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Nonreflecting stationary sets in $\mathcal{P}_\kappa\lambda$

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Abstract

Let κ be a regular uncountable cardinal and $\lambda \geqslant \kappa^+$. The principle of Stationary Reflection in $\mathcal{P}_\kappa\lambda$ has been successful in settling problems of infinitary combinatorics in the case $\kappa = \omega_1$. For $\kappa \geqslant \omega_2$ the principle is known to fail if λ is large enough. In this paper the principle is shown to fail for every $\lambda \geqslant \kappa^+$.

MSC: 03E05

Keywords: Stationary reflection; Nonstructure theory

1. Introduction

In [6] Foreman et al. introduced the following principle for a cardinal $\lambda \geqslant \omega_2$: If S is a stationary set in $\mathcal{P}_{\omega_1}\lambda$, $S \cap \mathcal{P}_{\omega_1} A$ is stationary in $\mathcal{P}_{\omega_1} A$ for some $\omega_1 \subset A \subset \lambda$ of size ω_1. Let us call the principle Stationary Reflection in $\mathcal{P}_{\omega_1}\lambda$. It follows from Martin's Maximum, and holds after a supercompact cardinal is Lévy-collapsed to ω_2 [6].

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For recent applications of reflection principles for stationary sets in $\mathcal{P}_{\omega_1}\lambda$, see e.g. [3,14,16,17].

What if ω_1 is replaced by a higher regular cardinal? Feng and Magidor [4] proved that Stationary Reflection in $\mathcal{P}_{\omega_2}\lambda$ fails if λ is large enough. Their argument shows in effect that Stationary Reflection in $\mathcal{P}_\kappa\lambda$ for some large enough λ implies that the club filter on κ is presaturated (see also [2]). It is known that the club filter on a successor cardinal $\geqslant\omega_2$ cannot be presaturated [10].

Extending the Feng–Magidor result, Foreman and Magidor [5] proved in effect that Stationary Reflection in $\mathcal{P}_\kappa\lambda$ fails if κ is regular $\geqslant\omega_2$ and λ is large enough. More precisely

Theorem 1. *Let κ be regular $\geqslant\omega_2$. Then Stationary Reflection in $\mathcal{P}_\kappa\lambda$ fails for every $\lambda\geqslant 2^{\kappa^+}$.*

We include a proof of Theorem 1 in §4. A further example of nonreflection, which is based on PCF Theory [11] can be found in [12].

This paper shows that for $\kappa\geqslant\omega_2$ Stationary Reflection in $\mathcal{P}_\kappa\lambda$ fails *everywhere*:

Theorem 2. *Let κ be regular $\geqslant\omega_2$. Then Stationary Reflection in $\mathcal{P}_\kappa\lambda$ fails for every $\lambda\geqslant\kappa^+$.*

In §3, we prove Theorem 2 in much greater generality.

2. Preliminaries

For background material we refer the reader to [7]. Throughout the paper, we use κ, λ, μ to denote an infinite cardinal. We write S^κ_λ for $\{\gamma < \lambda : \operatorname{cf}\gamma = \kappa\}$, and $[\lambda]^\mu$ for $\{x \subset \lambda : |x| = \mu\}$.

Let A be a set of ordinals. The set of limit points of A is denoted $\lim A$. It is easy to see $|\lim A|\leqslant|A|$. A is called σ-closed if every element of $\lim A$ of cofinality ω is in A.

Let κ be regular, $\omega_1\leqslant\kappa\leqslant\mu < \lambda$ and $f : [\lambda]^{<\omega} \to \mathcal{P}_\kappa\lambda$. We write $C(f)$ for $\{x \in \mathcal{P}_\kappa\lambda : f``[x]^{<\omega} \subset \mathcal{P}(x)\}$. For $x \in \mathcal{P}_\kappa\lambda$ the smallest superset of x in $C(f)$ is denoted $\operatorname{cl}_f x$. It is well-known that if C is club in $\mathcal{P}_\kappa\lambda$, there is $f : [\lambda]^{<\omega} \to \mathcal{P}_\kappa\lambda$ with $C(f) \subset C$.

