r})$ will then be an extension of (q^0, r^0). To obtain such $\boldsymbol{p}$ and $\boldsymbol{r}$, we work with an arbitrary $V[K_\beta]$-generic $K_\mu \subseteq P_\mu/K_\beta$ containing q, and we find, in $V[K_\beta, K_\mu]$, a $p \in P_\alpha/(K_\beta * K_\mu)$ and an $\boldsymbol{r}$ such that $p \Vdash$ "$\boldsymbol{r}$ is an extension of $\boldsymbol{r}_0$ in $\boldsymbol{Q}_\alpha$" and $(p, \boldsymbol{r})$ is $(N[K_\mu], P_{\alpha+1}/K_\beta{*}K_\mu)$-generic. Carrying out this construction in $V[K_\beta]^{(P_\mu/K_\beta)}$ with the canonical generic subset of P_μ/K_β as K_μ, we would then get the required names $\boldsymbol{p}$ and $\boldsymbol{r}$. (See [7, p. 91] for genericity of $(q, \boldsymbol{p}, \boldsymbol{r})$.)

Let $\langle \tau_i : i < \omega \rangle$ be an enumeration of all the $(P_{\alpha+1}/K_\mu)$-names in $N[K_\mu]$ of ordinals. Since $q^0 \Vdash$ "$\boldsymbol{r}^0 \in \boldsymbol{Q}_\alpha$" and K_μ contains the extension q of q^0, the K_μ-value r^0 of $\boldsymbol{r}^0$ is a member (w^0, T^0) of Q_α. Combining the fusion argument in the $\mathrm{cf}(\alpha) = \omega$ case above with the fusion argument in the proof of Proposition 2.4 (by interleaving the steps of the two constructions), we obtain an extension (w, S) of r such that $\mathrm{ps}(w, S) \subseteq^* W_\delta$ for all $\delta \in \alpha \cap N$ (countably many δ's) and such that the conclusion of 2.4 holds. As in the proof of Proposition 2.5, the fusion argument can be carried out so that every finite part of it takes place in $N[K_\mu]$ and therefore the particular values of the $\boldsymbol{\tau}_i$'s forced by extensions of (w, S) all lie in $N[K_\mu]$. If (w, S) were in Q_α, we could use it as r (i.e., use its standard name as $\boldsymbol{r}$) and take p to be trivial. Unfortunately, although we have $\mathrm{ps}(w, S) \subseteq^* W_\zeta$ for all $\zeta < \mu$ since $\alpha \cap N$ is cofinal in μ, we do not know that (w, S) has extensions with $\mathrm{ps}(w', S') \subseteq^* W_\zeta$ for $\mu \leq \zeta < \alpha$. In other words, (w, S) is in Q_μ, but it is not forced by the trivial condition in P_α/K_μ to be in $\boldsymbol{Q}_\alpha$. This difficulty is remedied by using a non-trivial condition as p, namely (w, S) itself. We define $p \in P_\alpha/K_\mu$ to be

$(w, S) \in Q_\mu = P_{\mu+1}/K_\mu \subseteq P_\alpha/K_\mu$ (so p is the sequence consisting of (w, S) in Q_μ followed by trivial conditions in Q_ζ for $\mu < \zeta < \alpha$) and we define r to be the standard (P_α/K_μ)-name for (w, S). Since p forces all W_ζ for $\mu \leq \zeta < \alpha$ to be subsets of $\mathrm{ps}(w, S)$ modulo finite sets, it forces $r \in Q_\alpha$, so p and r are as required to finish the proof of (a).

To prove (b), let $\mathcal{U}$ be a P-point in $V[K_\beta]$ and let A be a $(P_{\alpha+1}/K_\beta)$-name for a subset of ω. Starting with any condition (q^0, r^0) in $(P_{\alpha+1}/K_\beta)$, we first proceed as in the proof of (a), using an N that contains $\mathcal{U}$ and the name A, until we obtain (w, S) as there. Then we follow, in $N[K_\mu]$, the proof of Theorem 3.3 to obtain an extension (w', S') of (w, S) in Q_μ forcing A to be included in or disjoint from some set in the ultrafilter in $N[K_\mu]$ generated by $\mathcal{U}$. (It does generate an ultrafilter, by the induction hypothesis, the first paragraph of this proof, and the fact that $N[K_\mu]$ is an elementary submodel of $H(\chi)^{(V[K_\beta, K_\mu])}$ because K_μ contains the $(N, P_\mu/K_\beta)$-generic condition q.) Finally, as in the proof of (a), we extend q by putting (w', S') in the μth and αth components and trivial conditions between them. The resulting condition in $P_{\alpha+1}/K_\beta$ forces A to include or be disjoint from a set in $\mathcal{U}$.

The proof of (c) is just like that of (b) except that, instead of using the proof of Theorem 3.3 to extend (w, S) to (w', S'), we use the proof of Proposition 3.1. This completes the proof of Proposition 6.3. □

Corollary 6.4. (a) *P is proper.*

(b) *Every P-point in V generates a P-point in $V[G]$.*

Proof. See the first paragraph of the proof of Proposition 6.3. □

Proof of Theorem 6.1. Just as in Section 5, we see that P satisfies the $\aleph_2$ chain condition by [7, III.4.1]. Therefore each real in $V[G]$ is in $V[G_\alpha]$ for some $\alpha < \aleph_2$.

Since we are assuming the continuum hypothesis in the ground model V, there is a P-point, in fact (necessarily) a simple $P_{\aleph_1}$-point, $\mathcal{U}$ in V. By Corollary 6.4(b), $\mathcal{U}$ generates a P-point $\bar{\mathcal{U}}$ in $V[G]$, and $\bar{\mathcal{U}}$ is a simple $P_{\aleph_1}$-point because it is generated by the same almost decreasing $\aleph_1$-sequence as $\mathcal{U}$. ($\aleph_1$ is absolute by Corollary 6.4(a) or because an ultrafilter $\bar{\mathcal{U}}$ cannot be generated by countably many sets.)

Since every real in $V[G]$ is in some $V[G_\alpha]$, Proposition 6.3(c) implies that the sets W_α ($\alpha < \aleph_2$) and the cofinite sets generate an ultrafilter in $V[G]$. By Lemma 6.2, the W_α constitute a strictly almost decreasing $\aleph_2$-sequence, so this ultrafilter is a simple $P_{\aleph_2}$-point. □

As in Section 5, the model $V[G]$ is easily seen to satisfy $2^{\aleph_0} = \aleph_2$. We can also show that $b = \aleph_1$ and $d = s = \aleph_2$ by the same method as in Section 5, but in fact these equations follow from the existence of simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points, by work of Nyikos [5].

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