Stationary Reflection in $\mathcal{P}_\kappa\lambda$ states that if S is a stationary set in $\mathcal{P}_\kappa\lambda$, $S\cap\mathcal{P}_\kappa A$ is stationary in $\mathcal{P}_\kappa A$ for some $\kappa \subset A \subset \lambda$ of size κ. Let S be a stationary set in $\mathcal{P}_\kappa\lambda$. S is called nonreflecting if it witnesses the failure of Stationary Reflection, i.e. $S\cap\mathcal{P}_\kappa A$ is nonstationary in $\mathcal{P}_\kappa A$ for every $\kappa \subset A \subset \lambda$ of size κ. More generally S is called μ-nonreflecting if $S\cap\mathcal{P}_\kappa A$ is nonstationary in $\mathcal{P}_\kappa A$ for every $\mu \subset A \subset \lambda$ of size μ. If S is a μ-nonreflecting stationary set in $\mathcal{P}_\kappa\mu^+$ and $\mu^+\leqslant\lambda$, $\{x \in \mathcal{P}_\kappa\lambda : x\cap\mu^+ \in S\}$ is easily seen to be a μ-nonreflecting stationary set in $\mathcal{P}_\kappa\lambda$. In particular Stationary Reflection in $\mathcal{P}_\kappa\lambda$ fails for every $\lambda\geqslant\kappa^+$ iff Stationary Reflection in $\mathcal{P}_\kappa\kappa^+$ fails.

3. Main theorem

This section is devoted to the main Theorem 3 and its corollaries. We prove Theorem 3 using ideas from Nonstructure Theory [13]. Similar ideas can be found in the proof of Diamond for $\mathcal{P}_\kappa\lambda$ [10,15].

Theorem 3. *Let κ be regular $\geqslant\omega_2$ and μ a cardinal $\geqslant\kappa$. Assume there are $\{c_\xi : \xi < \mu\} \subset \mathcal{P}_\kappa\mu$ and a stationary $T \subset \mathcal{P}_\kappa\mu$ of size μ such that if $z \in T$ and $b \in [z]^\omega$, there is $\xi \in z$ with $b \subset c_\xi$. Then $\mathcal{P}_\kappa\lambda$ has a μ-nonreflecting stationary subset for every $\lambda \geqslant \mu^+$.*

Proof. It suffices to give a μ-nonreflecting stationary set in $\mathcal{P}_\kappa\mu^+$.

Let $\{c_\xi : \xi < \mu\}$ and T be as above. By Solovay's theorem we can split $S^\omega_{\mu^+}$ into μ disjoint stationary sets $\{S_z : z \in T\}$. For $\mu\leqslant\gamma < \mu^+$ fix a bijection $\pi_\gamma : \mu \to \gamma$.

Set $S = \{x \in \mathcal{P}_\kappa\mu^+ : \forall\gamma \in x - \mu(\pi_\gamma``(x \cap \mu) \subset x) \wedge x \cap \mu \in T \wedge \sup x \in S_{x\cap\mu}\}$.

Claim. *S is stationary in $\mathcal{P}_\kappa\mu^+$.*

Proof. Since $\{x \in \mathcal{P}_\kappa\mu^+ : \forall\gamma \in x - \mu(\pi_\gamma``(x \cap \mu) \subset x)\}$ is club, it suffices to show that $\{x \in \mathcal{P}_\kappa\mu^+ : x \cap \mu \in T \wedge \sup x \in S_{x\cap\mu}\}$ is stationary.

Fix $f : [\mu^+]^{<\omega} \to \mathcal{P}_\kappa\mu^+$. For $z \in T$ consider the following game $\mathcal{G}(z)$ of length ω between two players *I* and *II*:

In round n *I* chooses $\mu\leqslant\gamma_n < \mu^+$. Then *II* chooses $x_n \in C(f)$ with $\gamma_n < \sup x_n$. We further require $\sup x_n < \gamma_{n+1}$ and $x_n \subset x_{n+1}$. Finally *II* wins just in case $x_n \cap \mu = z$ for every $n < \omega$.

Set $T' = \{z \in T : II \text{ has no winning strategy in } \mathcal{G}(z)\}$.

Subclaim. *T' is nonstationary in $\mathcal{P}_\kappa\mu$.*

Proof. Suppose otherwise. Note that the game $\mathcal{G}(z)$ is closed for *II*, hence determined. Hence for $z \in T'$ we have a winning strategy σ_z for *I* in $\mathcal{G}(z)$. Set $D = \{\delta < \mu^+ : f``[\delta]^{<\omega} \subset \mathcal{P}_\kappa\delta\}$, which is club. By induction on $n < \omega$ we define $\beta_n \in S^\omega_{\mu^+} \cap D$ and x^z_n for $z \in T'$ so that $\langle x^z_n : n < \omega\rangle$ is a play of *II* in $\mathcal{G}(z)$ against σ_z and $\sup x^z_n = \beta_n$ for every $z \in T'$ as follows:

Assume we have β_i and $\{x^z_i : z \in T'\}$ for $i < n$ as above. Since $|T'|\leqslant|T| = \mu$, we have $\sup_{z\in T'} \sigma_z(\langle x^z_i : i < n\rangle) < \beta_n \in S^\omega_{\mu^+} \cap D$. Then $\beta_{n-1} = \sup x^z_{n-1} < \sigma_z(\langle x^z_i : i < n\rangle) < \beta_n$ for every $z \in T'$.

Fix $z \in T'$. Since $\sup x^z_{n-1} < \beta_n \in S^\omega_{\mu^+} \cap D$, $C^z_n = \{x \in \mathcal{P}_\kappa\beta_n : x^z_{n-1} \subset x \in C(f) \wedge \sup x = \beta_n\}$ is club. Let x^z_n be $\pi_{\beta_n}``z$ if $\pi_{\beta_n}``z \in C^z_n$, otherwise an element of C^z_n.

Set $\beta = \sup_{n<\omega} \beta_n$. Then $\mu\leqslant \sup_{z\in T'} \sigma_z(\emptyset) < \beta_0 < \beta$. Since $\beta_n \in S^\omega_{\mu^+} \cap D$ for every $n < \omega$, $C = \{x \in \mathcal{P}_\kappa\beta : \forall n < \omega\ (\pi_{\beta_n}``(x \cap \mu) = x \cap \beta_n \in C(f) \wedge \sup(x \cap \beta_n) = \beta_n)\}$ is club. Since T' is stationary in $\mathcal{P}_\kappa\mu$, we can take $x \in C$ so that $x \cap \mu \in T'$.

Set $z = x \cap \mu \in T'$. Since $x \in C$, we see by induction on $n < \omega$ that $\pi_{\beta_n}``z = \pi_{\beta_n}``(x \cap \mu) = x \cap \beta_n \in C^z_n$ and $x^z_n = x \cap \beta_n$. Hence $x^z_n \cap \mu = x \cap \mu = z$ for every

$n < \omega$. Thus *II* wins in $\mathcal{G}(z)$ against σ_z with the play $\langle x_n^z : n < \omega\rangle$. This contradicts that σ_z is a winning strategy for *I* in $\mathcal{G}(z)$, as desired. □

Fix $z \in T - T'$ and a winning strategy τ for *II* in $\mathcal{G}(z)$. Since S_z is stationary in μ^+, we have $\mu < \gamma \in S_z$ such that $\sup \tau(s) < \gamma$ for every $s \in \gamma^{<\omega}$. Since $\gamma \in S_z \subset S^{\omega}_{\mu^+}$, we have γ_n inductively so that $\gamma_0 = \mu$, $\sup \tau(\langle \gamma_i : i < n\rangle) < \gamma_n$ and $\sup_{n<\omega} \gamma_n = \gamma$. Then $\langle \gamma_n : n < \omega\rangle$ is a play of *I* in $\mathcal{G}(z)$ against τ.

For $n < \omega$ set $x_n = \tau(\langle \gamma_i : i \leqslant n\rangle)$. Since τ is a winning strategy, *II* wins in $\mathcal{G}(z)$ with the play $\langle x_n : n < \omega\rangle$. Hence $\{x_n : n < \omega\} \subset C(f)$ is increasing, $x_n \cap \mu = z$ and $\gamma_n < \sup x_n < \gamma_{n+1}$ for every $n < \omega$. Set $x = \bigcup_{n<\omega} x_n$. Then $x \in C(f)$, $x \cap \mu = z \in T$ and $\sup x = \sup_{n<\omega} \sup x_n = \sup_{n<\omega} \gamma_n = \gamma \in S_z = S_{x\cap\mu}$, as desired. □

Claim. *S is μ-nonreflecting.*

Proof. Suppose to the contrary $S \cap \mathcal{P}_\kappa A$ is stationary in $\mathcal{P}_\kappa A$ for some $\mu \subset A \subset \mu^+$ of size μ. Then $\{x \in \mathcal{P}_\kappa A : \forall \gamma \in x - \mu(\pi_\gamma``(x \cap \mu) \subset x)\}$ is unbounded in $\mathcal{P}_\kappa A$. Hence $\gamma = \pi_\gamma``\mu = \pi_\gamma``(A \cap \mu) \subset A$ for every $\gamma \in A - \mu$. Thus $A = \delta$ for some $\mu \leqslant \delta < \mu^+$.

Subclaim. $\mathrm{cf}\,\delta < \kappa$.

Proof. Since $\{x \in \mathcal{P}_\kappa\delta : \pi_\delta``(x \cap \mu) = x\}$ is club, $S' = \{x \in S \cap \mathcal{P}_\kappa\delta : \pi_\delta``(x \cap \mu) = x\}$ is stationary in $\mathcal{P}_\kappa\delta$. Fix $x \in S'$. Since $\sup x \in S_{x\cap\mu} \subset S^{\omega}_{\mu^+}$, we have $b_x \in [x]^\omega$ with $\sup b_x = \sup x$. Since $\pi_\delta^{-1}``b_x \in [x \cap \mu]^\omega$ and $x \cap \mu \in T$, we have $\xi \in x \cap \mu$ with $\pi_\delta^{-1}``b_x \subset c_\xi$.

Now we have $\xi^* < \mu$ and a stationary $S^* \subset S'$ such that $b_x \subset \pi_\delta``c_{\xi^*}$ for every $x \in S^*$. Since $S^*$ is unbounded in $\mathcal{P}_\kappa\delta$, $\delta = \sup_{x\in S^*} \sup x = \sup_{x\in S^*} \sup b_x \leqslant \sup \pi_\delta``c_{\xi^*} \leqslant \delta$. Since $|c_{\xi^*}| < \kappa$, $\delta = \sup \pi_\delta``c_{\xi^*}$ has cofinality $< \kappa$. □

Thus $\{x \in S \cap \mathcal{P}_\kappa\delta : \sup x = \delta\}$ is stationary in $\mathcal{P}_\kappa\delta$. Take x, y from this set so that $x \cap \mu \neq y \cap \mu$. Then $\delta = \sup x = \sup y \in S_{x\cap\mu} \cap S_{y\cap\mu}$. This contradicts $S_{x\cap\mu} \cap S_{y\cap\mu} = \emptyset$, as desired. □

Therefore $\mathcal{P}_\kappa\mu^+$ has a μ-nonreflecting stationary subset. □

Now Theorem 2 follows from Theorem 3 with $\mu = \kappa$: It is easy to check that the hypothesis of Theorem 3 is satisfied with $c_\xi = \xi$ for $\xi < \kappa$ and $T = S^{\omega_1}_\kappa$.

Theorem 3 with $\mu = \kappa^+$ yields the following:

Corollary 1. *Let κ be regular $\geqslant \omega_2$. Then $\mathcal{P}_\kappa\lambda$ has a κ^+-nonreflecting stationary subset for every $\lambda \geqslant \kappa^{++}$.*

Proof. It suffices to check that the hypothesis of Theorem 3 is satisfied.

For $\kappa \leqslant \gamma < \kappa^+$ we have a club $T_\gamma \subset \mathcal{P}_\kappa\gamma$ of size κ. List the elements of $\bigcup_{\kappa\leqslant\gamma<\kappa^+} T_\gamma$ as $\{c_\xi : \xi < \kappa^+\}$. Then $D = \{\delta < \kappa^+ : \bigcup_{\kappa\leqslant\gamma<\delta} T_\gamma = \{c_\xi : \xi < \delta\}\}$ is club. Set

$T = \{z \in \bigcup_{\kappa \leqslant \gamma < \kappa^+} T_\gamma : \forall b \in [z]^\omega \exists \xi \in z(b \subset c_\xi)\}$. Then $|T| \leqslant \kappa^+$. We show that T is stationary in $\mathcal{P}_\kappa \kappa^+$.

Fix $f : [\kappa^+]^{<\omega} \to \mathcal{P}_\kappa \kappa^+$. We have $\delta \in S^\kappa_{\kappa^+} \cap D$ with $f``[\delta]^{<\omega} \subset \mathcal{P}_\kappa \delta$. Then $T_\delta \cap C(f)$ is club in $\mathcal{P}_\kappa \delta$. Since $\mathrm{cf}\, \delta = \kappa$, $\{c_\xi : \xi < \delta\} = \bigcup_{\kappa \leqslant \gamma < \delta} T_\gamma$ is unbounded in $\mathcal{P}_\kappa \delta$. Hence we can build an increasing sequence $\{z_\alpha : \alpha < \omega_1\} \subset T_\delta \cap C(f)$ so that $z_\alpha \subset c_\xi$ for some $\xi \in z_{\alpha+1}$. Then $\bigcup_{\alpha < \omega_1} z_\alpha \in T \cap C(f)$, as desired. □

If $\mathrm{cf}\, \mu < \kappa$, $\mathcal{P}_\kappa \mu$ has no stationary subset of size μ. So we have nothing to say in this case. We have something to say, however, about a question of [8]:

Corollary 2. *Let κ be regular $\geqslant \omega_2$ and $\mu^{<\kappa} = \mu$. Then $\mathcal{P}_\kappa \lambda$ has a μ-nonreflecting stationary subset for every $\lambda \geqslant \mu^+$.*

Proof. Since $\mu^{<\kappa} = \mu$, we can list the elements of $\mathcal{P}_\kappa \mu$ as $\{c_\xi : \xi < \mu\}$. Then $T = \{z \in \mathcal{P}_\kappa \mu : \forall b \in [z]^\omega \exists \xi \in z(b \subset c_\xi)\}$ is stationary:

Fix $f : [\mu]^{<\omega} \to \mathcal{P}_\kappa \mu$. Build an increasing sequence $\{z_\alpha : \alpha < \omega_1\} \subset C(f)$ so that $z_\alpha = c_\xi$ for some $\xi \in z_{\alpha+1}$. Then $\bigcup_{\alpha < \omega_1} z_\alpha \in T \cap C(f)$, as desired. □

4. Proof of Theorem 1

This section presents the Foreman–Magidor example of a nonreflecting stationary set in $\mathcal{P}_\kappa \lambda$ as we understand it. Although the construction seems to work only for $\lambda \geqslant 2^{\kappa^+}$, the example has the feature that the intersection with $\{x \in \mathcal{P}_\kappa \lambda : \mathrm{cf}(x \cap \kappa) = \omega\}$ is stationary [5]. This is in contrast with our example of Theorem 2, which is a subset of $\{x \in \mathcal{P}_\kappa \lambda : \mathrm{cf}(x \cap \kappa) > \omega = \mathrm{cf}\sup(x \cap \kappa^+)\}$.

To prove the subclaims below, we invoke ideas from [9,1]. These ideas were crucial in showing that Chang's conjecture holds after a measurable cardinal is Lévy-collapsed to ω_2, and that $\mathcal{P}_\kappa \kappa^+$ has a club subset of size $\leqslant (\kappa^+)^{\omega_1}$, respectively.

Proof of Theorem 1. Since $\lambda \geqslant 2^{\kappa^+}$, we can list (possibly with repetition) the functions $: \kappa^+ \to \mathcal{P}_\kappa \kappa$ as $\{g_\xi : \xi < \lambda\}$. For $\kappa \leqslant \gamma < \kappa^+$ fix a bijection $\pi_\gamma : \kappa \to \gamma$. Define $h : \kappa \times (\kappa^+ - \kappa) \to \mathcal{P}_\kappa \kappa^+$ by $h(\alpha, \beta) = \lim \pi_\beta``\alpha$. Then $D = \{x \in \mathcal{P}_\kappa \lambda : \forall \xi \in x(g_\xi``(x \cap \kappa^+) \subset \mathcal{P}(x)) \wedge \forall \gamma \in x \cap (\kappa^+ - \kappa)(\pi_\gamma``(x \cap \kappa) = x \cap \gamma) \wedge h``((x \cap \kappa) \times (x \cap (\kappa^+ - \kappa))) \subset \mathcal{P}(x)\}$ is club.

Set $S = \{x \in \mathcal{P}_\kappa \lambda : \{\sup(y \cap \kappa^+) : x \subset y \in D \wedge y \cap \kappa = x \cap \kappa\}$ is nonstationary in $\kappa^+\}$.

Claim. *S is stationary in $\mathcal{P}_\kappa \lambda$.*

Proof. Suppose otherwise. By induction on $n < \omega$ we define $f_n : [\lambda]^{<\omega} \to \mathcal{P}_\kappa \lambda$ and $\xi_n : [\lambda]^{<\omega} \to \lambda$ as follows:

Since S is nonstationary, we have f_0 with $C(f_0) \subset D - S$. Assume next we have f_n. Define ξ_n and f_{n+1} by $g_{\xi_n(a)}(\gamma) = \mathrm{cl}_{f_n}(a \cup \{\gamma\}) \cap \kappa$ and $f_{n+1}(a) = f_n(a) \cup \{\xi_n(a)\}$. Finally define $f : [\lambda]^{<\omega} \to \mathcal{P}_\kappa \lambda$ by $f(a) = \bigcup_{n<\omega} f_n(a)$.

Subclaim. *Let* $x \in C(f)$. *Then* $x \in D$ *and* $\{\sup(z \cap \kappa^+) : x \subset z \in C(f) \wedge z \cap \kappa = x \cap \kappa\}$ *is unbounded in* κ^+.

Proof. To see the first claim, note that $C(f) \subset C(f_0) \subset D$.

To see the second claim, fix $\alpha < \kappa^+$. Since $x \in C(f) \subset C(f_0) \subset \mathcal{P}_\kappa\lambda - S$, $\{\sup(y \cap \kappa^+) : x \subset y \in D \wedge y \cap \kappa = x \cap \kappa\}$ is stationary in κ^+. Hence we have $x \subset y \in D$ such that $y \cap \kappa = x \cap \kappa$ and $\alpha < \sup(y \cap \kappa^+)$. Fix $\alpha < \gamma \in y \cap \kappa^+$. Then $z = \bigcup\{\mathrm{cl}_{f_n}(a \cup \{\gamma\}) : n < \omega \wedge a \in [x]^{<\omega}\}$ witnesses the subclaim:

Since $\gamma \in z$, $\alpha < \gamma \leqslant \sup(z \cap \kappa^+)$. By definition of f, it is easy to check $x \subset z \in C(f)$. To see $z \cap \kappa \subset x \cap \kappa$, fix $\beta \in z \cap \kappa$. Then $\beta \in \mathrm{cl}_{f_n}(a \cup \{\gamma\}) \cap \kappa = g_{\xi_n(a)}(\gamma)$ for some $n < \omega$ and $a \in [x]^{<\omega}$. Since $x \in C(f)$ and $a \in [x]^{<\omega}$, $\xi_n(a) \in f(a) \subset x \subset y$. Since $\xi_n(a), \gamma \in y \in D$, $\beta \in g_{\xi_n(a)}(\gamma) \subset y \cap \kappa = x \cap \kappa$, as desired. □

For $i = 0, 1$ we build an increasing sequence $\{x^i_\xi : \xi < \omega_1\} \subset C(f)$ so that $x^i_\xi \cap \kappa = x^0_0 \cap \kappa \in S^{\omega_1}_\kappa$, $\kappa < \sup(x^0_\xi \cap \kappa^+) \leqslant \sup(x^1_\xi \cap \kappa^+) < \sup(x^0_{\xi+1} \cap \kappa^+)$ but $x^1_0 \cap \kappa^+$ is not an initial segment of $x^0_1 \cap \kappa^+$ as follows:

First we have $x^0_0 \in C(f)$ such that $x^0_0 \cap \kappa \in S^{\omega_1}_\kappa$ and $\kappa < \sup(x^0_0 \cap \kappa^+)$. The subclaim allows us to take x^0_1 from $X = \{z \in C(f) : x^0_0 \subset z \wedge z \cap \kappa = x^0_0 \cap \kappa\}$ so that $\sup(x^0_1 \cap \kappa^+)$ is the κth element of $\{\sup(z \cap \kappa^+) : z \in X\}$. Since $x^0_1 \cap \kappa^+$ has $<\kappa$ initial segments, we have $x^1_0 \in X$ such that $\sup(x^1_0 \cap \kappa^+) < \sup(x^0_1 \cap \kappa^+)$ but $x^1_0 \cap \kappa^+$ is not an initial segment of $x^0_1 \cap \kappa^+$, as required above. The rest of the construction using the subclaim is routine.

Set $x^i = \bigcup_{\xi < \omega_1} x^i_\xi$. Since $\{x^i_\xi : \xi < \omega_1\} \subset C(f)$ is increasing and $\kappa \geqslant \omega_2$, $x^i \in C(f)$. By the subclaim $x^i_\xi, x^i \in D$. Hence $x^i_\xi \cap \kappa^+$ is an initial segment of $x^i \cap \kappa^+$: $x^i_\xi \cap \gamma = \pi_\gamma``(x^i_\xi \cap \kappa) = \pi_\gamma``(x^0_0 \cap \kappa) = \pi_\gamma``(x^i \cap \kappa) = x^i \cap \gamma$ for every $\gamma \in x^i_\xi \cap (\kappa^+ - \kappa)$. By construction of x^i_ξ's, $\sup(x^0 \cap \kappa^+) = \sup_{\xi < \omega_1} \sup(x^0_\xi \cap \kappa^+) = \sup_{\xi < \omega_1} \sup(x^1_\xi \cap \kappa^+) = \sup(x^1 \cap \kappa^+) \in S^{\omega_1}_{\kappa^+}$.

Subclaim. $x^i \cap \kappa^+$ *is* σ*-closed.*

Proof. Fix $\gamma \in \lim(x^i \cap \kappa^+)$ of cofinality ω. Then we have $b \subset x^i \cap \kappa^+$ of order type ω with $\sup b = \gamma$. Since $\kappa < \sup(x^i \cap \kappa^+) \in S^{\omega_1}_{\kappa^+}$, we have $b \subset \beta \in x^i \cap (\kappa^+ - \kappa)$. Since $\beta \in x^i \in D$, $\pi^{-1}_\beta``(x^i \cap \beta) = x^i \cap \kappa = x^0_0 \cap \kappa \in S^{\omega_1}_\kappa$. Since $\pi^{-1}_\beta``b \in [\pi^{-1}_\beta``(x^i \cap \beta)]^\omega$, we have $\pi^{-1}_\beta``b \subset \alpha \in x^i \cap \kappa$. Hence $b \subset \pi_\beta``\alpha$. Since $\alpha, \beta \in x^i \in D$, $\gamma = \sup b \in \lim \pi_\beta``\alpha = h(\alpha, \beta) \subset x^i$, as desired. □

Thus we have $\gamma \in x^0 \cap x^1 \cap \kappa^+$ with $\sup(x^1_0 \cap \kappa^+) < \sup(x^0_1 \cap \kappa^+) < \gamma$. Since $\gamma \in x^i \in D$, $x^0 \cap \gamma = \pi_\gamma``(x^0 \cap \kappa) = \pi_\gamma``(x^0_0 \cap \kappa) = \pi_\gamma``(x^1 \cap \kappa) = x^1 \cap \gamma$. This contradicts that $x^i_\xi \cap \kappa^+$ is an initial segment of $x^i \cap \kappa^+$ but $x^1_0 \cap \kappa^+$ is not an initial segment of $x^0_1 \cap \kappa^+$, as desired. □

Claim. *S is nonreflecting.*

Proof. Suppose to the contrary $S \cap \mathcal{P}_\kappa A$ is stationary in $\mathcal{P}_\kappa A$ for some $\kappa \subset A \subset \lambda$ of size κ. Fix a bijection $\pi : \kappa \to A$. Then $\{\gamma < \kappa : \pi``\gamma \in S\}$ is stationary. Since $\{\gamma < \kappa : (\pi``\gamma) \cap \kappa = \gamma\}$ is club, their intersection T is stationary in κ. Since $\{y \in D : \pi``(y \cap \kappa) \subset y\}$ is club in $\mathcal{P}_\kappa\lambda$, $\{y \in D : \pi``(y \cap \kappa) \subset y \wedge y \cap \kappa \in T\}$ is stationary in $\mathcal{P}_\kappa\lambda$. Hence $\{\sup(y \cap \kappa^+) : \pi``(y \cap \kappa) \subset y \in D \wedge y \cap \kappa \in T\}$ is stationary in κ^+.

Since $|T| = \kappa$, we have $\gamma \in T$ such that $\{\sup(y \cap \kappa^+) : \pi``(y \cap \kappa) \subset y \in D \wedge y \cap \kappa = \gamma\}$ is stationary in $\kappa^+$. Note that $(\pi``\gamma) \cap \kappa = \gamma$ by $\gamma \in T$. Hence $\{\sup(y \cap \kappa^+) : \pi``\gamma \subset y \in D \wedge y \cap \kappa = (\pi``\gamma) \cap \kappa\}$ is stationary in κ^+. But $\pi``\gamma \in S$ by $\gamma \in T$. Contradiction. □

Therefore Stationary Reflection in $\mathcal{P}_\kappa\lambda$ fails. □

Finally, we remark that the same proof goes through even if "nonstationary" is replaced by "bounded" in the definition of S.

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